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Classical Mechanics of Relativistic Particle with Colour

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Classical description of relativistic pointlike particle with intrinsic degrees of freedom such as isospin or colour is proposed. It is based on the Lagrangian of general form defined on the tangent bundle over a principal fibre bundle. It is shown that the dynamics splits into the external dynamics which describes the interaction of particle with gauge field in terms of Wong equations, and the internal dynamics which results in a spatial motion of particle via integrals of motion only. A relevant Hamiltonian description is built too.

1 Introduction

Wong equations of motion of classical pointlike relativistic particle with isospin or colour [1] permit various Lagrangian and Hamiltonian formulations. Two of them [2, 3] known to the author are based on geometric notions brought from gauge field theories. Namely, the configuration space of particle is a principal fibre bundle with the structure group being the gauge group; the interaction of particle with an external gauge field is introduced via the connection on this bundle. The only difference between two approaches consists in the choice of Lagrangian function. That proposed in [2] is linear in gauge potentials (as in classical electrodynamics) while the nonlinear Lagrangian [3, 2] arises naturally within the Kaluza–Klein theory. Nevertheless, both of them lead to the same Wong equations.

In the present paper, starting from the mentioned above geometrical treatment of the kinematics of relativistic particle with isospin or colour (Section 2), we construct in Section 3 the Lagrangian of a fairy general form. In particular cases this Lagrangian reduces to those of Refs. [2, 3]. The generalization is not trivial since it leads to two types of dynamics, according to what variables are used for the description of intrinsic degrees of freedom. The external dynamics is described by the Wong equations which include gauge potentials, but the form of which is indifferent to the choice of Lagrangian. On the contrary, the internal dynamics is governed by the particular choice of Lagrangian function while gauge potentials completely fall out of this dynamics. This fact becomes more transparent within the framework of the appropriate Hamiltonian formalism developed in Section 4. In Section 5 we sum up our results and discuss the perspectives of quantization.

2 Kinematics on a principal fibre bundle

Following Refs. [2, 3] we take for a configuration space of relativistic particle with isospin or colour the principal fibre bundle P over Minkowski space $\mathbb{M}$ with structure group G and projection $\pi : P \to \mathbb{M}$ (see Ref. [4] for these notions). A particle trajectory $\gamma : \mathbb{R} \to P$; $\tau \mapsto p(\tau) \in P$ is parameterized by evolution parameter τ. A state of particle is determined by $(p, \dot{p}) \in \mathrm{T}P$.

In this paper we are not interested in the global structure of P, and use its local coordinatization: $P \ni p = (x, g) = (x^\mu, g^i)$, $\mu = \overline{0,3}$, $i = \overline{1, \dim G}$, where $x = \pi(p) \in U \subset \mathbb{M}$ (U is an open subset of $\mathbb{M}$), $g = \varphi(p) \in G$, and $\varphi : P \to G$ defines the choice of gauge. Respectively, $\dot p = (\dot x, \dot g)$, where $\dot x \in \mathrm{T}_x\mathbb{M}$ and $\dot g \in \mathrm{T}_g G$. We call $(x, \dot x)$ and $(g, \dot g)$ the space and the intrinsic (local) variables of particle respectively.

Let the principal fibre bundle P be endowed with a connection defined by 1-form $\boldsymbol{\omega}$ on P which takes values in Lie algebra $\mathcal{G}$ of G. Locally it can be represented as follows [4]:

$$\boldsymbol{\omega} = \mathrm{Ad}_{g^{-1}}\pi^*(\mathbf{A}_\mu(x)dx^\mu) + g^{-1}dg \equiv g^{-1}\left(\pi^*(\mathbf{A}_\mu(x)dx^\mu)\right)g + g^{-1}dg, \tag{1}$$

where π^* is the pull back mapping onto P, Ad denotes the adjoint representation of G in $\mathcal{G}$, and $\mathcal{G}$-valued functions $\mathbf{A}_\mu(x)$ are gauge potentials. Under a right action of G defined in P by

$$R_h : p \mapsto p' \equiv R_h(x, g) = (x, gh), \qquad h \in G, \tag{2}$$

the connection form transforms via a pull back mapping R^*_h and is equivariant, i.e., $R^*_h\boldsymbol{\omega} = \mathrm{Ad}_{h^{-1}}\boldsymbol{\omega}$, $h \in G$.

A gauge transformation arises in a geometrical treatment as a bundle automorphism defined by

$$\Phi_{h(x)} : p \mapsto p' \equiv \Phi_{h(x)}(x, g) = (x, h(x)g), \qquad h(x) \in G. \tag{3}$$

It induces the transformation of the connection form defined by the inverse of the pull back mapping (actually, a push forward mapping), $\boldsymbol{\omega} \to \boldsymbol{\omega}' = [\Phi_{h^{-1}(x)}]^*\boldsymbol{\omega}$, so that the value of the connection form on each vector field is gauge-invariant by definition. The transformed form $\boldsymbol{\omega}'$ is also expressed by eqs.(1), but with new potentials

$$\mathbf{A}'_\mu(x) = h(x)\mathbf{A}_\mu(x)h^{-1}(x) + h(x)\partial_\mu h^{-1}(x). \tag{4}$$

The Minkowski metrics $\eta \equiv \eta_{\mu\nu}dx^\mu \otimes dx^\nu$, $\|\eta_{\mu\nu}\| = \mathrm{diag}\,(+,-,-,-)$ is defined on base space $\mathbb{M}$. It is invariant under the Poincaré group acting in $\mathbb{M}$. Being pulled back by π^* onto P it becomes also right- and gauge-invariant, but appears degenerate.

Here we suppose that the Lie algebra $\mathcal{G}$ of structure group G is endowed with non-degenerate Ad-invariant metrics $\langle\cdot,\cdot\rangle$. The example is the Killing–Cartan metrics in the case of semi-simple group. In terms of this metrics, the connection form, and the Minkowski metrics one can construct a nondegenerate metrics on the bundle P [3],

$$\Xi = \pi^*\eta - a^2\langle\boldsymbol{\omega}, \boldsymbol{\omega}\rangle \tag{5}$$

(a is a constant), which is right- and gauge-invariant but not Poincaré-invariant (the latter is broken by $\boldsymbol{\omega}$). In the case of bundle over a curved base space the Minkowski metrics on the right-hand side (r.h.s.) of eq.(5) is replaced by the Riemanian one. In this form the metrics Ξ arises in the Kaluza–Klein theory [5] which allows to unify the description of gravitational and Yang–Mills fields.

3 Lagrangian dynamics of particle with isospin or colour

The dynamical description of the relativistic particle with isospin or colour should, at least, satisfy the following conditions:

i) gauge invariance;

ii) invariance under an arbitrary change of evolution parameter;

iii) Poincaré invariance provided gauge potentials vanish.

These requrements can be embodied in the action $I = \int d\tau L(p, \dot{p})$ with the following Lagrangian

$$L = |\dot{x}| F(\mathbf{w}), \tag{6}$$

where $\mathbf{w} \equiv \boldsymbol{\omega}(\dot{p})/|\dot{x}|$, $|\dot{x}| \equiv \sqrt{\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} = \sqrt{\pi^*\eta(\dot{p}, \dot{p})}$, and $F : \mathcal{G} \to \mathbb{R}$ is an arbitrary function. We note that the quantities $|\dot{x}|$, $\boldsymbol{\omega}(\dot{p})$, and thus the variable $\mathbf{w}$ and the Lagrangian (6) are gauge-invariant.

In order to calculate the variation δI of the action I it is convenient to use, instead of intrinsic velocity $\dot{g}$ and variation δg, the following variables: $\mathbf{v} \equiv \dot{g}\, g^{-1}$ and $\boldsymbol{\delta} g \equiv \delta g\, g^{-1}$. They take values in Lie algebra $\mathcal{G}$ of group G. Then the argument $\mathbf{w}$ of F can be presented in the form

$$\mathbf{w} = \mathrm{Ad}_{g^{-1}}(\mathbf{v} + \mathbf{A}_\mu \dot{x}^\mu)/|\dot{x}|. \tag{7}$$

Using the formal technique (see Ref. [6] for rigorous substantiation)

$$\begin{aligned} &\delta g^{-1} = -g^{-1}\delta g\, g^{-1}, \\ &\delta \mathbf{v} = \delta(\dot{g}\, g^{-1}) = \delta\dot{g}\, g^{-1} + \dot{g}\delta g^{-1} = \frac{d}{d\tau}\boldsymbol{\delta} g - [\mathbf{v}, \boldsymbol{\delta} g], \\ &\delta(\mathrm{Ad}_{g^{-1}}\mathbf{V}) = \delta(g^{-1}\mathbf{V}\, g) = \mathrm{Ad}_{g^{-1}}(\delta\mathbf{V} + [\mathbf{V}, \boldsymbol{\delta} g]) \end{aligned} \tag{8}$$

etc., where $[\cdot,\cdot]$ are Lie brackets in $\mathcal{G}$ and $\mathbf{V}$ is an arbitrary $\mathcal{G}$-valued quantity, we obtain the following Euler–Lagrange equations:

$$\dot{p}_\mu = \mathsf{q} \cdot (\partial_\mu \mathbf{A}_\nu)\dot{x}^\nu, \tag{9}$$

$$\dot{\mathsf{q}} \;=\; \mathrm{ad}^*_{\mathbf{A}\cdot\dot{x}}\,\mathsf{q} \tag{10}$$

with

$$p_\mu \equiv \frac{\partial L}{\partial \dot{x}^\mu} = \left(F - \frac{\partial F}{\partial \mathbf{w}} \cdot \mathbf{w}\right)\frac{\dot{x}_\mu}{|\dot{x}|} + \mathsf{q} \cdot \mathbf{A}_\mu, \tag{11}$$

$$\mathsf{q} \equiv \frac{\partial L}{\partial \mathbf{v}} = \mathrm{Ad}^*_{g^{-1}}\frac{\partial F}{\partial \mathbf{w}}, \tag{12}$$

where p_μ are spatial momentum variables, and q is an intrinsic momentum-type variable which takes values in co-algebra $\mathcal{G}^*$. The linear operators ad^* and Ad^* define co-adjoint representations of $\mathcal{G}$ and G respectively, dot “ $\cdot$ ” denotes a contraction.

First of all we show that the function

$$M(\mathbf{w}) \equiv F - \frac{\partial F}{\partial \mathbf{w}} \cdot \mathbf{w} \tag{13}$$

is an integral of motion. For this purpose let us introduce the following $\mathcal{G}^*$-valued variable:

$$\mathsf{s} \equiv \frac{\partial F}{\partial \mathbf{w}} = \mathrm{Ad}^*_g\,\mathsf{q}. \tag{14}$$

In contrast to q, it is gauge-invariant. Taking into account the equations (14) and (10) we obtain after a bit calculation the equation:

$$\dot{\mathsf{s}} \;=\; \mathrm{ad}^*_{|\dot{x}|\mathbf{w}}\,\mathsf{s}. \tag{15}$$

Then $\dot{M} = -\dot{\mathsf{s}} \cdot \mathbf{w} = -\mathsf{s} \cdot [\mathbf{w}, \mathbf{w}]|\dot{x}| \equiv 0$ q.e.d.

Using this fact and (10) in (9) yields the equations of spatial motion

$$M\frac{d}{d\tau}\frac{\dot{x}_\mu}{|\dot{x}|} = \mathsf{q}\cdot\mathbf{F}_{\mu\nu}\dot{x}^\nu, \tag{16}$$

where $\mathbf{F}_{\mu\nu} \equiv \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + [\mathbf{A}_\mu, \mathbf{A}_\nu]$. Equations (16) together with the equation (10) or (15) of intrinsic motion determine the particle dynamics on a principal fibre bundle.

Now we suppose the existence in Lie algebra $\mathcal{G}$ of non-degenerate metrics $\langle\cdot,\cdot\rangle$. It allows to identify $\mathcal{G}^*$ and $\mathcal{G}$. In particular, for co-vector $\mathsf{q}\in\mathcal{G}^*$ we introduce the corresponding vector $\mathbf{q}\in\mathcal{G}$, such that $\mathsf{q} = \langle\mathbf{q},\cdot\rangle$. Then the equations of motion (16) and (10) take the form

$$M\frac{d}{d\tau}\frac{\dot{x}_\mu}{|\dot{x}|} = \langle\mathbf{q},\mathbf{F}_{\mu\nu}\rangle\dot{x}^\nu, \tag{17}$$

$$\dot{\mathbf{q}} \;=\; [\mathbf{q},\mathbf{A}_\mu]\dot{x}^\mu. \tag{18}$$

Besides, if this metrics is Ad-invariant, the quantity $\langle\mathbf{q},\mathbf{q}\rangle$ is an integral of motion.

At this stage we have obtained the well-known Wong equations (17)–(18) which describe the dynamics of a relativistic particle with mass M and isospin or colour $\mathbf{q}$. Despite that we started with the Lagrangian (6) of a fairy general form, this arbitrariness is obscured in the Wong equations. The reason resides in definitions of M and $\mathbf{q}$ which are, in general, complicated functions on $\mathrm{T}P$. This feature is better understood by analyzing equations (16) and (10) which are very similar to the Wong equations but do not involve the metrics in $\mathcal{G}$.

In general, the set of eqs.(16) and (10) is of the second order with respect to configuration variables x and g. In this regard it is quite equivalent to the set of eqs.(16) and (15). On the other hand, the equations (16) and (10) form a self-contained set in terms of variables x and q. They involve explicitly potentials of external gauge field, but their form is indifferent to a choice of Lagrangian.

The equation (15) is the closed first-order equation with respect to $\mathbf{w}$ or, if eq.(14) is invertible, with respect to s (the quantity $|\dot{x}|$ is not essential because of a parametric invariance of dynamics; we can put, for instance, $|\dot{x}| = 1$). In contrast to the set (16), (10), the equation (15) is determined by a choice of the Lagrangian, but, in terms of $\mathbf{w}$ or s, it does not include gauge potentials.

Hence, the dynamics of isospin particle splits into the *external dynamics* described by the equations (16), (10) in terms of variables x and q, and the *internal dynamics* determined by the equation (15) in terms of $\mathbf{w}$ or s. The only coupling of these realizations of dynamics is provided via integrals of motion, namely, the particle mass M and (if Ad-invariant metrics is involved) the isospin module $|\mathbf{q}| \equiv \sqrt{\langle\mathbf{q},\mathbf{q}\rangle} = |\mathbf{s}|$.

In the following examples we show that the general description of isospin particle includes, as particular cases, results known in the literature. Besides, we demonstrate some new features concerning the internal dynamics.

1. Linear Lagrangian. Electrodynamics. The simplest choice of Lagrangian (6) leading to non-trivial intrinsic dynamics corresponds to the following function $F(\mathbf{w})$:

$$F(\mathbf{w}) = m + \mathsf{k}\cdot\mathbf{w}, \tag{19}$$

where $m\in\mathbb{R}$ and $\mathsf{k}\in\mathcal{G}^*$ are constants. Up to notation it coincides with the Lagrangian proposed by Balachandran et al. in Ref. [2]. In this case the isospin $\mathsf{q} = \mathrm{Ad}^*_{g^{-1}}\mathsf{k}$ is purely configuration variable (it does not depend on velocities) and the equation (10) is truly the first-order Euler–Lagrange equation. Besides, M and s are constants, i.e., $M = m$, $\mathsf{s} = \mathsf{k}$, thus the internal dynamics completely degenerates.

The Lagrangian (6), (19) is linear with respect to gauge potentials $\mathbf{A}_\mu$. In the case of one-parametric gauge group $U(1)$ it reduces to that of electrodynamics. Indeed, in this case we have $g = \exp(\mathrm{i}\theta)$, $\mathbf{v} = \mathrm{i}\dot{\theta}$, $\mathbf{A}_\mu = \mathrm{i}A_\mu$. Choosing $\mathsf{k} = -\mathrm{i}e$, where e is the charge of electron, one can present the Lagrangian in the form:

$$L = m|\dot{x}| + eA_\mu \dot{x}^\mu + e\dot{\theta}. \tag{20}$$

The third term on r.h.s. of (20) is a total derivative and thus it can be omitted. Hence, intrinsic variables disappear in this Lagrangian, and the latter takes the standard form.

The similar situation occurs when considering an arbitrary Abelian gauge group.

2. Right-invariant Lagrangian. Kaluza–Klein theory. The following choice of the function $F(\mathbf{w})$:

$$F(\mathbf{w}) = f(|\mathbf{w}|), \qquad |\mathbf{w}| \equiv \sqrt{\langle \mathbf{w}, \mathbf{w} \rangle} = |\mathbf{v} + \mathbf{A}_\mu \dot{x}^\mu|/|\dot{x}|, \tag{21}$$

where $f : \mathbb{R} \to \mathbb{R}$ is an arbitrary function of $|\mathbf{w}|$, corresponds to Lagrangian which is invariant under the right action of G. Following the Noether theorem there exist corresponding integrals of motion. In the present case they form $\mathcal{G}^*$-valued co-vector s defined by eq.(14) or, equivalently, $\mathcal{G}$-valued vector

$$\mathbf{s} = |\mathbf{w}|^{-1} f'(|\mathbf{w}|)\mathbf{w}, \tag{22}$$

where $f'(|\mathbf{w}|) \equiv d\,f/d\,|\mathbf{w}|$. We note that the integrals of motion $\mathbf{s}$ and $M(\mathbf{w})$ are not independent. Indeed, if $f'(|\mathbf{w}|)$ is not constant, the mass M can be presented as a function of $|\mathbf{s}| = |\mathbf{q}|$. Otherwise both these quantities are constants. Thus the mass M and the isospin module $|\mathbf{q}|$ are completely determined by the external dynamics.

The right-invariant Lagrangian of special kind arises naturally in the framework of Kaluza–Klein theory [5]. It has the following form [3, 2]:

$$L = m\sqrt{\Xi(\dot{p}, \dot{p})}, \tag{23}$$

where the metrics Ξ on a principle bundle P is introduced by eq.(5). In our notations this Lagrangian corresponds to the choice:

$$f(|\mathbf{w}|) = m\sqrt{1 - a^2|\mathbf{w}|^2}. \tag{24}$$

This function determines the following relation between M and $|\mathbf{q}|$:

$$M(|\mathbf{q}|) = \sqrt{m^2 + |\mathbf{q}|^2/a^2}. \tag{25}$$

3. Isospin top. In the above two examples the internal dynamics does not affect the external dynamics. Here we consider a contrary example. Let

$$F(\mathbf{w}) = f(\nu), \qquad \nu \equiv \sqrt{\langle \mathbf{w}, T\mathbf{w} \rangle}, \tag{26}$$

where T is a self-adjoint (in the metrics $\langle \cdot, \cdot \rangle$) linear operator. In this case we have

$$\mathbf{s} = \nu^{-1} f'(\nu) T\mathbf{w}, \qquad M = f(\nu) - \nu f'(\nu). \tag{27}$$

If the function $f(\nu)$ is not linear, the quantity ν turns out to be an integral of motion which is independent of $|\mathbf{q}|$. Then using the parameterization $|\dot{x}| = 1$ one can reduce eq.(15) to the following equation of internal motion:

$$T\dot{\mathbf{w}} = [T\mathbf{w}, \mathbf{w}]. \tag{28}$$

This is nothing but the compact form of Euler equations (i.e., the equations of motion of a free top) generalized to the case of arbitrary group [7]. A solution of this equation is necessary for evaluation of the observable mass of particle.

The relation between the external dynamics and the internal one becomes more transparent within the Hamiltonian formalism which we consider in the next section.

4 Transition to Hamiltonian description

The Lagrangian description on the configuration space P enables a natural transition to the Hamiltonian formalism with constraints [8] on the cotangent bundle T^*P over P. Locally, $\mathrm{T}^*P \simeq \mathrm{T}^*U \times \mathrm{T}^*G$ and, in turn, $\mathrm{T}^*G \simeq G \times \mathcal{G}^*$. The latter isomorphism is established by right or left action of group G on T^*G (see, for instance, [9]). It is implicitly meant in our notation. Namely, we coordinatize T^*G by variables (g, q) or (g, s).

Let us introduce basis vectors $\mathbf{e}_i \in \mathcal{G}$ satisfying the Lie-bracket relations $[\mathbf{e}_i, \mathbf{e}_j] = c_{ij}^k \mathbf{e}_k$, where c_{ij}^k are the structure constant of G, and basis co-vectors $\mathsf{e}^i \in \mathcal{G}^*$ such that $\mathsf{e}^j \cdot \mathbf{e}_i = \delta_i^j$. Then the standard symplectic structure on the cotangent bundle T^*P over the manifold P can be expressed in terms of local coordinates x^μ, p_ν, g^i and $q_j \equiv \mathsf{q} \cdot \mathbf{e}_j$ by the following Poisson-bracket relations:

$$\{x^\mu, p_\nu\} = \delta_\nu^\mu, \qquad \{q_i, q_j\} = c_{ij}^k q_k, \qquad \{g^i, q_j\} = \zeta_j^i(g) \tag{29}$$

(other brackets are equal to zero), where $\zeta_i^j(g)$ are components of right-invariant vector fields on G. Equivalently, we can use variables $s_j \equiv \mathsf{s} \cdot \mathbf{e}_j$ instead of q_j. Then the Poisson-bracket relations take the form:

$$\{x^\mu, p_\nu\} = \delta_\nu^\mu, \qquad \{s_i, s_j\} = -c_{ij}^k s_k, \qquad \{g^i, s_j\} = \xi_j^i(g), \tag{30}$$

where $\xi_i^j(g)$ are the components of left-invariant vector fields on G.

Once the Poisson brackets are defined, we can use in calculations both sets of variables. In particular, taking into account the relation $\mathsf{s} = \mathrm{Ad}^*_g \mathsf{q}$ we obtain:

$$\{q_i, s_j\} = 0. \tag{31}$$

The transition from the Lagrangian description to the Hamiltonian one lies through the Legendre transformation defined by eqs.(11) and (12) or (14). It is degenerate and leads to vanishing canonical Hamiltonian, due to parametrical invariance. Instead, the dynamics is determined by constraints.

In order to obtain constraints explicitly let us consider the relations (13) and (14). They present, in fact, the Legendre mapping $\mathbf{w} \mapsto \mathsf{s}$ and thus allow to consider the mass M as a function of s only. Then eq.(11) reduces to

$$\Pi_\mu \equiv p_\mu - \mathsf{q} \cdot \mathbf{A}_\mu = M(\mathsf{s})\dot{x}_\mu/|\dot{x}| \tag{32}$$

which yields immediately the *mass-shell constraint*:

$$\phi \equiv \Pi^2 - M^2(\mathsf{s}) = 0. \tag{33}$$

There are no more constraints if

$$\det \left\| \frac{\partial^2 F(\mathbf{w})}{\partial w^i \partial w^j} \right\| \neq 0. \tag{34}$$

Otherwise, eq.(14) leads to additional constraints of the following general structure:

$$\chi_r(\mathsf{s}) = 0, \qquad r = \overline{1, \kappa} \leq \dim G \tag{35}$$

which together with the mass-shell constraint (33) form the set of primary constraints. Hence, the Dirac Hamiltonian is $H_D = \lambda_0 \phi + \lambda^r \chi_r$, where λ_0, λ^r are Lagrange multipliers. It is evident from eqs.(33), (35) and (29), (30), (31) that secondary constraints (should they exist) are of the same general structure as in eq.(35). At the final stage of analysis the mass-shell constraint can be considered as the first-class one. This is provided as soon the mass squared $M^2(\mathsf{s})$ is conserved. If it is not, there exists another integral of motion $\tilde{M}^2(\mathsf{s})$ such that $\tilde{M}^2(\mathsf{s})|_{\chi=0} = M^2(\mathsf{s})$. Then the first-class mass-shell constraint has the form (33) but with the function $\tilde{M}^2(\mathsf{s})$ instead of $M^2(\mathsf{s})$.

The total set of constraints is gauge-invariant. This follows from the transformation properties of gauge potentials (4) and variables:

$$x^{\mu\prime} = x^\mu, \qquad {p_\mu}' = p_\mu - \mathsf{q} \cdot \left(h^{-1}(x)\partial_\mu h(x)\right), \qquad \mathsf{q}' = \mathrm{Ad}^*_{h^{-1}(x)}\mathsf{q}, \qquad \mathsf{s}' = \mathsf{s}. \tag{36}$$

In particular, the variables Π_μ defined in eq.(32) are gauge-invariant.

At this stage the splitting of particle dynamics into the external and internal ones becomes obvious. Indeed, the Hamiltonian equations

$$(\dot{x}, \dot{p}, \dot{\mathsf{q}}) = \{(x, p, \mathsf{q}), H_D\} \approx \lambda_0\{(x, p, \mathsf{q}), \Pi^2\}, \tag{37}$$

where $\approx$ is Dirac's symbol of weak equality, are closed with respect to variables (x, p, q). They describe the external dynamics and can be reduced to the equations (16) and (10) by eliminating the variables p_μ and the multiplier λ_0. The equation

$$\dot{\mathsf{s}} = \{\mathsf{s}, H_D\} \approx -\lambda_0\{\mathsf{s}, M^2(\mathsf{s})\} + \lambda^r\{\mathsf{s}, \chi_r(\mathsf{s})\} \tag{38}$$

is closed in terms of s and can be reduced to the equation (15) of the internal dynamics. We note that the group variable g falls out of the equations (37) and (38) which is due to the structure of Poisson-bracket relations (29), (30), (31) and constraints (33), (35). Thus in the present formulation of isospin particle dynamics this variable can be considered as redundant unobservable quantity.

The further treatment of Hamiltonian dynamics, i.e., the classification of constrains as first- and second-class ones etc., demands a consideration of some specific examples.

5 Conclusions

In this paper we consider the formulation of classical dynamics of the relativistic particle in an external Yang–Mills field. We have deduced the Wong equations from the Lagrangian of rather general form defined on the tangent bundle over principle fibre bundle. Besides, we have shown that this Lagrangian leads to some internal particle dynamics. The only quantities coupling this dynamics with the Wong equations are the mass M and isospin (or colour) module $|\mathbf{q}|$, intrinsic characteristics of particle. In the present description they are integrals of internal motion.

The physical treatment of internal dynamics should become better understood within an appropriate quantum-mechanical description. It can be constructed on the base of Hamiltonian particle dynamics proposed in Section 4. Here we only suggest some features of such a description.

Following the procedure of canonical quantization one replaces dynamical variables $x, p, \mathsf{q}, \mathsf{s}$ etc. by operators $\hat{x}, \hat{p}, \hat{\mathsf{q}}, \hat{\mathsf{s}}$, and Poisson brackets by commutators. Let us suppose that the classical dynamics is determined by the only mass-shell constraint (33). Its quantum analogue determines physical states of the system. Eigenvalues q and M of operators $|\hat{\mathbf{q}}|^2 = |\hat{\mathsf{s}}|^2$ and $M^2(\hat{\mathsf{s}})$ which commute with mass-shell constraint and with one another can be treated as the isospin (or colour) and the rest mass of particle. In the case of right-invariant dynamics (as in Kaluza–Klein theory) M is unambiguous function of q. In the general case, the spectrum of $M^2(\hat{\mathsf{s}})$ can consist of few levels M_{qn} which correspond to the same value of q. Thus it is tempting to relate the quantum number n with a flavour or generation. Of course, this supposition is by no means substantiated and needs a following elaboration. It may suggest a phenomenological quantum description of relativistic particles with intrinsic degrees of freedom type of isospin, colour, flavour etc.

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