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On the Global Conjugacy of Smooth Flows

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We aim to give useful integral criteria for smooth flows to be globally conjugate when their infinitesimal generators are close to each other. We study and use the Möller wave operator which is familiar in quantum mechanics. Theorems on smooth global straightening out of nowhere zero vector fields and on smooth global linearization are given. The stability of trajectories at perturbations and properties of the adjoint operators are studied.

1 Preliminaries

The problem of conjugacy of differentiable flows is equally the problem of equivalence of complete vector fields or, as well, the problem of equivalence of differentiable dynamical systems. A part of the latter is the question of normal forms for a system of autonomous differential equations at a singular point.

The so far results, started by Poincaré and Siegel in the real analytic case, to those by Sternberg in the C^∞ setting, have been concentrated on the linearization of dynamical systems. All they have local character.

We wish to present a global treatment of the problem. If one also insist to work in the C^∞ setting, the Fréchet space calculus and relevant techniques seem to be adequate. In the noncompact case we consider manifolds which are countable at infinity and are endowed with a Riemannian metric. In R^n, having fixed a globally Lipschitz vector field X, perturbations $X + Z$ are performed only by vector fields Z which are globally bounded together with all derivatives. Such vector fields are globally Lipschitz and consequently they are complete.

Let D be the induced Riemannian covariant derivative operator. $\|D^k X\|$ will mean the operator norm of the k-th covariant derivative of X as a multilinear map on $(TM)^k$ valued in the tangent bundle TM. When $M = R^n$ it denotes the usual operator norm of the k-th derivative of vector function X.

For a differentiable map f on M, Tf will mean the induced linear tangent map defined on TM. A diffeomorphism f of M induces the adjoint linear operator

$$f_* X = (Tf.X) \circ f^{-1}$$

on the Lie algebra $\mathcal{X}$ of all C^∞ vector fields on M.

Since the smoothness is a local property, and since the boundedness with respect to the time t and the convergence of the considered improper time-integrals will be required to be uniform in x only on compact subsets of M, we may perform calculations in R^n. Eventually, we may glue the limits over different local charts. Therefore we assume $M = R^n$ throughout the paper, except the following basic definition

Definition 1. *Let X be a smooth, globally Lipschitz vector field on a manifold M. Let E be a subspace of $\mathcal{X}(M)$ equiped with a nondecreasing countable system of supremum seminorms $\|\cdot\|_k$, $k \geq 0$, related to the Riemannian norm $\|\cdot\|$.*

We shall say that the adjoint flow $(\phi_t)_*$ decays *on* E *(to infinite order) in a set* $\Omega \subset M$*, if for every integer* $k \geq 0$ *there is* $l_k \geq k$ *and a continuous function* $\nu_k(t,x) > 0$ *defined on* $(0,\infty) \times \Omega$ *such that for any vector field* $Z \in E$ *it holds*

$$\|D^k \phi Z(x)\| \leq \nu_k(t,x)\|Z\|_{l_k} \tag{1}$$

and

$$\int_0^\infty \nu_k(t,x)dt \tag{2}$$

converges uniformly with respect to x *on compact subsets of* Ω*.*

Alternatively, $(\phi_t)_*$ *in (1) can be replaced by* $\phi_t^* := (\phi_{-t})_*$*, depending on the asymptotic behavior of the flow* ϕ_t*.*

In particular ϕ *decays exponentially if* $\nu_k(t,x) \leq e^{-c_k t} M_k(\|x\|)$ *for some* $c_k > 0$ *and a positive continuous function* M_k*.*

Example 1. Let $\Omega = R^n$ and $E = \{Z \in \mathcal{X}^\infty(R^n);\ Z \in \Im^n\}$, where $\Im$ is the Schwartz space of all functions on R^n which are fast falling together with all derivatives. The norms in $\Im^n$ are

$$\|Z\|_r = \max_{i+k \leq r} \sup_{x \in R^n} (1+\|x\|^2)^{i/2}\|D^k Z(x)\|.$$

where $\|\cdot\|$ is the Euclidean norm. Take $X = v(\mathrm{const})$, $\|v\| = 1$. Then $\phi_t = \exp tv = id + tv$ and $\phi Z(x) = Z(x - tv)$. We have

$$\|D^k(\phi Z)(x)\| \leq \frac{1}{1+\|x-tv\|^2}\|Z\|_{k+2}. \tag{3}$$

Hence $l_k = k+2$ and

$$\nu_k(t,x) = \frac{1}{1+a(x)+(t-\langle x,v\rangle)^2} \qquad \text{for} \qquad a(x) = \|x\|^2 - \langle x,v\rangle^2 \geq 0, \tag{4}$$

where $\langle x,v\rangle$ means the scalar product. The convergence of (2) on compact sets is evident. Thus ϕ decays on E but not exponentially.

Example 2. Let $\Omega = R^n$ and let $C_b^\infty = C_b^\infty(R^n,R^n)$ be the space of vector fields in R^n with globally bounded derivatives

$$E = \{Z \in C_b^\infty(R^n,R^n), \qquad \|Z(x)\| = o(\|x\|) \text{ as } x \to 0\}.$$

We equip E with the standard operator norms in the spaces of bounded symmetric multilinear mappings.

Take $X(x) = -cx$ with $c > 0$. Then $\phi_t(x) = e^{-ct}x$, $\phi_t^* Z(x) = e^{ct}Z(e^{-ct}x)$ and

$$\|D^k \phi_t^* Z(x)\| = e^{(1-k)ct}\|(D^k Z)(e^{-ct}x)\| \leq e^{(1-k)ct}\|D^k Z\| \leq e^{(1-k)}\|Z\|_k$$

for $t > 0$, $x \in M$ and $k \geq 2$.

For $k = 0,1$ we have $\|\phi_t^* Z(x)\| \leq e^{-ct}\|x\|^2\|Z\|_2$ and $D\phi Z(x)\| \leq e^{-ct}\|x\|\|Z\|_2$. The integrals (2) are convergent for all $k \geq 0$ uniformly for x in any ball $\{\|x\| \leq r\}$.

2 Estimates of perturbations

The purpose of this section is to study the question of whether the property of decaying is preserved under small perturbations of X. We assume $M = R^n$. Let $Z \in E$ and let ϕ decay on Z in R^n. We consider the perturbed vector field $X + Z$. As both the X, Z are globally Lipschitz, the flow $\psi_t = \exp t(X + Z)$ is defined for all $t \in R$.

Lemma 1 (On orbital stability at perturbation). *Suppose that $\|Z\| < \infty$ and for some $L \geq 0$, $r > 0$ and all $x, y \in R^n$ such that $\|x - y\| \geq r$ we have*

$$\langle x - y, X(x) - X(y)\rangle \leq -L\|x - y\|^2 \tag{5}$$

If $L > 0$ then

$$\|\phi_t(x) - \psi_t(x)\| \leq r_1 = \max\{L^{-1}\|Z\|,\ r\} \tag{6}$$

for all $t \geq 0$ and $x \in R^n$.

If $X = v \neq 0$ is a constant vector field, and Z is fast falling to order 2 at infinity with small $\|Z\|$, then

$$\|\phi_t(x) - \psi_t(x)\| \leq C \qquad (t \geq 0) \tag{7}$$

on compact sets (where C is constant).

For $X = v$ and $Z \in C^1$ with $\|Z\| < \infty$ one gets the (sharp) estimate

$$\|\phi_t(x) - \psi_t(x)\| \leq \|Z\|t. \tag{8}$$

Proof. For the solutions of the Cauchy problems

$$x' = X(x), \qquad y' = X(y) + Z(y), \qquad x(0) = y(0)$$

the standard computation yields

$$\frac{1}{2}\left(\|x - y\|^2\right)' = \langle x - y, x' - y'\rangle = \langle x - y, X(x) - X(y)\rangle - \langle x - y, Z(y)\rangle,$$

hence

$$\left(\|x(t) - y(t)\|^2\right)' \leq 2(\|Z\| - L\|x(t) - y(t)\|)\|x(t) - y(t)\|.$$

For $L > 0$ this is possible only if $\|x(t) - y(t)\| \leq r_1$, which translates into (6).

Now assume $X(x) = v \neq 0$. Then it is easy to see that $\|x + tv - \psi_t(x)\| \leq \|Z\|t$. Thus for small $\|Z\|$ we have $\|\psi_t(x)\| \geq |\|x\| - bt|$ for some $b > 0$. But on the other hand we have for $X = v$

$$\psi_t(x) - (x + tv) = \int_o^t Z(\psi_s(x))ds \tag{9}$$

and (cf. Example 1)

$$\int_o^\infty \|Z(\psi_s(x))\|ds \leq \|Z\|_2 \int_o^\infty \frac{1}{1 + (\|x\| - bt)^2}dt \leq C < \infty$$

on compact sets. Therefore each trajectory $t \to \psi_t(x)$ has globally a finite distance from that of $\phi_t(x)$ uniformly in x on compact sets. In the last case the estimate (8) follows directly from (9). In particular, for $Z = w$ constant, (8) is equality.

Thus we see that the falling at infinity of Z is essential for the global proximity of the perturbation flow.

Remark. The assumption that both X and Z in Lemma 1 are globally Lipschitz can be relaxed. As we see from the proof, in all the cases considered in the Lemma, if X is complete then $\psi_t(x)$ can not be unbounded in finite time. Thus it can be defined over R^+ (i.e., the vector field $X+Z$ is positively semicomplete).

As a byproduct we obtain the following

Proposition 1. *If a vector field X in R^n satisfies condition (5) with $L > 0$, $X(x_o) \neq 0$, then it is positively semicomplete and for any fixed x_o and x in R^n and all $t \geq 0$*

$$\|\phi_t(x) - x_o\| \leq \max\{L^{-1}\|X(x_o)\|, r\}.$$

Moreover, if $X(x_o) = 0$ and $r = 0$ then

$$\|\phi_t(x) - x_o\| \leq \|x - x_o\|e^{-Lt}.$$

Proof. This time, for solutions of equation $x' = X(x)$, we have

$$\frac{1}{2}\left(\|x - x_o\|^2\right)' = \langle x - x_o, X(x) - X(x_o)\rangle + \langle x - x_o, X(x_o)\rangle.$$

Further arguments are analogous as in the proof of Lemma 1 or routine.

Definition 2. *We shall say that the adjoint flow ϕ C_m-decays on a subspace E if it decays on E and the functions $\nu_k(t,x)$ in (1) do not depend on x for $0 \leq k \leq m$, so that $\|D^k\phi Z(x)\| \leq \nu_k(t)\|Z\|_{l_k}$ for all x, and $\nu_k(t)$ is integrable over R^+.*

We say that ϕ decays C_∞ on E if it decays C_m for all integers $m \geq 0$.

This definition will apply also for the flow ϕ_t^* (generated by $-X$).

Lemma 2. *Suppose that ϕ generated by X decays on E and one of the following conditions is satisfied.*

(A.1) The vector field X fullfils the hypothesis (5) with $L > 0$, or

(A.2) $X = v$ (const), or

(A.3) ϕ C_o-decays on E.

Then also the adjoint flow ∂, where $\psi_t = \exp t(X+Z)$, decays on E provided Z is sufficiently small in the seminorm $\|\cdot\|_{l_1}$. Moreover, if X fulfills (A.1) then so does $X+Z$, and if ϕ C_1-decays on E then ∂ decays C_o on E.

A corresponding result is true for ϕ_t^*.

Proof. Put $f_t = \phi_t \circ \psi_{-t}$. Since $T\phi_t.X = X \circ \phi_t$, we get by differentiating in t

$$f_t' = -(T\phi_t.Z) \circ \psi_{-t} = -(\phi Z) \circ f_t, \qquad f_o = id, \tag{10}$$

we can integrate (10) to obtain

$$f_t = id - \int_o^t (\phi_s) * Z \circ f_s ds. \tag{11}$$

Now we aim to show that on compact subsets $f_t - id$ and all its derivatives $T^n(f_t - id)$ are bounded uniformly in $t \in (0, \infty)$.

First, suppose that X satisfies (A.1) or (A.2). By substituting $\psi_t^{-1}(x)$ in place of (x) in (6) or (7) we get $\|f_t(x) - x\| \leq C$ for all $t \geq 0$ uniformly for x in compact sets.

Similar result can be obtained when the condition (A.3) is satisfied. In fact, consider equation $x' = F(t,x)$, where $F(t,x) = -\phi Z(x)$. Putting $u = \|x\|$ we have

$$u\frac{du}{dt} = x^T F(t,x) \leq \|x\|\|\phi Z(x)\| \leq \nu_0(t)u\|Z\|_{l_0}.$$

Hence

$$u(t) - u(0) \le \left(\int_0^t \nu(s) ds \right) \|Z\|_{l_0},$$

where the integral is bounded as $t \to \infty$. Therefore $u(t) - u(0)$ remains bounded over R^+. For $x(t) = f_t$ it gives that $\|f_t(x)\| - \|x\|$ is bounded for $t \in R^+$. By the principle of the integral continuity of solutions it also remains bounded when x runs over a compact set.

Next we are going to prove the boundedness of the first derivative Tf_t. For this we differentiate (11) and take estimates

$$\|Tf_t(x)\| \le 1 + \int_o^t \|D(\phi_s)_* Z(f_s(x))\| \|Tf_s(x)\| ds \le \int_o^t \nu_1(s, f_s(x)) \|Z\|_{l_1} \|Tf_s(x)\| ds.$$

Using the Bellman's lemma, we get

$$\|Tf_t(x)\| \le \exp\left(\int_o^t \nu_1(s, f_s(x)) \|Z\|_{l_1} ds \right) = e^{B\|Z\|_{l_1}} < \infty,$$

where we put $B = \int_o^\infty \nu_1(s, f_s(x)) ds$. The integral is convergent since $\|f_s(x)\|$ differs from $\|x\|$ by a constant, so they may be placed in the same compact set for all $s \ge 0$.

In particular, if ϕ decays C_1 on E then ν_1 does not depend on x and then $Tf_t(x)$ is bounded globally for all $t \ge 0$ and $x \in M$.

Now, assuming that $T^{n-1} f_t$ is bounded for $n - 1 \ge 1$, we wish to show this for $T^n f_t$. We make use of the standard formulae for higher order derivatives of composition maps. We have

$$D^n(\phi Z \circ f_t)(x)) = (D\phi Z)(f_t(x)).T^n f_t(x) + R_n(t, x), \tag{12}$$

where

$$R_n(t, x) = \sum_{k=2}^n \sum_{j_1 + \cdots + j_k = n} C_{k, j_1, \ldots, j_k} D^k(\phi Z)(f_t(x)) \{T^{j_1} f_t(x), \ldots, T^{j_k} f_t(x)\}$$

with $j_1, \ldots, j_k \ge 1$. Passing to the estimates we have

$$\|D^n(\phi Z(f_t(x))\| \le \|D\phi Z\| \|T^n f_t(x)\| + \|R_n(t, x)\|,$$

where

$$\|R_n(t, x)\| \le C \sum_{k=2}^n \sum_{j_1 + \cdots + j_k = n} \nu_k(t, f_t(x)) \|Z\|_{l_k} \|T^{j_1} f_t(x)\| \cdots \|T^{j_k} f_t(x)\|$$

with $1 \le j_1, \ldots, j_k \le n - 1$. All this and (11) yields for the derivatives of order $n \ge 2$

$$\|T^n f_t\| \le \int_o^t \|R_n(s)\| ds + \|Z\|_{l_1} \int_o^t \nu_1(s, f_s(x)) \|T^n f_t\| ds.$$

Again by Bellman's inequality

$$\|T^n f_t\| \le \left(\int_o^t \|R_n(s, x)\| ds \right) \cdot \exp\left(\|Z\|_{l_1} \int_o^t \nu_1(s, f_s(x)) ds \right) < \infty$$

uniformly for $t \ge 0$. Thus $\|T^n f_t(x)\|_\infty$ is finite for $n \ge 1$.

Now, by the definition of f_t we have $\psi_t = f_t^{-1} \circ \phi_t$. Put $g_t = f_t^{-1}$. Then clearly $\|g_t(x)\| - \|x\|$ is also uniformly bounded in $t \ge 0$. We wish to prove that $g_t - id$ has all x-derivatives unformly bounded in t.

From (11) it follows

$$\|Tf_t - I\| \le \int_o^t \|D(\phi_s)_* Z\| \|Tf_s\| ds. \tag{13}$$

Put $\delta = \|Z\|_{l_1}$. Then, by (12), $\|Tf_t\| \leq e^{B\delta}$ for $t > 0$. Now the estimate (13) can be written

$$\|Tf_t - I\| \leq B\delta e^{B\delta}.$$

For δ sufficiently small, and B being independent of Z, we shall have $\|Tf_t - I\| < \epsilon < 1$. But if $\|Tf_t(x) - I\| < \epsilon$ then

$$\|Tg_t(f_t(x))\| = \|Tf_t(x)^{-1}\| \leq \frac{1}{1-\epsilon} < \infty$$

for all $t > 0$, uniformly in x in any compact set.

Now, since $f_t \circ g_t = id$, the uniform boundedness of higher order derivatives of g_t follows recurently from the relations

$$T^n g_t(x) = -Tg_t(x) \sum_{k=2}^{n} \sum_{j_1+\cdots+j_k=n} C_{k,j_1,\ldots,j_k} T^k f_t(g_t(x)) \left\{T^{j_1}g_t(x), \ldots, T^{j_k}g_t(x)\right\},$$

where $j_1, \ldots, j_k \leq n-1$, and hence the estimate

$$\|T^n g_t\| \leq C \sum_{k=2}^{n} \sum_{j_1+\cdots+j_k=n} \|T^k f_t\| \|T^{j_1} g_t\| \cdots \|T^{j_k} g_t\| < \infty.$$

Having $g_t - id$ uniformly bounded in t to infinite order, we deduce easily that for any smooth vector field Y with bounded derivatives it holds

$$\|D^n (g_t)_* Y(x)\| \leq C \sum_{k \leq n} \|D^k Y\| \qquad (n \geq 0)$$

for some constant $C > 0$ depending on n and independent of x in compact sets.

From $\psi_t = g_t \circ \phi_t$ it follows $\partial Z = (g_t)_*(\phi Z)$. In the above inequality we replace Y by ϕZ, which decays on Z. Consequently, $(\psi_t)_*$ decays on Z.

Finally, suppose that X satifies condition (5) with $L > 0$. Then for $X + Z$ we have whenever $\|x - y\| > r$

$$\begin{aligned}&\langle x - y, (X+Z)(x) - (X+Z)(y)\rangle \\ &\qquad \leq -L\|x-y\|^2 + \|x-y\| \|Z(x) - Z(y)\| \leq (-L+K)\|x-y\|^2,\end{aligned}$$

where K is the global Lipschitz constant of Z. If $\|Z\|_{l_1}$ is small then so is K and $-L + K < 0$, as required.

In the case where ϕ decays C_1 on E, then $Tf_t(x)$ and hence also $Tg_t(x)$ are bounded globally in t and x. Therefore we have

$$\|\partial Z(x)\| = \|(g_t)_*(\phi_t)_* Z(x)\| \leq C\|(\phi_t)_* Z\| \leq C\nu_o(t)\|Z\|_{l_1}$$

for all x. This completes the proof of the Lemma.

3 Conjugacy of flows

Lemma 3. *Let X, Z be C^k complete vector fields on a smooth manifold M. Suppose that the integrals*

$$f = id - \int_0^\infty (T \exp tX).Z \circ \exp(-t(X+Z))dt \tag{14}$$

converges to class C^k $(k \geq 1)$ and

$$g = id - \int_0^\infty (T \exp t(X+Z)).Z \circ \exp(-tX)dt \tag{15}$$

converges to class C^1, both uniformly on compact subsets of M.

Then f and g are C^k diffeomorphisms of M, $g = f^{-1}$, and

$$f_*(X + Z) = X.$$

Note that the assumptions are satisfied if ϕ_t fulfills the hypothesis of Lemma 2 and Z is sufficiently small.

Proof. In the proof we use the Möller wave operator [1] which is known in quantum mechanics. Put as before $\phi_t = \exp tX$ and $\psi_t = \exp t(X + Z)$. The idea is that if the diffeomorphisms $\phi_t \circ \psi_{-t}$ have the limit

$$\lim_{t\to\infty} \phi_t \circ \psi_{-t} = f \qquad \text{(wave operator)} \tag{16}$$

and f is invertible then $f^{-1} \circ \phi_t \circ f = \psi_t$. This is so because from (16) it follows $\phi_t \circ f \circ \psi_{-t} = f$.

We introduce the integral formulae for the wave operator in order to simplify the proof of its existence. For this again define

$$f_t = \phi_t \circ \psi_{-t} \qquad \text{and} \qquad g_t = \psi_t \circ \phi_{-t} = f_t^{-1}. \tag{17}$$

Hence

$$f'_t = -(T\phi_t.Z) \circ \psi_{-t}, \qquad g'_t = -(T\psi_t.Z) \circ \phi_{-t}. \tag{18}$$

The existence of both the limits $f = \lim_{t\to\infty} f_t$ and $g = \lim_{t\to\infty} g_t$ ensures the invertibility of f. Thus, we can integrate (18) in the interval $[0, t]$ and pass to limit as $t \to \infty$. It results in $\psi_t = f^{-1} \circ \phi_t \circ f$, so the flows are conjugate by f. From this, by differentiating in t, we get $(f^{-1})_*X = X + Z$ or equivalently $f_*(X + Z) = X$. It is well known that if f is C^k and has a C^1 inverse, then its inverse is C^k. So, g is also C^k.

Remark. Alternatevaly, by reversing time, we may look for the wave operator of the form $f = \lim_{t\to\infty} \phi_{-t}\psi_t$ which satisfies $\phi_{-t} \circ f \circ \psi_t = f$. It can be calculated from the integral formulae

$$f = id + \int_o^\infty (T\exp(-tX)).Z \circ \exp t(X + Z)dt,$$

$$g = id + \int_o^\infty (T\exp -t(X + Z)).Z \circ \exp tXdt.$$

If the integrals converge then $g = f^{-1}$ and $f_*(X + Z) = X$. This version may be used if the asymptotic behavior of $\exp(-tX)$ is more suitable than that of $\exp(tX)$.

Definition 3. *Let the subspace $E \subset C_b^\infty(R^n, R^n)$ be closed in the standard supremum norms $\|\cdot\|_k$ ($k \geq 0$). Let X be a globally Lipschitz vector field such that the X-flow ϕ_t leaves E invariant, i.e., $(\phi_t)_*E \subset E$ for any $t \in R$. We say that E has a hyperbolic structure for ϕ_t if there is a continuous splitting $E = E_1 + E_2$, such that E_2 is $\exp(X + E_1)$ invariant, $(\phi_t)_*$ fulfills the hypothesis (A.1) or (A.3) of Lemma 2 on E_1 and so does ϕ_t^* on E_2.*

Note that we do not assume any invariance of subspaces E_i individually.

Lemma 4. *Suppose that E has a hyperbolic structure for the X-flow. Let $Z = Z_1 + Z_2$, where Z_i are sufficiently small. Then $X + Z$ is C^∞ conjugate to X.*

Proof. By Lemma 3 for X and Z_1, there is a diffeomorphism f such that $f_*X = X + Z_1$. Since ϕ_t^* fulfills (A.1) or (A.3) on E_2, for small Z_1 also $(\exp t(X + Z_1))^*$ fulfills (A.1) or (A.3) on E_2. Therefore, for small Z_2 there exists a diffeomorphism h such that $h_*(X + Z_1) = (X + Z_1) + Z_2 = X + Z$. This completes the proof.

4 Main results

With the notations of preceding sections the integrals (14) and (15) can be written in the form

$$f = id - \int_0^\infty \phi Z \circ f_t dt \qquad \text{and} \qquad g = id - \int_0^\infty (\psi_t)_* Z \circ g_t dt$$

with $f_t = \phi_t \circ \psi_{-t}$ and $g_t = f_t^{-1}$.

Accordingly, we express the alternative formulae (14) and (15) by putting $-t$ in place of t.

Now we can apply the results of previous sections and Examples 1, 2 to prove the convergence of the integrals and of all their x-derivatives. This will result in the following theorems.

Theorem 1 (On conjugacy of perturbations). *Let $X \in \mathcal{X}(R^n)$ be a globally Lipschitz vector field and let E be a linear space of vector fields on R^n. Suppose that the adjoint flow generated by X decays on E with respect to a collection of seminorms $\{\|\cdot\|_k, k \in N\}$. Assume also that one of the conditions* (A.1) *to* (A.3) *is satisfied.*

Then there is a neighborhood $U = \{Z \in E;\ \|Z\|_{l_1} < \delta\}$ such that for every $Z \in U$ there exists a C^∞ diffeomorphism f of M which conjugate X to $X + Z$, that is $f_* X = X + Z$.

Theorem 2 (Global straightening out theorem). *Let X be a non-zero constant vector field on R^n. There is a $\delta > 0$ such that for every fast falling vector field Z on R^n with $\|Z\|_3 < \delta$ the vector fields X and $X + Z$ are C^∞ conjugate on R^n.*

Thus, any sufficiently small perturbation (as above) of $X = \frac{\partial}{\partial x_1}$ can be transformed globally to $\frac{\partial}{\partial x_1}$ by a C^∞ change of coordinates.

Theorem 3. *Let $X(x) = -cx$, $c > 0$, $x \in R^n$, and let Z be a vector field with globally bounded derivatives and satisfying $\|Z(x)\| = o(\|x\|)$ as $x \to 0$. Then the perturbed vector field $X + Z$ is C^∞ conjugate to X in R^n.*

This theorem can be easily generalized to the case where $X = Ax$ with negative real parts of the eigenvalues of the matrix A and without the familiar resonance relations. Thus the Sternberg [2] local linearization theorem for contractions can be given a global version.

Moreover, applying Lemma 4, we may also obtain the globalization of the Sternberg's linearization theorem for arbitrary hyperbolic point with no resonance. This will be subject to another article.

References

[1] Nelson E., Topics in Dynamics, I. Flows, Princeton University Press, Princeton 1969.

[2] Sternberg S., Local contractions and a theorem of Poincaré, *American Journal of Mathematics*, 1957, V.79, 809–824.