Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 447–455.

Supersymmetry and Supergroups in Stochastic Quantum Physics

Thomas GUHR

Max Planck Institut für Kernphysik, Postfach 103980, 69029 Heidelberg, Germany

Supersymmetry was first applied to high energy physics. In the early eighties, it found a second and very fruitful field of applications in stochastic quantum physics. This relates to Random Matrix Theory, the topic I focus on in this contribution. I review several aspects of more mathematical interest, in particular, supersymmetric extensions of the Itzykson–Zuber and the Berezin–Karpelevich group integral, supersymmetric harmonic analysis and generalized Gelfand–Tzetlin constructions for the supergroup $\mathrm{U}(k_1/k_2)$. The consequences for the representation theory for supergroups are also addressed.

1 Introduction

The spectral fluctuations of very different quantum systems show a remarkably high degree of similarity: if one measures the eigenenergies of the system in units of the local mean level spacing, the energy correlation functions of nuclei, atoms, molecules and disordered systems become almost indistinguishable. In many cases, the distribution $P(x)$ of the spacings x between adjacent levels on the local scale is well described by the Wigner surmise

$$P_{\mathrm{Wigner}}(x) = \frac{\pi}{2}\, x \exp\left(-\frac{\pi}{4}x^2\right). \tag{1}$$

We notice that the probability for finding small spacings is suppressed and vanishes linearly. This means that the levels are *correlated* and repel each other. A spectrum of uncorrelated levels behaves very differently. Such a spectrum can easily be modeled by producing a sequence of *uncorrelated* numbers with a random number generator. One then finds the Poisson law

$$P_{\mathrm{Poisson}}(x) = \exp\left(-x\right) \tag{2}$$

as the distribution of the spacings x. It is remarkable, that the wealth of conceivable correlations due to different types of interactions leads in so many cases to the distribution (1). This surprising universality is reflected in the simplicity of the phenomenological and statistical model, Random Matrix Theory (RMT), that quantitatively describes these correlations, or, their absence. It is easy to model the latter: the Hamiltonian in the energy bases is written as a diagonal matrix,

$$H = \mathrm{diag}\left(E_1, \ldots, E_N\right) \tag{3}$$

whose entries, the eigenvalues, are chosen as uncorrelated random numbers. To model the presence of correlations, one has to add off-diagonal matrix elements,

$$H = \begin{bmatrix} H_{11} & \cdots & H_{1N} \\ \vdots & & \vdots \\ H_{N1} & \cdots & H_{NN} \end{bmatrix}, \tag{4}$$

such that the eigenvalues of H are correlated. It is convenient to chose the matrix elements H_{nm} as Gaussian distributed random numbers.

RMT was founded by Wigner [1] about forty years ago and worked out mathematically by Mehta [2] and Dyson [3] in the following decade. Due to the general symmetry constraints, a time reversal invariant system with conserved or broken rotation invariance is modeled by the Gaussian Orthogonal (GOE) of real-symmetric matrices or the Gaussian Symplectic Ensemble (GSE) of self-dual matrices, respectively, while the Gaussian Unitary Ensemble (GUE) of Hermitean matrices models the fluctuation properties of a system under broken time reversal invariance. These generic fluctuation properties are referred to as Wigner–Dyson fluctuations. In 1984, a conceptually new element was brought into the discussion when Bohigas, Gianonni and Schmit [4] stated the following famous conjecture: *The quantization of a classical, conservative and fully chaotic system is expected to show Wigner–Dyson fluctuations, i.e. $P(x)$ is of the form (1). On the other hand, the quantization of a classical, conservative, integrable and regular system is expected to have significantly different spectral fluctuation properties, i.e. $P(x)$ is often of the form (2).* Although not yet rigorously proven, the Bohigas conjecture is supported by an overwhelming number of classical, semiclassical and quantal studies [5, 6].

It is now believed that RMT can be viewed as *thermodynamics for spectral fluctuations and related properties.* Based on little more than symmetry constraints and the assumption of sheer randomness, RMT grasps the crucial statistical features of a rich variety of systems. A detailed review was recently given in Ref. [7].

Unfortunately, the mathematical difficulties encountered in random matrix models were so serious that many interesting problems could only partially be solved. In 1983, Efetov [8] discovered, in condensed matter physics, a connection of paramount importance between RMT and supersymmetry in condensed matter physics. By supersymmetry we mean theories involving commuting and anticommuting degrees of freedom. We emphasize that these bosons and fermions have no direct physical interpretation as particles or so. They serve to map the stochastic model, without approximation, onto a model in superspace. The great merit of supersymmetry in stochastic quantum physics is a dramatic reduction of the number of integration variables due to this exact mapping. This can be viewed as an *irreducible representation* of the statistical system in question.

Verbaarschot, Weidenmüller and Zirnbauer [9] derived the same supersymmetric non-linear σ model for the statistical discussion of compound nuclear scattering starting from a random matrix model. Since then, the supersymmetric technique has experienced a burst of activities. The treatment of numerous, previously inaccessible, problems became possible.

2 Graded Eigenvalue Method

The mathematical solution of Efetov's supersymmetric non-linear σ models is still non-trivial. For purely spectral fluctuations, an alternative technique, the Graded Eigenvalue Method, was presented in Ref. [12], as a variant of Efetov's original approach.

The main motivation for this method will be given in the following section. The essence of the Graded Eigenvalue Method is the exact calculation of integrals over supergroups. The supersymmetric version of the Harish-Chandra–Itzykson–Zuber integral [13, 14] was evaluated in Ref. [12]. The Hermitean $2k \times 2k$ supermatrix σ has k eigenvalues s_{p1}, $p = 1, \ldots, k$ in the boson boson and k eigenvalues is_{p2}, $p = 1, \ldots, k$ in the fermion fermion sector ordered in the diagonal matrix s, it is diagonalized by a unitary supermatrix u such that $\sigma = u^{-1}su$. Moreover, we introduce a second supermatrix $\rho = v^{-1}rv$ with the same symmetries. The supersymmetric version of the Harish-Chandra–Itzykson–Zuber integral can then be written in the form [12]

$$\begin{aligned}\int \exp\left(i\operatorname{trg}\sigma\rho\right)\, d\mu(u) &= \int \exp\left(i\operatorname{trg} u^{-1}sur\right)\, d\mu(u)\\ &= \frac{\det\left[\exp(is_{p1}r_{q1})\right]_{p,q=1,\ldots,k}\det\left[\exp(is_{p2}r_{q2})\right]_{p,q=1,\ldots,k}}{B_k(s)B_k(r)},\end{aligned} \tag{5}$$

where trg stands for a properly defined trace in superspace. The function $B_k(s) = \det\left[1/(s_{p1} - is_{q2})\right]_{p,q=1,\ldots,k}$ is the square root of the Jacobian for the transformation of the volume element of the Hermitean supermatrix σ to eigenvalue-angle coordinates s and u.

With the Graded Eigenvalue Method, a quick rederivation of all GUE k-level correlation functions could be given [12].

3 Transitions towards Quantum Chaos

What is the merit of the Graded Eigenvalue Method for physics? – It is, for example, capable of exactly solving a fundamental problem of chaos theory, the regularity-chaos transition [15]. Within Efetov's original approach this problem can only asymptotically be studied. Consider the Hydrogen atom in a magnetic field and its classical analogue. The classical system is fully integrable for zero magnetic field, but becomes chaotic as the magnetic field grows because the spherical symmetry is broken. Following the Bohigas–Giannoni–Schmitt conjecture, it is easily conceivable that $P(x)$ undergoes a transition from the Poisson distribution to the Wigner surmise. Indeed, as Fig. 1 shows, this was confirmed in the experiment and in numerical calculations [10].

Similar transitions are encountered in many physical situations. In heavy ion reactions, a spreading of the electrical quadrupole transition strength has been observed which can be understood in terms of a regularity chaos transition [11]. In condensed matter physics, the phenomenon of localization can also be related to this crossover. Billiard systems [5] show similar transitions as well.

Naturally, the statistical model for this transition is the weighted sum of the two limiting Hamiltonians (3) and (4),

$$H(\alpha) = H^{(0)} + \alpha H^{(1)}, \tag{6}$$

where α is the dimensionless transition parameter. The matrices $H^{(1)}$ are drawn from a Gaussian Ensemble. Although the regularity chaos transition is our main interest, we make no assumptions yet for the probability distribution of the matrices $H^{(0)}$. Detailed numerical simulations of this transition can be found in Ref. [11] for two different ensembles of matrices $H^{(0)}$.

The key to the exact solution is the observation that the generating functions of the spectral correlators for the transition ensemble (6) obey an diffusive process. The fictitious time of the diffusion is related to the transition parameter through $\tau = \alpha^2/2$. The diffusion can be formulated in the curved space of the eigenvalues r of a supermatrix which provide the fictitious spatial coordinates. Moreover, this can be done not only for the GUE but for all three Gaussian Ensembles. For the generating functions $z_{\beta k}(r,\tau)$ the diffusive process reads

$$\frac{\beta}{4}\,\Delta_{\beta r}\, z_{\beta k}(r,\tau) = \frac{\partial}{\partial\tau} z_{\beta k}(r,\tau), \tag{7}$$

where $\beta = 1, 2, 4$ for the GOE, GUE and GSE, respectively. The operator $\Delta_{\beta r}$ is the radial part of the Laplacean in the space of supermatrices. It has $2k$ degrees of freedom for the

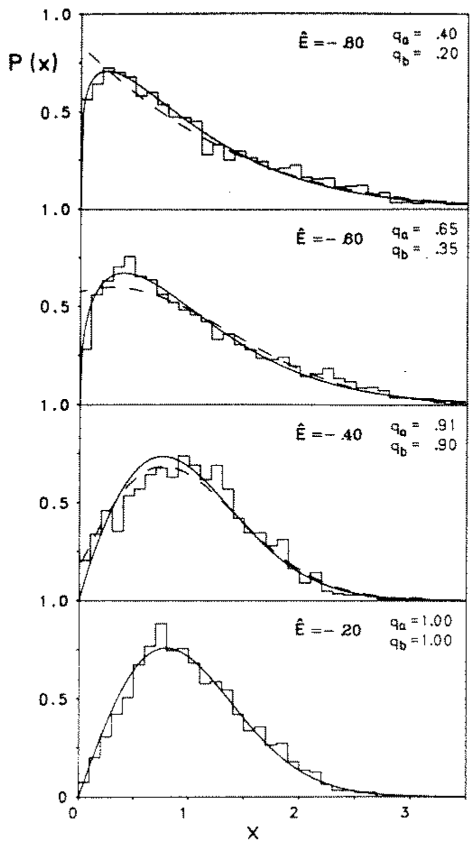

Figure 1. The nearest neighbor spacing distribution for the Hydrogen atom in a magnetic field. Since this system exhibits a certain scaling, the transition from regularity to chaos is governed by one single parameter $\widehat{E}$ which is a combination of energy and magnetic field. Taken from Ref. [10].

GUE and $4k$ for the GOE and the GSE. The initial condition is the generating function of the arbitrary correlations, $\lim_{\tau\to 0} z_{\beta k}(r,\tau) = z_{\beta k}^{(0)}(r)$. The diffusion kernel is, apart from trivial factors, the supersymmetric Harish-Chandra–Itzykson–Zuber integral for the three symmetry classes. It should be emphasized that the explicit knowledge of this kernel is not necessary to derive the diffusion process.

For the GUE, the k-level correlation functions can be expressed as a $2k$-fold integral over the eigenvalues s of a $2k \times 2k$ Hermitean supermatrix by using the integral (5). We arrive at the result

$$X_k(\xi_1,\ldots,\xi_k,\tau) = \frac{(-1)^k}{\pi^k} \int G_k(s-\xi,\tau)\, \Im z_k^{(0)}(s)\, B_k(s) d[s] \tag{8}$$

on the scale of the unfolded energies ξ_p, $p = 1,\ldots,k$. Here, the transition parameter τ is rescaled by the mean level spacing D such that $\tau \to \tau/D^2$. The pure GUE result is recovered in the limit $\tau \to \infty$. The function $G_k(s-\xi,\tau)$ is a normalized Gaussian with variance τ. The integral representation (8) is valid for an arbitrary ensemble of the matrices $H^{(0)}$. The function $\Im z_k^{(0)}(s)$ is the generating function of the corresponding correlations. In the case of two-level correlations, i.e. $k = 2$, two of the four integrals in Eq.(8) can be performed due to translation invariance for an arbitrary ensemble of the matrices $H^{(0)}$.

The regularity-chaos transition is obtained by choosing the matrices $H^{(0)}$ from the ensemble (3). This case is also discussed in detail in Refs. [15, 16].

4 Chiral Random Matrix Theory

While RMT was originally developed for non-relativistic quantum mechanics described by the Schrödinger equation, Shuryak and Verbaarschot [17] showed that this concept also works very well for relativistic systems where the Dirac equation applies. However, due to chiral symmetry, the RMT ansatz has to be modified. This leads to chiral RMT (chRMT).

For massless fermions, the Euclidean Dirac operator has the form

$$i\not{D} = i\not{\partial} + g \sum_a \frac{\eta^a}{2} \not{A}^a , \tag{9}$$

where g is the coupling constant, η^a are the generators of the gauge group and $\not{A}^a$ are the gauge fields. Physically, the gauge fields represent the gluons, i.e. the exchange particles of the strong interaction. The eigenfunctions of the Dirac operator are the *constituent quarks* which eventually form pion, proton, neutron, etc. In the chiral basis, the Dirac operator has an off-diagonal matrix structure, indicated in the relation (10). The main idea of chRMT is the replacement of the operator $i\not{D}$ in relation (10) by a random matrix W, such that

$$i\not{D} \quad \longrightarrow \quad \begin{bmatrix} 0 & i\not{D} \\ (i\not{D})^\dagger & 0 \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} 0 & W \\ W^\dagger & 0 \end{bmatrix} , \tag{10}$$

where W is a complex $N \times N$ matrix which has no further symmetries. The chirality of the Dirac operator implies that all eigenvalues come in pairs $(-\lambda, +\lambda)$. Thus, the center of the spectrum, where the eigenvalues are zero is distinguished. The existence of this region of zero virtuality states a fundamental difference to ordinary RMT.

What is the relevance of chRMT for QCD? – First, chRMT correctly reproduces low energy sum rules of QCD. Second, detailed studies have shown that the spectra from lattice gauge calculations indeed exhibit the correlations predicted by chRMT, see the review in Ref. [7]. To calculate the spectral correlators in non-trivial cases, it is highly convenient to extend the Graded Eigenvalue Method to chiral symmetry. The generating function is mapped onto superspace and expressed as an integral over $2k \times 2k$ complex supermatrices σ. Due to chirality, and in contrast to the cases discussed in the previous sections, σ has no further symmetries, in particular, it is not Hermitean. We proceed by introducing polar coordinates $\sigma = us\bar{v}$, where $u \in \mathrm{U}(k/k)$, $\bar{v} \in \mathrm{U}(k/k)/\mathrm{U}^{2k}(1)$, and $s = \mathrm{diag}\,(s_1, is_2)$ with $s_j = \mathrm{diag}\,(s_{1j}, \ldots, s_{kj})$ for $j = 1, 2$. The s_{pj} are real and non-negative. The transformation of the Cartesian volume element to radial and angular coordinates involves the Jacobian $B_k^2(s^2)$. The integral over the supergroups is non-trivial. It is the supersymmetric extension [18] of the Berezin–Karpelevich integral and reads

$$\int d\mu(u) \int d\mu(\bar{v}) \exp\left(i \mathrm{Re}\,\mathrm{trg}\, us\bar{v}r\right) = \frac{1}{2^{2k^2}(k!)^2} \frac{\det[J_0(s_{p1}r_{p'1})]\det[J_0(s_{q2}r_{q'2})]}{B_k(s^2)B_k(r^2)} , \tag{11}$$

where r is diagonal and where J_0 is a Bessel function. We stress that this integral is not contained in the supersymmetric analogue of the celebrated Harish-Chandra formula.

By using the supergroup integral (11), we succeeded in presenting exact calculations of the spectral correlators in the presence of arbitrarily many Matsubara frequencies [19]. We found a remarkable scaling property of all correlators near zero virtuality. Moreover, we also calculated the correlators for the massive Dirac operator, i.e. if non-zero *sea quark* masses are taken into account [20]. Recently, we managed to combine these two scenarios which is the physically realistic case. Again, we could give an exact and complete solution [21].

5 Representation Theory for Supergroups

The use of supersymmetric techniques in physical applications raises numerous non-trivial mathematical questions. Most of them lead directly to the representation theory of supergroups. Interestingly, a previously unknown class of representations emerges as a natural consequence of supersymmetry in stochastic quantum physics.

To begin with, we work out the simplest case of an harmonic analysis in a matrix space by studying 2×2 Hermitean matrices [22]. First, we consider two ordinary matrices H and K with eigenvalue matrices x and k, respectively. The expansion of the plane wave in this space reads trivially

$$\exp(i \operatorname{tr} HK) = \sum_{LM} T_L(x,k) Y^*_{LM}(\Omega_H) Y_{LM}(\Omega_K), \tag{12}$$

where L and M are the usual angular momentum and its magnetic projection. The function $T_L(x,k)$ is related to the spherical Bessel function $j_L(z)$ and the $Y_{LM}(\Omega)$ are the usual spherical harmonics depending on the solid angle Ω.

Surprisingly, a completely analogous expansion can be found in superspace. For two supermatrices σ and ρ with eigenvalue matrices s and r, respectively, we find

$$\exp(i \operatorname{trg} \sigma\rho) = \int t_{|\mu|}(s,r) y^*_{\mu\mu^*}(\omega_\sigma) y_{\mu\mu^*}(\omega_\rho) d[\mu] \tag{13}$$

such that the summation over L and M is replaced by an integral over an anticommuting variable μ and its complex conjugate μ^*. The graded Bessel function $t_{|\mu|}(s,r)$ depends only on the *length* $|\mu|^2 = \mu\mu^*$ of the anticommuting variable similar to the fact that $T_L(x,k)$ depends only on L. The graded spherical harmonics $y_{\mu\mu^*}(\omega)$ depend on a solid angle ω consisting of anticommuting variables which can be viewed as the analogue of Euler angles. Remarkably and crucially, these functions span something like a Hilbert space whose states are labeled by something like anticommuting angular momentum quantum numbers μ and μ^*. There are orthogonality and completeness relations. This is of fundamental importance for an application of the expansion (13), particularly for a Fourier–Bessel analysis in superspace.

The occurrence of this Hilbert-space like object spanned by the graded spherical harmonics raises the question whether one can construct representation functions of the supergroup U(1/1) which involve these anticommuting quantum numbers. The answer is affirmative [23]. Completely analogous to the Wigner representation functions of the ordinary group SU(2), graded Wigner representation functions can be constructed for U(1/1). The anticommuting variables μ and μ^* label these representations.

For the general case of the supergroup $\mathrm{U}(k_1/k_2)$, the construction of something like Euler angles is completely out of question. Thus, a method has to be devised which incorporates a recursion in the dimension of the supergroup. For the ordinary unitary group $\mathrm{U}(k)$, Gelfand [25] constructed such representations based on a recursive embedding in the group chain

$$\mathrm{U}(k) \supset \mathrm{U}(k-1) \supset \cdots \supset \mathrm{U}(2) \supset \mathrm{U}(1).$$

Moreover, Gelfand and Tzetlin [26, 27] constructed explicit coordinates on the manifold of $\mathrm{U}(k)$ which are closely related to these representations. The recursive structure of the representations is also reflected in the coordinates implying several most useful features. A detailed discussion can be found in Ref. [28].

The original, more geometric construction [26, 27] for ordinary unitary matrices needs to be modified to a more algebraic procedure in the case of supermatrices for reasons which will become clear in the following. We consider a Hermitean supermatrix σ with k_1 bosonic and k_2 fermionic dimensions. Let u be a unitary supermatrix in the supergroup $\mathrm{U}(k_1/k_2)$ such that $\sigma = u^{-1}su$ with the eigenvalues ordered in a diagonal matrix s. Define $u_p, p = 1, \ldots, k_1 + k_2$ as the columns of u. Since u_1 is an unit vector, the number of independent variables is $2(k_1 + k_2) - 1$, and the elements of u_1 cannot be used directly as independent variables. The idea is to project onto the $k_1 + k_2 - 1$ dimensional subspace spanned by the vectors $u_2, \ldots, u_{k_1+k_2}$. The corresponding projection $(1 - u_1u_1^\dagger)s(1 - u_1u_1^\dagger)$ of the eigenvalue matrix s has has $k_1 + k_2 - 1$ eigenvalues $s_p^{(1)}, p = 2, \ldots, k_1 + k_2$. We refer to them as the generalized Gelfand–Tzetlin eigenvalues. The ensuing system of equations involves a new type of singularity. It can be solved and yields explicit, comparatively simple expressions for the elements of u_1 in terms of the eigenvalues s, the generalized Gelfand–Tzetlin eigenvalues $s^{(1)}$ and phases. The eigenvalues in the fermion fermion block have the important property $|\xi_p^{(1)}|^2 = is_{p2}^{(1)} - is_{p2}$, $p = 1, \ldots, k_2$ where $\xi_p^{(1)}$ is a complex anticommuting variable and $\xi_p^{(1)*}$ its complex conjugate. By making appropriate basis rotations, this coordinate system is recursively continued to $k_1 + k_2$ levels with Gelfand–Tzetlin eigenvalues $s^{(m)}$, $m = 1, \ldots, k_1 + k_2$ and anticommuting variables $\xi^{(m)}$, $m = 1, \ldots, k_1$. On each level, the number of Gelfand–Tzetlin eigenvalues is lowered by one. Thus, we arrive at a complete, explicit coordinate system for $\mathrm{U}(k_1/k_2)$.

In view of the discussion in the previous section, we are led to conclude that the construction of the representations of the unitary supergroup $\mathrm{U}(1/1)$ can now be generalized to the unitary supergroup $\mathrm{U}(k_1/k_2)$. Analogously to the Gelfand–Tzetlin representations of the ordinary unitary group $\mathrm{U}(k)$, we interpret the commuting Gelfand–Tzetlin eigenvalues $s_{p1}^{(m)}$ and $is_{p2}^{(m)}$ as positive integers subject to a *betweenness condition.* Naturally, there is no interpretation of this sort for the anticommuting variables $\xi_p^{(m)}$ and $\xi_p^{(m)*}$. We obtain the generalized Gelfand pattern

$$
\begin{array}{ccccccccc}
s_{11} & & s_{21} & & s_{31} & \cdots & s_{k_1 1} \\
& s_{21}^{(1)} & & s_{31}^{(1)} & \cdots & s_{k_1 1}^{(1)} & \\
& & & \vdots & & & \\
& & s_{(k_1-1)1}^{(k_1-2)} & & s_{k_1 1}^{(k_1-2)} & & \\
& & & s_{k_1 1}^{(k_1-1)} & & &
\end{array}
$$

$$
\begin{array}{cccc}
|\xi_1^{(1)}|^2 & |\xi_2^{(1)}|^2 & \cdots & |\xi_{k_2}^{(1)}|^2 \\
|\xi_1^{(2)}|^2 & |\xi_2^{(2)}|^2 & \cdots & |\xi_{k_2}^{(2)}|^2 \\
\vdots & & & \vdots \\
|\xi_1^{(k_1)}|^2 & |\xi_2^{(k_1)}|^2 & \cdots & |\xi_{k_2}^{(k_1)}|^2
\end{array}
$$

$$
\begin{array}{ccccccccc}
is_{12}^{(k_1)} & & is_{22}^{(k_1)} & & is_{32}^{(k_1)} & \cdots & is_{k_2 2}^{(k_1)} \\
& is_{22}^{(k_1+1)} & & is_{32}^{(k_1+1)} & \cdots & is_{k_2 2}^{(k_1+1)} & \\
& & & \vdots & & & \\
& & is_{(k_2-1)2}^{(k_1+k_2-2)} & & is_{k_2 2}^{(k_1+k_2-2)} & & \\
& & & is_{k_2 2}^{(k_1+k_2-1)} & & &
\end{array}
$$

which labels representations of the unitary supergroup $\mathrm{U}(k_1/k_2)$. The generalized Gelfand pattern consists of two triangular sub-patterns for the commuting and one rectangular sub-pattern for the anticommuting variables. The two triangular sub-patterns label irreducible bases of the ordinary unitary groups $\mathrm{U}(k_1)$ and $\mathrm{U}(k_2)$ and hence both together label irreducible bases for the direct product $\mathrm{U}(k_1) \otimes \mathrm{U}(k_2)$ which is a subgroup of the supergroup $\mathrm{U}(k_1/k_2)$. The remaining coset $\mathrm{U}(k_1/k_2)/(\mathrm{U}(k_1) \otimes \mathrm{U}(k_2))$ is represented by the rectangular pattern of anticommuting variables.

From the construction, we may conclude that the generalized Gelfand pattern labels the basis which corresponds to the supergroup chain

$$\mathrm{U}(k_1/k_2) \supset \mathrm{U}(k_1-1/k_2) \supset \cdots \supset \mathrm{U}(1/k_2) \supset \mathrm{U}(k_2) \supset \cdots \supset \mathrm{U}(2) \supset \mathrm{U}(1)$$

and that the basis functions are eigenfunctions of the complete set of commuting operators in this chain.

There is already a theory [29] of finite-dimensional representations of the superalgebras $\mathrm{gl}(k_1/k_2)$ and $\mathrm{u}(k_1/k_2)$ involving a Gelfand pattern. Although anticommuting variables do not appear in those, there ought to be a connection to the generalized Gelfand pattern for the supergroup $\mathrm{U}(k_1/k_2)$ which contains anticommuting variables explicitly. Moreover, Balantekin and Bars [30] constructed representations of the unitary supergroup in terms of extended Young supertableaux. Again, anticommuting variables do not appear explicitly in those tableaux and it remains an open problem to find the relation to the generalized Gelfand pattern.

6 Summary and Outlook

The Graded Eigenvalue Method has proven to be a powerful technique for the exact calculation of various problems in stochastic quantum physics such as the regularity-chaos transition. Importantly, this method could be extended to chiral RMT where it also made several exact calculations feasible. The method is based on the computation of certain supergroup integrals, the supersymmetric versions of the Harish-Chandra–Itzykson–Zuber and the Berezin–Karpelevich integral.

Very naturally, these studies lead to a new representation theory for supergroups. A complete harmonic analysis on the supergroup $\mathrm{U}(1/1)$ and a representation in terms of graded Wigner functions are constructed involving anticommuting labels of the representations. The Gelfand–Tzetlin method was generalized for the supergroup $\mathrm{U}(k_1/k_2)$ in arbitrary dimensions k_1 and k_2.

Work on various physical applications of these results is in progress. Moreover, some interesting mathematical questions are still unanswered yet, such as the precise mathematical interpretation of the anticommuting group labels. This will also be studied in the near future.

Acknowledgments

I thank the Heisenberg foundation for financial support.

References

[1] Wigner E.P., *Proc. Cambridge Phil. Soc.*, 1951, V.47, 790; Conference on Neutron Physics by Time-of-Flight, Gatlinburg, Tennessee, 1956.

[2] Mehta M.L., Random Matrices, 2nd ed., Academic Press, New York – London, 1991

[3] Dyson F.J., *J. Math. Phys.*, 1962, V.3, 140; 157; 166; 1199.

[4] Bohigas O., Giannoni M.J. and Schmit C., *Phys. Rev. Lett.*, 1984, V.52, 1.

[5] Gutzwiller M.C., Chaos in Classical and Quantum Mechanics, Springer, New York, 1990.

[6] Haake F., Quantum Signatures of Chaos, Springer, Berlin, 1991.

[7] Guhr T., Müller-Groeling A. and Weidenmüller H.A., *Phys. Rep.*, 1998, V.299, 189, `cond-mat/9707301`.

[8] Efetov K.B., *Adv. Phys.*, 1983, V.32, 53.

[9] Verbaarschot J.J.M., Weidenmüller H.A. and Zirnbauer M.R., *Phys. Rep.*, 1985, V.129, 367.

[10] Friedrich H. and Wintgen D., *Phys. Rep.*, 1989, V.183, 37.

[11] Guhr T. and Weidenmüller H.A., *Ann. Phys. (NY)*, 1989, V.193, 472.

[12] Guhr T., *J. Math. Phys.*, 1991, V.32, 336.

[13] Itzykson C. and Zuber J.B., *J. Math. Phys.*, 1980, V.21, 411.

[14] Harish-Chandra, *Amer. J. Math.*, 1958, V.80, 241.

[15] Guhr T., *Phys. Rev. Lett.*, 1996, V.76, 2258; *Ann. Phys. (NY)*, 1996, V.250, 145.

[16] Frahm K., Guhr T. and Müller-Groeling A., *Ann. Phys. (NY)*, 1998, V.270, 292, `cond-mat/9801298`.

[17] Shuryak E. and Verbaarschot J.J.M., *Nucl. Phys. A*, 1993, V.560, 306.

[18] Guhr T. and Wettig T., *J. Math. Phys.*, 1996, V.37, 6395.

[19] Guhr T. and Wettig T., *Nucl. Phys. B*, 1997, V.506, 589, `hep-th/9704055`.

[20] Wilke T., Guhr T. and Wettig T., *Phys. Rev. D*, 1998, V.57, 6486, `hep-th/9711057`.

[21] Seif B., Wettig T. and Guhr T., *Nucl. Phys. B*, 1999, V.548, 475, `hep-th/9811044`.

[22] Guhr T., *J. Math. Phys.*, 1993, V.34, 2523.

[23] Guhr T., *J. Math. Phys.*, 1993, V.34, 2541.

[24] Guhr T., *Commun. Math. Phys.*, 1996, V.176, 555.

[25] Gelfand I.M., *Mat. Sb.*, 1950, V.26, 103.

[26] Gelfand I.M. and Tzetlin M.L., *Dokl. Akad. Nauk*, 1950, V.71, 825.

[27] Gelfand I.M. and Naimark M.A., *Trudy MIAN*, 1950, V.36, 1.

[28] Barut A.O. and Raczka R., Theory of Group Representations and Applications, Polish Scientific Publishers, Warszawa, 1980.

[29] Palev T.D., Stoilova N.I. and Van der Jeugt J., *Commun. Math. Phys.*, 1994, V.166, 367 (and references therein).

[30] Balantekin A.B. and Bars I., *J. Math. Phys.*, 1981, V.22, 1149; 1810; 1982, V.23, 1239.