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Models of Covariant Quantum Theories of Extended Objects from Generalized Random Fields

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The problem of supplying nontrivial models of local covariant and obeying the spectral condition quantum theories of extended objects is discussed. In particular, it was demonstrated that starting from sufficiently regular generalized random fields the construction of the corresponding quantum dynamics describing extended objects is possible. Several particular examples of such generalized random fields are presented.

1 Introduction

The importance of nonlocal, gauge invariant functionals was firstly recognized in local quantum field theories of gauge type [28]. In such theories the compatibility of standard positivity, locality and covariance is hard to achieve if at all, see e.g. [26, 25]. The restrictions of the allowed set of observables to the set of gauge invariant observables and the arising space of states seem to be correct choice of subspace of physical states. Also the role played by certain nonlocal order parameters in studying the phase structure (the complicated vacuum structure) of gauge quantum field theories must to be pointed out [4, 6, 23, 27, 29]. The still continued attempts [14] to formulate local, covariant and positive quantum theories of extended objects like strings, membranes, etc. also justify the importance of searching new mathematical techniques for constructing nontrivial models of this type. Let us recall also that the recent attempts to formulate quantum gravity in terms of loop variables seem to be very attractive idea [2]. Finally let us mention the application of the loop variables in the topological quantum field theories to the classical problems of geometry [3].

An interesting approach to the construction physically reasonable models of extended objects was proposed in [23] in the context of quantum field theories of gauge type. The approach presented in [23] can be called the Euclidean approach and is of axiomatic type. However there are not too many nontrivial models obeying the system of axioms proposed in [23]. To our best knowledge the Wilson loop Schwinger functions in the continuum limit of QCD_2, and in the free QED_d are the only examples discussed explicitely in the literature [23], see also [18, 22]. It is the main aim of the present contribution to provide some new examples of theories obeying the proposed axiomatic scheme of [23] and to outline a general constructive approach for constructing models of this sort from the generalized random fields.

2 The Fröhlich–Osterwalder–Seiler axiomatic approach

Let $\mathcal{C}_k(d)$ be a variety of k-dimensional piecewise C^1 cycles in the space $\mathbf{R}^d$, i.e. elements Γ of $\mathcal{C}_k(d)$, a k-dimensional boundaryless piecewise C^1 compact submanifolds of the d-dimensional

Euclidean space $\mathbf{R}^d$. The allowed topologies τ on $\mathcal{C}_k(d)$ are such that only small C^∞ local deformations are allowed and they define a basis of neighborhoods of a given $\Gamma \in \mathcal{C}_k(d)$, in particular local continuous but not differentiable deformations $\delta\Gamma$ of Γ send $\delta\Gamma$ far from Γ. The allowed topologies (as above) on the variety $\mathcal{C}_k(d)$ can be prescribed explicitly in the metric form (an example in the case of loops is provided in [23]).

From now on we shall assume that τ is an allowed topology on $\mathcal{C}_k(d)$.

A system $\mathbf{S} = \{S_n\}_{n\geq 0}$ of functionals, where each S_n is jointly τ-continuous functional on the space $(\mathcal{C}_k(d), \tau)^{\times n}_{\#}$, where $(\Gamma_1, \ldots, \Gamma_n) \in \mathcal{C}_k(d)^{\times n}_{\#}$ iff $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$, is called k-cycles Schwinger functional iff it fulfills the following conditions:

FOS0-1 Let $\Gamma_i^{a_i}$, $i = 1, \ldots, n$ be a translation of $\Gamma_i \in \mathcal{C}_k(d)$ by the vector $a_i \in \mathbf{R}^d$ and let $\delta(\Gamma_1^{a_1}, \ldots, \Gamma_n^{a_n}) = \inf_{i,j}\{\mathrm{dist}(\Gamma_1^{a_i}, \Gamma_j^{a_j})\}$. If $\delta(\Gamma_1^{a_1}, \ldots, \Gamma_n^{a_n}) > 0$, then there exist constants K_n, c_n, p such that:

$$|S_n(\Gamma_1^{a_1}, \ldots, \Gamma_n^{a_n})| \leq K_n \exp c_n \delta^{-p}.$$

FOS0-2 Let

$$\delta_t(\Gamma_1, \Gamma_2) = \inf\left\{|t_1 - t_2|; t_1 : \ \bigvee(t_1, \mathbf{x}_1) \in \Gamma_1, \quad t_2 : \ \bigvee(t_2, \mathbf{x}_2) \in \Gamma_2\right\}$$

be a temporal distance between Γ_1 and Γ_2. Then there exist constants $K_{\Gamma_i,\epsilon}$ (depending on Γ_i and $\epsilon > 0$) such that:

$$|S_n(\Gamma_1, \ldots, \Gamma_n)| \leq K_{\Gamma_1,\epsilon} \cdot \cdots \cdot K_{\Gamma_n,\epsilon}$$

providing $\delta_t(\Gamma_i, \Gamma_j) \geq \epsilon$, $i, j = 1, \ldots, n$.

FOS1 For any $n \geq 1$, any ensemble $\{\Gamma_1, \ldots, \Gamma_n\} \subset \mathcal{C}_k(d)^{\times n}_{\#}$ and any permutation $\pi \in s_n$ ($\equiv$ the symmetric group):

$$S_n(\Gamma_1, \ldots, \Gamma_n) = S_n(\Gamma_{\pi(1)}, \ldots, \Gamma_{\pi(n)}).$$

FOS2 For any Euclidean motion $(a, \Lambda) \in T \lhd O(d)$ (where $O(d)$ stands for the orthogonal group, T are translations and $\lhd$ means the standard semidirect product) and any ensemble $\Gamma_1, \ldots, \Gamma_n \in \mathcal{C}_k(d)$ we have:

$$S_n(\Gamma_1, \ldots, \Gamma_n) = S_n\left(\Gamma_1^{(a,\Lambda)}, \ldots, \Gamma_n^{(a,\Lambda)}\right),$$

where $\Gamma^{(a,\Lambda)} = \left\{\Lambda^{-1}(x - a) \mid x \in \Gamma\right\}$.

FOS3 Reflection Positivity. Let $\mathbf{V}_{+(-)}$ be a subset of $\mathbf{C}_k(d) \equiv \bigcup_{n\geq 0} \mathcal{C}_k(d)^{\times n}_{\#}$ consisting of the ensembles of families of nonintersecting cycles

$$\left(\emptyset, \Gamma^1, \left(\Gamma_1^2, \Gamma_2^2\right), \ldots, \left(\Gamma_1^n, \ldots, \Gamma_n^n\right), \ldots\right)$$

that are supported in $\mathbf{R}^d_{+(-)} = \{(t, \mathbf{x}) \in \mathbf{R}^d \mid t > 0 (< 0)\}$. Let Θ be a natural involution from $\mathbf{V}_+$ onto $\mathbf{V}_-$. Then for any

$$\underline{\Gamma} \equiv \left(\emptyset, \Gamma^1, \left(\Gamma_1^2, \Gamma_2^2\right), \ldots, \left(\Gamma_1^n, \ldots, \Gamma_n^n\right), \ldots\right) \in \mathbf{V}_+$$

we have

$$\mathbf{S}(\underline{\Gamma}\Theta\underline{\Gamma}) = \sum_{l,m} c_l \bar{c}_m S_{l+m}(\Gamma_1^l, \ldots, \Gamma_l^l, \Theta\Gamma_1^m, \ldots, \Theta\Gamma_m^m) \geq 0$$

and for any $\underline{c} = (c_0, c_1, \ldots)$ (finite sequence of complex numbers).

FOS4 For any $n = k + l$, $k, l > 0$ and $|a| \to \infty$

$$\lim_{|a|\to\infty} S_n(\Gamma_1, \ldots, \Gamma_k, \Gamma'^a_1, \ldots, \Gamma'^a_l) = S_k(\Gamma_1, \ldots, \Gamma_k) S_l(\Gamma'_1, \ldots, \Gamma'_l).$$

It was demonstrated (originally for the case of 1-cycles but the arguments are easily extendable to the case of k-cycles with $1 \leq k \leq d-1$) in [23] that certain real time quantum dynamical system can be reconstructed from any system of Schwinger functions obeying **FOS**0-**FOS**4.

Theorem 2.1 *Let* $\mathbf{S}$ *be a system of* k*-cycles Schwinger functions on* $(\mathbf{C}_k(d), \tau)$*. Then there exists: a separable Hilbert space* $\mathcal{H}$*, a continuous unitary representation of the universal covering group of the proper orthochronous Poincaré group* $\mathcal{P}_+^\uparrow(d)$ *obeying a spectral condition (i.e. the joint spectrum of the generators of translations is included in the closed forward light cone). Moreover there exists a unique vector* $\Omega \in \mathcal{H}(\mathbf{S})$ *which is invariant under the action of* $\mathcal{P}_+^\uparrow(d)$*.*

In particular, with any time-ordered ensemble of k-cycles $\{\Gamma_1, \ldots, \Gamma_n\}$ and such that $\inf_{i,j}\{d_t(\Gamma_i, \Gamma_j)\} > 0$ one can associate (in a unique manner) a system of holomorphic functionals $\mathcal{W}_{(\Gamma_1,\ldots,\Gamma_n)}(z_1, \ldots, z_n)$ in the tubular region

$$\mathcal{T}_n = \left\{(z_1, \ldots, z_n) \in \mathbf{C}^{dn} \mid \Im m(z_i - z_{i-1}) \in V_+^d\right\},$$

where $V_+^d = \{x \in \mathbf{R}^d \mid x \cdot x > 0, x^0 > 0\}$ (where $x \cdot x = (x^0)^2 - \mathbf{x}^2$ means Minkowski space scalar product) and such that

(i) restriction of $\mathcal{W}^n_{(\Gamma_1,\ldots,\Gamma_n)}(z_1, \ldots, z_n)$ to the "Euclidean" piece of the boundary $\partial_E \mathcal{T}_n$ of $\mathcal{T}_n$ defined as:

$$\partial_E \mathcal{T}_n = \left\{\underline{z} \in \mathbf{C}^{nd} \mid \Re z_i^0 = 0, \Im m\mathbf{z}_i = 0, \Im m z_i^0 < \Im m z_{i+1}^0\right\}$$

is equal to $\mathbf{S}_n(\Gamma_1, \ldots, \Gamma_n)$, i.e.

$$\mathcal{W}^n_{(\Gamma_1,\ldots,\Gamma_n)}\left(\left(ia_1^0, \mathbf{a}_1\right), \ldots, \left(ia_n^0, \mathbf{a}_n\right)\right) = \mathbf{S}_n\left(\Gamma_1^{(ia_1^0,\mathbf{a}_1)}, \ldots, \Gamma_n^{(ia_n^0,\mathbf{a}_n)}\right);$$

(ii) for any collection $(\Gamma_1, \ldots, \Gamma_n)$ of k-cycles located in space-like hyperplanes there exists

$$\lim_{\substack{z_l = x_l + i\eta_l \to 0 \\ \eta_l - \eta_{l-1} \in V_+^d}} \mathcal{W}^n_{(\Gamma_1,\ldots,\Gamma_n)}(z_1, \ldots, z_n) = \mathcal{W}^n_{(\Gamma_1,\ldots,\Gamma_n)}(x_1, \ldots, x_n)$$

in the space of ultradistributions of Jaffe type and with the corresponding indicator function compatible with the singularity behaviour of **FOS**0–**FOS**1.

The boundary ultradistributions $\mathcal{W}^n_{(\Gamma_1,\ldots,\Gamma_n)}(x_1, \ldots, x_n)$ are called k-cycles Wightman ultradistributions corresponding to Schwinger functional $\mathbf{S}$. The problem of formulating conditions on the system of $\mathcal{W}$ of Wightman ultradistributions that lead to k-cycles Schwinger functions $\mathbf{S}$ obeying **FOS**0-**FOS**4 still seems to be open.

3 The scalar models

Let μ_λ be an infinite-volume limit of the so called $P(\Phi)_2$ interaction [13, 24], where $\lambda > 0$ refers to the major coupling constant. The case $\lambda = 0$ corresponds to the Nelson free field measure, i.e. μ_0 stands for the centered Gaussian measure on the space of (real valued) tempered distributions $\mathcal{S}'(\mathbf{R}^2)$ defined by

$$\mu_0(\exp(i(\varphi, f))) \equiv \exp\left\{-\frac{1}{2}\|f\|_{-1}^2\right\}, \tag{1}$$

where $\|f\|_{-1}^2 = (-\triangle + 1)^{-1}(f \otimes f)$, $S_0 \equiv (-\triangle + 1)^{-1}$ being a principial Green function of the operator $(-\triangle+1)$. Let Γ be a Jordan type curve which is assumed to be sufficiently smooth (see below). We would like first to give a rigorous mathematical meaning to the (formal) expression $\oint_\Gamma \varphi$. For this goal we use a theory of Lions–Magenes traces of distributions [20] together with some arguments from [1].

Lemma 3.1 *Let Γ be a Jordan type 1-cycle in $\mathbf{R}^2$. If the generalized random field μ_λ on $\mathcal{S}'(\mathbf{R}^2)$ obeys the estimate*

$$\mu_\lambda(\varphi^2(f)) \le c_{-1}\|f\|_{-1}^2 + c_p\|S_0 * f\|_{L^p} + c_1\|S_0 * f\|_{L^1}, \tag{2}$$

where $p \in [2, \infty)$ and c_{-1}, c_p, c_1 are some nonnegative constants then for μ_λ a.e. $\varphi \in \mathcal{S}'(\mathbf{R}^2)$ there exists a trace of φ on Γ in the Lions–Magenes sense, denoted as $\varphi|_\Gamma$ and moreover $\varphi|_\Gamma \in \bigcap_{\alpha>0}\mathcal{H}^{-\alpha}(\Gamma)$, where $\mathcal{H}^{-\alpha}(\Gamma)$ are negative-order Sobolev spaces on Γ (defined as in [20]).

Using the fact that χ_Γ($\equiv$ the characteristic function of Γ) belongs to $\bigcap_{\alpha>0}\mathcal{H}^{+\alpha}(\Gamma)$ (as being a constant function) it follows easily by dualization that for any μ obeying the estimate (2) we can define $\langle\chi_\Gamma, \varphi\rangle$ and this number is defined to be $\oint_\Gamma \varphi$. Proceeding in this way we can define for any collection $\{\Gamma_1, \ldots, \Gamma_n\}$ a measurable and defined μ a.e. function

$$\mathcal{L}^*_{\Gamma_1,\ldots,\Gamma_n}(\varphi) \equiv \prod_{j=1}^n \mathrm{e}^{i\oint_{\Gamma_j}\varphi}.$$

This is an almost sure version of the result on the existence of random loop function for models of Euclidean Quantum Field Theory obeying (2).

However, due to the problem of exceptional sets the above a.e. result is not sufficient and certain computable L^p version of the random loop functions has to be given.

Proposition 3.2 *Let μ be generalized random field on $\mathcal{S}'(\mathbf{R}^2)$ obeying the following estimate:*

$$|\mu(\varphi^2(f))| \le c\|f\|_{-1}^2 \tag{3}$$

for any f with compact support. Let $(\chi_\epsilon)_{\epsilon>0}$ be any smooth mollifier i.e. $0 \le \chi_\epsilon \in C_0^\infty(\mathbf{R}^2)$ for any $\epsilon > 0$, $\int \chi_\epsilon(x)d^2x = 1$ and $\lim_{\epsilon\downarrow 0}\chi_\epsilon = \delta$ ($\equiv$ Dirac delta) in the sense of weak convergence. Let $\{\Gamma_1, \ldots, \Gamma_n\}$ be any ensemble of nonintersecting loops of Jordan type.

Then for any $p \ge 1$ the unique limit

$$\lim_{\epsilon\downarrow 0}\mathcal{L}^\mu_\epsilon(\Gamma_1, \ldots, \Gamma_n)(\varphi) \equiv \prod_{j=1}^n e^{i\oint_{\Gamma_j}\varphi_\epsilon} \equiv \mathcal{L}^\mu(\Gamma_1, \ldots, \Gamma_n)(\varphi)$$

exists in $L^p(d\mu)$ sense.

Thus, defining the loop Schwinger functions

$$S^\mu(\Gamma_1,\ldots,\Gamma_n) = \int_{\mathcal{S}'(\mathbf{R}^2)} \mathcal{L}^\mu(\Gamma_1,\ldots,\Gamma_n)(\varphi)d\mu(\varphi)$$

for any generalized random field μ obeying (3), we can expect that they are good candidates for nontrivial models obeying the systems of axioms proposed in Section 2.

Theorem 3.3 *Let μ be a Euclidean homogeneous generalized random field obeying the estimate (3). Then the corresponding system of loop Schwinger functions $\mathbf{S}^\mu$ obeys the system of* **FOS**0–**FOS**2 *axioms with the possible exception of reflection positivity. If moreover μ is a reflection positive random field then the corresponding loop Schwinger functions obey the reflection positivity axiom too.*

It is well known that many of the constructed two-dimensional scalar models of Euclidean Quantum Field Theory [13, 24] obey the estimates like (2) with the values of p as indicated in (2) and it is known that the following estimates are valid (see e.g. Lemma 2.1 in [1]):

$$c_{-1}\|f\|_{-1}^2 + c_p\|S_0 * f\|_{L^p} + c_1\|S_0 * f\|_{L^1} \le c\|f\|_{-1}^2$$

for any f with compact support and some $c > 0$.

A similar theorem is valid for the case of renormalized ϕ_3^4 theory [24] and 2-cycles of Jordan type in $\mathbf{R}^3$. The proof being similar to that above.

However, the weak point of these examples is that the corresponding quantum systems reproduce the basic quantum field theoretical structures.

Theorem 3.4 *Let $({}^c\mathcal{H}^{\mu_\lambda}; {}^c\Omega^{\mu_\lambda}; {}^cU_t^{\mu_\lambda})$ be a quantum dynamical system obtained from the $P(\varphi)_2$ loop Schwinger functions and let $(\mathcal{H}^{\mu_\lambda}; \Omega^\lambda; U_t^{\mu_\lambda})$ be the corresponding quantum dynamical system obtained from the point (field theoretical) Schwinger functions [13, 24]. Then there exists a unitary map J:*

$$J : {}^c\mathcal{H}_\lambda^\mu \to \mathcal{H}_\lambda^\mu$$

such that $J : {}^c\Omega^{\mu_\lambda} \to \Omega^\lambda$ and $J^{-1}U_t^\lambda J = {}^cU_t^\lambda$.

For a complete proof see [12].

4 Regular, covariant, generalized random fields

Let $(A_0, \mathbf{A})$ be a generalized random field indexed by $\mathcal{S}(\mathbf{R}^d) \otimes \mathbf{R}^d$, where $d \ge 2$ and $\mathbf{A}$ stands for the space components of A according to the decomposition

$$\mathbf{R}^d = \left\{(x^0, \mathbf{x}) \mid x^0 \in \mathbf{R}, \quad \mathbf{x} \in \mathbf{R}^{d-1}\right\}.$$

Let us denote by μ the corresponding law of A, i.e. the probability, Borel, cylindric measure on $\mathcal{S}'(\mathbf{R}^d) \otimes \mathbf{R}^d$. Here $\mathcal{S}'(\mathbf{R}^d)$ stands for the space of tempered distributions. A field A is called vector field iff for any pair (a, Λ), where $a \in \mathbf{R}^d$, $\Lambda \in SO(d)$ the following equality $(A, f_{(a,\Lambda)}) \cong (A, f)$ in law holds, where $f_{(a,\Lambda)}(x) = \sum_{j=0}^{d-1} \Lambda_i^j f_j(\Lambda^{-1}(x-a))$. A vector field A is called reflection invariant iff $(A, rf) \cong (A, f)$ (in law), where $(rf)^0(x^0, \mathbf{x}) = -f^0(-x^0, \mathbf{x})$ and $(rf)^i(x^0, \mathbf{x}) = f^i(-x^0, \mathbf{x})$ for $i = 1, \ldots, d-1$. Let us recall that a vector field A which is Markoff and reflection invariant is reflection positive. The main question addressed in this section is now to find sufficient conditions on the field A that enable us to define a family of loop Schwinger functions obeying the system of axioms **FOS**0–**FOS**4. Let $\omega \in C_0^\infty(\mathbf{R}^d)$ be a non-negative function with support in the unit ball $\{x: \ \|x\| \le 1\}$ and such that $\int \omega(x)dx = 1$. Then the we define $\omega^N(x) = N^d\omega(Nx)$ and we note that $\lim_{N\to\infty} \omega^N(x) = \delta(x)$. For any loop Γ, parametrized by $\gamma(t)$, $t \in [0,1]$, we define the following family of test functions from $C_0^\infty(\mathbf{R}^d) \otimes \mathbf{R}^d$:

$$\Delta^N_{\Gamma,k}(x) = \oint_\Gamma \omega^N(x-y)dy^k = \int_0^1 \omega^N(\gamma(t)-x)\dot{\gamma}^k(t)dt.$$

For a given ensemble $\{\Gamma_1, \dots, \Gamma_n\}$ of loops we define the sequence of functionals

$${}^N\mathcal{L}(\Gamma_1,\dots,\Gamma_n)(A) = \prod_{l=1}^n \exp\{i\langle \Delta^N_{\Gamma_l}, A\rangle)\}$$

and the corresponding Schwinger functions

$${}^NS(\Gamma_1,\dots,\Gamma_n) = \mathbf{E}\,{}^N\mathcal{L}(\Gamma_1,\dots,\Gamma_n)(A).$$

Theorem 4.1 *Let A be a vector, reflection positive generalized random field on the space $\mathcal{S}'(\mathbf{R}^d) \otimes \mathbf{R}^d$, $d \ge 2$ and let $\{\mathcal{G}_{ij}(x-y)\}$ be a two-point Schwinger function of A. Assume that for any loop $\Gamma \in \mathcal{C}_1^g(\mathbf{R}^d)$ the following integrals*

$$\oint_\Gamma \oint_\Gamma |\mathcal{G}_{ii}(x-y)|dx^i dy^i \tag{4}$$

and for all $i = 0, \dots, d-1$ do exist. Then, there exists a system of loop Schwinger functions $\{S_n\}$ on $\bigcup_{n\ge 0} \mathcal{C}_1^g(\mathbf{R}^d)^{\times n}$ obeying the system of axioms **FOS**0–**FOS**3, *where $\mathcal{C}_1^g$ means globally $\mathcal{C}_1$-curves.*

In particular, the assumptions of Theorem 4.1 are valid for the 2-dimensional versions of Abelian, free *QED*. In higher dimensions we should expect some infinite renormalizations connected to the divergence of the integrals (4) see e.g. [18, 22]. A suitable version of Theorem 4.1 to handle this case can also be formulated [12].

Proof of Theorem 4.1. Using

$$\mathbf{E}\left|{}^N\mathcal{L}(\Gamma)(A) - {}^{N'}\mathcal{L}(\Gamma)(A)\right| \le \mathbf{E}\left|\langle \Delta^N_\Gamma - \Delta^{N'}_\Gamma, A\rangle\right| \le \left\{\mathbf{E}\left|\langle \Delta^N_\Gamma - \Delta^{N'}_\Gamma, A\rangle\right|^2\right\}^{\frac{1}{2}},$$

but

$$\mathbf{E}\left|\langle \Delta^N_\Gamma - \Delta^{N'}_\Gamma, A\rangle\right|^2 = \sum_{i,j} G_{ij}\left(\Delta^N_{\Gamma,i} - \Delta^{N'}_{\Gamma,i}, \Delta^N_{\Gamma,j} - \Delta^{N'}_{\Gamma,j}\right),$$

where $G_{ij}(x,y) = \mathbf{E}A_i(x)A_j(y)$. We see that the problem of $L^1(d\mu)$-convergence of functionals ${}^N\mathcal{L}(\Gamma)$ is reduced to the question of existence of $\lim_{N\to\infty} G_{ii}\left(\Delta^N_{\Gamma,i}, \Delta^N_{\Gamma,i}\right)$. For this

$$\begin{aligned}\left|G_{ij}\left(\left(\Delta^N_\Gamma - \Delta^{N'}_\Gamma\right)_i, \left(\Delta^N_\Gamma - \Delta^{N'}_\Gamma\right)_j\right)\right| &= \left|\mathbf{E}A_i\left(\Delta^N_\Gamma - \Delta^{N'}_\Gamma\right)_i A_j\left(\Delta^N_\Gamma - \Delta^{N'}_\Gamma\right)_j\right| \\ &\le \left\{\mathbf{E}\langle A_i, \left(\Delta^N_{\Gamma,i} - \Delta^{N'}_{\Gamma,i}\right)\rangle^2\right\}^{\frac{1}{2}} \left\{\mathbf{E}\langle A_j, \left(\Delta^N_{\Gamma,j} - \Delta^{N'}_{\Gamma,j}\right)\rangle^2\right\}^{\frac{1}{2}}.\end{aligned}$$

Thus, we need to prove that $\lim_{N\to\infty} G_{ii}\left(\Delta^N_{\Gamma,i}, \Delta^N_{\Gamma i}\right)$ exists for all i and then

$$\lim_{N\to\infty} G_{ii}\left(\Delta^N_{\Gamma,i}, \Delta^N_{\Gamma i}\right) = \lim_{N\to\infty} \int_{\mathbf{R}^4} dx \int_{\mathbf{R}^4} dy G_{ii}(x-y)$$

$$\int\int_{[0,1]^{\times 2}} \omega^N(\gamma(t_1)-x)\dot{\gamma}^i(t_1)\omega^N(\gamma(t_2)-y)\dot{\gamma}^i(t_2)dt_1dt_2$$

formally is equal to:

$$\oint_\Gamma \oint_\Gamma G_{ii}(x-y)dx^i dy^i = \int_0^1 dt_1 \int_0^1 dt_2 G_{ii}(\gamma(t_1)-\gamma(t_2))\dot{\gamma}^i(t_1)\dot{\gamma}^i(t_2)$$

so we need to justify only the change of limit operation $\lim_{N\to\infty}$ with integral but this is allowed by the Lebesgue dominated convergence theorem. ■

5 Some solvable interacting models

A large class of covariant, Markovian generalized random fields can be obtained as a solution of systems of covariant partial differential stochastic equations [7, 9, 10, 11].

For this let (τ, τ') be a pair of real representations of the special orthogonal transformation group $SO(d)$, where d is the dimension of the Euclidean space-time. We assume that dimension of τ (resp. τ') is equal to n_τ (resp. $n_{\tau'}$) and we denote the natural lifting of τ to the space $\mathcal{S}(\mathbf{R}^d)\otimes\mathbf{R}^{n_\tau}$ (resp. $\mathcal{S}(\mathbf{R}^d)\otimes\mathbf{R}^{n_{\tau'}}$) as T_τ (resp. $T_{\tau'}$). A first order differential operator $\mathcal{D} = \sum_{\mu=0}^{3} B_\mu\partial_\mu + \mathbf{M}$, where $B_\mu, \mathbf{M} \in \mathrm{Hom}(\mathbf{R}^{n_\tau}, \mathbf{R}^{n_{\tau'}})$ is called (τ,τ')-covariant operator iff the following diagram

$$\begin{array}{ccc} \mathcal{S}'(\mathbf{R}^d)\otimes\mathbf{R}^{\dim\tau} & \xrightarrow{\mathcal{D}} & \mathcal{S}'(\mathbf{R}^d)\otimes\mathbf{R}^{\dim\tau'} \\ T_\tau \big\downarrow & & \big\downarrow T_{\tau'} \\ \mathcal{S}'(\mathbf{R}^d)\otimes\mathbf{R}^{\dim\tau} & \xrightarrow{\mathcal{D}} & \mathcal{S}'(\mathbf{R}^d)\otimes\mathbf{R}^{\dim\tau'} \end{array} \tag{5}$$

commutes. The complete list of such operators for the case $d = 2, 3, 4$ is well known for any pair (τ, τ'). See, e.g. [8, 19, 11, 21].

Let $\underline{\alpha} = (\alpha_0, \ldots, \alpha_{d-1})$ be any multiindex of length d, i.e. $\alpha_\mu \in \mathbf{N}\cap\{0\}$, $\mu = 0, \ldots, d-1$ and let $|\underline{\alpha}| = \alpha_0 + \cdots + \alpha_{d-1}$. We denote by $\mathbf{I}_K(d)$ (for a given integer $K > 0$) the set of all multiidices $\underline{\alpha}$ as above and such that $|\underline{\alpha}| \le K$ and let $\mathbf{C}_K(d)$ be a cardinality of the set $\mathbf{I}_K(d)$. For a given $\underline{\alpha}$, let $D^{\underline{\alpha}} = \frac{\partial^{\alpha_0+\cdots+\alpha_{d-1}}}{\partial x_0^{\alpha_0}\cdots\partial x_{d-1}^{\alpha_{d-1}}}$.

Let us consider the operator $\mathbf{D}$ defined as

$$(\mathbf{D}^{jl}_{\underline{\alpha}}) \equiv \sum_{\underline{\beta}\in\mathbf{I}_K(d)} E^{jl}_{\underline{\alpha}\underline{\beta}} D^{\underline{\beta}} \tag{6}$$

for $\underline{\alpha}, \underline{\beta} \in \mathbf{I}_K(d)$, $j, l = 1, \ldots, N$, where $E^{jl}_{\underline{\alpha}\underline{\beta}}$ are some real numbers. The endomorphism E of the space $\mathbf{R}^{N\mathbf{C}_K(d)}$ corresponding to $E^{jl}_{\underline{\alpha}\underline{\beta}}$ in the canonical basis of $\mathbf{R}^{N\mathbf{C}_K(d)}$ will be useful in the following. For $f \in \mathcal{S}(\mathbf{R}^d)\otimes\mathbf{R}^N$ the operator $\mathbf{D}$ corresponding to (6) is given by $(\mathbf{D})^j_{\underline{\alpha}}(x) = \sum_l (\mathbf{D})^{jl}_{\alpha} f^l(x)$, so $\mathbf{D}$ maps $\mathcal{S}(\mathbf{R}^d)\otimes\mathbf{R}^N$ into $\mathcal{S}(\mathbf{R}^d)\otimes\mathbf{R}^{N\mathbf{C}_K(d)}$. We fix a pair $(\mathbf{D}_G, \mathbf{D}_P)$ of operators defined as above.

A noise corresponding to the pair $(\mathbf{D}_G, \mathbf{D}_P)$ (a general noise of order K) is defined as a generalized random field ν on the space $\mathcal{S}(\mathbf{R}^d) \otimes \mathbf{R}^N$ the characteristic functional Γ_ν of which is given by the product:

$$\Gamma_\nu(f) = \Pi_\nu^G(f)\Pi_\nu^P(f), \tag{7}$$

where the characteristic functional (of Gaussian part of ν) Π_ν^G is defined

$$\Pi_\nu^G(f) = \exp\left\{-\frac{1}{2}\int_{\mathbf{R}^d}\langle \mathbf{D}_G f, \mathbf{A}\mathbf{D}_G f\rangle(x)dx\right\}, \tag{8}$$

where $\mathbf{A}$ $\in$End$(\mathbf{R}^{N\mathbf{C}_K(d)})$, $\mathbf{A} \geq 0$, and the characteristic functional (of the Poisson part of ν) Π_ν^P is explicitly displayed as:

$$\Pi_\nu^P(f) = \exp\left\{-\int_{\mathbf{R}^d}\Psi^P(\mathbf{D}_P f(x))dx\right\}, \tag{9}$$

where

$$\Psi^P(y) = -\int_{\mathbf{R}^{N\mathbf{C}_K}\setminus\{0\}}\left[\mathrm{e}^{i\langle\Lambda,y\rangle} - 1 - i\langle\Lambda, y\rangle\right] dL(\Lambda) \tag{10}$$

or

$$\Psi^P(y) = -\int_{\mathbf{R}^{N\mathbf{C}_K}\setminus\{0\}}\left[\mathrm{e}^{i\langle\Lambda,y\rangle} - 1\right] dL(\Lambda) \tag{11}$$

for some Borel measure dL on the space $\mathbf{R}^{N\mathbf{C}_K}\setminus\{0\}$ with all finite moments.

It is easy to observe that a given noise ν on the space $\mathcal{S}'(\mathbf{R}^d) \otimes \mathbf{R}^{n_\tau}$ is T_τ-covariant iff the following covariance conditions are fulfilled

$$\left(\tau^T \otimes \gamma\right)(g)\mathbf{B}\left(\tau \otimes \gamma^T\right)(g) = \mathbf{B}, \tag{12}$$

$$dL_{(E_P^T)^{-1}}(\tau\otimes\gamma)(g)(\Lambda) = dL_{(E_P^T)^{-1}}(\Lambda), \tag{13}$$

where $\mathbf{B} \equiv E_G^T\mathbf{A}E_G$, $dL_{(E_P^T)^{-1}}$ is the transport of the Levy measure dL by the endomorphism E_P^T and finally γ is an orthogonal representation of the group $SO(d)$ in the space $\mathbf{R}^{\mathbf{C}_K(d)}$ defined explicitly as:

$$\gamma_{\underline{\alpha}\underline{\beta}}(g) = \sum_{\Pi_{\mu\nu}}\prod_{\mu,\nu=1}^{d-1} g_{\mu\nu}^{\Pi_{\mu\nu}}, \tag{14}$$

where the sum $\sum\limits_{\Pi_{\mu\nu}}$ runs over all matrices $(\Pi_{\mu\nu})_{\mu,\nu=0}^{d-1}$ built from the elements of $\{1, \ldots, K\}$ and chosen in such a way that $\alpha_\mu = \sum\limits_{\nu=0}^{d-1}\Pi_{\nu\mu}$, $\beta_\mu = \sum\limits_{\nu=0}^{d-1}\Pi_{\mu\nu}$ for $\underline{\alpha}, \underline{\beta} \in \mathbf{I}_K(d)$.

The interesting class of non-Gaussian covariant generalized (Markovian) in a suitable sense, see e.g. [17, 15], random fields is obtained as a solution of covariant SPDE's of the type

$$\mathcal{D}\varphi = \eta, \tag{15}$$

where $\mathcal{D}$ is some (τ, τ')-covariant operator which obeys certain additional conditions for the existence of not too singular Green function (from the infrared divergencies point of view, see [11, 21] for details), η is a noise of order K which is assumed to be $T_{\tau'}$-covariant noise.

It was proven in [9, 10, 11, 21] that under these conditions the solutions of the equation (15) do exist in certain sense and give rise to a new T_τ-covariant, generalized Markovian random fields, the moments of which can be analytically continued to Minkowski space-time yielding a system of covariant Wightman distributions obeying the spectral conditions (in the weak form) and the quantum field theoretical locality principle as well (see [7, 11] for details).

We would like to address here the question whether with solutions of (15) obtained in [9, 10, 11] one can associate systems of k-loop Schwinger functions on $\mathbf{R}^d$ that might be good candidates for explicit models obeying **FOS**0-**FOS**2. The important question on the existence of the reflection positive solutions of equations of the type (15) being still unsolved in general, presses the necessity to develop a weaker scheme for obtaining results on the real-time dynamics of extended objects from the corresponding Euclidean data of the spirit as in the general indefinite metric quantum field theory [16].

The following localization property of the noise ν is crucial for the existence of the almost sure version of the corresponding k-cycles Schwinger functionals.

Proposition 5.1 *Let $\Gamma(\mathbf{R}^d)$ be the space of locally finite configurations of the space $\mathbf{R}^d$ and let $\ni$ be a Poisson noise with the characteristics $(\mathbf{D}_P, E_p)$. Then, the set*

$$\left\{ \eta \in \mathcal{D}'(\mathbf{R}^d) \otimes \mathbf{R}^N \mid \eta = \sum_{k=1}^{N} \sum_{\underline{\alpha} \in \mathbf{I}_K(d)} \sum_{\delta_{k,\underline{\alpha}}=1}^{\infty} (-1)^{|\underline{\alpha}|} D^{\underline{\alpha}} \delta_{x_{\delta_{k,\underline{\alpha}}}} \otimes \left(E_P^T \Lambda_{\delta_{k,\underline{\alpha}}} \right)^k_{\underline{\alpha}} \right\}. \tag{16}$$

As a corollary we obtain

Theorem 5.2 *Let φ be a solution of (15) in the sense explained in [7, 9, 11] and let us assume that the underlying Green function $\mathcal{G}_{\mathcal{D}}$ of the operator $\mathcal{D}^T$ has a decay at least as $\frac{1}{|x|^{d+\epsilon}}$ if $|x| \to \infty$ and such that τ contains the appropriate subrepresentation corresponding to k-skew symmetric tensor. Then, for any fixed configuration $(\Gamma_1, \ldots, \Gamma_n)$ of k-cycles on $\mathbf{R}^d$, there exists a measureable functional defined:*

$$\mathcal{S}'(\mathbf{R}^d) \otimes \mathbf{R}^{n_\tau} \ni \varphi \longrightarrow \prod_{l=1}^{n} e^{i \oint_{\Gamma_l} \varphi|_{\tau(k)}}, \tag{17}$$

where $\phi_{\tau(k)}$ is the corresponding stochastic differential k-form which is perfectly well defined for μ_φ-a.e. $\varphi \in \mathcal{S}'(\mathbf{R}^d) \otimes \mathbf{R}^{n_\tau}$.

By simple argumentation, the existence of the unique measurable, defined μ_φ-a.e. maps

$$\mathcal{S}'(\mathbf{R}^d) \otimes \mathbf{R}^{n_\tau} \times \mathcal{C}_k(d)^{\times n} \ni (\varphi, (\Gamma_1, \ldots, \Gamma_n)) \longrightarrow \prod_{l=1}^{n} \mathrm{e}^{i \oint_{\Gamma_l} \varphi|_{\tau(k)}}$$

can be proven.

The computable, i.e. L^1-version of the above result is provided by the following theorem.

Theorem 5.3 *Let $\mathcal{D}$, $\tau^{(k)}$ be as in the previous theorem. We impose the following estimates on the behaviour of the Green function $G|_{\tau^{(k)}}$ and its first deriviatives:*

$$|G|_{\tau^{(k)}}|(x) \leq \frac{\underline{c}}{|x|^{\underline{p}}} \qquad \text{for} \quad 0 < |x| < 1 \quad \text{and} \quad 0 < \underline{p} < 4,$$

$$|G|_{\tau^{(k)}}|(x) \leq \frac{\overline{c}}{|x|^{\overline{p}}} \qquad \text{for} \quad 1 < |x| \quad \text{and} \quad 0 < \overline{p},$$

$$\left| \frac{\partial}{\partial x^\mu} G \right|_{\tau^{(k)}} |(x) \leq \frac{\overline{d}}{|x|^{\overline{q}}} \qquad \text{for} \quad 1 < |x|, \quad \mu = 0,1,2,3 \quad \text{and} \quad 0 < \overline{q}$$

and the estimates on the behaviour of the characteristic function ψ (negative defined function, see e.g. [5]) of the noise η.

$$|\psi(y)| \le \underline{M}|y|^{1+\underline{\eta}} \qquad \text{for} \quad |y|<1 \quad \text{and} \quad (-1+\frac{4}{\underline{q}},1].$$

In the case of k-cycles, if we demand the estimate

$$|\psi(y)| \le \overline{M}|y|^{1+\overline{\eta}} \qquad \text{for} \quad 1<|y| \quad \text{with} \quad \overline{\eta} \in \left(-1,-1+\frac{4-k}{\underline{p}}\right) \cap (-1,1]$$

then for any collection $\left\{\Gamma_1^{(k)},\dots,\Gamma_n^{(k)}\right\}$ of k-cycles there exists a Cauchy sequence of functionals $\left\{{}^N\hat{\mathbf{S}}_n\left(\Gamma_1^{(k)},\dots,\Gamma_n^{(k)}\right)\right\}_{N=1}^{+\infty} \subset L^p(\mathcal{S}'(\mathbf{R}^4)\otimes\mathbf{R}^{n_\tau},\mu_\varphi)$ for all $p\in[1,+\infty)$.

Let $\hat{\mathbf{S}}_n\left(\Gamma_1^{(k)},\dots,\Gamma_n^{(k)}\right)$ denote the limit of that sequence treated as an element in the space $L^1(\mathcal{S}'(\mathbf{R}^4)\otimes\mathbf{R}^{n_\tau},\mu_\varphi)$ $(p=1)$ and let us define

$$\mathbf{S}_n\left(\Gamma_1^{(k)},\dots,\Gamma_n^{(k)}\right) = \int_{\mathcal{S}'(\mathbf{R}^4)\otimes\mathbf{R}^{n_\tau}} \hat{\mathbf{S}}_n\left(\Gamma_1^{(k)},\dots,\Gamma_n^{(k)}\right)(T)\mu_\varphi(T).$$

If, in addition, the condition $\underline{\eta}\in\left(-1+\frac{4}{\underline{p}},1\right]$ is fulfilled then

$$\mathbf{S}_n\left(\Gamma_1^{(k)},\dots,\Gamma_n^{(k)}\right) = \exp\left\{-\int_{\mathbf{R}^4}\psi\left(\sum_{l=1}^n G_{\Gamma_l^{(k)}}|_{\tau^{(k)}}(x)\right)d^4x\right\},$$

where we introduced the auxiliary function

$$G_{\Gamma^{(k)}}|_{\tau^{(k)}}(x) = \begin{cases} \int_{\Gamma^{(k)}} G|_{\tau^{(k)}}(\Omega-x)d\Omega & \text{for} \quad x\notin\Gamma^{(k)} \\ 0 & \text{for} \quad x\in\Gamma^{(k)} \end{cases}$$

with integration in the sense of k-forms.

The proof of the above results follows the chain of arguments as presented in our earlier paper [11], where the case of the Wilson loops is discussed. All the details can be found in [21].

Theorem 5.4 *Let $\mathcal{D}$, η, $\tau^{(k)}$ be as in Theorem 5.3. Then the correspondinig k-loop Schwinger functionals:*

$$\mathbf{S}(\Gamma_1,\dots,\Gamma_n) = \int_{\mathcal{S}'(\mathbf{R}^d)\otimes\mathbf{R}^{n_\tau}} \prod_{l=1}^n e^{i\oint_{\Gamma_l}\varphi|_{\tau^{(k)}}} d\mu(\varphi) \tag{18}$$

obey the axioms **FOS**0–**FOS**2 *and also* **FOS**4.

The important problem to reconstruct the corresponding quantum, real-time dynamics from the data contained in the k-loop Schwinger functionals and the existence of the corresponding Wightman functions is left to another publication [12].

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