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A Unitary Parallel Filter Bank Approach to Magnetic Resonance Tomography

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The purpose of the present paper is to establish a unitary parallel multichannel filter bank approach to clinical magnetic resonance tomography and magnetic resonance microscopy. The approach which is based on the Stern–Gerlach filter explains the high resolution capabilities of these non-invasive cross-sectional imaging modalities which revolutionized the field of clinical diagnostics.

To a physicist, the 20*th century begins in* 1895*, with* Wilhelm Conrad Röntgen*'s unexpected discovery of* X*-rays.*

Steven Weinberg

One morning about the 10 *July* 1925 *I suddenly saw light:* Heisenberg*'s symbolic manipulation was nothing but the matrix calculus well-known to me since my student days.*

Max Born

The amount of theoretical work one has to cover before being able to solve problems of real practical value is rather large, but this circumstance is an inevitable consequence of the fundamental part played by transformation theory and is likely to become more pronounced in the theoretical physics of the future.

Paul Adrien Maurice Dirac

Magnetic resonance imaging has the potential of totally replacing computed tomography. If history was rewritten, and CT *invented after* MRI*, nobody would bother to pursue* CT.

Philip Drew

From the beginning, it was evident that all of the powerful and sophisticated methods of nuclear MR *spectroscopy could be used in imaging, and those branches in the field of* MR *have continued to enrich one another.*

Paul Christian Lauterbur

1 The Stern–Gerlach filter

In 1921 and 1922, the pioneers of nuclear magnetic resonance (NMR), Otto Stern (Nobel Prize 1943) and Walther Gerlach, came up with a technique to test Arnold Sommerfeld's theory of spatial quantization using molecular beams. The basic idea of the molecular beam technique – first demonstrated by L. Dunoyer in 1911 – was to vaporize an element by heating it in a furnace and then allowing atoms or molecules to escape through a narrow slit located at one end of the furnace adjacent to a vacuum chamber. As the atoms or molecules entered the chamber, they were collimated into a narrow beam of isolated, non-colliding particles via a series of apertures and directed toward the opposite end of the vacuum chamber where they could be detected by a flat piece of cold glass that caused the heated molecules or atoms to condense and be deposited for post-flight analysis.

About 25 years earlier, Pieter Zeeman (Nobel Prize 1902) had found by a spectroscopic experiment that sodium's single band of yellow spectral light split into multiple, narrow bands when placed in a strong magnetic field. To experimentally verify the spatial quantization phenomenon, Stern and Gerlach added a strong magnetic field to a molecular beam – actually, in this case, a beam of silver atoms – and found that they were deposited on the beam detector in two distinct and well defined separate bands. However, two separate beam channels were not what they had expected.

Midway through the normal flight path of the beam, Stern and Gerlach had placed a C-shaped electromagnet, positioned in such a way that a beam of silver atoms traveling along and between its north and south poles would encounter, perpendicular to the beams's direction of motion, a strong magnetic field acting upon the magnetic moments of the atoms in the beam. How would the external magnetic field affect the individual magnetic moments? The answer they expected, based on classical physics, was that their degree of reorientation would be determined by their initial orientation. In other words, they expected that as the randomly oriented atoms encountered the magnetic field, those atoms whose magnetic moments tended toward parallel orientation with the external field would be attracted toward the south pole of the external magnet, those that tended toward anti-parallel orientation with the external field would be repelled or pushed toward the north pole of the external field, and those that were basically at right angles to the field direction would proceed to the destination unaffected. Under that scenario, the magnetic field would tend to widen the narrow beam, because of the reorientations, resulting in a fairly solid smear of silver being deposited in the center of the detector and decreasing in density toward the edge of the beam.

That is not what happened. The silver atoms did initially enter the chamber randomly oriented, as expected, but when they encountered the magnetic field, they were given only two choices, either parallel or antiparallel, either beam channel 1 or beam channel 2. No other options were allowed. And apparently all the magnetic moments complied, as two distinct arcing bands of silver were deposited on the detector glass with nothing in between. Where the heaviest deposit should have accumulated, according to conventional understanding, there was nothing! According to George E. Uhlenbeck and Samuel A. Goudsmit, the spatial quantization phenomenon is due to the spin of the farthest electron of the silver atom.

It is noteworthy, incidentally, that the two ways of looking at quantum effects – Bohr's quantum transitions between energy levels and Sommerfeld's spatial quantization theory – would foreshadow the two approaches to magnetic resonance taken by Isidor I. Rabi (Nobel Prize 1944) nearly two decades later as well as the nearly-simultaneous, but independent discoveries of NMR in condensed matter in late 1945 and early 1946. Rabi and his collaborators would use both approaches. Edward M. Purcell (Nobel Prize 1952, jointly with Felix Bloch) would focus on quantum transitions between energy levels and Felix Bloch, former student of Werner Heisenberg, on physical reorienting of magnetic moments. Of course, neither Purcell nor Bloch had digital computers available which were powerful and fast enough to apply the *dynamics* inherent to NMR to the purposes of non-invasive clinical imaging. Finally, Nicolaas Bloembergen (Nobel Prize 1981) helped set stage for human MRI scanning and Richard R. Ernst (Nobel Prize 1990) provided the Fourier transform method to the realm of NMR.

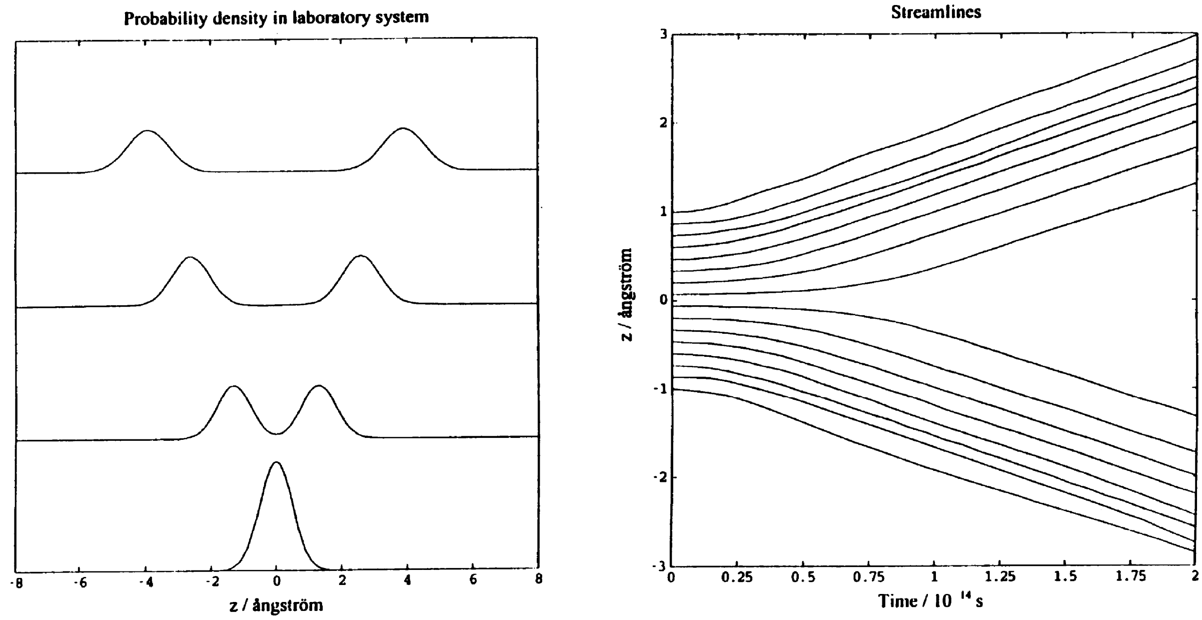

Figure 1. Splitting of a wave packet caused by an impulsive magnetic field. The figure on the left displays the probability density, the figure on the right displays streamlines.

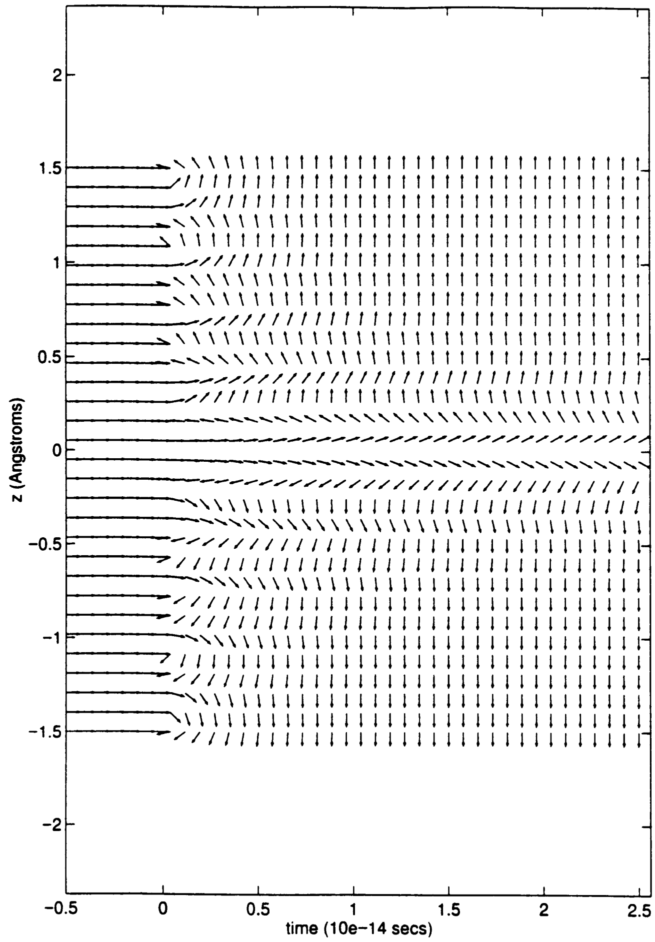

Figure 2. Evolution of the spin vector. Immediately after the shock the spin vectors point in all directions, but after about $2 \times 10^{-14} s$, they sort themselves into the wave packets of the two channels.

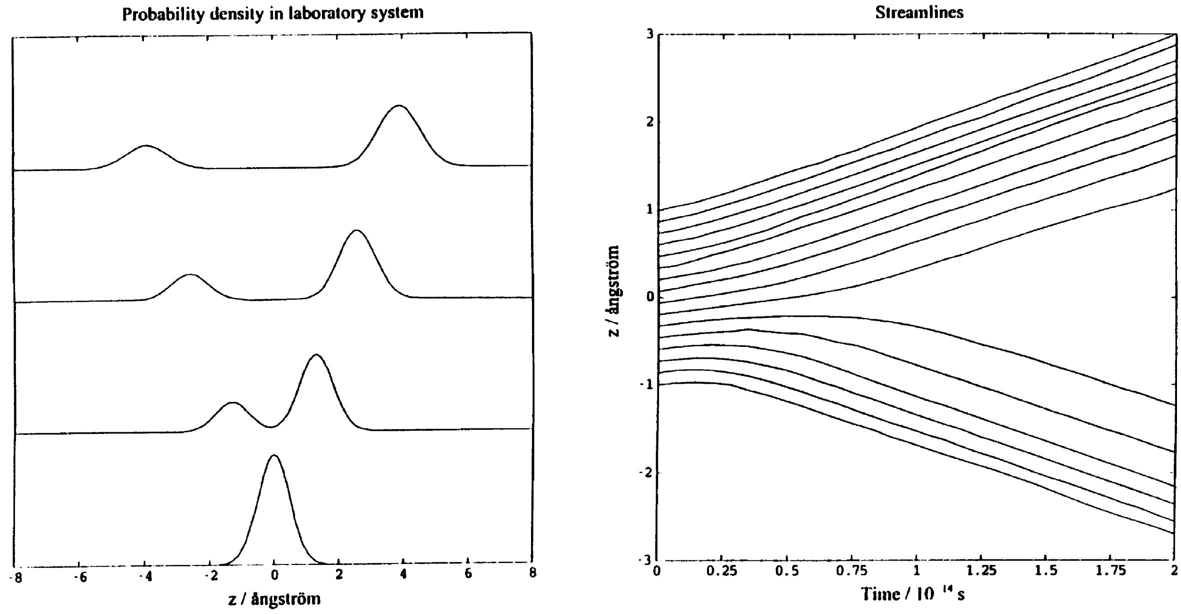

Figure 3. Splitting of a wave packet with different mixtures of spin-up and spin-down components.

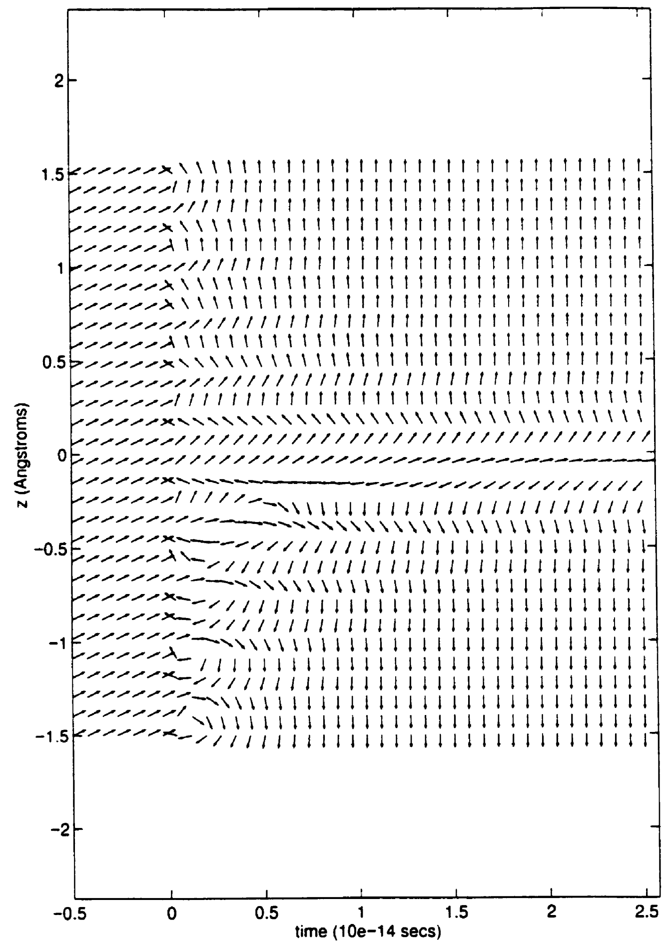

Figure 4. The evolution of the spin vector when different mixtures of spin-up and spin-down components are initially present.

2 Magnetic resonance tomography

The Stern–Gerlach experiment admits a high tech system application to clinical magnetic resonance tomography [22, 18] and magnetic resonance spectroscopy [10] which is suggested by the holographic viewpoint of high resolution radar imaging [16]. Actually clinical MRI is based on a generalization of the Stern–Gerlach filter.

Magnetic resonance tomography has changed the practice of medicine perhaps more than any other clinical technology developed over the past quarter of a century. This cross-sectional imaging modality allows to generate high precision images inside a tomographic slice when the patient is placed in a strong magnetic flux density of high uniformity [12, 17, 21, 25]. The spins of the protons in the tissue start to precess in a coherent way. Because spin ensembles are objects of quantum physics [23], the orbital plane W must be interpreted in terms of quantum physics. This can be done by means of projective spinor quantization which does *not* consider the spin as an intrinsic angular momentum but an orientation of the quantum field information channel. The ensuing method is based on the concept of three-dimensional real Heisenberg Lie group [19]

$$G \hookrightarrow \mathrm{Loop}(\mathbf{T})$$

and its natural fibration

$$G \longrightarrow \mathbf{C}.$$

The planar affine symplectic coadjoint orbits of G are realizations of the affine symplectic plane W not containing the origin, within the dual vector space $\mathrm{Lie}(G)^\star$ of the Lie algebra $\mathrm{Lie}(G)$ [22]. The embedding

$$W \hookrightarrow \mathrm{Lie}(G)$$

makes it apparent that the affine symplectic plane W is also self-dual with respect the Jacobi bracket of the Lie algebra $\mathrm{Lie}(G)$ and that G forms a central *extension* of the horizontal plane $\exp_G W$. The one-dimensional center of G pointed by the origin carries a logarithmic scale along the longitudinal channel direction. The scale comes from the homeomorphism

$$\exp_G : \mathrm{Lie}(G) \longrightarrow G.$$

By duality, the moment map

$$\mu : W \longrightarrow \mathrm{Lie}(G)^\star$$

is injective and defines a geometric quantization procedure. This quantization procedure allows to derive Keppler's Third Law from the Second Law via hyperbolic geometry. Moreover, the moment map μ associates with the coordinate swapping

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightsquigarrow \begin{pmatrix} -y \\ x \end{pmatrix}$$

the Fourier cotransform

$$\bar{\mathcal{F}}_{\mathbf{R}} : \mathrm{L}^2(\mathbf{R}) \longrightarrow \mathrm{L}^2(\mathbf{R}).$$

Thus

$$\bar{\mathcal{F}}_{\mathbf{R}} = \mu(J).$$

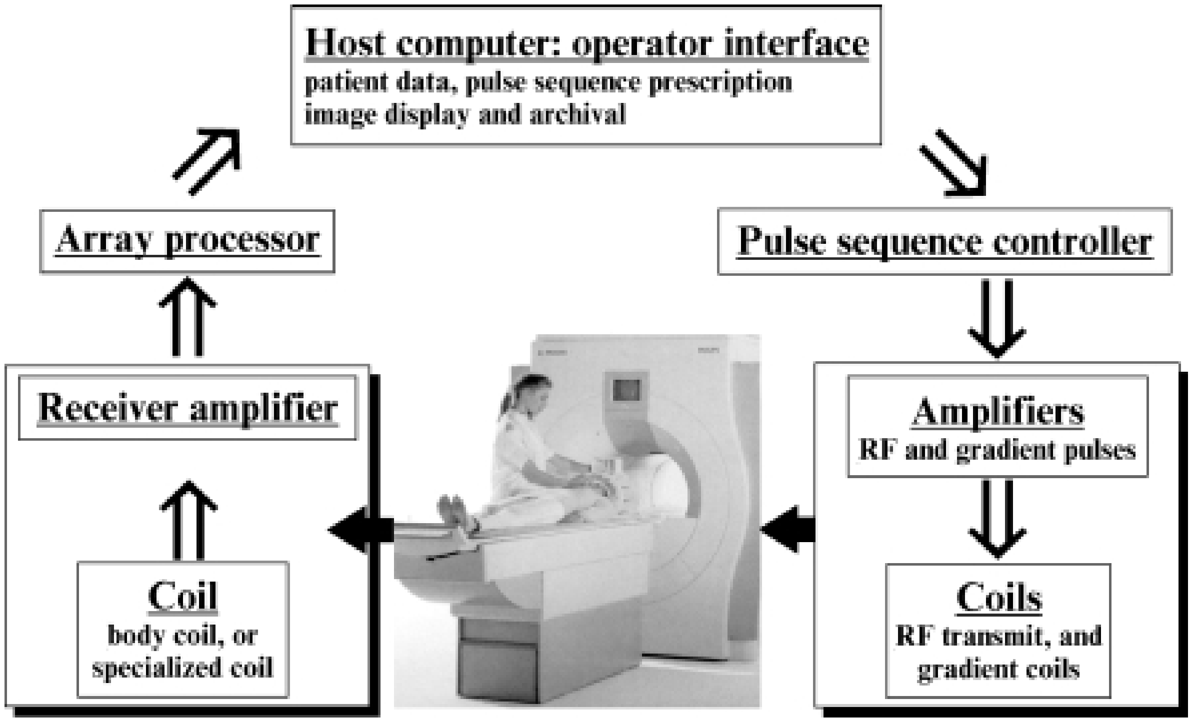

Figure 5. Architecture of a clinical MRI scanner. The system block diagram shows the main hardware components of magnetic resonance tomography organized according to function. The system includes the main magnet, a set of gradient coils, a radiofrequency (RF) coil, a transmitter based on a pulse shape generator of hard and soft RF pulse trains, a receiver endowed with a superheterodyne circuitry, and a work station for image processing. Also shown is the pathway followed by operator commands, pulse sequences, and image data. The imaging subsystem displayed on the right consists of amplifiers (RF and gradients), the excitation RF coil, and the gradient coils used to encode the signals. The detection subsystem displayed on the left consists of the receiver coil, and the hardware involved in the signal amplification pathway.

Due to the moment map μ, the affine symplectic structures of W and the planar coadjoint orbits of G in the dual vector space $\mathrm{Lie}(G)^\star$ are compatible. Due to the action of the one-dimensional compact group

$$\mathbf{U}(1, \mathbf{C}) = \mathbf{T}$$

of gauge transformations, which leaves the one-dimensional center of G invariant, they are suitable structures for describing the *resonance* phenomena of NMR. By making the flux density non-uniform via the controlled application of $\mathbf{R}$-linear distortions, called magnetic field *gradients*, the synchronized phase-frequency coordinates of the points in a spin ensemble inside the selected tomographic slice can be spatially separated. The switching of orthogonal gradients converts the Fourier cotransform $\bar{\mathcal{F}}_{\mathbf{R}}$ via affine $\mathbf{R}$-linear Möbius transformations of the Keppler flow within the affine symplectic plane $W \hookrightarrow \mathbf{P}(\mathbf{R} \times W)$ into a unitary parallel multichannel filter bank. The bank of polyphase filters implemented in the *ruled* plane W transforms the spin density into a clinically valuable planar image (Fig. 7). One of the various advantages of the MRI procedure over X-ray imaging modalities such as CT scanning is that it works in a non-invasive manner [12, 17, 21, 25]. Unlike CT, which is uniplanar, MRI can produce cross-sectional images from any plane.

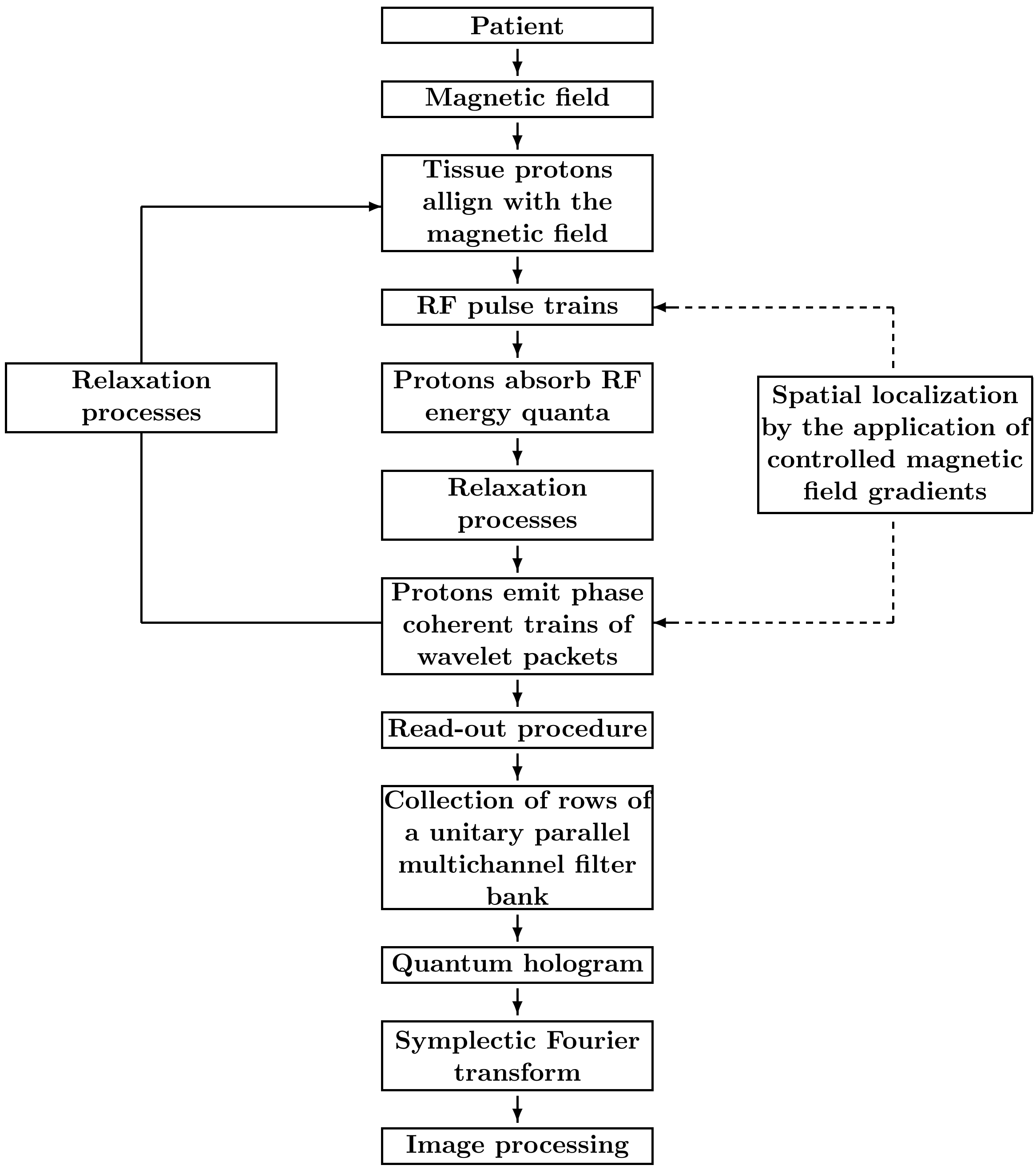

Figure 6. Simplified flowchart displaying the main principles on which a clinical MRI scanner operates. The reconstruction process from a collection of multichannel phase histories is through probing the magnetic moments of nuclei employing strong magnetic flux densities and radiofrequency (RF) electromagnetic radiation. The whole process of clinical MRI is based on a controlled perturbation of the equilibrium magnetization of the object with a train of RF pulses and observing the resulting time-evolving phase coherent response wavelet packet trains in a coil. In terms of line geometry of dimension 3, the collection of a series of 256–512 views establishes a unitary parallel multichannel filter bank which reduces the quantum entropy [8]. In the read-out procedure of the quantum hologram [20], the algorithm of the symplectic Fourier transform reflects the basic structural feature of the clinical MRI modality. The HASTE (half-Fourier acquisition single-shot turbo spin-echo) technique takes advantage of the symmetry properties of the symplectic Fourier transform.

Figure 7. Clinical magnetic resonance tomography: A high resolution sagittal image of the head. In terms of line geometry of dimension 3, the unitary parallel multichannel filter bank generalizes the Stern–Gerlach filter to clinical MRI. It generates the high resolution image which is simulated by a phased-array coil on the receiver side.

Keppler's geometric method of planet tracking by synchronized clockworks, however, admits a completely unexpected high tech system application to the field of non-invasive clinical imaging by NMR [7, 11, 12, 13, 17, 18, 21, 22, 25]. Spin ensembles are excited by the Keppler flow and combine to a planar image within the affine symplectic plane $W \hookrightarrow \mathbf{P}(\mathbf{R} \times W)$ that can be decoded by a unitary parallel multichannel filter bank. In terms of line geometry of dimension 3, the filter bank generalizes the Stern–Gerlach filter (Fig. 7 supra).

Within ten years, clinical MRI had become a billion-dollar business and continues to expand rapidly. Although already well established in the clinical diagnostic centers all over the world, the future of the MRI modality continues to be bright [11, 12, 21, 25]. MRI will become the most important clinical imaging modality by the year 2010. It will be the major morphological staging method for the abdomen and pelvis. MR-guided surgery will be commonplace, and thermal ablation of disc and soft-tissue tumors will be performed under MR control. Morphology, early stroke detection, and even emergency trauma cases will be done by clinical MRI. In most cases, the clinical MRI modality will replace conventional CT whole-body scan interpretation, as well as helical CT scanning [15]. Specifically, brain imaging will no longer make use of CT by 2010 [9].

Quantum teleportation shows that projective spinor quantization is not in conflict with projective relativity [6]. The method is not restricted to RF pulse trains. It extends to the light-in-flight recording by ultrafast laser pulse trains in the pico- and femtosecond regime [1, 2, 3, 4, 5]. Moreover, the technique of projective spinor quantization provides a new approach to the Becchi–Rouet–Stora–Tyutin (BRST) quantization procedure [14, 7]). This underlines the power of the projective viewpoint adopted for the matching of quantum physics and special relativity.

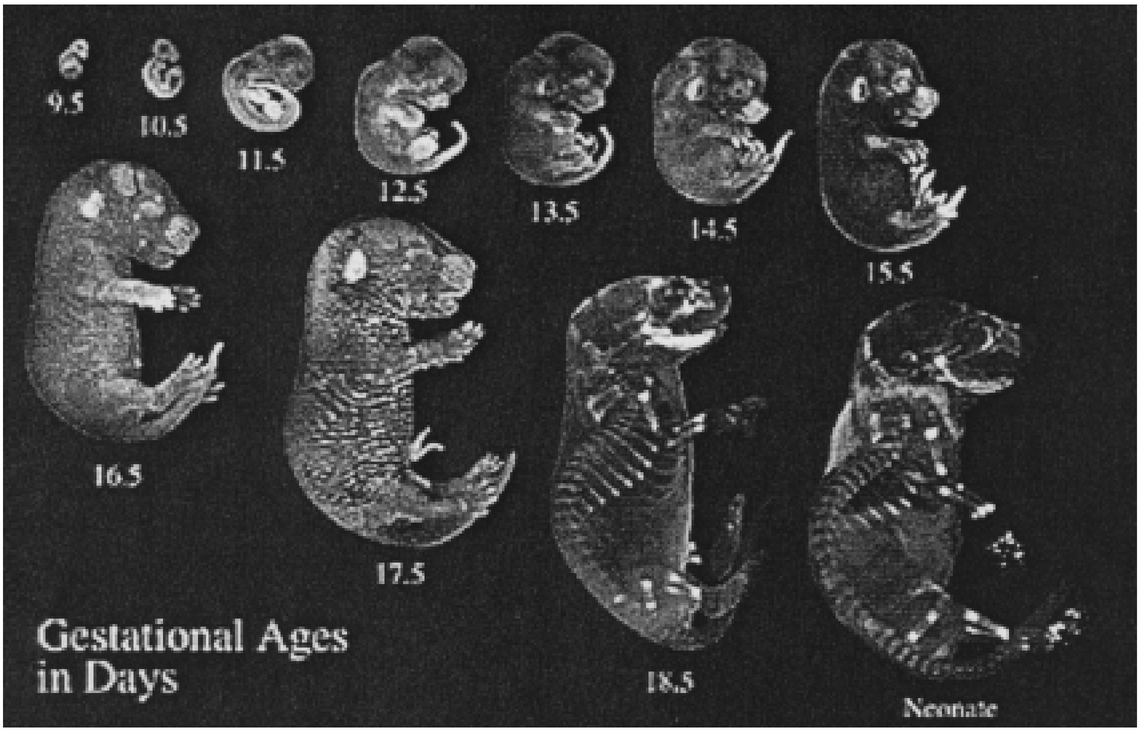

Figure 8. Magnetic resonance microscopy: An embryologic study.

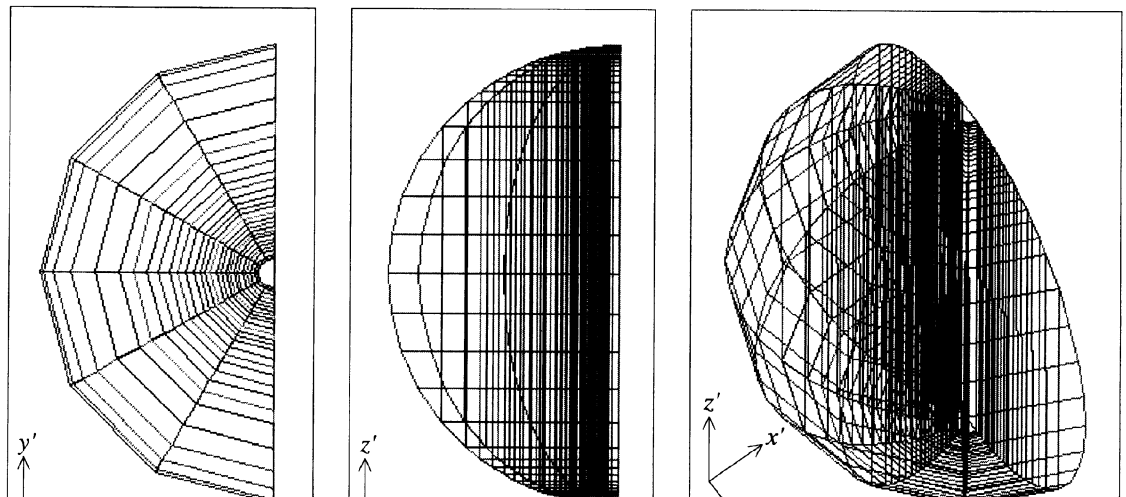

Figure 9. Finite element analysis provides the two types of projectively dual totally orthogonal planes occurring in line geometry of dimension 3. The two planes are cohomologically of different type. The bundle of radial lines as well as the ruled planes FEM visualize the one-dimensional compact group $\mathbf{U}(1, \mathbf{C})$ of gauge transformations acting on $\mathbf{C}^2$. The bank polyphase filters in the ruled planes transform the spin density into a clinically valuable image.

3 Conclusion

As a conclusion, the paper establishes via the Stern–Gerlach filter and gauge group analysis that the higt resolution capability of clinical magnetic resonance tomography is due to a unitary parallel multichannel filter bank generated by quantum holography.

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