Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 68–72.

New Evolution Completely Integrable System

M.Ju. BALAKHNEV, I.V. KULEMIN and A.G. MESHKOV

Oryol State University, 95 Komsomolskaya Str., Oryol, 302015, Russia

We present a detailed algebraic investigation of the evolution system $u_t = u_3 + u_1 v_1 + \delta u_1 + u v_2/2$, $v_t = u^2$ that was obtained in the recent paper by two of the authors. We present the zero curvature representation, the infinite sequences of the conserved densities and Lie–Bäcklund symmetries for the system under consideration. We also found the Noether operator, the Hamiltonian form, and the inverse Noether operator. The one-soliton solution is also obtained.

In our previous paper [1] we presented the classification of evolution systems satisfying the necessary conditions of integrability. This classification was obtained with the help of the conserved canonical densities approach. Here we present more detailed investigation of one of that systems

$$u_t = u_3 + u_1 v_1 - \delta u_1 + \frac{1}{2} u v_2, \qquad v_t = u^2. \tag{1}$$

Here $u_i = (\partial^i u/\partial x^i)$, $u_t = (\partial u/\partial t)$. We found the zero curvature representation, the infinite sequences of the conserved densities and Lie–Bäcklund symmetries for system (1). We also found the Noether operator Θ and the inverse Noether operator J. The operator Θ is implectic and provides the Hamiltonian form of system (1) and the product $\Theta J = \Lambda$ is the recursion operator for system (1). The one-soliton solution was also obtained.

1. To find the linear system realizing the zero curvature representation

$$\Psi_x = U\,\Psi, \qquad \Psi_t = V\,\Psi \tag{2}$$

we assumed that $U = U(u_0, v_0)$. Then we solved the compatibility equation for system (2)

$$U_t - V_x + [\,U,\,V\,] = 0, \tag{3}$$

where $[\,U,\,V\,]$ is the commutator, and obtained the matrices U and V in the following form

$$\begin{aligned} U &= A_1 + A_2\,u + A_3\,v + A_4\,v^2,\\ V &= A_2\,u_2 + 1/2\,A_2\,u\,v_1 + A_5\,u_1 + 1/2\,v\,A_2\,u_1 + 1/8\,v^2\,A_2\,u\\ &\quad +1/2\,u^2\,A_7 + u\,A_6 + 1/2\,u\,v\,A_5 - \delta\,u\,A_2 + A_8, \end{aligned} \tag{4}$$

where A_i are constant unknown matrices satisfying the following commutation relations:

$$\begin{gathered} [\,A_4,\,A_7\,] = 0, \quad [\,A_3,\,A_5\,] = 0, \quad [\,A_2,\,A_8\,] + [\,A_1,\,A_6\,] = \delta\,A_5, \quad [\,A_1,\,A_8\,] = 0,\\ [\,A_2,\,A_7\,] = 0, \quad [\,A_3,\,A_7\,] = -4\,A_4 - A_7, \quad [\,A_3,\,A_8\,] = 0, \qquad [\,A_4,\,A_8\,] = 0,\\ [\,A_4,\,A_6\,] = -1/8\,A_5, \quad [\,A_3,\,A_6\,] = -1/2\,A_6 + \delta/2\,A_2, \quad [\,A_1,\,A_2\,] = A_5,\\ [\,A_1,\,A_7\,] + 2\,[\,A_2,\,A_6\,] = -2\,A_3, \quad [\,A_2,\,A_4\,] = 0, \quad [\,A_1,\,A_5\,] = A_6,\\ [\,A_2,\,A_5\,] = A_7, \quad [\,A_2,\,A_3\,] = -1/2\,A_2, \quad [\,A_3,\,A_5\,] = 0, \quad [\,A_4,\,A_5\,] = -1/8\,A_2. \end{gathered}$$

This table of commutators is obviously not closed, and the first problem is to obtain all commutators $[\,A_i,\,A_j\,]$. Following ideas of Wahlquist and Estabrook [2] we consider the unknown commutators as new elements of Lie algebra. For example, we set $[A_1, A_6] = A_9$ and so on. Then using the Jacobi identity we found some commutation relations for the new elements A_i. But in general case this process is infinite. To make it finite we assume a linear dependence between the elements A_i. It is important that system (1) satisfies sufficiently many conditions of the integrability and representation (3) exists. Therefore if one introduces sufficiently many new elements A_i then the linear constraint provides the closed nontrivial algebra. To close the presented algebra we were forced to consider 19-dimensional Lie algebra. But when we obtained the complete table of the commutators we found a 4-dimensional ideal I. We set the elements of the ideal to be zeros and obtained the 15-dimensional Lie algebra. This new algebra is isomorphic to the factor algebra with respect to ideal I and is simple. We cannot write here the final table of the commutators because it consists of 105 equations.

To construct the representation of the obtained simple algebra we found the Cartan–Weyl basis and the Dynkin diagram for it. It was the diagram of $sl(4)$ algebra. Hence the minimal dimension of a representation of the algebra is 4. The final result takes the following form

$$U = \begin{pmatrix} -\dfrac{v}{2} & \dfrac{1}{2} & 0 & 0 \\ \delta - \dfrac{v^2}{4} & 0 & 0 & \dfrac{1}{2} \\ \dfrac{2\,u}{3} & 0 & 0 & 0 \\ 4\,\mu & \delta - \dfrac{v^2}{4} & -2\,u & \dfrac{v}{2} \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 & -\dfrac{u}{2} & 0 \\ \dfrac{u^2}{3} & 0 & -u_1 - \dfrac{u\,v}{2} & 0 \\ \dfrac{f}{3} & -\dfrac{u_1}{3} - \dfrac{u\,v}{6} & -\mu & \dfrac{u}{6} \\ 0 & \dfrac{u^2}{3} & -f & 0 \end{pmatrix}, \tag{5}$$

where μ is the spectral parameter and

$$f = 2\,u_2 + u\,v_1 + v\,u_1 + 1/4\,v^2\,u - \delta\,u.$$

2. To check whether the obtained zero curvature representation is nontrivial we constructed from the matrix U the sequence of the conserved densities following to J.M. Alberty, T. Koikawa and R. Sasaki's algorithm [3]. Let c be a constant vector and (c, ψ) be the Euclidean scalar product. Setting $\varphi = \psi/(c, \psi)$ one can obtain from system (2) the following nonlinear system

$$\varphi_x = U\,\varphi - \varphi\,(c\,U\varphi), \qquad \varphi_t = V\varphi - \varphi\,(c\,V\varphi). \tag{6}$$

It is easy to check that the following continuity equation

$$(c\,U\varphi)_t = (c\,V\varphi)_x$$

follows from equation (3). Hence the function

$$\rho = (c\,U\varphi) \tag{7}$$

is the generating function for conserved densities of system (1). Setting

$$c = (0,\, 0,\, 1,\, 0),$$

$$\varphi_1 = \sum_{i=1}^{\infty} f_i\,k^i, \qquad \varphi_2 = \sum_{i=1}^{\infty} g_i\,k^i, \qquad \varphi_4 = \sum_{i=1}^{\infty} h_i\,k^i, \qquad k = 1/(4\,\mu),$$

we obtained from (6) the following recursion formulas

$$g_i = 2\,D\,f_i + v\,f_i + \frac{4}{3}\,u\sum_{j=1}^{i-1} f_j\,f_{i-j},$$
$$h_i = 2\,D\,g_i + \left(\frac{1}{2}v^2 - 2\,\delta\right)\,f_i + \frac{4}{3}u\sum_{j=1}^{i-1} f_j\,g_{i-j}, \tag{8}$$
$$f_{i+1} = D\,h_i + \left(\frac{1}{4}\,v^2 - \delta\right)\,g_i - \frac{1}{2}v\,h_i + \frac{2}{3}\,u\sum_{j=1}^{i-1} f_j\,h_{i-j},$$

where $D = \partial/\partial x$, $f_1 = 2\,u$ and $i > 0$. Formula (7) is reduced now to the form

$$\rho = \frac{2}{3}\sum_{i=1}^{\infty} \rho_i\,k^i, \qquad \rho_i = u\,f_i, \tag{9}$$

and provides an infinite sequence of conserved densities ρ_i. It is obvious that the conserved densities provided by equations (8) are local. It can be easily verified that some first even densities are trivial, but the odd densities are nontrivial. Hence the presented zero curvature representation is nontrivial. The first two nontrivial densities take the following form

$$\rho_1 = u^2,$$
$$\rho_3 = u_3^2 + u_3\,(u\,v_2 + 2\,u_1\,v_1) + 2\delta\,u_2^2 + \frac{1}{4}\,u^2\,v_2^2$$
$$+u\,u_2\,v_1\left(\delta - \frac{1}{2}\,v_1\right) + \left(\delta^2 + \frac{7}{3}\,u^2 + \frac{1}{2}\,v_1^2 - \delta\,v_1\right)u_1^2 + \frac{1}{3}\,u^4\,(\delta - v_1).$$

The subsequent densities are very cumbersome and we do not give them here. Let us note that system (1) possesses other conserved densities, that are not expressed by formula (9). For example, the function

$$\rho = \frac{1}{4}\,{v_2}^2 - \delta\,{v_1}^2 + \frac{1}{3}\,{v_1}^3 + 3\,{u_1}^2 - 2\,v_1\,u^2$$

is a conserved density as well.

3. Let us denote by K the vector field that determines system (1), that is, $K = \{u_3 + u_1\,v_1 - \delta\,u_1 + 1/2\,u\,v_2,\ u^2\}$. And let K' be the Fréchet derivative of K and K'^{+} be the adjoint of the operator K'. It is well known (see [4, 5] or [6], for instance) that the equation

$$(D_t - K')\,\sigma = 0,$$

is the determining equation for the Lie–Bäcklund symmetries σ of system (1). And the gradients of conserved densities ($\gamma_\alpha = E_\alpha\,\rho \equiv \{\delta\,\rho/\delta\,u,\ \delta\,\rho/\delta\,v\}$) satisfy the equation

$$(D_t + K'^{+})\,\gamma = 0.$$

In the papers [7] and [8] two following operators were introduced. An operator Θ satisfying the equation

$$(D_t - K')\,\Theta = \Theta\,(D_t + K'^{+}), \tag{10}$$

maps the set of the gradients of the conserved densities Γ into the set of the Lie–Bäcklund symmetries Σ. It is called the Noether operator. And an operator J satisfying the equation

$$(D_t + K'^{+})\, J = J\,(D_t - K'), \tag{11}$$

provides the inverse map $\Sigma \to \Gamma$. It is called the inverse Noether operator. An elementary computation shows that the operator $\Lambda = \Theta J$ is the recursion one. That is, Λ solves the equation

$$[\, D_t - K',\, \Lambda\,] = 0. \tag{12}$$

We found the operators Θ and J for system (1) in the following form:

$$\Theta = \begin{pmatrix} D^3 + (v_1 + \delta)\, D + \frac{1}{2}\, v_2 & -\frac{4}{3}\, u - \frac{2}{3}\, u_1\, D^{-1} \\ \frac{4}{3}\, u - \frac{2}{3}\, D^{-1}\, u_1 & -\frac{4}{3}\, D - \frac{2}{3}\,(D^{-1}\, v_1 + v_1\, D^{-1}) + \frac{4}{3}\,\delta\, D^{-1} \end{pmatrix},$$

$$J = \begin{pmatrix} 36\, D^3 + w\, D + 18\, v_2 + 32\, u\, D^{-1}\, u & 20\, u\, D^2 + 30\, u_1\, D + 8\, u\, w + 12\, u_2 \\ -20\, u\, D^2 - 10\, u_1\, D - 8\, u\, w - 2\, u_2 & D^5 + 5\, w\, D^3 + \frac{15}{2}\, v_2\, D^2 + p\, D + q \end{pmatrix},$$

where $p = 4\, w^2 + 9/2\, v_3 - 2\, u^2$, $q = 4\, v_2\, w + v_4 - 8\, u\, u_1$, $w = v_1 - \delta$. It may be checked that the operator Θ is implectic . Hence it yields the Hamiltonian form of system (1):

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \Theta\, E\, H, \qquad H = \frac{1}{2}\, u^2 = \frac{1}{2}\, \rho_1,$$

where E is the Euler operator. The Noether operator Θ generates an infinite sequence of Lie–Bäcklund symmetries

$$\sigma_n = \Theta\, E\, \rho_n, \qquad n \geq 0, \qquad \rho_0 = 1; \qquad \sigma_0 = \begin{pmatrix} c_1\, u_1 \\ c_1\, v_1 + c_2 \end{pmatrix}, \qquad \sigma_1 = K, \quad \ldots$$

These symmetries can be constructed by means of the recursion operator Λ. One can easily see that the differential part of Λ has order 6. Therefore $\Lambda\, \sigma_0$ is the 7th-order symmetry. But system (1) possesses the lower order symmetries σ_0, σ_1 and σ_2:

$$\begin{aligned} \sigma_2^u &= u_5 + 5/3\, u_3\, v_1 + 5/2\, u_2\, v_2 + 5/9\, u_0\, v_4 + 35/18\, u_1\, v_3 + 5/9\, v_1\, u_0\, v_2 \\ &\quad + 5/18\, u_0\, \delta\, v_2 + 5/9\, u_1\, v_1^2 + 5/9\, u_1\, \delta\, v_1 - 5/9\, \delta^2\, u_1 + 10/9\, u_0^2\, u_1, \\ \sigma_2^v &= -1/9\, v_5 + 20/9\, u_0\, u_2 + 5/9\, u_1^2 - 5/9\, v_1\, v_3 + 5/9\, \delta\, v_3 + 10/9\, v_1\, u_0^2 \\ &\quad + 5/9\, u_0^2\, \delta - 5/12\, v_2^2 + 5/9\, v_1^2\, \delta - 5/27\, v_1^3. \end{aligned}$$

Hence, we have the triple sequence of symmetries: $\sigma_{3n} = \Lambda^n\, \sigma_0$, $\sigma_{3n+1} = \Lambda^n\, K$, $\sigma_{3n+2} = \Lambda^n \sigma_2$.

So, system (1) possesses the nontrivial zero curvature representation and is exactly solvable. We present the one-soliton solution of system (1):

$$u = \frac{k^2\, \sqrt{3}}{\cosh(k\, x + k^3\, t)}, \qquad v = 3\, k \tanh(k\, x + k^3\, t) + \delta\, x.$$

Here are the plots of this solution for $t = 0$, $k = 2$ and two values of δ:

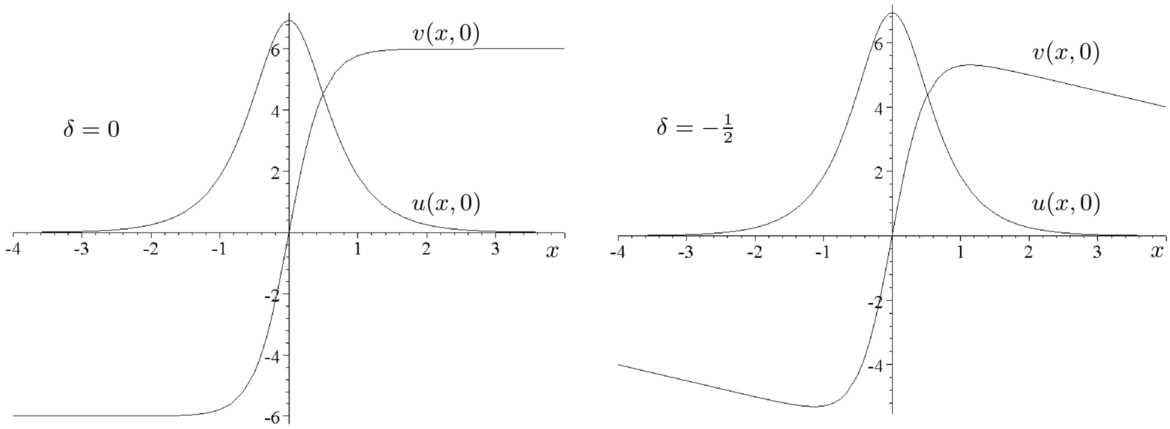

Fig. 1, 2. Soliton solution of system (1).

It is obvious that the u-curve has the typical soliton form and the v-curve has the kink form. The plots of the function v have two asymptotics $v = \delta\, x \pm 3\, k$.

In conclusion we note that system (1) can be reduced to the following single equation

$$v_{tt} = \frac{\partial}{\partial x}\left[v_{txx} - \frac{3}{4}\frac{v_{tx}^2}{v_t} + v_t\, v_x + \delta\, v_t \right]$$

that is integrable of course.

All calculations were performed with the help of an IBM computer and the JET package presented in the separate paper.

References

[1] Kulemin I.V. and Meshkov A.G., To the classification of the integrable systems in $1+1$ dimensions, Proc. of Second International Conference "Symmetry in Nonlinear Mathematical Physics", 1997, V.1, 115–121.

[2] Wahlquist H.D. and Estabrook F.B., Prolongation structures of nonlinear evolution equations, *J. Math. Phys.*, 1975, V.16, 1–7.

[3] Alberty J.M., Koikawa T. and Sasaki R., Canonical structure of soliton equations, *Physica D*, 1982, V.5, N 1, 43–74.

[4] Ibragimov N.H., Transformation Groups in Mathematical Physics, Moscow, Nauka, 1983.

[5] Olver P.J., Applications of Lie Groups to Differential Equations, New York, Springer-Verlag, 1986.

[6] CRC Handbook of Lie Group Analysis of Differential Equations, ed. N.H. Ibragimov, London, Tokyo, CRC Press, 1994, 1995, etc.

[7] Fokas A.S. and Fuchssteiner B., *Lett. Nuovo Cimento*, 1980, V.28, 299.

[8] Fuchssteiner B. and Fokas A.S., *Physica D*, 1981, V.4, 47.