Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 411–416.

Exact Solutions of the Inhomogeneous Problems for Polyparabolic Operator

Alexander N. ZHUKOVSKY and Victor P. KARAGODOV

Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivska Street, Kyiv, Ukraine

The theorem establishing the correctness of the inhomogeneous problem for polyparabolic equation with righ-hand side belonging to set of bounded functions in $\mathbf{R}^n$ is proved. Exact formulas for constants of evaluations for potentials with the these densities are represented; exact solutions for particular cases are obtained.

We consider inhomogeneous problem for a linear partial differential equation

$$T^{m+1}u \equiv \sum_{j=0}^{m+1}(-1)^j\binom{m+1}{j}\frac{\partial^{m-j+1}}{\partial t^{m-j+1}}\nabla^{2j}u(t,\mathbf{x}) = f(t,\mathbf{x}), \tag{1}$$

where $t \in \mathbf{R}^1_+$, $\mathbf{x} \in \mathbf{R}^n$ $(n \in \mathbf{N})$, ∇^2 is the Laplace operator, $f(t,\mathbf{x}) \in L^1_{\mathrm{loc}}(\mathbf{R}^{1+n})$, $\binom{m+1}{j}$ are binomial coefficients.

In the case $m = 0$ this equation is transformed to the classical heat transfer equation. Let us introduce an arbitrary exact solution for equation $T^{m+1}u = 0$, which is defined at a domain of space $\mathbf{R}^{n+1}$, the polycaloric function [1, 2], which takes the form

$$u(t,\mathbf{x}) = u_0(t,\mathbf{x}) + tu_1(t,\mathbf{x}) + \cdots + t^m u_m(t,\mathbf{x}), \tag{2}$$

where $u_k(t,\mathbf{x})$ are solutions of the equation $Tu = 0$. We find by induction

$$T^{m+1}(t^k u) = \sum_{j=0}^{m+1} j!\binom{m+1}{j}\binom{k}{j}t^{k-j}T^{m-j+1}u. \tag{3}$$

The fundamental solution for operator T^{m+1} from space $\mathcal{D}'(\mathbf{R}^{1+n})$ is [3]

$$\mathcal{E}_{m,n}(t,\mathbf{x}) = \frac{\theta(t)t^{m-n/2}}{(2\sqrt{\pi})^n m!}e^{-\frac{|\mathbf{x}|^2}{4t}}. \tag{4}$$

It is positive, vanishing for $t < 0$, infinitely differentiable for $(t,\mathbf{x}) \neq 0$ and has additional properties

$$\int\limits_{\mathbf{R}^n} \mathcal{E}_{m,n}(t,\mathbf{x})\,d^n\mathbf{x} = \frac{t^m}{m!}, \tag{5}$$

$$\frac{\partial^k}{\partial t^k}\mathcal{E}_{m,n}(+0,\mathbf{x}) = 0 \quad (0 \le k \le m-1), \qquad \frac{\partial^m}{\partial t^m}\mathcal{E}_{m,n}(+0,\mathbf{x}) = 1. \tag{6}$$

From $\mathcal{E}_{m,n} \in L^1_{\mathrm{loc}}(\mathbf{R}^{1+n})$ the solution of the problem (1) can be written as convolution [4]

$$u(t,\mathbf{x}) = \mathcal{E}_{m,n}(t,\mathbf{x}) * f(t,\mathbf{x}), \tag{7}$$

which define a polycaloric potential with density $f(t, \mathbf{x})$. Then $u \in L^1_{\mathrm{loc}}(\mathbf{R}^{1+n})$, if

$$h(t, \mathbf{x}) = [\mathcal{E}_{m,n}(t, \mathbf{x}) * |f(t, \mathbf{x})|] \in L^1_{\mathrm{loc}}(\mathbf{R}^{1+n}). \tag{8}$$

The following theorem gives one of density classes with convolution (7). For simplification we do not write further indices m and n.

Let us denote a class of functions vanishing for $t < 0$ and bounded in the sphere $0 \le t \le t_0$: $|f| \le A_f = \sup |f(\tau, \xi)|$ $(0 \le \tau \le t\ ,\ \xi \in \mathbf{R}^n)$ as K_0.

Theorem. *If $f(t, \mathbf{x}) \in K_0$, a polycaloric potential $U(t, \mathbf{x})$ of m-th order is in K_0, can be written in the form (7) and satisfies the following estimates:*

$$|U| \le A_f \frac{t^{m+1}}{(m+1)!}, \tag{9}$$

$$\left| \frac{\partial^k U}{\partial t^k} \right| \le A_f a^{(k)}_{m,n} t^{m-k+1}, \tag{10}$$

$$\left| \nabla^{2p} U \right| \le A_f b^{(p)}_{m,n} t^{m-p+1}, \tag{11}$$

Here $a^{(k)}_{m,n}$ and $b^{(p)}_{m,n}$ are positive constants, and initial conditions are

$$U(+0, \mathbf{x}) = 0, \tag{12}$$

$$\left. \frac{\partial^k U}{\partial t^k} \right|_{t=+0} = 0 \quad (1 \le k \le m), \qquad \nabla^{2p} U|_{t=+0} = 0 \quad (1 \le p \le m), \tag{13}$$

$$T^s U(+0, \mathbf{x}) = 0 \quad (1 \le s \le m). \tag{14}$$

Proof. Under Fubini theorem from (5) we get

$$h(t, \mathbf{x}) \le A_f \frac{t^{m+1}}{(m+1)!}.$$

As $|U| \le h$, since $U = 0$ for $t < 0$, estimate (9) is satisfied and thus $U \in K_0$.

Using the formula of convolution differentation with respect to t and property (6), for $t > 0$ we come to

$$\frac{\partial^k U}{\partial t^k} = \int\limits_0^t \int\limits_{\mathbf{R}^n} f(\tau, \xi) \frac{\partial^k}{\partial t^k} \mathcal{E}(t - \tau, \mathbf{x} - \xi)\, d^n\xi\, d\tau.$$

Then

$$\left| \frac{\partial^k U}{\partial t^k} \right| \le A_f \int\limits_{\mathbf{R}^n} \frac{\partial^{k-1}}{\partial t^{k-1}} \mathcal{E}(t, \xi)\, d^n\xi. \tag{15}$$

Further

$$\frac{\partial^{k-1} \mathcal{E}}{\partial t^{k-1}} = \frac{1}{(2\sqrt{\pi})^n m!} \sum_{l=0}^{k-1} \binom{k-1}{l} \frac{d^{k-l-1} t^{m-n/2}}{dt^{k-l-1}} \frac{\partial^l}{\partial t^l} e^{-\frac{|\xi|^2}{4t}},$$

and

$$\frac{d^{k-l-1}t^{m-n/2}}{dt^{k-l-1}} = \frac{\Gamma\left(m-\frac{n}{2}+1\right)}{\Gamma\left(m-\frac{n}{2}-k+l+2\right)} t^{m-\frac{n}{2}-k+l-1},$$

$$\frac{\partial^l}{\partial t^l} e^{-\frac{|\xi|^2}{4t}} = t^{-l} e^{-\frac{|\xi|^2}{4t}} \sum_{j=0}^{l-1} (-1)^j (l-1)\dots(l-j) \binom{l}{j} \left(\frac{|\xi|^2}{4t}\right)^{l-j},$$

follows term in (15)

$$\int\limits_{\mathbf{R}^n} \frac{\partial^{k-1}}{\partial t^{k-1}} \mathcal{E}(t,\xi)\, d^n\xi = \frac{t^{m-\frac{n}{2}+k+1}}{(2\sqrt{\pi})^n m!} \sum_{l=0}^{k-1} \binom{k-1}{l} \frac{\Gamma\left(m-\frac{n}{2}+1\right)}{\Gamma\left(m-\frac{n}{2}-k+l+2\right)}$$
$$\times \sum_{j=0}^{l-1} (-1)^j (l-1)(l-2)\dots(l-j) \binom{l}{j} \int\limits_{\mathbf{R}^n} \left(\frac{|\xi|^2}{4t}\right)^{l-j} e^{-\frac{|\xi|^2}{4t}}\, d^n\xi.$$

Here $\Gamma(z)$ is the Gamma function [5].

But inserting

$$\begin{aligned} &\int\limits_{\mathbf{R}^n} |\xi|^{2l-2j} e^{-\frac{|\xi|^2}{4t}}\, d^n\xi \\ &= \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int\limits_0^\infty \rho^{2(l-j)+n-1} e^{-\frac{\rho^2}{4t}}\, d\rho = \frac{(2\sqrt{\pi})^n}{\Gamma\left(\frac{n}{2}\right)} 2^{2l-2j} \Gamma\left(l-j+\frac{n}{2}\right) t^{l-j+\frac{n}{2}}, \end{aligned} \tag{16}$$

into (15) yields estimate (10), where a constant $a_{m,n}^{(k)}$ depends on the order of polycaloric potential, order of its derivative with respect to time, dimension of space and is calculated exactly:

$$a_{m,n}^{(k)} = \frac{\Gamma\left(m-\frac{n}{2}+1\right)}{m!\Gamma\left(\frac{n}{2}\right)} \sum_{l=1}^{k-1} \sum_{j=0}^{l-1} \frac{\binom{k-1}{l} \left(\binom{l}{j}\right)^2 \Gamma\left(l-j+\frac{n}{2}\right)(l-j)j!}{l\,\Gamma\left(m-\frac{n}{2}-k+l+2\right)}.$$

Following the usual procedure for finding estimates let us consider operator

$$\nabla^{2p} U = \nabla^{2p}[\mathcal{E}(t,\mathbf{x}) * f(t,\mathbf{x})] = f(t,\mathbf{x}) * \nabla^{2p}\mathcal{E}(t,\mathbf{x}).$$

Since $f \in K_0$, then

$$\left|\nabla^{2p} U\right| \le A_f \int\limits_0^t \int\limits_{\mathbf{R}^n} \nabla^{2p}\mathcal{E}(\tau,\mathbf{x})\, d^n\mathbf{x}\, d\tau = \frac{A_f}{(2\sqrt{\pi})^n m!} \int\limits_0^t \tau^{m-n/2} \int\limits_{\mathbf{R}^n} \nabla^{2p} e^{-\frac{|\mathbf{x}|^2}{4\tau}}\, d^n\mathbf{x}\, d\tau. \tag{17}$$

We can easily prove that

$$\nabla^{2p} e^{-\frac{|\mathbf{x}|^2}{4\tau}} = 2^{-2p} e^{-\frac{|\mathbf{x}|^2}{4\tau}} \sum_{j=0}^{p} \frac{(-1)^j (2j)! \binom{2p}{2j}}{j!\, \tau^{2p-j}} |\mathbf{x}|^{2p-2j},$$

hence in (17)

$$\int\limits_{\mathbf{R}^n} \nabla^{2p} e^{-\frac{|\mathbf{x}|^2}{4\tau}}\, d^n\mathbf{x} = 2^{-2p} \sum_{j=0}^{p} \frac{(-1)^j (2j)! \binom{2p}{2j}}{j!\, \tau^{2p-j}} \int\limits_{\mathbf{R}^n} |\mathbf{x}|^{2p-2j} e^{-\frac{|\mathbf{x}|^2}{4\tau}}\, d^n\mathbf{x}. \tag{18}$$

Here the integral in the right hand side exists and is calculated according to the formula (16). Inserting the result of calculation into inequality (17), we obtain final result (11). Here a constant $b_{m,n}^{(p)}$ can be written in the form:

$$b_{m,n}^{(p)} = \frac{(2p)!}{(m-p+1)m!\,\Gamma\left(\frac{n}{2}\right)} \sum_{j=0}^{p} \frac{\Gamma\left(p-j+\frac{n}{2}\right)}{2^{2j}(2p-2j)!}.$$

So we proved the estimates (9)–(11). Hence the polycaloric potential satisfies conditions (12) and (13), and the initial condition (14) follows from previous formula (3).

From Theorem we can apply the results for equation (1) with inhomogeneous initial conditions corresponding conditions (14):

$$T^k u(+0,\mathbf{x}) = \varphi_k(\mathbf{x}) \quad (1 \le k \le m).$$

We look for exact solutions of problem (1) in the form corresponding (7):

$$u(t,\mathbf{x}) = \frac{1}{(2\sqrt{\pi})^n m!} \int_0^t \tau^{m-n/2} e^{-\frac{|\mathbf{x}|^2}{4\tau}}\, d\tau \int_{\mathbf{R}^n} f(t-\tau,\xi) e^{-\frac{|\xi|^2-2(\mathbf{x}\cdot\xi)}{4\tau}}\, d^n\xi. \tag{19}$$

Namely, we consider cases of the general formula (19). If $f = f(t,|\mathbf{x}|)$, then for $n \ge 2$ from (19) we obtain

$$u(t,\mathbf{r}) = \frac{1}{2^{n-1}\sqrt{\pi}\,\Gamma\left(\frac{n-1}{2}\right)} \int_0^t \tau^{m-n/2} e^{-\frac{|\mathbf{r}|^2}{4\tau}}\, d\tau$$

$$\times \int_0^\infty \rho^{n-1} e^{-\frac{\rho^2}{4\tau}} f(t-\tau,\rho) d\rho \int_0^\pi e^{\frac{\mathbf{r}\rho}{2\tau}\cos\varphi} \sin^{n-2}\varphi\, d\varphi,$$

where $\mathbf{r}^2 = x_1^2 + x_2^2 + \cdots + x_n^2$. Since for $\nu > 0$

$$\int_0^\pi e^{\pm z\cos\varphi} \sin^{2\nu}\varphi\, d\varphi = \sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)\left(\frac{2}{z}\right)^\nu I_\nu(z),$$

then for $n \ge 2$

$$u(t,\mathbf{r}) = \frac{1}{2\mathbf{r}^{\frac{n}{2}-1} m!} \int_0^t \tau^{m-1} e^{-\frac{|\mathbf{r}|^2}{4\tau}}\, d\tau \int_0^\infty \rho^{\frac{n}{2}} e^{-\frac{\rho^2}{4\tau}} I_{\frac{n}{2}-1}\left(\frac{\mathbf{r}\rho}{2\tau}\right) f(t-\tau,\rho)\, d\rho. \tag{20}$$

Here $I_\nu(z)$ is the Bessel function [5].

The degenerate case $n = 1$ has the following solution

$$u(t,\mathbf{x}) = \frac{1}{2\sqrt{\pi} m!} \int_0^t \tau^{m-\frac{1}{2}} e^{-\frac{\mathbf{x}^2}{4\tau}}\, d\tau \int_{-\infty}^\infty e^{-\frac{\xi^2-2\mathbf{x}\xi}{4\tau}} f(t-\tau,\xi)\, d\xi. \tag{21}$$

If the dimension of space is odd, then the integral with respect to ρ in (16) yields the integral from elementary functions. Namely, since for $n = 3$

$$I_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\mathrm{sh} z,$$

solution (20) is

$$u(t,\mathbf{r}) = \frac{1}{\sqrt{\pi} m!\,\mathbf{r}} \int\limits_0^t \tau^{m-\frac{1}{2}} e^{-\frac{\mathbf{r}^2}{4\tau}}\,d\tau \int\limits_0^\infty \rho^2 e^{-\frac{\rho^2}{4\tau}} \operatorname{sh}\left(\frac{\mathbf{r}\rho}{2\tau}\right) f(t-\tau,\rho)\,d\rho. \tag{22}$$

Separately, we consider the case when $f(t,\mathbf{x})$ is a finite function in $\mathbf{R}^n$. If

$$f(t,\mathbf{x}) = A\omega(t) F(|\mathbf{x}|)\,\theta(R^2 - |\mathbf{x}|^2) \qquad (A, R = \text{const} > 0,\ n \geq 2).$$

then for same density from (20) we obtain

$$u(t,\mathbf{r}) = \frac{1}{2\mathbf{r}^{\frac{n}{2}-1} m!} \int\limits_0^t \tau^{m-1} e^{-\frac{\mathbf{r}^2}{4\tau}} \omega(t-\tau)\,d\tau \int\limits_0^R \rho^{\frac{n}{2}} e^{-\frac{\rho^2}{4\tau}} I_{\frac{n}{2}-1}\left(\frac{\mathbf{r}\rho}{2\tau}\right) F(\rho)\,d\rho.$$

Let us consider $n = 1$ and $f(t,\mathbf{x}) = A\theta(t)\theta(R - |\mathbf{x}|)$. From (21) we get solution in the form

$$u(t,\mathbf{x}) = \frac{A}{2m!} \int\limits_0^t \tau^m \left[\operatorname{erf}\left(\frac{R+\mathbf{x}}{2\sqrt{\tau}}\right) + \operatorname{erf}\left(\frac{R-\mathbf{x}}{2\sqrt{\tau}}\right)\right] d\tau, \tag{23}$$

where $\operatorname{erf}(z)$ is the probabilistic integral [5]. Then using

$$\int\limits_0^t \tau^m \operatorname{erf}\left(\frac{z}{2\sqrt{\tau}}\right) d\tau = 2\left(\frac{z}{2}\right)^{2m+2} \int\limits_{\frac{z}{2\sqrt{t}}}^\infty \xi^{-2m-3} \operatorname{erf}(\xi)\,d\xi,$$

and the formula [3]

$$\int\limits_\xi^\infty \xi^{-2m-3} \operatorname{erf}(\xi)\,d\xi = \frac{1}{2(m+1)} \left\{ \frac{(-1)^{m+1}\sqrt{\pi}}{\Gamma\left(m+\frac{3}{2}\right)} \operatorname{erfc}(\xi) \right.$$

$$\left. + \xi^{-2m-2} \left[\operatorname{erf}(\xi) - \frac{1}{\pi} e^{-\xi^2} \sum_{k=1}^{m+1} (-1)^k \frac{\Gamma\left(m-k+\frac{3}{2}\right)}{\Gamma\left(m+\frac{3}{2}\right)} \xi^{2k+1} \right] \right\},$$

we obtain exact solution from (23)

$$u(t,\mathbf{x}) = \frac{At^m}{2(m+1)!} \left[\Phi_m\left(\frac{R+\mathbf{x}}{2\sqrt{t}}\right) + \Phi_m\left(\frac{R-\mathbf{x}}{2\sqrt{t}}\right)\right], \tag{24}$$

where

$$\Phi_m(z) = \operatorname{erf}(z) + \frac{(-1)^{m+1}\sqrt{\pi}}{\Gamma\left(m+\frac{3}{2}\right)} z^{2m+2} \operatorname{erfc}(z) - \frac{1}{\pi} e^{-z^2} \sum_{k=1}^{m+1} (-1)^k \frac{\Gamma\left(m-k+\frac{3}{2}\right)}{\Gamma\left(m+\frac{3}{2}\right)} z^{2k+1}.$$

We can verify easily that reduced exact solutions of problem (1) for density from K_0 satisfied proved above theorem.

Thus we can apply the results for a problem

$$\sin Tu = F(t,\mathbf{x}). \tag{25}$$

Using the expansion of left side in series and generalizing (2) and (3), we can write the fundamental solution of operator $\sin T$ from space $\mathcal{D}'(\mathbf{R}^{1+n})$ in the form of expansion into series

$$\begin{aligned}\mathcal{S}_n(t,\mathbf{x}) &= \sum_{m=0}^{\infty} \frac{(-1)^m \mathcal{E}_{m,n}(t,\mathbf{x})}{(2m+1)!} \\ &= \frac{\theta(t) e^{-\frac{|\mathbf{x}|^2}{4t}}}{(2\sqrt{\pi})^n} \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m-n/2}}{(2m)!(2m+1)!} = \frac{\theta(t) e^{-\frac{|\mathbf{x}|^2}{4t}} t^{-n/2-1}}{(2\sqrt{\pi})^n} \int_0^t \operatorname{ber} 2\sqrt{\tau}\, d\tau.\end{aligned} \tag{26}$$

This expansion is absolutely convergent for $t > 0$ and has the following properties:

$$\int_{\mathbf{R}^n} \mathcal{S}_n(t,\mathbf{x})\, d^n\mathbf{x} = \sum_{m=0}^{\infty} \frac{(-1)^m t^{2m}}{(2m)!(2m+1)!},$$

$$\frac{\partial^k}{\partial t^k}\mathcal{S}_n(+0,\mathbf{x}) = 0 \quad (k = 2p-1), \qquad \frac{\partial^k}{\partial t^k}\mathcal{S}_n(+0,\mathbf{x}) = \frac{(-1)^p}{(2p+1)!} \quad (k = 2p).$$

Then we obtain a solution of problems (25) for $f(t,\mathbf{x}) \in \mathbf{K_0}$ in the form of convolution

$$u(t,\mathbf{x}) = \mathcal{S}_n(t,\mathbf{x}) * f(t,\mathbf{x}), \tag{27}$$

that has form similarly to (20) and (21) for $n \geq 2$

$$\begin{aligned} u(t,\mathbf{r}) &= \frac{1}{2\mathbf{r}^{\frac{n}{2}-1}} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!(2m+1)!} \int_0^t \tau^{2m-1} e^{-\frac{|\mathbf{r}|^2}{4\tau}}\, d\tau \\ &\times \int_0^{\infty} \rho^{\frac{n}{2}} e^{-\frac{\rho^2}{4\tau}} I_{\frac{n}{2}-1}\left(\frac{\mathbf{r}\rho}{2\tau}\right) f(t-\tau,\rho)\, d\rho, \end{aligned} \tag{28}$$

and for $n = 1$

$$u(t,\mathbf{x}) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2\sqrt{\pi}(2m)!(2m+1)!} \int_0^t \tau^{2m-\frac{1}{2}} e^{-\frac{\mathbf{x}^2}{4\tau}}\, d\tau \int_{-\infty}^{\infty} e^{-\frac{\xi^2-2\mathbf{x}\xi}{4\tau}} f(t-\tau,\xi)\, d\xi. \tag{29}$$

If $n = 1$ and $f(t,\mathbf{x}) = A\theta(t)\theta(R - |\mathbf{x}|)$, we find the exact solution of problem (25)

$$u(t,\mathbf{x}) = \sum_{m=0}^{\infty} \frac{(-1)^m A t^{2m}}{2((2m+1)!)^2} \left[\Phi_{2m}\left(\frac{R+\mathbf{x}}{2\sqrt{t}}\right) + \Phi_{2m}\left(\frac{R-\mathbf{x}}{2\sqrt{t}}\right)\right]. \tag{30}$$

References

[1] Tichonov A.N., Theoremes d'unicite pour l'equation de la chaleur, *Matem. sbornik*, 1935, V.42, 199–216.

[2] Nicolescu M., Ecuatia iterata a caldurii, *Studii si Cercetari Matematice*, 1954, V.5, N 3–4, 243–332.

[3] Galitsyn A.S., Fundamental solution of polyparabolic operator, *Proc. Acad. Sci. Ukraine*, 1995, N 6, 36–38.

[4] Vladimirov V.S., Generalised Functions in Mathematical Physics, Moscow, Nauka, 1976 (in Russian).

[5] Bateman H. and Erdelyi A., Higher Transcendental Functions, Vol.2, New York, Mc Graw-Hill Book Company, 1953.