Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 406–410.

Explicit Solutions of Nonlinear Evolution Equations via Nonlocal Reductions Approach

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We consider a system of nonlinear partial differential equations admitting the operator Zakharov–Shabat representation. By means of nonlocal reductions approach explicit solutions of the equations under consideration are found.

In the last 30 years a great progress in the investigation of nonlinear partial differential equations of mathematical and theoretical physics was achieved due to application of different approaches mainly based on modern functional and algebraic-geometric methods [1–3]. It gave possibility to study different properties of solutions important for applications in physics, mechanics and other fields of knowledge and to construct exact formulae for solutions of many systems of nonlinear differential equations having numerous applications.

In the present communication we consider an approach connected with studying of a system of nonlinear partial differential equations possessing so-called operator Zakharov–Shabat representation which is based on reduction principle. The formula for exact solutions of a system of nonlinear partial differential equations describing resonance interaction of M waves is given.

1. Let Φ be a function of variables $x, y, t \in \mathbf{R}^1$ satisfying to a system of linear partial differential equations

$$\alpha\frac{\partial\Phi}{\partial y} = L_1\Phi, \qquad \beta\frac{\partial\Phi}{\partial t} = L_2\Phi, \qquad \alpha,\beta\in\mathbf{R}^1, \tag{1}$$

where differential operators

$$L_1 = \sum_{i=0}^{p} u_i\frac{\partial^i}{\partial x^i}, \qquad u_i = u_i(x,y,t), \quad i=\overline{0,p}, \tag{2}$$

$$L_2 = \sum_{0}^{q} v_j\frac{\partial^j}{\partial x^j}, \qquad v_j = v_j(x,y,t), \quad j=\overline{0,q}, \tag{3}$$

are defined in the corresponding functional space.

A system (1)–(3) is compatible if the following Frobenius (operator) condition

$$\beta\frac{\partial L_1}{\partial t} - \alpha\frac{\partial L_2}{\partial y} + [L_1, L_2] = 0 \tag{4}$$

takes place, where in the operator equation $[L_1, L_2] = L_1L_2 - L_2L_1$ is a commutator of differential operators L_1 and L_2.

The operator relation (4) is called the Zakharov–Shabat representation or the generalized Lax representation [1, 2].

In general the equality (4) is fulfilled if the potentials (coefficients of operators L_1 and L_2)

$$u_i = u_i(x, y, t), \quad i = \overline{0, p}, \qquad v_j = v_j(x, y, t), \quad j = \overline{0, q}, \tag{5}$$

satisfy a system of partial differential equations usually written in the form

$$K_l[\mathbf{u}, \mathbf{v}] = 0, \qquad l = \overline{0, p+q}, \tag{6}$$

where $\mathbf{u} = \{u_1, u_2, \ldots, u_p\}$, $\mathbf{v} = \{v_1, v_2, \ldots, v_q\}$.

The equations (4) and (6) are equivalent in the certain sense [1, 2].

The given compatible linear system (1)–(3) has an important significance and practical applications when the coefficients of the operators (2), (3) satisfy to some additional conditions, which are called *reductions.*

One of the important problem of the modern theoretical and mathematical physics and, in particular, the theory of nonlinear dynamical system [2, 3], is the problem of classification and description of the reductions under which the system (1)–(3) is compatible.

There is an important type of restrictions under which the system of differential equations (1)–(3) is compatible. This restrictions are called nonlocal reductions [4, 5].

Definition. *A system of equations of general form*

$$F[\mathbf{u}, \mathbf{v}; \Phi; x, y, t] = 0 \tag{7}$$

is called a nonlocal reduction for the system (1)–(3) if the equations (7) and (1)–(3) are compatible.

If nontrivial functions

$$u_i = F_i[\Phi; x, y, t], \quad i = \overline{0, p}, \qquad v_j = G_j[\Phi; x, y, t], \quad j = \overline{0, q}, \tag{8}$$

satisfy the equations (7), then after substitution of functions (8) into system (1)–(3) we obtain a system of nonlinear partial differential equations of the following form

$$\alpha \frac{\partial \Phi}{\partial y} = \sum_{i=0}^{p} F_i[\Phi; x, y, t] \frac{\partial^i \Phi}{\partial x^i}, \tag{9}$$

$$\beta \frac{\partial \Phi}{\partial t} = \sum_{j=0}^{q} G_j[\Phi; x, y, t] \frac{\partial^j \Phi}{\partial x^j}. \tag{10}$$

The system (9), (10) is compatible due to the conditions (4). In other words, the functions (8) satisfy the system of partial differential equations (6).

Thus, the problem of solving the system of nonlinear differential equations (7) equivalent to the operator equation (4) in $(2+1)$-dimensions is reduced to the corresponding problem for the system (9), (10) in $(1+1)$-dimensions.

2. Nonlocal reductions in linear hyperbolic systems. Explicit solutions to a system of nonlinear differential equations describing resonance interaction of M waves. Let us consider a hyperbolic system of linear partial differential equations of first order

$$\frac{\partial \Phi}{\partial y} = A \frac{\partial \Phi}{\partial x} + P\Phi, \tag{11}$$

$$\frac{\partial \Phi}{\partial t} = B\frac{\partial \Phi}{\partial x} + Q\Phi, \tag{12}$$

$$\frac{\partial \Phi^*}{\partial y} = A\frac{\partial \Phi^*}{\partial x} - \bar{P}^T\Phi^*, \tag{13}$$

$$\frac{\partial \Phi^*}{\partial y} = B\frac{\partial \Phi^*}{\partial x} - \bar{Q}^T\Phi^*, \tag{14}$$

where $A = \operatorname{diag}(a_1, a_2, \ldots, a_n)$ and $B = \operatorname{diag}(b_1, b_2, \ldots, b_n)$ are diagonal $(n \times n)$-matrices, elements of which are real numbers satisfying to the conditions $a_i \neq a_j$, $b_i \neq b_j$ when $i \neq j$, $i, j = \overline{1, n}$.

Here $\Phi = \Phi(x, y, t)$ and $\Phi^* = \Phi^*(x, y, t)$ are $(n \times m)$-matrix functions, elements of which are second-degree integrable with respect to variable x, i.e.,

$$\int_s^{+\infty} |\Phi_{km}(x, y, t)|^2 \, dx < +\infty, \qquad \int_s^{+\infty} |\Phi^*_{km}(x, y, t)|^2 \, dx < +\infty, \tag{15}$$

where $k = \overline{1, n}$, $j = \overline{1, m}$ and s is an arbitrary (fixed) real number.

The property (15) of the matrices $\Phi = \Phi(x, y, t)$ and $\Phi^* = \Phi^*(x, y, t)$ described above we will denote in the following way

$$\Phi = \Phi(x, y, t), \qquad \Phi^* = \Phi^*(x, y, t) \in \mathrm{Mat}_{N\times nN}(\mathbf{R}^3; L_2^x(s, +\infty)). \tag{16}$$

The matrix potentials $P = P(x, y, t)$ and $Q = Q(x, y, t)$ belong to the space $\mathrm{Mat}_{n\times m}(\mathbf{R}^3; L_2^x(s, +\infty))$, i.e., they are $(n \times m)$-matrix functions, elements of which are second-degree integrable with respect to variable x. In addition, we suppose that the diagonal elements of matrices P and Q are equal to zero, i.e.,

$$P_{ii}(x, y, t) \equiv 0, \qquad Q_{ii}(x, y, t) \equiv 0, \qquad i = \overline{1, n}. \tag{17}$$

The compatibility condition (4) for the system (11), (12) as well as for the conjugate system (13), (14), implies the following relation

$$[A, Q] = [B, P], \tag{18}$$

$$P_t - Q_y + AQ_x - BP_x + [P, Q] = 0. \tag{19}$$

It is easy to verify that the matrix-functions

$$P = [V, A], \qquad Q = [V, B] \tag{20}$$

satisfy the condition (18) for some arbitrary $(n \times n)$-matrix-function $V = V(x, y, t)$.

Thus the corresponding system of partial differential equations of the form (6) can be written as follows

$$[V_t, A] - [V_y, B] + AV_xB - BV_xA + [[V, A], [V, B]] = 0. \tag{21}$$

In the case when V is an Hermitian matrix, the equation (21) is one of fundamental nonlinear models of theoretical physics since it describes a resonance interaction of M waves, where $M = n(n-1)/2$ waves [1–3]. The equation (21) is a basic system of differential equations of nonlinear optics [6] when $n = 3$.

The equality (21) is considered as a system of nonlinear partial differential equations, solutions of which should be found.

To find the exact formula for the solutions of equation (21) let us consider now the unperturbed system of the form (11), (12) (or with zero potential P and Q) of the following form

$$\frac{\partial \varphi}{\partial y} = A\frac{\partial \varphi}{\partial x}, \qquad \frac{\partial \varphi}{\partial t} = B\frac{\partial \varphi}{\partial x}. \tag{22}$$

where the matrix function $\varphi = \varphi(x, y, t) \in \mathrm{Mat}_{n\times m}(\mathbf{R}^3, L_2^x(s, +\infty))$.

The following theorem is valid.

Theorem 1. *Let C is $(n \times n)$-matrix with constant and real elements such that $\bar{C}^T = C$ and* $\det C \neq 0$.

Then under the mapping

$$\varphi \to \Phi = \varphi\Omega^{-1}, \tag{23}$$

where $(m \times m)$-matrix

$$\Omega = C + \int_x^{+\infty} \bar{\varphi}^T(x, y, t)\varphi(x, y, t)\, dx \tag{24}$$

has in some domain $\sigma = \{(y, t) \in \mathbf{R}^2\}$ nonzero determinant, the system of differential equations (23) is transformed into the following system

$$\frac{\partial \Phi}{\partial y} = A\frac{\partial \Phi}{\partial x} + \left[\Phi\tilde{\Omega}^{-1}\bar{\Phi}^T, A\right]\Phi, \tag{25}$$

$$\frac{\partial \Phi}{\partial t} = B\frac{\partial \Phi}{\partial x} + \left[\Phi\tilde{\Omega}^{-1}\bar{\Phi}^T, B\right]\Phi, \tag{26}$$

where $(m \times m)$-matrix $\tilde{\Omega} = \tilde{\Omega}(x, y, t)$ is represented by the formula

$$\tilde{\Omega} = C^{-1} - \int_x^{+\infty} \bar{\Phi}^T(x, y, t)\Phi(x, y, t)\, dx. \tag{27}$$

Theorem 1 is proved by the direct calculation and by using the following lemma.

Lemma 1. *The matrix $\tilde{\Omega}$ is inverse to the matrix Ω, i.e.,*

$$\Omega\tilde{\Omega} = \tilde{\Omega}\Omega = E, \tag{28}$$

where E is identity $(m \times m)$-matrix.

To prove the lemma 1 it is sufficient to note that $\det\tilde{\Omega} \neq 0$ if $(y, t) \in \sigma$ and x is enough large and to consider the derivative of matrix $\tilde{\Omega}$ with respect to variable $x \in \mathbf{R}^1$, i.e.,

$$\frac{\partial}{\partial x}\tilde{\Omega} = \bar{\Phi}^T(x, y, t)\Phi(x, y, t). \tag{29}$$

Taking into consideration formula (23) and compairing the value (29) with relations

$$\frac{\partial}{\partial x}\tilde{\Omega}^{-1} = -\Omega^{-1}\Omega_x\Omega^{-1} = \Omega^{-1}\bar{\varphi}^T\varphi\Omega^{-1} = \bar{\Phi}^T\Phi, \tag{30}$$

it is easy to conclude the Lemma 1.

From the argument mentioned above we deduce the following theorem.

Theorem 2. *The identity*

$$\Phi\tilde{\Omega}^{-1}\bar{\Phi}^T \equiv \varphi\Omega^{-1}\bar{\varphi}^T \tag{31}$$

is true.

The system (25), (26) is compatible since the system (22) has the same property. It is a simple implication from compatibility conditions (18) and (19).

By comparison of equations (11), (12) and (22) it is easy to conclude that the constraints

$$P = [V, A], \qquad Q = [V, B], \tag{32}$$

where

$$V = \Phi\tilde{\Omega}^{-1}\bar{\Phi}^T = \Phi\left(C^{-1} - \int_x^{+\infty} \bar{\Phi}^T\Phi\, dx\right)^{-1} \bar{\Phi}^T \tag{33}$$

are admissible nonlocal reductions for the system of linear partial differential equations (11), (12).

Thus, it allows us to state the following result.

Theorem 3. *The solution of partial differential equations (21) are represented with the following formula*

$$V = \Phi\left(C^{-1} - \int_x^{+\infty} \bar{\Phi}^T\Phi\, dx\right)^{-1} \bar{\Phi}^T \equiv \varphi\left(C + \int_x^{+\infty} \bar{\varphi}^T\varphi\, dx\right)^{-1} \bar{\varphi}^T$$

or in component form

$$V_{ij} = \varphi_i\left(C + \int_x^{+\infty} \bar{\varphi}_k^T \otimes \varphi_k\, dx\right)^{-1} \bar{\varphi}_j^T,$$

where $\varphi_i = \varphi_i(x + a_i y + b_i t)$ *is* i*-tuple of matrix function* $\varphi(x, y, t)$.

Matrix function $\varphi(x, y, t)$ *is a solution of unperturbed systems (22) and has elements of the following form* $\varphi_{km} = f_{km}(x + a_k y + b_k t)$, *where* $f_{km}(\tau)$, $k = \overline{1,n}$, $m = \overline{1,m}$, *are arbitrary continuos differentiable functions of variable* $\tau \in \mathbf{R}^1$.

References

[1] Lax P.D., Integrals for nonlinear equations and solitary waves, *Commun. Pure and Appl. Math.*, 1968. V.21, N 2, 467–490.

[2] Zakharov E.V., Manakov S.V., Novikov S.P. and Pitaevskiy L.P., Theory of Solitons, Moscow, Nauka, 1980.

[3] Solitons, eds. R.K. Bullough and P.J. Caudrey, Springer-Verlag, 1980.

[4] Sidorenko Yu. and Strammp W., Multicomponents integrable reductions in Kadomtsev–Petviashvilli hierarchy, *J. Math. Phys.*, 1993, V.34, N 4, 1429–1446.

[5] Samoylenko V.Hr. and Sidorenko Yu., Hierarchy of matrix Burgers equations and integrable reductions in Devi–Stewardson system, *Ukrainian Math. J.*, 1998, V.50, N 2, 252–263 (in Ukrainian).

[6] Bloembergen N., Nonlinear Optics, Lecture Notes, W.A. Benjamin, New York – Amsterdam, 1965.