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On Hyperelliptic Solutions of Spectral Problem for the Two-Dimensional Schrödinger Equation

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On the basis of the special Abelian 2-differentials of the second kind corresponding to hypereliptic curves of the g genus addition formulae for hyperelliptic functions determined on the Jacobi manifold is considered. In the case of hyperelliptic curves of the second genus these formulae yield both relations for hypereliptic $\wp$-functions which can be rewritten as integrable nonlinear differential equations and 2-dimensional differential relations which are the generalization of the one-dimensional two-gap Schrödinger equation with potentials which have the form of the linear combinations of hyperelliptic $\wp$-functions with shift arguments.

Introduction

The hyperelliptic Abel functions expressed as derivatives from the hyperelliptic sigma function σ (which is proportional to the n-dimensional Riemann theta function) are an n-dimensional generalization of the elliptic one-dimensional Weierstrass functions [1, 2, 3]. First and second derivatives of these hypereliptic functions are hyperelliptic ζ- and $\wp$-functions dependent on vector arguments $\boldsymbol{u}$ which is the Abel map of corresponding hypereliptic curve V to the Jacobi manifold $\mathrm{Jac}(V)$, where $V = \left\{(y,x) \in \boldsymbol{C}^2 : y^2 - \sum\limits_{i=0}^{2g} \lambda_i x^i = 0\right\}$ means the hyperelliptic curve.

An algebraic curve V is characterized by canonical differential 1-forms including holomorphic, meromorhic differentials of second and third kinds and the special differential 2-form of second kind. The fundamental relation between the differential 2- and 1-forms which is established with a help of the Riemann vanishing theorem for the theta functions leads to the fundamental Baker relations between hyperelliptic σ- and $\wp$-functions. This permits to construct special multi-dimensional linear differential equations with known (see [4, 5]) solutions. Also, the Baker relation leads to the relation connecting derivatives of $\wp$-functions part of which can be rewritten in the form of known integrable equations (see [5]).

1 Relations between differential 1- and 2-forms

The hyperelliptic curve V with cuts connecting branching points realizes the hyperelliptic Riemann curve which is characterized by the canonical system of differential 1-forms holomorphic and meromorphic on the Reimann surface. Holomorphic differentials $du_i = x^{i-1}dx/y$, $i = \overline{1,g}$ on the Riemann curve with the canonical basis of cycles $\overline{a_1, a_g}$ and $\overline{b_1, b_g}$ determine $g \times g$-matrices of a- and b-periods

$$2\omega = \left(\oint_{a_k} u_l\right), \qquad 2\omega' = \left(\oint_{b_k} du_l\right).$$

The relations

$$\mathrm{d}\mathbf{v} = (2\omega)^{-1}\mathrm{d}\mathbf{u},$$

and

$$\boldsymbol{\tau} = \oint_{b_k} \mathrm{d}v_l = \omega^{-1}\omega'$$

determine the normalized g-dimensional vector $\mathbf{v}$ and the $\boldsymbol{\tau}$-matrix of the Riemann surface Γ, respectively. These vector and matrix determine the the Riemann theta function

$$\theta[^{\epsilon'}_{\epsilon''}](\tilde{\boldsymbol{z}}|\boldsymbol{\tau}) = \sum_{\boldsymbol{n}\in Z^g} \exp\left\{\imath\pi\left(\boldsymbol{n} + \frac{1}{2}\epsilon', \tau\boldsymbol{n} + \frac{1}{2}\epsilon\right) + 2\imath\pi\left(\tilde{\boldsymbol{z}} + \frac{1}{2}\epsilon'', \boldsymbol{n} + \frac{1}{2}\epsilon'\right)\right\},$$

$$\tilde{\boldsymbol{z}} = \int_{x_0}^{x_k} d\boldsymbol{v} - \sum_{k=1}^{g}\int_{x_0}^{x_k} d\boldsymbol{v} + \boldsymbol{K}, \qquad K_j = \frac{1+\tau_{jj}}{2} - \sum_{i\neq j}\int_{a_i}\left(dv_i(x)\int_{x_0}^{x} dv_j\right),$$

at $x \in \Gamma$ where Γ is the Riemann surface corresponding to the hyperelliptic curve V. Here $\epsilon'^{('')} = \left(\epsilon_1'^{('')}, \ldots, \epsilon_g'^{('')}\right)$, $\epsilon_i'^{('')} \in (0,1)$. The vanishing property of this theta function in g points of the Riemann surface which constitutes the essence of the Riemann vanishing theorem (see [6]) is used for calculating principal relations between proposed by Klein [1] above mentioned hyperelliptic Abel functions.

The meromorphic differentials of second kind of the form

$$\mathrm{d}r_j = \sum_{k=j}^{2g+1-j}(k+1-j)\lambda_{k+1+j}\frac{x^k\mathrm{d}x}{4y}, \qquad j = \overline{1,g} \tag{1}$$

determine η-matrices of a and b-periods

$$2\eta = \left(-\oint_{a_k}\mathrm{d}r_l\right), \qquad 2\eta' = \left(-\oint_{b_k}\mathrm{d}r_l\right).$$

The latter together with ω-matrices enter in definition of the above mentioned basis hyperelliptic Abel function

$$\sigma(\boldsymbol{u}) = C(\boldsymbol{\tau})\exp\left\{\boldsymbol{u}^T\boldsymbol{\kappa}\boldsymbol{u}\right\}\theta[\varepsilon]\left((2\omega)^{-1}\boldsymbol{u} - \boldsymbol{K}_a|\boldsymbol{\tau}\right). \tag{2}$$

Here $\boldsymbol{\kappa} = (2\omega)^{-1}\boldsymbol{\eta}$, $\boldsymbol{K}_a$ is the vector of Riemann constants with the base point a and $C(\boldsymbol{\tau})$ is the constant which is determined by the parameters of the hyperelliptic curve V (see [5]).

Principal relations between the hyperelliptic σ-functions (refsithet) and ζ and $\wp$-functions are provided with help of the fundamental 2-differential

$$\mathrm{d}\omega(z_1, z_2) = \frac{2y_1y_2 + F(x_1,x_2)}{4(x_1-x_2)^2}\frac{\mathrm{d}x_1}{y_1}\frac{\mathrm{d}x_2}{y_2}, \tag{3}$$

where

$$F(x_1,x_2) = 2\lambda_{2g+2}x_1^{g+1}x_2^{g+1} + \sum_{i=0}^{g}x_1^ix_2^i(2\lambda_{2i} + \lambda_{2i+1}(x_1+x_2)). \tag{4}$$

Taking into account (4) we can rewrite (3) in the form

$$\mathrm{d}\omega(x_1,x_2) = \frac{\partial}{\partial x_2}\left(\frac{y_1+y_2}{2y_1(x_1-x_2)}\right)\mathrm{d}x_1\mathrm{d}x_2 + \mathrm{d}\boldsymbol{u}^T(x_1)\mathrm{d}\boldsymbol{r}(x_2) \tag{5}$$

of the Abelian 2-differential with the pole of the second order.

Applying the Abel map (defined by the equality$\hat{\boldsymbol{A}}(\cdots) = \sum_{k=1}^{g} \int_{x_{0k}}^{x_k} dx(\cdots)$) to the fundamental 2-differential (3) with respect to the variable x_2, integrating over the variable x_1 taking into account (5) and the Riemann vanishing theorem [6] we can obtain an expression in the form of rations of logarithm of the Riemann theta functions [2] (also see [5]). Then, a substitution the theta-representation of σ functions (2) leads to the fundamental relation

$$\int_{\mu}^{x} \sum_{i=1}^{g} \int_{\mu_i}^{x_i} \frac{2yy_i + F(x,x_i)}{4(x-x_i)^2} \frac{\mathrm{d}x}{y} \frac{\mathrm{d}x_i}{y_i}$$

$$= \ln \left\{ \frac{\sigma\left(\int_{a_0}^{x} \mathrm{d}\mathbf{u} - \sum_{i=1}^{g} \int_{a_i}^{x_i} \mathrm{d}\mathbf{u}\right)}{\sigma\left(\int_{a_0}^{x} \mathrm{d}\mathbf{u} - \sum_{i=1}^{g} \int_{a_i}^{\mu_i} \mathrm{d}\mathbf{u}\right)} \right\} - \ln \left\{ \frac{\sigma\left(\int_{a_0}^{\mu} \mathrm{d}\mathbf{u} - \sum_{i=1}^{g} \int_{a_i}^{x_i} \mathrm{d}\mathbf{u}\right)}{\sigma\left(\int_{a_0}^{\mu} \mathrm{d}\mathbf{u} - \sum_{i=1}^{g} \int_{a_i}^{\mu_i} \mathrm{d}\mathbf{u}\right)} \right\}, \tag{6}$$

where the F-function is defined above.

By definition, ζ and $\wp$ hyperelliptic functions are determined via σ functions by differential relations

$$\zeta_i(\boldsymbol{u}) = \frac{\partial}{\partial_{u_i}} \ln \sigma(\boldsymbol{u}), \qquad \wp_{ij}(\boldsymbol{u}) = -\frac{\partial^2}{\partial_{u_j}\partial_{u_j}} \ln \sigma(\boldsymbol{u}), \qquad i,j = \overline{1,g},$$

where the vector $\boldsymbol{u}$ belongs to the Jacobian $\mathrm{Jac}(V)$ of the hyperelliptic curve. A differentiation of (6) with respect to variables u_j leads to relation for ζ and $\wp$ hyperelliptic functions corresponding to hyperelliptic curves with the arbitrary genus g.

Differentiating $\partial^2/\partial_{x_i}\partial_{x_j}$ the relation (6) yields the equality

$$P(x;\boldsymbol{u}) = 0, \qquad P(x;\boldsymbol{u}) = \sum_{j=0}^{g-1} \wp_{g,j+1} x^j \tag{7}$$

which gives the solution of the inverse Jacobi problem consisting in calculating points x_i, $i = \overline{1,g}$ of the Riemann surface via the values of the vector $\boldsymbol{u}$.

Differentiating both sides (6) with respect to ∂/∂_{x_i} from both sides of (6) we can come to the relations

$$-\zeta_j\left(\int_a^{x_0} \mathrm{d}\mathbf{u} + \boldsymbol{u}\right) = \int_a^{x_0} \mathrm{d}r_j + \sum_{k=1}^{g} \int_{a_k}^{x_k} \mathrm{d}r_j - \frac{1}{2}\sum_{k=0}^{g} y_k \left(\left.\frac{D_j(R'(z))}{R'(z)}\right|_{z=x_k} \right), \tag{8}$$

where $R(z) = \prod_0^g (z - z_j)$ and $R'(z) = (\partial/\partial_z)R(z)$ and

$$-\zeta_j(\boldsymbol{u}) = \sum_{k=1}^{g} \int_{a_k}^{x_k} \mathrm{d}r_j - \frac{1}{2}\wp_{gg,j+1}(\boldsymbol{u}). \tag{9}$$

The relations (8) and (9) are the basis for obtaining principal relations between hyperelliptic functions.

2 Basis relations for hyperelliptic functions

A differentiation of the relations (8) and (9) with the respect to u_j leads to the fundamental Baker addition formula ([3], see [5])

$$\frac{\sigma(\boldsymbol{u}+\boldsymbol{v})\sigma(\boldsymbol{u}-\boldsymbol{v})}{\sigma^2(\boldsymbol{u})\sigma^2(\boldsymbol{v})} = M(\boldsymbol{u},\boldsymbol{v}), \tag{10}$$

where $M(\boldsymbol{u},\boldsymbol{v})$ is a polynomial in $\wp$-functions. Here M-function is determined by differential equation [5]

$$\left\{\left(\frac{\partial^2}{\partial_{u_g^2}} - \frac{\partial^2}{\partial_{v_g^2}}\right)\ln M_{k-1}(\boldsymbol{u},\boldsymbol{v}) + 2M_1(\boldsymbol{u},\boldsymbol{v})\right\}M_{k-1}^2(\boldsymbol{u},\boldsymbol{v}) - 4M_k(\boldsymbol{u},\boldsymbol{v})M_{k-1}(\boldsymbol{u},\boldsymbol{v}) = 0. \tag{11}$$

This equation is a recursive relation for M_k-functions where the subscript k means the genus of the corresponding hyperelliptic curve V.

In the case $g = 2$ under consideration it can be shown that

$$M(\boldsymbol{u},\boldsymbol{v}) = \wp_{22}(\boldsymbol{u})\wp_{12}(\boldsymbol{v}) - \wp_{12}(\boldsymbol{u})\wp_{22}(\boldsymbol{v}) + \wp_{11}(\boldsymbol{v}) - \wp_{11}(\boldsymbol{u}). \tag{12}$$

On the basis of the Baker addition formula and the recursive relation (11) we can obtain the system of possible relations between derivatives of $\wp$-functions with respect to variables u_i, $i = \overline{1,g}$.

Taking logarithm of both sides of the equality (10) and with help of differentiating ith respect to u_j and v_j we can obtain well known addition formulae for the hyperelliptic ζ-functions of the form

$$\zeta_j(\boldsymbol{u}+\boldsymbol{v}) - \zeta(\boldsymbol{u}) - \zeta_j(\boldsymbol{v}) = \frac{1}{2}\frac{1}{M(\boldsymbol{u},\boldsymbol{v})}\left(\frac{\partial}{\partial_{u_j}} + \frac{\partial}{\partial_{u_j}}\right)M(\boldsymbol{u},\boldsymbol{v}) \tag{13}$$

and

$$\zeta_j(\boldsymbol{u}+\boldsymbol{v}) + \zeta_j(\boldsymbol{u}-\boldsymbol{v}) = \zeta(\boldsymbol{u}) + \zeta_j(\boldsymbol{v}) + \frac{1}{2}\frac{1}{M(\boldsymbol{u},\boldsymbol{v})}\left(\frac{\partial}{\partial_{u_j}} + \frac{\partial}{\partial_{u_j}}\right)M(\boldsymbol{u},\boldsymbol{v}), \tag{14}$$

$$\zeta_j(\boldsymbol{u}+\boldsymbol{v}) - \zeta_j(\boldsymbol{u}-\boldsymbol{v}) = 2\zeta(\boldsymbol{v}) + \frac{1}{M(\boldsymbol{u},\boldsymbol{v})}\frac{\partial}{\partial_{u_j}}M(\boldsymbol{u},\boldsymbol{v}). \tag{15}$$

Then, expanding the functions

$$\Omega_+ = \ln\sigma(\boldsymbol{u}+\boldsymbol{v}) + \ln\sigma(\boldsymbol{u}-\boldsymbol{v}), \qquad \Omega_- = \ln\sigma(\boldsymbol{u}+\boldsymbol{v}) - \ln\sigma(\boldsymbol{u}-\boldsymbol{v})$$

into power series in the small value $\boldsymbol{v}$ and taking into account the relations (14) and (15) we can obtain all possible differential relations between $\wp$-functions. In the case of the genus $g = 2$ such power expansion leads to relations

$$\wp_{2222} = 6\wp_{22}^2 + \frac{1}{2}\lambda_3 + \lambda_4\wp_{22} + 4\wp_{12},$$

$$\wp_{1111} = 6\wp_{11}^2 - 3\lambda_0\wp_{22} + \lambda_1\wp_{12} + \lambda_2\wp_{11} - \frac{1}{2}\lambda_0\lambda_4 + \frac{1}{8}\lambda_1\lambda_3,$$

$$\wp_{2221} = 6\wp_{22}\wp_{12} + \lambda_4\wp_{12} - 2\wp_{11},$$

$$\wp_{2111} = 6\wp_{12}\wp_{11} + \lambda_2\wp_{12} - \frac{1}{2}\lambda_1\wp_{22} - \lambda_0,$$

$$\wp_{2211} = 2\wp_{22}\wp_{11} + 4\wp_{12}^2 + \frac{1}{2}\lambda_3\wp_{12}.$$

Here the first equation can be rewritten as the two-gap KdV equation with respect to $\wp_{22}$ and the last equation can be rewritten as sine-Gordon equation with respect to $\ln \wp_{12}$. Analogously we can obtain another differential relations between $\wp$ functions (see [5]).

The relation (13) can be rewritten as a differential equation

$$\frac{\partial}{\partial_{u_j}}\Phi(\boldsymbol{u}) = \Lambda\Phi(\boldsymbol{u}), \qquad \Lambda = \frac{1}{2}\frac{1}{M(\boldsymbol{u},\boldsymbol{v})}\left(\frac{\partial}{\partial_{u_j}} + \frac{\partial}{\partial_{u_j}}\right) M(\boldsymbol{u},\boldsymbol{v}) \tag{16}$$

with respect to the Bloch function

$$\Phi(u_0, \boldsymbol{u}; (y,x)) = \frac{\sigma(\boldsymbol{\alpha} - \boldsymbol{u})}{\sigma(\boldsymbol{\alpha})\sigma(\boldsymbol{u})} \exp\left(-\frac{1}{2}yu_0 + \boldsymbol{\zeta}^T(\boldsymbol{\alpha})\boldsymbol{u}\right),$$

for which the hyperelliptic curve $V(y,x)$ is the spectral variety. Here $\boldsymbol{\zeta}^T(\boldsymbol{\alpha}) = (\zeta_1(\boldsymbol{\alpha}), \ldots, \zeta_g(\boldsymbol{\alpha}))$ and $(y,x) \in V$, $\boldsymbol{u}$ and $\boldsymbol{\alpha} = \int_a^x \mathrm{d}\boldsymbol{u} \in \mathrm{Jac}(V)$. Using the equation (refDE) we can construct the system of linear differential equations of the second order both for the Φ-function and for linear combinations of their derivatives. This system yields the solution of the spectral problem for linear differential equations of the second order which can considerate as generalization of Schrödinger equation.

In the case of the genus $g = 2$ this system of differential equations with respect to Φ-function has the form [5]

$$\begin{aligned}
&\left(\partial_2^2 - 2\wp_{22}\right)\Phi = \frac{1}{4}(4x + \lambda_4)\Phi,\\
&\left(\partial_2\partial_1 - \wp_{22}\partial_2^2 + \frac{1}{2}\wp_{222}\partial_2 + \wp_{22}^2 - 2\wp_{12}\right)\Phi = \frac{1}{4}\left(4x^2 + \lambda_4 x + \lambda_3\right)\Phi,\\
&\left(\partial_1^2 - 2\wp_{12}\partial_2^2 + \wp_{122}\partial_2 + 2\wp_{22}\wp_{12}\right)\Phi = \frac{1}{4}\left(4x^3 + \lambda_4 x^2 + \lambda_3 x + \lambda_2\right)\Phi,
\end{aligned}$$

where λ_i denotes coefficients of the hyperelliptic curve V. Analogously spectral problem is solved for the function in the form of linear combinations of different derivatives of Φ-function with different shifts of argument $\boldsymbol{u}$. In doing so, corresponding linear differential equations have a similar form and after reduction from the hyperelliptic to elliptic curve V turn to be the one-dimension Schrödinger equations with the Treibich–Verdier potentials [7].

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