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Asymptotic Integration of System of the Differential Equations of Third Order

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We present the theorems about the asymptotic expansion in ε for the solution of the three-component system $\varepsilon \frac{dx}{dt} = A(t)x$ for the case, when 3×3 matrix $A(t)$ has multiple roots.

1. Development of asymptotic methods of solution of differential equations with variable coefficients started in 19 century in Fourier's, Liouville's and Sturm's papers. The systematic research of the linear differential equations with slowly variable coefficients bigin from fifty yeares after Feschenko papers were published. However for new, more accurate and rational the serch methods of solutions such equations and their systems continus. For the systems which roots of characteristic equations are simple the solutions are easily found by classical Birkhoff method. The first result in the case of multiple roots of characteristics equations in general case was obtained in 60–70 years in Shkil's works.

In my work I suggest a method for reduction of identical multiple roots of characteristic equations to simple roots of some algebraic equations. It enables one to make use of well-known formula for the construction of the asymptotic solution of the system under study what considerably simplifies calculations.

2. Consider the system of first order linear differential equations

$$\varepsilon \frac{dx}{dt} = A(t)x, \tag{1}$$

where x is three dimensional vector, $A(t) = \|a_{ij}\|$, $i, j = 1, 2, 3$ is matrix, whose elements are infinitely differentiable on the segment $[0, L]$, $\varepsilon > 0$ is the small parameter.

Let us build the characteristic equation

$$\det \|A(t) - \lambda E\| = 0, \tag{2}$$

(where E is unit matrix), that is

$$\begin{pmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{pmatrix} = -\lambda^3 + \lambda^2(a_{11} + a_{22} + a_{33}) - \lambda(a_{11}a_{33} + a_{22}a_{33}$$

$$+a_{11}a_{22} - a_{31}a_{13} - a_{32}a_{23} - a_{12}a_{21}) + a_{21}a_{32}a_{13} + a_{12}a_{23}a_{31}$$

$$+a_{11}a_{22}a_{33} - a_{31}a_{13}a_{22} - a_{12}a_{21}a_{33} - a_{32}a_{23}a_{11} = 0.$$

The substitution

$$\lambda = \nu + \frac{a_{11} + a_{22} + a_{33}}{3}$$

transforms it into the cubic equation of the canonical form

$$\nu^3 + p(t)\nu + g(t) = 0,$$

where $p(t)$, $q(t)$ is linear combination of the matrix $A(t)$ coefficients.

Let the matrix $A(t)$ be such that

$$D(t) = -27q^2(t) - 4p^3(t) = 0, \qquad \forall\, t \in [0, L].$$

Then equation (1) has multiple roots. There are two possible cases

a) $p(t) \neq 0$, $q(t) \neq 0$ – root of multiplicity two and one root of multiplicity one;

b) $p(t) = q(t) = 0$ – one root of multiplicity three.

We don't consider the ase a), since then the methods of [1] allow to split the system under study into subsystem, for which the characteristic equations has one simple root and one root of multiplicity two. This case will be considered separately. In case b) we have one elementary divisor

$$\lambda_1(t) \equiv \lambda_2(t) \equiv \lambda_3(t) \equiv \lambda_0(t) \equiv \frac{a_{11}(t) + a_{22}(t) + a_{33}(t)}{3} = \frac{\operatorname{tr} A(t)}{3}. \tag{3}$$

of multiplicity three.

Then, as it is shown in [2], there exists the nondegenerate matrix of the transformation of similarity $V(t)$ that transforms the matrix $A(t)$ to the quasidiagonal canonical form $W(t) = V^{-1}(t)A(t)V(t)$, where

$$W(t) = \begin{pmatrix} \lambda_0(t) & 1 & 0 \\ 0 & \lambda_0(t) & 1 \\ 0 & 0 & \lambda_0(t) \end{pmatrix},$$

The substitution $x = V(t)y$ transforms system (1) into the system

$$\varepsilon \frac{dy}{dt} = D(t, \varepsilon) y, \tag{4}$$

where

$$D(t, \varepsilon) = \begin{pmatrix} \lambda_0(t) & 1 & 0 \\ 0 & \lambda_0(t) & 1 \\ 0 & 0 & \lambda_0(t) \end{pmatrix} + \varepsilon \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = V^{-1}(t)V'(t).$$

Using Cardano formula, we obtain the eidenvalues of system (4).

$$\rho_1 = \lambda_0 + \frac{\varepsilon b_1}{3} + \sqrt[3]{\varepsilon^3 a + \varepsilon^2 b + \varepsilon c + \sqrt{\varepsilon^6 d_6 + \varepsilon^5 d_5 + \cdots + \varepsilon^2 d_2}}$$
$$+ \sqrt[3]{\varepsilon^3 a + \varepsilon^2 b + \varepsilon c - \sqrt{\varepsilon^6 d_6 + \varepsilon^5 d_5 + \cdots + \varepsilon^2 d_2}} = \lambda_0 + O\left(\varepsilon^{1/3}\right),$$

$$\rho_2 = \lambda_0 + \frac{\varepsilon b_1}{3} + \sqrt[3]{\varepsilon^3 a + \varepsilon^2 b + \varepsilon c + \sqrt{\varepsilon^6 d_6 + \varepsilon^5 d_5 + \cdots + \varepsilon^2 d_2}} \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$
$$+ \sqrt[3]{\varepsilon^3 a + \varepsilon^2 b + \varepsilon c - \sqrt{\varepsilon^6 d_6 + \varepsilon^5 d_5 + \cdots + \varepsilon^2 d_2}} \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = \lambda_0 + O\left(\varepsilon^{1/3}\right),$$

$$\rho_3 = \lambda_0 + \frac{\varepsilon b_1}{3} + \sqrt[3]{\varepsilon^3 a + \varepsilon^2 b + \varepsilon c + \sqrt{\varepsilon^6 d_6 + \varepsilon^5 d_5 + \cdots + \varepsilon^2 d_2}}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$+\sqrt[3]{\varepsilon^3 a + \varepsilon^2 b + \varepsilon c - \sqrt{\varepsilon^6 d_6 + \varepsilon^5 d_5 + \cdots + \varepsilon^2 d_2}}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \lambda_0 + O\left(\varepsilon^{1/3}\right),$$

where $b_1, a, b, c, d_6, d_5, \ldots, d_2$ are linear combinations of the coefficient of matrix $A(t)$. Assume that $\varepsilon^6 d_6 + \cdots + \varepsilon^2 d_2 \neq 0$. When ρ_i are different and the eigenvectors $\mu_i(t,\varepsilon)$, $\mu_i^*(t,\varepsilon)$ of the matrix $D(t,\varepsilon)$ and conjugate matrix $D^*(t,\varepsilon)$ may be chosen in such a way that their scalar products have the form

$$(\mu_i \cdot \mu_j^*) = \begin{cases} 1, & i = j, \\ 0, & i \neq j,\ i,j = 1,2,3. \end{cases} \tag{5}$$

For example take

$$\mu_i(t,\varepsilon) = \begin{pmatrix} 1 \\ \dfrac{\varepsilon b_{31}(1+\varepsilon b_{23}) - \varepsilon b_{21}(\lambda_0 + \varepsilon b_{33} - \rho_i)}{(\lambda_0 + \varepsilon b_{22} - \rho_i)(\lambda_0 + \varepsilon b_{33} - \rho_i) - \varepsilon b_{32}(1+\varepsilon b_{23})} \\ \dfrac{\varepsilon^2 b_{32} b_{21} - \varepsilon b_{31}(\lambda_0 + \varepsilon b_{22} - \rho_i)}{(\lambda_0 + \varepsilon b_{22} - \rho_i)(\lambda_0 + \varepsilon b_{33} - \rho_i) - \varepsilon b_{32}(1+\varepsilon b_{23})} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ \dfrac{\varepsilon b_{31}(1+\varepsilon b_{23}) - \varepsilon b_{21}\left(\varepsilon b_{33} - O\left(\varepsilon^{1/3}\right)\right)}{\left(\varepsilon b_{22} - O\left(\varepsilon^{1/3}\right)\right)\left(\varepsilon b_{33} - O\left(\varepsilon^{1/3}\right)\right) - \varepsilon b_{32}(1+\varepsilon b_{23})} \\ \dfrac{\varepsilon^2 b_{32} b_{21} - \varepsilon b_{31}\left(\varepsilon b_{22} - O\left(\varepsilon^{1/3}\right)\right)}{\left(\varepsilon b_{22} - O\left(\varepsilon^{1/3}\right)\right)\left(\varepsilon b_{33} - O\left(\varepsilon^{1/3}\right)\right) - \varepsilon b_{32}(1+\varepsilon b_{23})} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ \varepsilon^{1/3}\dfrac{b_{31}(1+\varepsilon b_{23}) - \varepsilon^{1/3} b_{21}\left(\varepsilon^{2/3} b_{33} - O\left(\varepsilon^{0}\right)\right)}{\left(\varepsilon^{2/3} b_{22} - O\left(\varepsilon^{0}\right)\right)\left(\varepsilon^{2/3} b_{33} - O\left(\varepsilon^{0}\right)\right) - \varepsilon^{1/3} b_{32}(1+\varepsilon b_{23})} \\ \varepsilon^{2/3}\dfrac{\varepsilon^{2/3} b_{32} b_{21} - b_{31}\left(\varepsilon^{2/3} b_{22} - O\left(\varepsilon^{0}\right)\right)}{\left(\varepsilon^{2/3} b_{22} - O\left(\varepsilon^{0}\right)\right)\left(\varepsilon^{2/3} b_{33} - O\left(\varepsilon^{0}\right)\right) - \varepsilon^{\frac{1}{3}} b_{32}(1+\varepsilon b_{23})} \end{pmatrix} = \begin{pmatrix} O\left(\varepsilon^{0}\right) \\ O\left(\varepsilon^{1/3}\right) \\ O\left(\varepsilon^{2/3}\right) \end{pmatrix}.$$

As we can see, vector μ_i can be write in form

$$\mu_i = \left(1, \varepsilon^{1/3}\mu_{i2}^{a}, \varepsilon^{2/3}\mu_{i3}^{a}\right). \tag{6}$$

Here and below index a denotes analitical in point $\varepsilon = 0$ function. So, it is easy to notice that for the condition (5) to hold, the coordinates of the vector μ_i^* must have the form

$$\mu_i^* = \left(\mu_{i1}^{a*}, \varepsilon^{-1/3}\mu_{i2}^{a*}, \varepsilon^{-2/3}\mu_{i3}^{a*}\right). \tag{6'}$$

The following theorems hold true.

Theorem 1. *If the functions $a_{ij}(t)$, $i,j = 1,2,3$ are infinitely differentiable on the segment $[0,L]$, then system (4) has the formal particular solution*

$$y(t,\varepsilon) = U(t,\varepsilon,\varepsilon)\exp\left(\frac{1}{\varepsilon}\int_0^t \lambda(\tau,\varepsilon,\varepsilon)\,d\tau\right), \tag{7}$$

where $U(t,\varepsilon,\varepsilon)$ is an 3-component vector, and $\lambda(t,\varepsilon,\varepsilon)$ is a scalar function which are represented by the following formal series

$$U(t,\varepsilon,\varepsilon)=\sum_{s=0}^{\infty}\varepsilon^s U_s(t,\varepsilon),\qquad \lambda(t,\varepsilon,\varepsilon)=\sum_{s=1}^{\infty}\varepsilon^s\lambda_s(t,\varepsilon)+\rho_1(t,\varepsilon). \tag{8}$$

Proof. In order to prove this theorem let us substitute the vector y, given by the relation (7), into system (4). We have

$$\varepsilon U'(t,\varepsilon,\varepsilon)\equiv(D(t,\varepsilon)-\lambda(t,\varepsilon,\varepsilon)E)U(t,\varepsilon,\varepsilon). \tag{9}$$

The coefficients of series (8) are to be determined from the following system of algebraic equations

$$(D(t,\varepsilon)-\rho_1(t,\varepsilon)E)U_0(t,\varepsilon)=0, \tag{10}$$

$$(D(t,\varepsilon)-\rho_1(t,\varepsilon)E)U_m(t,\varepsilon)=U'_{m-1}(t,\varepsilon)+\sum_{j=1}^{m}\lambda_j(t,\varepsilon)U_{m-j}(t,\varepsilon),\quad m=1,2,\dots. \tag{11}$$

Here $(\ )'$ denotes the derivative with respect to t.

Consider vector equation (10). It is obvious that

$$U_0(t,\varepsilon)=\mu_1(t,\varepsilon). \tag{12}$$

Let us turn to the system (11). Since $\det\|D(t,\varepsilon)-\rho_1(t,\varepsilon)E\|=0$, for the existence of a solution of the inhomogeneous system of algebraic equations of such form it is necessary and sufficient that the scalar product of the vector on the right hand side with any solution of the associated system, that is the system

$$(D(t,\varepsilon)-\rho_1(t,\varepsilon)E)^*y=0,$$

vanishes [3]. That's why

$$\left(\left(U'_{m-1}(t,\varepsilon)+\sum_{j=1}^{m}\lambda_j(t,\varepsilon)U_{m-j}(t,\varepsilon)\right)\cdot\mu_1^*(t,\varepsilon)\right)=0,\quad m=1,2,\dots.$$

From this, making use of properties of the scalar product and formulas (5), (12), we obtain

$$\lambda_m(t,\varepsilon)=-(U'_{m-1}(t,\varepsilon)\cdot\mu_1^*(t,\varepsilon))-\sum_{j=1}^{m-1}\lambda_j(t,\varepsilon)(U_{m-j}(t,\varepsilon)\cdot\mu_1^*(t,\varepsilon)),\quad m=1,2,\dots. \tag{13}$$

We shall look for the vector $U_m(t,\varepsilon)$ in the form

$$U_m(t,\varepsilon)=c_1^{(m)}(t,\varepsilon)\cdot\mu_1(t,\varepsilon)+c_2^{(m)}(t,\varepsilon)\cdot\mu_2(t,\varepsilon)+c_3^{(m)}(t,\varepsilon)\cdot\mu_3(t,\varepsilon), \tag{14}$$

where $c_k^{(m)}(t,\varepsilon)$, $k=1,2,3$ are scalar functions. Substituting (14) into (11), we obtain that the functions $c_1^{(m)}(t,\varepsilon)$ are arbitrary. Let $c_1^{(m)}(t,\varepsilon)\equiv 0$. Then

$$\begin{aligned}U_m(t,\varepsilon)&=\sum_{k=2}^{3}c_k^{(m)}(t,\varepsilon)\cdot\mu_k(t,\varepsilon)\\&=\sum_{k=2}^{3}\frac{(U'_{m-1}(t,\varepsilon)\cdot\mu_k^*(t,\varepsilon))+\sum\limits_{j=1}^{m}\lambda_j(t,\varepsilon)U_{m-j}(t,\varepsilon)\cdot\mu_k^*(t,\varepsilon))}{\rho_k(t,\varepsilon)-\rho_1(t,\varepsilon)}\cdot\mu_k(t,\varepsilon).\end{aligned} \tag{15}$$

Taking into account (5) and (15), formula (13) can be rewritten in the form

$$\lambda_m(t,\varepsilon) = -(U'_{m-1}(t,\varepsilon)\cdot \mu_1^*(t,\varepsilon)), \quad m=1,2,\dots. \tag{16}$$

So, the solution of the system (10), (11) can be written in the form of recurrent formulas (15), (16). Construction of the formulas for the coefficientes of series (8) completes the proof.

3. Let's evaluate the coefficients of series (8). Estimate of the $U_0(t,\varepsilon)$ give in (6). That's why, taking into account (12), write

$$U_0(t,\varepsilon) = \left(1, \varepsilon^{1/3}U_{02}^a(t,\varepsilon), \varepsilon^{2/3}U_{03}^a(t,\varepsilon)\right). \tag{17}$$

The diferentiation with respect to t have not an influence on the function's analitical in ε. If ε is function's zero, then we carry out it from the differentiation sign as a const, that is ε will not become a function's pole. The differentiation differential can increase only the order of the functions zero in the point $\varepsilon = 0$, but it only can improve the rezult. Look for

$$\lambda_1(t,\varepsilon) = \begin{pmatrix} 0 \\ \varepsilon^{1/3}U_{02}^{a'}(t,\varepsilon) \\ \varepsilon^{2/3}U_{03}^{a'}(t,\varepsilon) \end{pmatrix} \cdot \begin{pmatrix} \mu_{11}^{a*}(t,\varepsilon) \\ \varepsilon^{-1/3}\mu_{12}^{a*}(t,\varepsilon) \\ \varepsilon^{-2/3}\mu_{13}^{a*}(t,\varepsilon) \end{pmatrix} = \lambda_1^a(t,\varepsilon). \tag{18}$$

Then

$$U_1(t,\varepsilon) = \sum_{k=2}^{3} \frac{(U_0'(t,\varepsilon)\cdot\mu_k^*(t,\varepsilon)) + \lambda_1(t,\varepsilon)(U_0(t,\varepsilon)\cdot\mu_k^*(t,\varepsilon))}{\rho_k(t,\varepsilon)-\rho_1(t,\varepsilon)}\mu_k(t,\varepsilon).$$

Proceed from (6′), (17), (18) we can conclude that the numerator of the fraction of this sum is analitical function. Evaluate the denominator.

$$\rho_2-\rho_1 = \sqrt[3]{\varepsilon^3 a+\varepsilon^2 b+\varepsilon c+\sqrt{\varepsilon^6 d_6+\varepsilon^5 d_5+\dots+\varepsilon^2 d_2}}\left(-\frac{3}{2}+i\frac{\sqrt{3}}{2}\right)$$
$$+\sqrt[3]{\varepsilon^3 a+\varepsilon^2 b+\varepsilon c-\sqrt{\varepsilon^6 d_6+\varepsilon^5 d_5+\dots+\varepsilon^2 d_2}}\left(-\frac{3}{2}-i\frac{\sqrt{3}}{2}\right) = O\left(\varepsilon^{1/3}\right),$$

$$\rho_3-\rho_1 = \sqrt[3]{\varepsilon^3 a+\varepsilon^2 b+\varepsilon c+\sqrt{\varepsilon^6 d_6+\varepsilon^5 d_5+\dots+\varepsilon^2 d_2}}\left(-\frac{3}{2}-i\frac{\sqrt{3}}{2}\right)$$
$$+\sqrt[3]{\varepsilon^3 a+\varepsilon^2 b+\varepsilon c-\sqrt{\varepsilon^6 d_6+\varepsilon^5 d_5+\dots+\varepsilon^2 d_2}}\left(-\frac{3}{2}+i\frac{\sqrt{3}}{2}\right) = O\left(\varepsilon^{\frac{1}{3}}\right).$$

So

$$U_1(t,\varepsilon) = \sum_{k=2}^{3} \frac{f^a(t,\varepsilon)\cdot\left(1,\varepsilon^{1/3}\mu_{k2}^a(t,\varepsilon),\varepsilon^{2/3}\mu_{k3}^a(t,\varepsilon)\right)}{\varepsilon^{1/3}(\rho_k(t,\varepsilon)-\rho_1(t,\varepsilon))^a} = \left(\varepsilon^{-1/3}U_{11}^a, \varepsilon^0 U_{12}^a, \varepsilon^{1/3}U_{13}^a\right).$$

Then

$$\lambda_2(t,\varepsilon) = -\begin{pmatrix} \varepsilon^{-1/3}U_{11}^{a'} \\ \varepsilon^0 U_{12}^{a'} \\ \varepsilon^{1/3}U_{13}^{a'} \end{pmatrix} \cdot \begin{pmatrix} \mu_{11}^{a*} \\ \varepsilon^{-1/3}\mu_{12}^{a*} \\ \varepsilon^{-2/3}\mu_{13}^{a*} \end{pmatrix} = \varepsilon^{-1/3}\lambda_2^a(t,\varepsilon).$$

Assume that for all $U_j(t,\varepsilon)$, $\lambda_j(t,\varepsilon)$, $j<m$ next formulas are true

$$U_j(t,\varepsilon)=\left(\frac{U^a_{1j}(t,\varepsilon)}{\varepsilon^{j/3}},\frac{U^a_{2j}(t,\varepsilon)}{\varepsilon^{(j-1)/3}},\frac{U^a_{3j}(t,\varepsilon)}{\varepsilon^{(j-2)/3}}\right)=\frac{U^a_j(t,\varepsilon)}{\varepsilon^{j/3}},$$
$$\lambda_j(t,\varepsilon)=\frac{\lambda^a_j(t,\varepsilon)}{\varepsilon^{(j-1)/3}},\qquad j=1,2,\ldots,m-1, \tag{19}$$

m is some fixed natural number.

Let's show the correctness this assumption for $j=m$, $m\in N$. Whis usage of (19), (6′), formula (16) can be rewritten as

$$\lambda_m(t,\varepsilon)=-\left(\frac{U^{a'}_{1(m-1)}(t,\varepsilon)}{\varepsilon^{(m-1)/3}},\frac{U^{a'}_{2(m-1)}(t,\varepsilon)}{\varepsilon^{(m-2)/3}},\frac{U^{a'}_{3(m-1)}(t,\varepsilon)}{\varepsilon^{(m-3)/3}}\right)$$
$$\times\left(\mu^{a*}_{11}(t,\varepsilon),\varepsilon^{-1/3}\mu^{a*}_{12}(t,\varepsilon),\varepsilon^{-2/3}\mu^{a*}_{13}(t,\varepsilon)\right)=\frac{\lambda^a_m(t,\varepsilon)}{\varepsilon^{(m-1)/3}}.$$

So, $\lambda_m(t,\varepsilon)$ have pole in ε and its order is $(m-1)/3$, what confirms the correctness of our assumption for $j=m$.

Likewise, rewrite (15):

$$U_m(t,\varepsilon)=\sum_{k=2}^{3}\left(\left(\frac{U^{a'}_{1(m-1)}}{\varepsilon^{(m-1)/3}},\frac{U^{a'}_{2(m-1)}}{\varepsilon^{(m-2)/3}},\frac{U^{a'}_{3(m-1)}}{\varepsilon^{(m-3)/3}}\right)\cdot\left(\mu^{a*}_{k1},\varepsilon^{-1/3}\mu^{a*}_{k2},\varepsilon^{-2/3}\mu^{a*}_{k3}\right)\right.$$
$$\left.+\sum_{j=1}^{m}\frac{\lambda^a_j(t,\varepsilon)}{\varepsilon^{(j-1)/3}}\left(\frac{U^a_{1(m-j)}}{\varepsilon^{(m-j)/3}},\frac{U^a_{2(m-j)}}{\varepsilon^{(m-j-1)/3}},\frac{U^a_{3(m-j)}}{\varepsilon^{(m-j-2)/3}}\right)\cdot\left(\mu^{a*}_{k1},\varepsilon^{-1/3}\mu^{a*}_{k2},\varepsilon^{-2/3}\mu^{a*}_{k3}\right)\right)$$
$$\times\frac{\left(\mu^a_{k1},\varepsilon^{1/3}\mu^a_{k2},\varepsilon^{2/3}\mu^a_{k3}\right)}{\varepsilon^{1/3}(\rho_k-\rho_1)^a}=\sum_{k=2}^{3}\frac{f^a_1}{\varepsilon^{(m-1)/3}}\cdot\frac{\left(\mu^a_{k1},\varepsilon^{1/3}\mu^a_{k2},\varepsilon^{2/3}\mu^a_{k3}\right)}{\varepsilon^{1/3}(\rho_k-\rho_1)^a}$$
$$=\left(\frac{\mu^a_{k1}}{\varepsilon^{m/3}},\frac{\mu^a_{k2}}{\varepsilon^{(m-1)/3}},\frac{\mu^a_{k3}}{\varepsilon^{(m-2)/3}}\right)\cdot\frac{f^a_1}{(\rho_k-\rho_1)^a}=\frac{U^a_m(t,\varepsilon)}{\varepsilon^{j/3}}.$$

So, using the mathematical induction, we conclude that the formulas (19) hold true for all $j\in N$.

Then the following expansions for series (8) take place

$$U(t,\varepsilon,\varepsilon)=\sum_{s=0}^{\infty}\varepsilon^{2s/3}U^a_s(t,\varepsilon),\qquad \lambda(t,\varepsilon,\varepsilon)=\sum_{s=1}^{\infty}\varepsilon^{(2s+1)/3}\lambda^a_s(t,\varepsilon)+\rho_1(t,\varepsilon). \tag{20}$$

Here $U^a_s(t,\varepsilon)$, $\lambda^a_s(t,\varepsilon)$ are analytical in ε.

4. Next Theorem 2 proves the asymptotic property of formal solution in the sense of [4].

Theorem 2. *If the conditions of Theorem 1 are fulfilled and*

$$\operatorname{Re}(\rho_1(t,\varepsilon))\le 0\qquad \textit{for}\quad \forall\, t\in[0,L],\ 0<\varepsilon\le\varepsilon_0, \tag{21}$$

on the segment $[0,L]$ *m-th approximations satisfies the diferential system (4) up to* $O\left(\varepsilon^{(2m+3)/3}\right)$.

Proof. For the proof, similarly to [1], let us introduce the vector

$$y_m(t,\varepsilon,\varepsilon)=U_m(t,\varepsilon,\varepsilon)\exp\left(\frac{1}{\varepsilon}\int_0^t\lambda_m(\tau,\varepsilon,\varepsilon)\,d\tau\right), \tag{22}$$

where

$$U_m(t,\varepsilon,\varepsilon)=\sum_{s=0}^{m}\varepsilon^s U_s(t,\varepsilon)=\sum_{s=0}^{m}\varepsilon^{2s/3}U_s^a(t,\varepsilon),$$
$$\lambda_m(t,\varepsilon,\varepsilon)=\sum_{s=1}^{m}\varepsilon^s\lambda_s(t,\varepsilon)+\rho_1(t,\varepsilon)=\sum_{s=1}^{m}\varepsilon^{(2s+1)/3}\lambda_s^a(t,\varepsilon)+\rho_1(t,\varepsilon),\qquad m\geq 1. \tag{23}$$

Substitute (22) to the differential expression

$$L(y_m)=\varepsilon\frac{dy_m}{dt}-D(t,\varepsilon)y_m. \tag{24}$$

We have

$$L(y_m(t,\varepsilon))=[\varepsilon U_m'(t,\varepsilon,\varepsilon)-D(t,\varepsilon)U_m(t,\varepsilon,\varepsilon)$$
$$+\lambda_m(t,\varepsilon,\varepsilon)U_m(t,\varepsilon,\varepsilon)]\exp\left(\frac{1}{\varepsilon}\int_0^t\lambda_m(\tau,\varepsilon,\varepsilon)\,d\tau\right). \tag{25}$$

The magnitude of the function $\exp\left(\frac{1}{\varepsilon}\int_0^t\lambda_m(\tau,\varepsilon,\varepsilon)\,d\tau\right)$ is limited on the set $\{K:0<\varepsilon\leq\varepsilon_0;t\in[0,L]\}$. In fact

$$\left|\exp\left(\frac{1}{\varepsilon}\int_0^t\lambda_m(\tau,\varepsilon,\varepsilon)\,d\tau\right)\right|=\exp\left(\frac{1}{\varepsilon}\int_0^t\alpha_m(\tau,\varepsilon,\varepsilon)\,d\tau\right)$$
$$=\exp\left(\frac{1}{\varepsilon}\int_0^t\left[\alpha_0(\tau,\varepsilon)+\varepsilon\alpha_1(\tau,\varepsilon)+\ldots+\varepsilon^{(2m+1)/3}\alpha_m(\tau,\varepsilon)\right]\,d\tau\right) \tag{26}$$

(here $\alpha_m(\tau,\varepsilon,\varepsilon)=\sum_{s=1}^{m}\varepsilon^{\frac{2}{3}s+\frac{1}{3}}\alpha_s(\tau,\varepsilon)$ is the real part of the function $\lambda_m(\tau,\varepsilon,\varepsilon)$, defined by (23), $\alpha_0(\tau,\varepsilon)$ is the real part of the function $\rho_1(\tau,\varepsilon)$, $\alpha_s(\tau,\varepsilon)$ are analytical, $s=1,2,\ldots$).

Since the functions $\alpha_1(\tau,\varepsilon),\ldots,\alpha_m(\tau,\varepsilon)$ according to theorem 1 are infinitely differentiable on the segment $[0,L]$, (26) can be rewritten as

$$\exp\left(\frac{1}{\varepsilon}\int_0^t\alpha_0(\tau,\varepsilon)\,d\tau\right)\exp\left(\frac{1}{\varepsilon}\int_0^t\left[\varepsilon\alpha_1(\tau,\varepsilon)+\cdots+\varepsilon^{(2m+1)/3}\alpha_m(\tau,\varepsilon)\right]\,d\tau\right)$$
$$\leq\exp\left(\frac{1}{\varepsilon}\int_0^t\alpha_0(\tau,\varepsilon)\,d\tau\right)\exp\left(\int_0^t\sum_{s=1}^{m}\varepsilon^{(2s-2)/3}\mid\alpha_s(\tau,\varepsilon)\mid\,d\tau\right)$$
$$\leq\exp\left(\frac{1}{\varepsilon}\int_0^t\alpha_0(\tau,\varepsilon)\,d\tau\right)\exp\left(M\int_0^t\left(1+\varepsilon^{2/3}+\ldots+\varepsilon^{(2m-2)/3}\right)d\tau\right)$$
$$\leq\exp\left(\frac{1}{\varepsilon}\int_0^t\alpha_0(\tau,\varepsilon)\,d\tau\right)\exp\left(ML\frac{1-\varepsilon_0^{m/3}}{1-\varepsilon_0^{2/3}}\right),$$

here $(M = \max \mid \alpha_s(t,\varepsilon) \mid)$, $s = 1,\dots,m$. In virtue of (21) we can state that this value is limited.

Let's evaluate the vector which is the multiplier at $\exp\left(\frac{1}{\varepsilon}\int\limits_0^t \lambda_m(\tau,\varepsilon,\varepsilon)\,d\tau\right)$ in right hand side of the equality (25). Since when we determined coeficients of series $U_m(t,\varepsilon,\varepsilon), \lambda_m(t,\varepsilon,\varepsilon)$ we compared the coefficientes at the external powers of ε parameter up to order including m itself, it is clear that this vector will have nonzero coefficients only at the powers $\varepsilon^{m+1}, \varepsilon^{m+2}, \dots, \varepsilon^{2m}$. That is why, taking into account said above on derivative with respect to t of function $U_m(t,\varepsilon,\varepsilon)$ and (19), we obtain

$$\varepsilon(U_0'(t,\varepsilon)+\varepsilon U_1'(t,\varepsilon)+\dots+\varepsilon^m U_m'(t,\varepsilon)) - D(t,\varepsilon)(U_0(t,\varepsilon)+\varepsilon U_1(t,\varepsilon)+\dots+\varepsilon^m U_m(t,\varepsilon))$$
$$+(\rho_1(t,\varepsilon)+\varepsilon\lambda_1(t,\varepsilon)+\dots+\varepsilon^m\lambda_m(t,\varepsilon))(U_0(t,\varepsilon)+\varepsilon U_1(t,\varepsilon)+\dots+\varepsilon^m U_m(t,\varepsilon))$$
$$=\varepsilon^{m+1}U_m'(t,\varepsilon)+\sum_{j=1}^{m}\varepsilon^{m+j}\sum_{s=j}^{m}\lambda_s(t,\varepsilon)U_{m+j-s}(t,\varepsilon)=\varepsilon^{m+1}\frac{U_m^{a'}(t,\varepsilon)}{\varepsilon^{m/3}}$$
$$+\sum_{j=1}^{m}\varepsilon^{m+j}\sum_{s=j}^{m}\frac{\lambda_s^a(t,\varepsilon)}{\varepsilon^{(s-1)/3}}\frac{U_{m+j-s}^a(t,\varepsilon)}{\varepsilon^{(m+j-s)/3}}=\varepsilon^{(2m+3)/3}U_m^{a'}(t,\varepsilon)+\sum_{j=1}^{m}\varepsilon^{(2m+2j+1)/3}\sum_{s=j}^{m}\lambda_s^a(t,\varepsilon)$$
$$\times U_{m+j-s}^a(t,\varepsilon)=\varepsilon^{(2m+3)/3}U_m^{a'}(t,\varepsilon)+\varepsilon^{(2m+3)/3}\sum_{j=1}^{m}\varepsilon^{(2j-2)/3}\sum_{s=j}^{m}\lambda_s^a(t,\varepsilon)U_{m+j-s}^a(t,\varepsilon)$$
$$=\varepsilon^{(2m+3)/3}\left(U_m^{a'}(t,\varepsilon)+\sum_{j=1}^{m}\varepsilon^{(2j-2)/3}\sum_{s=j}^{m}\lambda_s^a(t,\varepsilon)U_{m+j-s}^a(t,\varepsilon)\right)=O\left(\varepsilon^{(2m+3)/3}\right),\ m\in N.$$

The theorem is proved.

5. If system (1) is system of the linear differential equations of the second order, then theorem likewise Theorem 1, 2 take place. Formulas (20) for $n = 2$ have form

$$U(t,\varepsilon,\varepsilon)=\sum_{s=0}^{\infty}\varepsilon^{s/2}U_s^a(t,\varepsilon),\qquad \lambda(t,\varepsilon,\varepsilon)=\sum_{s=1}^{\infty}\varepsilon^{(s+1)/2}\lambda_s^a(t,\varepsilon)+\rho_1(t,\varepsilon).$$

The solution, found by the method of Theorem 1, satisfies the system (4) up to $O\left(\varepsilon^{(m+2)/2}\right)$.

The advantage of this method in contradistinction to well know M.I. Shkil's method [1], is the possibility bringing the solution of the equation with multiple roots to the classical theory of the simple roots.

Let us illustrate it on the example

$$\varepsilon\frac{d^2y}{dt^2}+p(t)y=0,$$

here $p(t)\neq 0, t\in[0,L]$.

Let us write this equation in the form of system (4). We use here the following notations $y=\varepsilon y_1$, $\frac{dy}{dt}=y_2$.

Then we obtain the system

$$\varepsilon\frac{dy_1}{dt}=y_2,$$
$$\varepsilon\frac{dy_2}{dt}=\varepsilon\frac{d^2y}{dt}=-p(t)\varepsilon y_1$$

or

$$\varepsilon\frac{dy}{dt} = \begin{pmatrix} 0 & 1 \\ -\varepsilon p(t) & 0 \end{pmatrix} y,$$

where y is a two dimensional vector.

In this case the roots of the characteristic equation are different.

$$\rho_1(t,\varepsilon) = \sqrt{-\varepsilon p(t)}, \qquad \rho_2(t,\varepsilon) = -\sqrt{-\varepsilon p(t)}.$$

Then the conditions of Theorem 1 are satisfied. That's why using the recurrent formulas for $U_m(t,\varepsilon)$, $\lambda_m(t,\varepsilon)$, we obtain

$$U_0(t,\varepsilon) = \left(1, \sqrt{-\varepsilon p(t)}\right), \quad \lambda_1(t,\varepsilon) = -\frac{p'(t)}{4p(t)}, \quad U_1(t,\varepsilon) = \left(\frac{p'(t)}{8\sqrt{\varepsilon}(-p(t))^{3/2}}, -\frac{p'(t)}{8p(t)}\right),$$

$$\lambda_2(t,\varepsilon) = \frac{p'^2(t)}{32\sqrt{\varepsilon}(-p(t))^{5/2}}, \quad U_2(t,\varepsilon) = \left(\frac{p''}{16\varepsilon p^2} - \frac{3p'^2}{32\varepsilon p^3}, \frac{p''}{16\sqrt{\varepsilon}(-p)^{3/2}} - \frac{3p'^2}{32\sqrt{\varepsilon}(-p)^{5/2}}\right).$$

Substitute the vector

$$y_1(t,\varepsilon,\varepsilon) = \left((1,\sqrt{-\varepsilon p}) + \varepsilon\left(\frac{p'}{8\sqrt{\varepsilon}(-p)^{3/2}}, -\frac{p'}{8p}\right)\right)\exp\left(\frac{1}{\varepsilon}\int_0^t\left(\sqrt{-\varepsilon p} - \varepsilon\frac{p'}{4p}\right)d\tau\right)$$

to differential expression (24). We have

$$L(y_1(t,\varepsilon,\varepsilon)) = \varepsilon\sqrt{\varepsilon}\left(-\frac{p''}{8p^{3/2}} + \frac{7p'^2}{32(-p)^{5/2}}, \frac{\sqrt{\varepsilon}p''}{8p} - \frac{3\sqrt{\varepsilon}p'^2}{64p^2}\right)\exp\left(\frac{1}{\varepsilon}\int_0^t\left(\sqrt{-\varepsilon p} - \varepsilon\frac{p'}{4p}\right)d\tau\right).$$

As you can see, the vector $y_1(t,\varepsilon,\varepsilon)$ satisfies the system of the differential equations up to $O\left(\varepsilon^{3/2}\right)$.

Likewise

$$y_2 = \left(\left(1,\sqrt{-\varepsilon p}\right) + \varepsilon\left(\frac{p'}{8\sqrt{\varepsilon}(-p)^{3/2}}, -\frac{p'}{8p}\right) + \varepsilon^2\left(\frac{p''}{16\varepsilon p^2} - \frac{3p'^2}{32\varepsilon p^3},\right.\right.$$

$$\left.\left.\frac{p''}{16\sqrt{\varepsilon}(-p)^{3/2}} - \frac{3p'^2}{32\sqrt{\varepsilon}(-p)^{5/2}}\right)\right)\exp\left(\frac{1}{\varepsilon}\int_0^t\left(\sqrt{-\varepsilon p} - \varepsilon\frac{p'}{4p} + \varepsilon^2\frac{p'^2}{32\sqrt{\varepsilon}(-p)^{5/2}}\right)d\tau\right),$$

and asymptotical evaluation

$$L(y_2) = \varepsilon^2\left(\frac{77p'^3}{2^8p^4} - \frac{21p'p''}{2^6p^3} + \frac{\sqrt{\varepsilon}p'^2p''}{2^9(-p)^{9/2}} - \frac{3\sqrt{\varepsilon}p'^4}{2^{10}(-p)^{11/2}},\right.$$

$$\left.-\frac{7p'^2}{2^6p^2} - \frac{19\sqrt{\varepsilon}p'p''}{2^6(-p)^{5/2}} - \frac{33\sqrt{\varepsilon}p'^3}{2^7(-p)^{7/2}} + \frac{\sqrt{\varepsilon}p''}{2^4(-p)^{3/2}} - \frac{\varepsilon p'^2p''}{2^9p^4} + \frac{3\varepsilon p'^4}{2^{10}p^5}\right) = O\left(\varepsilon^2\right)$$

holds true.

References

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