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PP-Test for Integrability of Some Evolution Differential Equations

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The connection between transcendents of Painlevé and evolution equations is discussed. The calculus PP-procedure is proposed.

The Painlevé singularity analysis is one of the systematic and powerful method to identify the integrability conditions of nonlinear partial differential equations (NPDEs). In recent years, this method has been applied to a very large number of NPDEs and systematically established the complete integrability properties like Lax pair, Bäcklund, Darboux and Miura transformations, bilinear transformation, soliton solutions and so on.

In the last decade of the nineteenth century some mathematicians focused their attention on the classification of ordinary differential equations (ODEs) on the basis of the type of singularity their of solutions.

It is essential to distinguish between two types of singularities. Fixed singularities determined by the coefficients of the equation and its location do not therefore depend on initial conditions. Movable singularities are such whose location on the complex plane does indeed depend on the initial conditions.

The beginning of the study of singularities in the complex plane for differential equations was always attributed to Cauchy, whose idea was to consider local solutions on the complex plane and to use methods of analytical continuation to obtain general solutions. For this procedure to work a complete knowledge of singularities of the equation and its location in the complex plane is required.

Some French mathematicians (Painlevé, Gambier, Garnier and Chazy), following the ideas of Fuchs, Kovalevskaya, Picard and other, completely classified first order equations and studied second order differential equations. In this case, Paul Painlevé [1] found 50 types of second order equations whose only movable singularities were ordinary poles. This special analytical property now carries his name and in what follow will be referred to as the Painlevé Property (PP). Of these 50 types of equations 44 can be integrated in terms of known functions (Riccati equations, elliptic functions, linear equations) and the other six in spite of having meromorphic solutions do not have algebraic integrals that would allow to reduce the equation to quadratures. Today these are known as Painlevé Transcendents:

$$P_1: \quad w''(z) = 6w^2(z) + az;$$

$$P_2: \quad w''(z) = 2w^3(z) + zw(z) + b;$$

$$P_3: \quad w''(z) = \frac{w'^2}{w} + e^z(aw^2 + b) + e^{2z}\left(cw^3 + \frac{d}{w}\right);$$

$$P_4: \quad w''(z) = \frac{w'^2}{2w} + \frac{3w^3}{2} - 4zw^2 + 2\left(z^2 - a\right)w + \frac{b}{w};$$

$$P_5: \quad w''(z) = w'^2\left(\frac{1}{2w} + \frac{1}{w-1}\right) - \frac{w'}{z} + \frac{(w-1)^2(aw + b/w)}{z^2} + \frac{cw}{z} + \frac{dw(w+1)}{w-1};$$

$$P_6: \quad w''(z) = \frac{w'^2}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right) - w'\left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right) + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(a + \frac{bz}{w^2} + c(z-1)(w-1)^2 + dz(z-1)(w-z)^2\right).$$

P_1, P_2, P_3, P_4 being simple meromorphic functions; P_5 has fixed transcendent critical points $z = 0$, $z = \infty$; P_6 has fixed transcendent critical points $z = 0$, $z = 1$, $z = \infty$.

The main contribution of Paul Painlevé lies in that he established the basis for a theory that would allow one a priory, by singularity analysis, to decide on integrability of the partial differential equations (PDEs) without previously solving them. Singularity analysis turns out to be a test of integrability for an equation.

An ordinary differential equation (ODE) is said to possess the Painlevé property if all of its movable singularities are poles. The relation between integrability and the absence of movable critical points was made more explicit through the work [2] in which it was established such ARS (Ablowitz, Ramani, Segur) conjecture: every ODE obtained by similarity reduction of a partial differential equation (PDE) solvable with the inverse scattering method posses the Painlevé property. For the equations that do not have symmetries the ARS conjucture is quite useless as it is not possible to obtain similarity reductions from usual group-theory procedures.

The definition of the PP for PDEs was proposed in [3]. According to these authors, we say that a PDE has the PP if its solutions are singlevalued in a neighbourhood of the manifold of movable singularities. When this manifold depends on the initial conditions it is called a movable singularity manifold.

It is known that the singularities of a function $f(z_1, z_2, \ldots, z_n)$ of $n > 1$ complex variables cannot be isolated; rather they occur along analytic manifolds of (complex)-dimension $n - 1$ determined by equation of the form

$$\chi(z_1, z_2, \ldots, z_n) = 0, \tag{1}$$

being an analytical function of its variables in a neighbourhood of the singularity manifold defined by (1).

To test for the presence of PP one assumes that a solution $u(z_1, z_2, \ldots, z_n)$ of a PDE can be expanded around the singularity manifold (1) as following Laurent series of the form

$$u = \chi^{-\alpha} \sum_{k=0}^{\infty} u_k \chi^k, \tag{2}$$

where the coefficients $u_k(z_1, z_2, \ldots, z_n)$ are analytical in a neighbourhood of $\chi = 0$.

It is possible in any to truncate the expansion series at a certain term in order to obtain particular solutions of the equation. If the expansion is truncated at the constant term, expression (2) reduces to:

$$u = u_0\varphi^{-\alpha} + u_1\varphi^{1-\alpha} + \cdots + u_\alpha. \tag{3}$$

Substitution of (3) in the corresponding PDE leads to an overdetermined system of equations for φ, u_j and their derivatives. The truncation of the Painlevé series is the basis of a method called as Singular Manifold Method (SMM). One then substitutes the above expansion (2) in the PDE to determine the value of α and the recurrence relations among the u_k's. If all the allowed values of α turn out to be integers and the set of recurrence relations consistently allows for the arbitrariness of initial conditions, then the given PDE is said to posses the PP and is conjectured to integrable.

An algorithmic procedure has recently been put forward for determining similarity reductions for PDEs. The essence of the procedure is to study the Lie symmetries.

It has been found that Painlevé Transcendents often appear in similarity reductions of the evolution equations with solitons.

The soliton is an object describing solitary wave solutions interacting among themselves without any change in shape except for a small change in its phase. The solitary waves were studied and were described in hydrodynamics problems by Scott Rassel (1844), Boussinesq (1872), Korteveg-de-Vries (1895), M.A. Lavrentjev (1945), Friedrichs (1954). But the concept of "soliton" emerged for the first time in 1965 with Zabusky and Kruskal [4] and the Korteveg-de-Vries (KDV) equation reappeared between 1955 and 1960 in the context of plasma physics.

The Hirota's bilinear method [5] is known as a powerful procedure for generating multisoliton solutions for PDEs. It is essentially consists in bilinearizing the differential equation by an transformation reminiscent of the Painlevé truncated expansion. The WTC (Weiss, Tabor, Carnevale) method also provides an iterative procedure for generating solitons from the Lax pair and from the corresponding auto-Bäcklund transformation,where the corresponding singularity manifold φ is determined in each step and after n-steps the solution can be expressed in terms of the product $\varphi_1, \varphi_2, \dots, \varphi_n$ from which it is then possible to construct the Hirota τ-function associated with the solution with n solitons.

The Inverse Scattering Method (ISM) was developed initially allowing one to solve many integrable evolution equations with soliton solutions,in particular, the KdV equation:

$$6u_t = 3uu_x - \frac{1}{2}u_{xxx}, \tag{4}$$

and its different modifications:

$$6v_t = 3v^2v_x - \frac{1}{2}v_{xxx}, \tag{5}$$

$$u_t + u^pu_x + u_{xxx} = 0; \tag{6}$$

the sine-Gordon equation:

$$u_{xt} = \sin u; \tag{7}$$

the Kline–Gordon equation:

$$u_{xt} = f'(u), \qquad f(u) = -\cos u; \tag{8}$$

the Schrödinger equation:

$$iu_t = u_x^2 - 4iu^2u_x + 8|u|^4u; \tag{9}$$

the Boussinesq equation:

$$u_{tt} = u_{xx} + 6\left(u^2\right)_{xx} - u_{xxx}; \tag{10}$$

and the Born–Infeld equation:

$$\left(1 - u_t^2\right)u_{xx} + 2u_xu_{xt} - \left(1 + u_x^2\right)u_{tt} = 0. \tag{11}$$

In recent years Miura transformation

$$f(z) = w'(z) + w^2(z), \qquad w(z) \equiv P_2 \tag{12}$$

widely was applied to find the automodel solutions of evolution equations of the type (4)–(11).

There is a known connection between first Painlevé transcendent P_1 and the automodel solution of KdV equation of such type

$$u_t + u_{xxx} - 6u_x u = 0, \qquad u(x,t) \equiv u, \tag{13}$$

which is received by following relation:

$$u(x,t) = 2[w(z) - t], \qquad z = x - 6t^2, \quad w(z) \equiv P_1. \tag{14}$$

The substitution

$$u(x,t) \equiv w(z), \qquad z = x - t \tag{15}$$

relates P_1 with automodel solution of Boussinesq equation (10). Second Painlevé transcendent P_2 relates with modifications of KdV equation (6) and

$$v_t + v_{xxx} - 6v_x v^2 = 0; \qquad v \equiv v(x,t). \tag{16}$$

In fact, if we make such transformation

$$u(x,t) = [3(t - t_0)]^{-2/3} f(z), \qquad z = [3(t - t_0)]^{-1/3}(x - x_0) \tag{17}$$

and then carry out some mathematical procedure of differential calculus using Miura transformation (12) we obtain automodel solution (17) for equation (13) where $w(z)$ being P_2. KdV evolution equation of type (16) has the automodel solution

$$v(x,t) = [3(t - t_0)]^{-1/3} w(z), \qquad z = [3(t - t_0)]^{-1/3}(x - x_0), \qquad w(z) \equiv P_2. \tag{18}$$

Other evolution equations also have relation with Painlevé transcendents, particularly, the sine–Gordon equation (7) relates with P_3, the Schrödinger equation (9) relates with P_4, the KdV equation of type (5) relates with P_5 and the Born–Infeld equation (11) relates with P_6.

We apply two types of expansions, described in [6], to construct the Painlevé transcendents at explicit form.

In brief, the calculus of Painlevé Property Procedure comes to the following:

1. The initial Cauchy problem for Painlevé equations $P_1 - P_6$ solves, by exact initial conditions, in a holomorphic neighbourhood with help of truncated expansions to consider local solutions on the complex plane and use the method of analytical continuation for obtaining general solution. Unknown coefficients a_n of the power series can be obtained from recurrence relations [7].
2. In order to find a location of unknown poles of m-th order (to the point, Painlevé transcendents have the poles of the different orders for every number $(1, 2, \ldots, 6)$ we apply the algorithm of isolation of pole, proposed by the author [8].
3. On the one hand, the meromorphic integrals of $[P_1 - P_6]$ can be represented by superposition of finite polynomial $\mathcal{P}_\nu(z)$ and some general summarized geometrical progression $\mathcal{R}_\nu(z)$ in a neighbourhood of singularities for Painlevé transcendents. Corresponding correlations for the coefficients of the regular power series, m-orders of poles and value $q = 1/R$, defining location of poles, were found.
4. On the other hand, meromorphic integrals of $[P_1 - P_6]$ can be expanded around poles in form of Laurent series in a neighbourhood of the found poles and then both type of expansions (regular, described in 1, and irregular, described in 3) stick together.
5. Transition across the pole realizes with help of procedure of the analytical continuations, which also is used in the case of realization of procedure for isolation of poles.

6. All algebraic operations with the power series and Laurent series and obtaining of recurrence relations were made according to the Method of generalized power series, proposed by Prof. P.F. Fil'chakov for solving of wide classes of linear and nonlinear problems and described in his book [9]. This method is based on Euler's method with using of Cauchy's formula for multiplying of power series.

We hope that there will be further study in this direction.

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