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On Asymptotic Formulae for Solutions of Differential Equations with Summable Coefficients

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Using the technique of asymptotic expansions, we prove the theorem about the form of system $\frac{dx}{dt} = A(t)x$, when the eigenvalues of $A(t)$ are multiple and the corresponding elementary divisors have constant multiplicity.

In this paper we consider a system of differential equations

$$\frac{dx}{dt} = A(t)x, \qquad t \geq t_0 \tag{1}$$

in the case when eigenvalues of the matrix $A(t)$ are multiple and the corresponding elementary divisors are of constant multiplicity.

The problem of asymptotic behaviour of solutions of the system (1) was partially solved by I.M. Rappoport [1] and M.I. Shkil [2] by reducing it to the generalized L-diagonal form

$$\frac{dx}{dt} = (\Lambda(t) + C(t))x, \tag{2}$$

where $\Lambda(t) = \operatorname{diag}\{W_1(t), \ldots, W_m(t)\}$, $W_k(t) = ||w_{ij}(t)||_1^{n_k}$, $w_{ii}(t) = w_i(t)$, $w_{ii+1}(t) = 1$, $w_{ij}(t) = 0$ $(i \neq j,\ j \neq i+1)$.

In particular I.M. Rappoport, having imposed some restrictions on elements of the matrices $\Lambda(t)$ and $C(t)$ [1], obtained asymptotic formulae for solutions of the system (2). He also adduced the simplest substitutions by means of which the system (1) can be reduced to the system (2). M.I. Shkil and his students in investigation of asymptotic properties of the system (1) used the previously developed methods of asymptotic integration for systems of differential equations with slowly changing coefficients

$$\frac{dx}{dt} = A(\tau, \varepsilon)x,$$

where

$$A(\tau, \varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s A_s(\tau), \qquad \tau \in [0; L], \quad \tau = \varepsilon t$$

and ϵ is a small parameter, in the case of multiple eigenvalues of the matrix $A_0(\tau)$. Here constant multiplicity elementary divisors of $A_0(\tau)$ caused substantial difficulties.

In this paper we suggest a method for construction of asymptotic solutions of the system (1) by reducing it to the L-diagonal form

$$\frac{dx}{dt} = (\Lambda(t) + C(t))x, \tag{3}$$

where $\Lambda(t) = \operatorname{diag}\{\lambda_1(t), \ldots, \lambda_n(t)\}$.

It follows from conditions imposed on the matrix $A(t)$ that there exist non-degenerate for all $t \geq t_0$ matrix $T(t)$ such that

$$T^{-1}(t)A(t)T(t) = \Lambda(t),$$

where $\Lambda(t)$ is a Jordan matrix corresponding to the matrix $A(t)$. In the system (1) we set

$$x = T(t)y$$

and multiply the obtained system by $T^{-1}(t)$. We get

$$\frac{dy}{dt} = \left(\Lambda(t) - T^{-1}T^{'}(t)\right) y. \tag{4}$$

In the following we will assume that the matrix $D(t) = \Lambda(t) - T^{-1}T^{'}(t)$ has simple eigenvalues at the interval $[t_0; +\infty)$. Instead of the system (4) we will consider the system

$$\varepsilon \frac{dy}{dt} = D(t)y, \tag{5}$$

where $\varepsilon > 0$ is a real parameter (the system (5) with $\varepsilon = 1$ coincides with system (4)). When we assume

$$y = U_m(t, \varepsilon)z, \qquad U_m(t, \varepsilon) = \sum_{s=0}^{m} \varepsilon^s U_s(t),$$

where z is an n-dementional vector and $U_s(t)$ are square matrices of the order n, we obtain

$$\varepsilon U_m(t, \varepsilon)\frac{dz}{dt} = (D(t)U_m(t, \varepsilon) - \varepsilon U_m'(t, \varepsilon))z. \tag{6}$$

We will construct the matrices $U_s(t)$ $(s = 0, 1, \dots, m)$ for the following matrix equality to be satisfied:

$$D(t)U_m(t, \varepsilon) - \varepsilon U_m'(t, \varepsilon) = U_m(t, \varepsilon) \left(\Lambda_m(t, \varepsilon) + \varepsilon^{m+1} C_m(t, \varepsilon)\right), \tag{7}$$

where $\Lambda_m(t, \varepsilon)$ is diagonal matrix of the form $\Lambda_m(t, \varepsilon) = \sum\limits_{s=0}^{m} \varepsilon^s \Lambda_s(t)$ and $C_m(t, \varepsilon)$ is a square matrix of the order n to be determined.

Matrices $U_m(t, \varepsilon), \Lambda_m(t, \varepsilon)$ we will determine from the equality (7) where we require coefficients at $\varepsilon^0, \varepsilon^1, \dots, \varepsilon^m$ to be equal. Then we obtain the following system of matrix equations

$$D(t)U_0(t) - U_0(t)\Lambda_0(t) = 0, \tag{8}$$

$$D(t)U_s(t) - U_s(t)\Lambda_0(t) = U_{s-1}'(t) + \sum_{j=1}^{s} U_{s-j}(t)\Lambda_j(t). \tag{9}$$

Let us write the matrix equation (8) in the vector form. For this purpose we designate the columns of the matrix $U_0(t)$ as $u_{0i}(t)$ $(i = 1, 2, \dots, n)$ and write the matrix $\Lambda_0(t)$ in the form

$$\Lambda_0(t) = \operatorname{diag}\{\lambda_1(t), \lambda_2(t) \dots, \lambda_n(t)\},$$

where $\lambda_i(t)$ $(i = 1, 2, \dots, n)$ are eigenvalues of the matrix $D(t)$. Then we obtain the following system of equations from (8):

$$(D(t) - \lambda_i(t)E)u_{0i}(t) = 0. \tag{10}$$

Thus, if $\mu_i(t)$ $(i = 1, 2, \ldots, n)$ are eigenvectors of the matrix $D(t)$, we can set

$$u_{0i}(t) = \mu_i(t).$$

Let us note that we showed in [3] that it is possible to construct $\mu_i(t)$ $(i = 1, 2, \ldots, n)$ so that

$$(\mu_i(t), \psi_j(t)) = \begin{cases} 1, & i = j; \\ 0, & i \neq j, \ i, j = 1, \ldots, n. \end{cases}$$

where $\psi_j(t)$ $(j = 1, 2, \ldots, n)$ are elements of the zero space of the matrices $(D(t) - \lambda_j(t)E)^*$.

Let us consider the system of matrix equations (9) with $s = 1$, having written it in the vector form

$$(D(t) - \lambda_i(t)E)u_{1i}(t) = g_{1i}(t), \qquad i = 1, 2, \ldots, n, \tag{11}$$

where $u_{1i}(t)$ are columns of the matrix $U_1(t)$ and vector $g_{1i}(t)$ is determined as follows:

$$g_{1i}(t) = u'_{0i}(t) + u_{0i}(t)\lambda_{1i}(t). \tag{12}$$

The equation (11) is solvable with respect to $u_{1i}(t)$ if and only if the vector $g_{1i}(t)$ $(i = 1, 2, \ldots, n)$ is orthogonal to all vectors that are solutions of the corresponding homogeneous associated system. Thus, for the system (11) to have a solution, it is necessary and sufficient that for all $t \geq t_0$ the following equality is satisfied:

$$(g_{1i}(t), \psi_i(t)) = 0, \qquad i = 1, 2, \ldots, n.$$

Substituting to the latter equation the value of the vector $g_{1i}(t)$, we obtain a scalar equation with respect to $\lambda_{1i}(t)$

$$(\mu'_i(t), \psi_i(t)) + (\mu_i(t)\lambda_{1i}(t), \psi_i(t)) = 0.$$

Whence we get that

$$\lambda_{1i}(t) = -(\mu'_i(t), \psi_i(t)), \qquad i = 1, 2, \ldots, n. \tag{13}$$

Therefore we can set

$$\Lambda_1(t) = \operatorname{diag}\{\lambda_{11}(t), \lambda_{12}(t), \ldots, \lambda_{1n}(t)\}.$$

Then, substituting the values of $\lambda_{1i}(t)$ into (11), we get a system that has a solution with respect to the vector $u_{1i}(t)$. We will look for this solution in the form

$$u_{1i}(t) = \sum_{r=1}^{n} c_{ri}^{(1)}(t)\mu_r(t), \qquad i = 1, 2, \ldots, n, \tag{14}$$

where $c_{ri}^{(1)}(t)$ are functions that have to be determined for the vector (14) to satisfy the system (11). For this purpose we substitute (14) into the system (11) and scalarly multiply the obtained equality by the vector $\psi_j(t)$ $(j = 1, 2, \ldots, n)$. We obtain

$$c_{ji}^{(1)}(t)(\lambda_j(t) - \lambda_i(t)) = (g_{1i}(t), \psi_j(t)), \qquad j = 1, 2, \ldots, n.$$

When $i = j$ we obtain the equality

$$c_{jj}^{(1)}(t) \cdot 0 \equiv 0.$$

Thus we can take an arbitrary function $c_{jj}^{(1)}(t)$, e.g.

$$c_{jj}^{(1)}(t) \equiv 0, \qquad t \geq t_0.$$

In case when $i \neq j$ we get that

$$c_{ji}^{(1)}(t) = \frac{(g_{1i}(t), \psi_j(t))}{\lambda_j(t) - \lambda_i(t)}.$$

Then the vector $u_{1i}(t)$ is as follows:

$$u_{1i}(t) = \sum_{r=1, r\neq i}^{n} \frac{(g_{1i}(t), \psi_r(t))}{\lambda_r(t) - \lambda_i(t)} \mu_r(t).$$

So we determined the matrices $U_1(t)$ and $\Lambda_1(t)$.

Using the method of mathematical induction we can show that similarly it is possible to find all further matrices $U_s(t)$ and $\Lambda_s(t)$ $(s = 2, 3, \ldots, n)$. from the equations (9) [2, 3].

Let us proceed to the finding of the matrix $C_m(t, \varepsilon)$.

Taking into account that arbitrary elements of the matrix $U_m(t, \varepsilon)$ can be chosen so that [3]

$$\det U_m(t, \varepsilon) \neq 0, \qquad t \geq t_0,$$

then we get from the system (7) with $\varepsilon = 1$

$$C_m(t, 1) = -U_m^{-1}(t, 1) \left(U_m'(t) + \sum_{k=1}^{m} \sum_{j=k}^{m} U_j(t) \Lambda_{m+k-j}(t) \right).$$

Thus, the system (4) is reduced to the system of the form

$$\frac{dz}{dt} = (\Lambda_m(t, 1) + C_m(t, 1))z. \tag{15}$$

If also

a) neither of the differences

$$\operatorname{Re} \lambda_i(t, 1) - \operatorname{Re} \lambda_j(t, 1) \tag{16}$$

changes the sign for all $t \geq t_1 \geq t_0$, where $\lambda_i(t, 1)$ $(i = 1, 2, \ldots, n)$ are diagonal elements of the matrix $\Lambda_m(t, 1)$;

b)

$$\int_{t_0}^{\infty} ||C_m(t, 1)|| dt < \infty, \tag{17}$$

then for the vector $x_s(t)$, which is solution of the system (1), we have the formula

$$x_s(t) = \mu_{sj}(t) \exp \int_{t_0}^{t} \lambda_j(t, 1) dt, \qquad s, j = 1, 2, \ldots, n, \tag{18}$$

where $\mu_{sj}(t)$ are continuous functions in the interval $[t_0; +\infty)$.

Thus, the following theorem hold true.

Theorem. *Let the matrix $D(t)$ of the system (4) on the segment $[t_0; +\infty)$ have simple eigenvalues and the condidtions (16), (17) be fulfilled. Then n solutions of the system (1) have the form (18).*

References

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