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C^*-Algebras Associated with Quadratic Dynamical System

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In this paper we consider enveloping C^*-algebras of $*$-algebras given by generators and defining relations of the following form $A = \mathbb{C}\langle X, X^* |\ XX^* = f(X^*X)\rangle$, where f is a Hermitian mapping. Some properties of these algebras associated with simple dynamical systems $(f, \mathbb{R})$ are studied. As an example quadratic dynamical systems are considered.

1 Introduction

It is well known that there is close connection between the representation theory of C^*-algebras and structure of dynamical systems $(f(), X)$. In the case when f is one-to-one mapping, the C^*-algebra associated with the transformation group had been studied by many authors, for example by Glimm, Effros and Hahn. The general theory of cross-products of C^*-algebras was elaborated by Doplicher, Kastler and Robinson.

In recent papers (see [8] and references given there in) a special class of $*$-algebras given by generators and relations was considered and some of the results from the theory of cross-product C^*-algebras were transferred into non-bijective settings, which may be important in studying of multi-dimensional non-linear deformation (see [11, 10, 3]), such as Witten's first deformation of $su(2)$, Quesne and Beckers non-linear deformation of $su(2)$ etc. Examples were studied in connection with different quantum deformations of algebras, such as Quantum Unit Disc (Klimek and Lesnievski), one-dimensional q-CCR and their non linear transformation, ets see [7, 8].

Thus, for example, for one-parameter Quantum Unit Disc there corresponds the dynamical system: $f(\lambda) = \frac{(q+\mu)\lambda-\mu}{\mu\lambda+1-\mu}$, where μ is a parameter of deformation, for two-parameter Quantum Unit Disc there corresponds $f(\lambda) = \frac{(q+\mu)\lambda+1-q-\mu}{\mu\lambda+1-\mu}$, for Witten's first deformation of $su(2)$ there corresponds two-dimensional quadratic map $f(x, y) = (p^{-1}(1 + p^{-1}x), g(gy - x + (p - p^{-1})x^2))$, where $g = \pm 1$ depending on the chosen real form and p is a parameter of deformation.

In the present paper we will deal with a one-dimensional polynomial map $f : \mathbb{R} \to \mathbb{R}$ and consider $*$-algebra $\mathcal{A}_f = \mathbb{C}\langle X, X^* |\ XX^* = f(X^*X)\rangle$. Under condition of simplicity of the dynamical system $(f, \mathbb{R})$ we prove that the enveloping C^*-algebra is GCR (type I) C^*-algebra and investigate some other properties. We also discuss the question what relation between dynamical systems $(f_1, \mathbb{R})$ and $(f_2, \mathbb{R})$ corresponds to the isomorphism of enveloping C^*-algebras of $*$-algebras $\mathcal{A}_{f_1}$ and $\mathcal{A}_{f_2}$.

In the last section we consider an example: Unharmonical Quantum Oscillator, i.e. the two-parametric family of $*$-algebras $\mathcal{A}_{a,b} = \mathbb{C}\langle X, X^* |\ XX^* = 1 + aX^*X - b(X^*X)^2\rangle$, where a and b are real parameters with $b > 0$. Some partitioning of parametric domain into parts depending on isomorphism class of C^*-enveloping algebra are given.

2 Simple dynamical systems

For the convenience of the reader we repeat the relevant material from [12, 8] without proofs, thus making our exposition self-contained. By the dynamical system we mean a continuous map $f : \mathbb{R} \to \mathbb{R}$ or $f : I \to I$, where $I \subset \mathbb{R}$ is a closed bounded interval. By the orbit of dynamical system $(f, \mathbb{R})$ we mean a sequence $\delta = (x_k)_{k \in P}$, where P is one of the sets $\mathbb{Z}$, $\mathbb{N}$, such that $f(x_k) = x_{k+1}$. But sometimes we will consider orbit as the set $\{x_k |\ k \in P\}$. The set of all orbits will be denoted by $\mathrm{Orb}(f)$. For $x \in \mathbb{R}$ denote by $\mathcal{O}_+(x)$ the forward orbit, i.e. $(f^k(x))_{k \geq 0}$. For every orbit $\delta \in \mathrm{Orb}(f)$ define $\omega(\delta)$ be the set of accumulation points of forward half-orbit and $\alpha(\delta)$ be the set of accumulation points of backward half-orbit.

By the positive orbit of $(f(), \mathbb{R})$ we mean a sequence $\omega = (x_k)_{k \in \mathbb{Z}}$ such that $f(x_k) = x_{k+1}$ and $x_k > 0$ for all integer k. Unilateral positive orbit is a sequence $\omega = (x_k)_{k \in \mathbb{N}}$ (Fock-orbit) such that $x_1 = 0$ and $f(x_k) = x_{k+1}$, $x_k > 0$ for $k > 1$ or $\omega = (x_{-k})_{k \in \mathbb{N}}$ (anti-Fock-orbit) such that $x_{-1} = 0$ and $f(x_k) = x_{k+1}$, $x_k > 0$ for $k < -1$. Define $\mathrm{Orb}_+(f)$ be the set of all positive orbits which are either periodic (cycles) or contain no cycles. Note that $\omega(\delta) = \emptyset$ for any anti-Fock orbit δ and $\alpha(\delta_1) = \emptyset$ for the Fock orbit δ_1.

Cycle $\beta = \{\beta_1, \ldots, \beta_m\}$ is called attractive if there is a neighborhood U of β such that $f(U) \subseteq U$ and $\cap_{i>0} f^i(U) = \beta$.

Point $x \in \mathbb{R}$ is called non-wandering if for every its neighborhood U there exists a positive integer m such that $f^m(U) \cap U \neq \emptyset$.

Since we will consider only bounded from above functions f and positive orbits we can always consider our dynamical system on closed interval $[0, \sup f]$.

In this paper we will deal with a simple dynamical system which possesses one of the equivalent properties listed in the following theorem:

Theorem 1 ([12], 3.14]). *Let $(f(), I)$ be dynamical system with $f \in C(I)$, ($I \subset \mathbb{R}$ is closed bounded interval). The following conditions are equivalent:*

1) for every $x \in I$ $\omega(x) = \omega(\mathcal{O}_+(x))$ is cycle;

2) $\mathrm{Per}(f)$ is closed;

3) every non-wandering point is periodic.

f is called partially monotone, if I decomposes into a finite union of sub intervals, on which f is monotone.

Let us mention the following statement from [8].

Theorem 2. *Let (f, I) be a dynamical system with partially monotone and continuous f. Then the following conditions are equivalent:*

1) $\mathrm{Per}(f)$ is closed;

2) for some positive integer m the relation $\mathrm{Fix}(f^{2^{m+1}}) = \mathrm{Fix}(f^{2^m})$ holds;

3) any quasi-invariant ergodic measure is concentrated on a single element of the trajectory decomposition.

The class of dynamical systems which satisfies equivalent conditions 1–3 of Theorem 2 is denoted by $\mathcal{F}_{2^m}$. Let us note that when $\mathrm{Per}(f)$ is closed Theorem ([12], 3.12) implies that the length of every cycle is a power of 2 and there no homoclinical orbits (i.e. orbit δ such that $\alpha(\delta) = \omega(\delta)$ is a cycle).

We will need the following lemma:

Lemma 1. *Let $(f, \mathbb{R})$ be simple dynamical system with bounded f and the set of periodic points which are not the points of attractive cycles, i.e. the set $[0, \sup f] \cap \mathrm{Per}(f) \setminus \cup_{\beta\text{-attractive cycle}} \beta$ be finite then for every orbit $\delta \in \mathrm{Orb}_+(f)$ the α-boundary $\alpha(\delta)$ is cycle which is not attractive.*

Proof. 1. Let us show that every α-boundary point is non-wandering: if $x \in \alpha(\delta)$ then for arbitrary $\epsilon > 0$ and positive integer n there is $y \in B_\epsilon(x)$ and integer $l \geq n$ such that $f^l(y) \in B_\epsilon(x)$. Indeed, if $\delta = (x_k)_{k\in\mathbb{Z}}$ then there is subsequence $x_{-n_k} \to x$. For a given $\epsilon > 0$ we can find an integer k_0 such that $x_{-n_k} \in B_\epsilon(x)$ for all $k \geq k_0$. Take $k_1 > k_0$ such that $n_{k_1} \geq n_{k_0} + n$ and put $y = x_{-n_{k_1}} \in B_\epsilon(x)$ and $l = n_{k_1} - n_{k_0} \geq n$. Then $f^l(y) = x_{-n_{k_0}} \in B_\epsilon(x)$.

2. Since for a simple dynamical system every non-wandering point is periodic we obtain that $\alpha(f) = \cup_{\delta\in\mathrm{Orb}_+(f)}\alpha(\delta)$ is contained in $\mathrm{Per}(f) \cap [0, \sup f]$.

3. Let β be an attractive cycle and assume that $\beta \subseteq \alpha(\delta)$, then $\alpha(\delta) = \beta$. Indeed, let $\beta_1 \in \alpha(\delta)$ be another cycle, then there is $\epsilon > 0$ such that $\beta_1 \cap B_\epsilon(\beta) = \emptyset$. Since β is an attractive cycle there is $\eta > 0$ and $\eta < \epsilon$ such that for arbitrary $y \in B_\eta(\beta)$ we have $\mathcal{O}_+(y) \subseteq B_\epsilon(\beta)$. Let $\delta = (x_k)_{k\in\mathbb{Z}}$. Since $\beta_1 \in \alpha(\delta)$ there is a positive integer k_0 such that $x_{-k_0} \in B_{\epsilon_1}(\beta_1)$, where $\epsilon_1 > 0$ chosen such that $B_{\epsilon_1}(\beta_1) \cap B_\epsilon(\beta) = \emptyset$. But $\beta \in \alpha(\delta)$ so there is a positive integer $k_1 > k_0$ with the property $x_{-k_1} \in B_\eta(\beta)$ then $\mathcal{O}_+(x_{-k_1}) \subseteq B_\epsilon(\beta)$ and, obviously, $x_{-k_0} \in \mathcal{O}_+(x_{-k_1})$. This is a contradiction. Thus we have proved that $\alpha(\delta) = \beta$. But this implies that $\alpha(\delta) = \omega(\delta) = \beta$. So δ is a homoclinical orbit and so $(f, \mathbb{R})$ is not simple. Which is a contradiction. Thus $\alpha(\delta)$ has no attractive cycles.

4. Let us prove that $\alpha(\delta)$ is a single cycle for every orbit δ. We already know that $\alpha(\delta) \subset [0, \sup f] \cap \mathrm{Per}(f) \setminus \cup_{\beta\text{-attractive cycle}} \beta$. Since the last one is a finite set there is $\epsilon > 0$ such that for arbitrary distinct cycles $\beta_1, \beta_2 \in \alpha(\delta)$ we have $B_\epsilon(\beta_1) \cap B_\epsilon(\beta_2) = \emptyset$ and $f(B_\epsilon(\beta_1)) \cap B_\epsilon(\beta_2) = \emptyset$ (the last is a possible since $f(\beta_1) = \beta_1$ and so $f(\beta_1) \cap \beta_2 = \emptyset$). There is a positive integer n such that $x_{-k} \in B_\epsilon(\alpha(\delta))$ for every $k > n$. Thus for every cycle $\beta \in \alpha(\delta)$ there is positive $\epsilon_1 < \epsilon$ such that $f(y) \in B_{\epsilon_1}(\beta_1)$ for every $y \in B_{\epsilon_1}(\beta)$, where $\beta_1 \neq \beta$ (in the opposite case would be $\alpha(\delta) = \beta$). This contradicts $f(B_\epsilon(\beta)) \cap B_\epsilon(\beta_1) = \emptyset$. ■

3 Representation theory of $*$-algebras associated with $\mathcal{F}_{2^m}$ dynamical systems

The following theorem is due to Samoilenko and Ostrovskii [8].

Theorem 3. *Let f be partially monotone continuous map and $(f, \mathbb{R})$ be $\mathcal{F}_{2^m}$ dynamical system. Let $A = \mathbb{C}\langle X, X^* |\ XX^* = f(X^*X)\rangle$ be corresponding $*$-algebra.*

1. To every positive non-cyclic orbit $\omega(x_k)_{k\in Z}$ there corresponds an irreducible representation π_ω in Hilbert space $l_2(Z)$ given by the formulae: $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k \in Z$ and $X = UC$ is a polar decomposition.

2. To positive Fock-orbit $\omega = (x_k)_{k\in N}$ there corresponds an irreducible representation π_ω in Hilbert space $l_2(N)$ given by the formulae: $Ue_0 = 0$, $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k > 1$ and $X = UC$.

3. To positive anti-Fock-orbit $\omega = (x_{-k})_{k\in N}$ there corresponds an irreducible representation π_ω in Hilbert space $l_2(N)$ given by the formulae: $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k > 1$ and $X = UC$.

4. To cyclic positive orbit $\omega = (x_k)_{k\in N}$ of length m there corresponds a family of m-dimensional irreducible representation $\pi_{\omega,\phi}$ in Hilbert space $l_2(\{1, \dots, m\})$ given by the formulae: $Ue_0 = e^{i\phi}e_{m-1}$, $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k = 1, \dots, m$; $0 \leq \phi \leq 2\pi$ and $X = UC$.

This is a complete list of unequivalent irreducible representation of a given $*$-algebra.

4 Enveloping C^*-algebra

Let f be a bounded from above Hermitian polynomial (hence f is always partially monotone and continuous). Let $A_f = \mathbb{C}\langle X, X^* | \; XX^* = f(X^*X)\rangle$ be $*$-algebra given by generators and relations which has at least one representation. Let $C = \sup f$. Then for any representation π of $*$-algebra A_f we have $\|X\| \le \sqrt{C}$. Thus there is an enveloping C^*-algebra, which we denote by $\mathcal{E}_f$. Let us note that by Theorem 3.3 [12] for $f \in C^1(I, I)$ simplicity of a dynamical system is equivalent to $(f, I) \in \mathcal{F}_{2^m}$ for some integer m.

Theorem 4. *Let a dynamical system $(f, \mathbb{R})$ be simple and $\delta \in \mathrm{Orb}_+(f)$.*

1. If δ is non-cyclic bilateral orbit than $C^*(\pi_\delta) = Z \times_\delta C(\overline{\delta})$ is a cross-product of C^*-algebra, where $\overline{\delta} = \delta \cup \omega(\delta) \cup \alpha(\delta)$.

The set of irreducible representation $\mathrm{Irr}(C^*(\pi_\delta))$ is $\pi_\delta, \pi_{\omega(\delta),\phi}, \pi_{\alpha(\delta),\phi}$, where $0 \le \phi \le 2\pi$.

2. Assume that 0 is not a periodic point. If δ is a Fock-orbit then $C^*(\pi_\delta) \cong M_m(\mathcal{T}(C(\mathbb{T})))$ is a matrix algebra of dimension $m =| \omega(\delta) |$ over C^*-algebra $\mathcal{T}(C(\mathbb{T}))$ of the Toeplitz operators.

The same is true for anti-Fock orbit with $m =| \alpha(\delta) |$.

Proof. Let $\pi \in \mathrm{Irr}(C^*(\pi_\delta))$ then $\sigma(\pi(C^2)) \subseteq \sigma(\pi_\delta(C^2)) = \delta \cup \omega(\delta) \cup \alpha(\delta)$. Since every irreducible representation of $C^*(\pi_\delta)$ is also an irreducible representation of $*$-algebra $\mathcal{A}$ there is orbit δ' such that $\pi = \pi_{\delta'}$. Then we will have $\delta' \subseteq \delta \cup \omega(\delta) \cup \alpha(\delta)$. Since no cycle can be properly contained in an orbit δ' we conclude that $\delta' = \delta$ or $\delta' = \omega(\delta)$ or $\delta' = \alpha(\delta)$. So π must be one of the representations listed in the theorem. Let us prove that all of them are actually representations of the algebra $C^*(\pi_\delta)$. Let $A = \mathrm{diag}(\ldots, x_{-1}, x_0, x_1, \ldots)$ be diagonal operator in Hilbert space $l_2(\mathbb{Z})$ with orthonormal basis $\{e_k\}$, where $\delta = (\ldots, x_1, x_0, x_1, \ldots)$. Let U be a bilateral shift operator $Ue_k = e_{k-1}$. The equality $UAU^* = f(A)$ implies $UAU^* \in C^*(A)$. Since $B = U^*AU$ is a diagonal operator $Be_k = x_{k+1}e_k$ and the mapping $x_k \to x_{k+1}$ is mutually continuous on the closure of δ(which is $\sigma(A)$) $B \in C^*(A)$. Thus the mapping $\rho(D) = UDU^*$ is a automorphism of $C^*(A)$. Let us prove that $C^*(\pi(\delta))$ is a cross-product C^*-algebra. Consider the linear subspace $L_t = \{\sum\limits_{-m}^{n} A_i u^i e_t | A_i \in C^*(A);\; m, n \ge 0\}$. Then L_t is dense in H and L_t is isomorphic to a dense subspace in $L_2(\mathbb{Z}, \mathbb{C})$ via isomorphism $\sum A_i u^i e_t \to f()$, where $f(i) = (A_i e_{t+i}, e_{t+i})$. Direct computations show that $A(\sum \alpha_i e_i) = \sum \alpha_i \phi_t(U^{-i}AU^{*-i})$, where $\phi_t(D) = (De_t, e_t)$ for all $D \in C^*(A)$ is a one-dimensional representation of $C^*(A)$. Thus π_δ is a regular representation, λ_ϕ, associated with the representation ϕ_t. Since $\phi = \oplus_{t\in\mathbb{Z}}\phi_t$ is a faithful representation of $C^*(A)$ we conclude that λ_ϕ is faithful on the cross-product $\mathbb{Z} \times_\rho C^*(A)$ (see [9], Theorems 7.7.5 and 7.7.7). Since all representations λ_{ϕ_t} are isomorphic to π_δ we conclude that $C^*(\pi_\delta)$ is $\mathbb{Z} \times_\rho C^*(A)$.

Consider the case of unilateral orbits. Since point 0 is not periodic and the dynamical system is simple we conclude that for every orbit δ there is $\eta > 0$ such that $\delta \subseteq [\eta, \sup f]$, i.e. δ is separeted from zero. Let $\delta = (x_k)_{k\in\mathbb{N}}$ be the Fock orbit. Then $\pi_\delta(X)$ is a weighted shift operator with all weights separated from zero. If $X = UC$ is polar decomposition of X then $U, C \in C^*(X)$ and the algebra of compact operators $\mathcal{K} \subseteq C^*(X)$ (see [1], Lemma 2.1). We know that $\omega(\delta) = (y_k)_{k\in\mathbb{N}}$ is periodic orbit and $x_k - y_k \to 0$. By Theorem 3 $C = \mathrm{diag}(\sqrt{x_k})$. Let us put $C_1 = \mathrm{diag}(\sqrt{y_k})$ then since $C - C_1 \in \mathcal{K}$ and $\mathcal{K} \subseteq C^*(U, C)$ and $\mathcal{K} \subseteq C^*(U, C_1)$ we conclude that $C^*(U, C) = C^*(U, C_1)$ as operator algebras. If $m =| \omega(\delta) |$ then $C^*(U, C_1) = C^*(UC_1)$ is an algebra generated by m-periodic weighted shift. It is known that C^*-algebra generated by all m-periodic weighted shifts in a given separable Hilbert space is isomorphic to $\mathcal{T}(C(\mathbb{T}))$ (the Toeplitz operators) and there is m-periodic shift which generate this algebra. But if $D = \mathrm{diag}(d_k)$ and $D_1 = \mathrm{diag}(d_k^1)$ are diagonal operators with m-periodic coefficients and m is least possible, then

$C^*(U,D) = C^*(U,D_1)$ (since the map $g:\ d_k \to d_k^1$ is continuous on $\sigma(D)$ and by functional calculus $f(D) = D_1$ and obviously, $f(UDU^*) = Uf(D)U^*$). From these facts follows that $C^*(X) = \mathcal{T}(C(\mathbb{T}))$. For anti-Fock orbits arguments are the same. ∎

Define support of the dynamical system $(f,\mathbb{R})$ to be the union $X = \bigcup_{\delta\in\mathrm{Orb}_+(f)}\delta$ and the finite support X_{fin} to be union of positive cycles.

Theorem 5. *If a dynamical system $(f(),\mathbb{R})$ is simple then the C^*-algebra $\mathcal{E}_f$ is GCR (type I C^*-algebra), and the finite spectrum is homeomorphic to $X_{\mathrm{fin}}/\sim$, where $\sim$ is an orbit equivalence relation.*

Proof. First let us show that the finite-dimensional spectrum $\mathrm{Irr}(\mathcal{E}_f) \simeq (X_{\mathrm{fin}}/\ )\times S^1$ (X_{fin} is finite and so compact set). Indeed, it is obvious that $f : X_{\mathrm{fin}} \to X_{\mathrm{fin}}$ is one-to-one map. Thus we can apply the results from [5]. If $\delta \in \mathrm{Orb}_+(f)$ is a bilateral non-periodic orbit then by previous theorem, the set of irreducible representation $\mathrm{Irr}(C^*(\pi_\delta))$ is $\pi_\delta, \pi_{\omega(\delta),\phi}, \pi_{\alpha(\delta),\phi}$, where $0 \le \phi \le 2\pi$. Since we know the topology on finite dimensional representations we conclude that $\mathrm{Irr}(C^*(\pi_\delta))$ is T_0 space. Hence $C^*(\pi_\delta)$ is GCR-C^*-algebra. It is known that $\mathcal{T}(C(\mathbb{T}))$ is also a GCR algebra. Cyclic orbits generate finite-dimensional and so GCR algberas. Hence $\mathcal{E}_f$ is GCR C^*-algebra. ∎

The question of isomorphism of enveloping C^*-algebras may turn to be very difficult. Even in one-to-one case there are only fragmentary results in this direction, for example it is known that for minimal dynamical systems on Cantor sets the isomorphism of cross-product C^*-algebras equivalent to orbit-equivalence of corresponding dynamical systems (with condition on K_0-groups) see [4], Theorem 4. However, in particular "discrete" case we have the following:

Theorem 6. *If dynamical systems $(f_1,\mathbb{R})$ and $(f_2,\mathbb{R})$ are simple and for every non-cyclic orbit $\delta \in \mathrm{Orb}_+(f_k)$ there exists a point $x \in \delta$ isolated in the support space of a dynamical system $(f_k,\mathbb{R})$, then $\mathcal{E}_{f_1} \cong \mathcal{E}_{f_2}$ if and only if there is a one-to-one map $\phi : \mathrm{Orb}_+(f_1)/\sim \to \mathrm{Orb}_+(f_2)/\sim$, such that $|\phi(\delta)| = |\delta|$ and $\phi(\omega(\delta)) = \omega(\phi(\delta))$, $\phi(\alpha(\delta)) = \alpha(\phi(\delta))$. Moreover in this case the topology on $\mathrm{Irr}(\mathcal{E}_{f_k})$ is given by its base consisting of closed sets $\{\pi_\delta, \pi_{\omega(\delta),\phi}, \pi_{\alpha(\delta),\phi} | \phi \in S^1\}$, where $\delta \in \mathrm{Orb}_+(f_k)\backslash\mathrm{Cyc}(f)$ and $\{\pi_{\beta,\phi}|\phi \in M\}$, where β is a positive cycle and M is a closed subset in S^1. Thus $\mathcal{E}_{f_1} \cong \mathcal{E}_{f_2}$ if and only if their dual spaces are homeomorphic.*

Proof. Let $\psi : \mathcal{E}(f_1) \to \mathcal{E}(f_2)$ be an isomorphism. Then ψ induces a homeomorphism of spectra spaces $\psi^* : \mathcal{E}(\hat{f_2}) \to \mathcal{E}(\hat{f_1})$, i.e the spaces of irreducible representation with Jacobson's topology. We know that $\mathcal{E}(\hat{f_1})$ can be identified with $\mathrm{Orb}_+(f_1)/$. With this identification we have one-to-one map $\psi^* : \mathrm{Orb}_+(f_2) \to \mathrm{Orb}_(f_1)$. Let $\delta \in \mathrm{Orb}_+(f_2)$. Since ψ^* is homeomorphism $\psi^*(\overline{\pi_\delta}) = \overline{\psi^*(\pi)}$. As we know that $\overline{\pi_\delta} = \{\pi_{\omega(\delta),\phi}, \pi_{\alpha(\delta),\phi}\}$ we have proved necessity of conditions of the theorem.

Let ω_1 be a non-attractive cycle or an empty set and ω_2 be non-repellent cycles or empty sets. Denote $\Omega_{\omega_1}^{\omega_2}(f_1) = \{\delta \in \mathrm{Orb}_+(f_1) | \alpha(\delta) = \omega_1,\ \omega(\delta) = \omega_2\}$. Then $\mathrm{Orb}_+(f_1)$ is the disjoint union of these sets. For all $\delta \in \Omega_{\omega_1}^{\omega_2}(f_1)$ we will realize the corresponding representation π_δ in the same Hilbert space $H_{\omega_1}^{\omega_2}$. Consider an atomic representation of $\mathcal{E}(f_1)$ which is realized in Hilbert space $H = \otimes_{\omega_1,\omega_2}(H_{\omega_1}^{\omega_2})^{\otimes n(\omega_1,\omega_2)}$, where $n(\omega_1,\omega_2) = |\Omega_{\omega_1}^{\omega_2}(f_1)|$. Then $\mathcal{E}(f_1)$ is isomorphic to the algebra generated by the diagonal operator $C = \mathrm{diag}(C_{\omega_1,\omega_2})$, where $C_{\omega_1,\omega_2} = \mathrm{diag}(C_\delta | \delta \in \Omega_{\omega_1,\omega_2})$ and $C_\delta = \pi_\delta((XX^*)^{1/2})$ and block-diagonal with respect to direct sum decomposition of H operator $U = \mathrm{diag}(I \otimes U_{\omega_1,\omega_2})$. Discreteness of the dynamical system implies that all block-diagonal with respect to an expanded direct sum decomposition $H = \oplus_{\omega_1,\omega_2} \oplus_{n(\omega_1,\omega_2)} H_{\omega_1}^{\omega_2}$ compact operators belong to $\mathcal{E}(f_1)$. We will denote this subalgebra of compact operators by $\mathcal{K}_1$. Modulo this compact operators C is $C_{\mathrm{normal}} = \mathrm{diag}(I \otimes A_{\omega_1,\omega_2})$, where $A_{\omega_1,\omega_2} e_k = x_k^1$ if $k < 0$ and $A_{\omega_1,\omega_2} e_k = x_k^2$ if $k > 0$ and $\omega_1 = (x_{-k}^1)_{k\in\mathbb{N}}$ and $\omega_2 = (x_k^2)_{k\in\mathbb{N}}$ regarded as periodic orbit.

Moreover, it is obvious that $C^*(C_{\mathrm{normal}}, U, \mathcal{K}_1) = C^*(C, U)$. If there is ϕ which satisfies all conditions of the theorem then we can consider $\mathcal{E}(f_1)$ and $\mathcal{E}(f_2)$ in the same Hilbert space and using functional calculus obtain $\phi^*(C_{\mathrm{normal}}(f_1)) = C_{\mathrm{normal}}(f_2)$ (where ϕ^* is continuous map $\mathrm{Per}(f_1)_+ \to \mathrm{Per}(f_2)_+$ which is lifting of ϕ). and so $\mathcal{E}(f_1) = \mathcal{E}(f_2)$ as operator algebras. This completes the proof. ■

5 Quadratic dynamical system

In this section we consider an example of one-dimensional quadratic dynamical system. Let $f_{a,b}(x) = 1 + ax - bx^2$ with $\{a, b\} \in \mathbb{R}$ and $b > 0$ to provide boundedness. Since when $a < 0$ dynamical system is one-to-one on $\mathbb{R}_+$ (and so all irreducible representations are one-dimensional) we assume that $a > 0$. This dynamical system is conjugated to $f_\mu(x) = \mu x(1 - x)$, where $\mu = 1 + \sqrt{a^2 - 2a + 1 + 4b}$. The values of parameter μ when bifurcations of cycles of one parametric family $\{f_\mu\}$ occur are given in [12]. However a conjugacy relation does not preserve positiveness, i.e. $\mathrm{Orb}_+(f_{a,b})$ may not map into $\mathrm{Orb}_+(f_\mu)$.

If (a, b) belong to domain $D = \{(a, b)|\ b < \frac{1}{2} - \frac{a^2}{4} + \frac{a}{2} + \frac{\sqrt{1+2a}}{2}\}$ bounded by curve G (see Fig. 1) then for every $x \in [0; \sup f_{a,b}]$ $\mathcal{O}_+(x) \subset [0; \sup f_{a,b}]$. Thus for such (a, b) algebra $A_{a,b}$ has Fock representation and as it easily can be shown has no anti-Fock representations. In the complement of D algebra $A_{a,b}$ has anti-Fock representations.

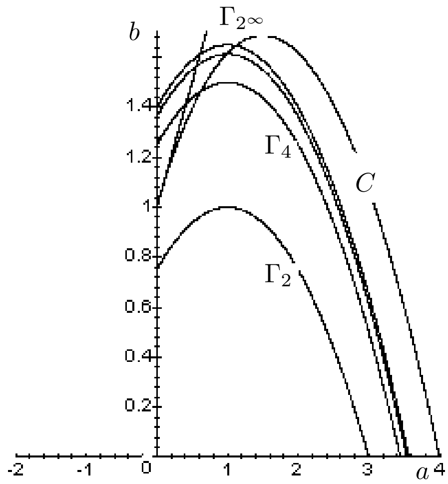

Figure 1.

Proposition 1. *If (a, b) belong to domain $P_1 = \{(a, b)|\ b < 1 - \frac{(a-1)^2}{4}\}$ bounded by curve Γ_2 (see picture) then $\mathcal{E}_{a,b}$ has one dimensional and Fock irreducible representations only. Moreover $\mathcal{E}_{a,b} \simeq \mathcal{T}(C(\mathbb{T}))$.*

Proof. For $(a, b) \in P_1$ dynamical system has two fix point $\beta_+ > 0$, $\beta_- < 0$ but has no other cycles.

Let us show that $\mathrm{Orb}_+(f) = \{\beta_+,\ \delta_1\}$, where δ_1 is the Fock orbit. If $\delta \in \mathrm{Orb}_+(f)$ and $\delta \neq \beta_+$, $\delta \neq \delta_1$ then $\alpha(\delta)$ is a cycle which cannot be the attractive point β_+ (see Lemma 1). Hence $\alpha(\delta) = \beta_- < 0$ which is contradiction.

Since $P_1 \subset D$ then δ_1 is positive orbit. And Theorem 4 implies that $\mathcal{E}_{a,b} \simeq \mathcal{T}(C(\mathbb{T}))$. ■

Proposition 2. *Let $P_2 = \{(a, b)|\ 1 - \frac{(a-1)^2}{4} < b < -\frac{a^2}{4} + \frac{a}{2} + \frac{5}{4}\}$ bounded by curves Γ_2 and Γ_4. Domain P_2 is divided into three domains P_2^1, P_2^2, P_2^3: $P_2^1 = P_2 \cap D$; $P_2^2 = \{(a, b) \in P_2 \setminus D|\ b < a + 1\}$; $P_2^3 = \{(a, b) \in P_2 \setminus D|\ b > a + 1\}$.*

Then for $(a,b) \in P_2^1$ C^*-algebra $\mathcal{E}_{a,b}$ has the family of one-, two-dimensional and the Fock irreducible representations, but has no anti-Fock representation. For $(a,b) \in P_2^2$ C^*-algebra $\mathcal{E}_{a,b}$ has one-, two-dimensional and anti-Fock irreducible representations, but has no Fock representation. For $(a,b) \in P_2^3$ C^*-algebra $\mathcal{E}_{a,b}$ has the family of one-dimensional representations and anti-Fock irreducible representations, but has no two-dimensional and Fock representations.

For (a,b) from domain bounded by curves Γ_4 and Γ_8 the dynamical system has 4-cycle, 2-cycle, two fix points and no other cycles.

The curve $\Gamma_{2^\infty} \approx 1.651225 - \frac{(a-1)^2}{4}$ separates the domain, where the dynamical system is simple from the one (including Γ_{2^∞}), where the dynamical system is not simple.

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