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On Generalization of the Cuntz Algebras

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We study the representations of generalisation of the Cuntz algebra O_n. The algebra $O_{n,\{\alpha_k\}_{k=1}^n}$ is a C^*-algebra generated by isometries $s_1, \dots, s_n$ such that $\sum\limits_{k=1}^n \alpha_k s_k s_k^* = e$, where $0 < \alpha_k < 1, k = 1, \dots, n$. The fact that some algebra is $*$-wild implies that the problem of unitary description of all representations of the algebra is very complicated. We show that the algebra $O_{4,\{\alpha_k=1/2\}_{k=1}^4}$ is $*$-wild and establish the criterion of $*$-wildness of the algebra $O_{3,\{\alpha_k\}_{k=1}^3}$.

This paper is concerned with the complexity problem of unitary description of representations for C^*-algebras generated by isometries connected with some relation on an infinite-dimensional Hilbert space. These algebras were suggested by Yuriĭ Samoilenko.

The fact that some algebra is $*$-wild implies that the problem of unitary description of all representations is very complicated.

On C^*-algebra $O_n, \{\alpha_k\}_{k=1}^n$

As in [1, 2], we consider a C^*-algebra $\mathfrak{P}_n = \mathfrak{P}(\alpha_1, \dots, \alpha_n)$ generated by orthoprojectors $p_1, \dots,$ p_n such that

$$\sum_{k=1}^n \alpha_k p_k = e, \tag{1}$$

where e is the identity of the algebra and $0 < \alpha_k < 1$, $k = 1, \dots, n$. Let us note that the condition $0 < \alpha_k < 1$, $k = 1, \dots, n$ is not a restriction. It was shown in [6] that it is always possible to reduce the values of the coefficients $\alpha_1, \dots, \alpha_n$ by a linear change of the variables $p_1, \dots, p_n$ to this form.

For the same set α_k we deal with the C^*-algebra $O_{n,\{\alpha_k\}_{k=1}^n}$ generated by isometries $s_1, s_2, \dots,$ s_n such that

$$\sum_{k=1}^n \alpha_k s_k s_k^* = e. \tag{2}$$

For $\alpha_k = 1, k = 1, \dots, n$ it is the Cuntz algebra O_n. The Cuntz algebra O_n is nuclear, simple and non type I (see [3]).

In this paper we prove that the algebra $O_{4,\{\alpha_k=1/2\}_{k=1}^4}$ is $*$-wild and establish the criterion of $*$-wildness of the algebra $O_{3,\{\alpha_k\}_{k=1}^3}$. In [4] we considered $O_{3,\{\alpha_k\}_{k=1}^3}$ when $\alpha_1 + \alpha_2 + \alpha_3 = 2$.

In [5, 6] all irreducible representations of the algebras $\mathfrak{P}_3 = \mathfrak{P}(\alpha_1, \alpha_2, \alpha_3)$ and $\mathfrak{P}_4 = \mathfrak{P}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ were described. The algebra $\mathfrak{P}_3$ has one-dimensional and two-dimensional irreducible representations. All irreducible representations of the algebras $\mathfrak{P}_4$ are finite-dimensional Jacobian matrices. In case $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2$ all irreducible representations of the algebra $\mathfrak{P}_4$ are one-dimensional and two-dimensional. It means that the algebras $\mathfrak{P}_3$ and $\mathfrak{P}_4$ are tame.

For $n \geq 5$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$; $\alpha_4 = \alpha_5 = \beta$, $\alpha + \beta = 1$, the problem of description of the collection of orthoprojectors $\{P_k\}_{k=1}^5$ such that

$$\alpha \sum_{k=1}^{3} P_k + \beta \sum_{i=4}^{5} P_i = I, \tag{3}$$

is $*$-wild (see [7]).

On majorization of C^*-algebras

Let us give definitions of majorization of C^*-algebras and $*$-wildness following to [1, 8].

The problem of description of pairs of self-adjoint (or unitary) operators up to unitary equivalence (representations of the $*$-algebra $\mathfrak{S}_2$ (or $\mathfrak{U}_2$) generated by a pair of free self-adjoint (or unitary) generators) was choosen as the standard $*$-wild problem in the theory of $*$-representations in [7].

The problem of unitary classification of representation of pairs of self-adjoint operators contains as a subproblem the problem of unitary classification of representation of any $*$-algebra with a countable number of generators (see [1]).

A problem containing the standard $*$-wild problem is called $*$-wild.

A number of $*$-wild algebras have been studied in recent years ([1, 7, 9]).

Let $\mathfrak{A}$ be a C^*-algebra. We will denote by $\operatorname{Rep}\mathfrak{A}$ the category of representations of $\mathfrak{A}$. The objects of this category are the representations $\mathfrak{A}$ to $L(H)$ (the algebra of linear bounded operators in a Hilbert space H), the morphisms are intertwining operators. Let $\mathfrak{N}$ be a nuclear C^*-subalgebra of $L(H_0)$. Let $\pi : \mathfrak{A} \to L(H)$ be a representation of $\mathfrak{A}$. It induces the representation

$$\tilde{\pi} = \pi \otimes id \colon \mathfrak{A} \otimes \mathfrak{N} \mapsto L(H \otimes H_0)$$

of the algebra $\mathfrak{A} \otimes \mathfrak{N}$.

Definition 1. *We say that a C^*-algebra $\mathfrak{B}$ majorizes a C^*-algebra $\mathfrak{A}$ (and denote it by $\mathfrak{B} \succ \mathfrak{A}$), if there exist a nuclear C^*-algebra $\mathfrak{N}$ and a unital $*$-homomorphism $\psi\colon \mathfrak{B} \mapsto \mathfrak{A} \otimes \mathfrak{N}$ such that the functor $F\colon \operatorname{Rep}\mathfrak{A} \mapsto \operatorname{Rep}\mathfrak{B}$ defined by the following rule:*

$$F(\pi) = \tilde{\pi} \circ \psi \quad \text{for any } \pi \in \operatorname{Rep}\mathfrak{A}, \tag{4}$$

$$F(A) = A \otimes I \quad \text{for any operator } A \text{ intertwining } \pi_1 \text{ and } \pi_2, \tag{5}$$

is full.

Denote by $\pi(\mathfrak{A})'$ a commutant of $\pi(\mathfrak{A})$.

Remark 1. In order to verify whether F is full it is enough to check for any representation $\pi \in \operatorname{Rep}(\mathfrak{A})$ in $L(H)$ that the condition $\mathcal{A} \in F(\pi)(\mathfrak{B})'$ implies $\mathcal{A} = A \otimes I \in \pi(\mathfrak{A})$ and $A \in \pi(\mathfrak{A})'$.

Remark 2. To prove that functor F is full it is enough to show that the $*$-homomorphism ψ is a surjection (see [1]).

The proofs of these remarks see in [1].

Let F_2 denote the free group on two generators u, v. Denote by $C^*(F_2)$ an enveloping C^*-algebra of F_2.

Definition 2. *A C^*-algebra is called $*$-wild if $\mathfrak{A} \succ C^*(F_2)$.*

Let us repeat that the fact that some algebra is $*$-wild implies that the problem of unitary description of all representations is very complicated.

On representations of the algebra $O_{4,\{\alpha_k=1/2\}_{k=1}^4}$

The C^*-algebra $\mathfrak{P}_4 = \mathfrak{P}(1/2, 1/2, 1/2, 1/2)$ has such irreducible representations (see [5]):

1) one-dimensional representation is

$$P_1 = P_2 = I, \qquad P_3 = P_4 = 0;$$

2) two-dimensional representation is

$$P_1 = \begin{pmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{pmatrix}, \qquad P_2 = \begin{pmatrix} \sin^2\phi & -\cos\phi\sin\phi \\ -\cos\phi\sin\phi & \cos^2\phi \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P_4 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

here $0 < \phi < \pi/2$.

Let us consider the corresponding C^*-algebra $O_{4,\{\alpha_k=1/2\}_{k=1}^4}$.

Theorem 1. *The C^*-algebra $O_{4,\{\alpha_k=1/2\}_{k=1}^4}$ is $*$-wild.*

We will prove three lemmas for the proof of this theorem. In accordance by the definition of $*$-wildness to prove $*$-wildness of the algebra $O_{4,\{\alpha_k=1/2\}_{k=1}^4}$, we give a $*$-homomorphism

$$\psi : O_{4,\{\alpha_k=1/2\}_{k=1}^4} \to M_2(C^*(F_2)) \otimes \mathfrak{N}.$$

Here $\mathfrak{N}$ is a nuclear C^*-algebra, $\mathfrak{N} \subset L(H_0)$. As $\mathfrak{N}$ we take the Cuntz algebra

$$O_2 = \mathbb{C}\langle T_1, T_2, T_1^*, T_2^* \mid T_1^*T_1 = T_2^*T_2 = I_0,\ T_1T_1^* + T_2T_2^* = I_0\rangle.$$

We take the operators T_1, T_2 acting in a separable Hilbert space H_0 such that

$$T_1 : e_j \to e_{2j-1}, \qquad T_2 : e_j \to e_{2j}, \tag{6}$$

where $\{e_j\}_{j=1}^\infty$ is an orthonormal basis of H_0.

We set

$$\psi(s_1) = S_1 = \begin{pmatrix} (\cos\phi)u & 0 & 0 & 0 & 0 & 0 & \dots \\ (\sin\phi)e & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & (\cos\phi)v & 0 & 0 & 0 & 0 & \dots \\ 0 & (\sin\phi)e & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & (\cos\phi)u & 0 & 0 & 0 & \dots \\ 0 & 0 & (\sin\phi)e & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & (\cos\phi)v & 0 & 0 & \dots \\ 0 & 0 & 0 & (\sin\phi)e & 0 & 0 & \dots \\ \vdots & & \ddots & & \ddots & & \ddots \end{pmatrix},$$

$$\psi(s_2) = S_2 = \begin{pmatrix} (\sin\phi)u & 0 & 0 & 0 & 0 & 0 & \dots \\ -(\cos\phi)e & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & (\sin\phi)v & 0 & 0 & 0 & 0 & \dots \\ 0 & -(\cos\phi)e & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & (\sin\phi)u & 0 & 0 & 0 & \dots \\ 0 & 0 & -(\cos\phi)e & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & (\sin\phi)v & 0 & 0 & \dots \\ 0 & 0 & 0 & -(\cos\phi)e & 0 & 0 & \dots \\ \vdots & & \ddots & & \ddots & & \ddots \end{pmatrix}, \tag{7}$$

$$\psi(s_3) = S_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ e & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & e & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & e & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & e & 0 & 0 & \dots \\ \vdots & & \ddots & & \ddots & & \ddots \end{pmatrix},$$

$$\psi(s_4) = S_4 = \begin{pmatrix} e & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & e & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & e & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & e & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & & \ddots & & \ddots & & \ddots \end{pmatrix},$$

here $0 < \phi < \pi/2$.

Lemma 1. *The map ψ defined by (7) is a $*$-homomorphism from $O_{4,\{\alpha_k=1/2\}_{k=1}^4}$ to $M_2(C^*(F_2)) \otimes O_2$.*

Proof. It is easy to check that S_1, S_2, S_3, S_4 satisfy the relations of the C^*-algebra $O_{4,\{\alpha_k=1/2\}_{k=1}^4}$.

One can see that the map ψ has the form:

$$S_1 = \begin{pmatrix} (\cos\phi)u & 0 \\ (\sin\phi)e & 0 \end{pmatrix} \otimes T_1 + \begin{pmatrix} 0 & (\cos\phi)v \\ 0 & (\sin\phi)e \end{pmatrix} \otimes T_2,$$

$$S_2 = \begin{pmatrix} (\sin\phi)u & 0 \\ -(\cos\phi)e & 0 \end{pmatrix} \otimes T_1 + \begin{pmatrix} 0 & (\sin\phi)v \\ 0 & -(\cos\phi)e \end{pmatrix} \otimes T_2, \tag{8}$$

$$S_3 = \begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix} \otimes T_1 + \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} \otimes T_2, \qquad S_4 = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \otimes T_1 + \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix} \otimes T_2,$$

here T_1, T_2 are the same as in (6). ■

Let us note that $M_2(C^*(F_2)) \otimes O_2 \simeq (C^*(F_2)) \otimes O_2$ because $O_2 \simeq M_2(O_2)$ [10]. Therefore the $*$-homomorphism ψ is the needed homomorphism for the proof of $*$-wildness of the algebra.

Let π be a representation of $C^*(F_2)$ in a Hilbert space $\hat{H}$. Then the map ψ induces the representation $F(\pi)$ of $O_{4,\{\alpha_k=1/2\}_{k=1}^4}$ in a Hilbert space H.

Lemma 2. *If $\pi \in \operatorname{Rep} C^*(F_2)$ in $L(\hat{H})$ and $\mathcal{A} \in (F_\psi(\pi)(O_{4\{\alpha_k=1/2\}_{k=1}^4}))'$ then $\mathcal{A} = A \otimes I$ and $A \in \pi(C^*(F_2))'$ (here I is the identity in $L(H)$).*

The proof follows by direct computation.

Lemma 3. *The $*$-homomorphism $\psi : O_{4,\{\alpha_k=1/2\}_{k=1}^4} \to M_2(C^*(F_2)) \otimes O_2$ is a surjection.*

Proof. In the algebra $M_2(C^*(F_2)) \otimes O_2$ we choose the following generators:

$$a_{11} = \begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix} \otimes I_0, \qquad a_{12} = \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix} \otimes I_0,$$

$$a_{21} = \begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix} \otimes I_0, \qquad a_{22} = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} \otimes I_0, \tag{9}$$

$$b = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \otimes I_0, \qquad c_1 = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \otimes T_1, \qquad c_2 = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \otimes T_2.$$

It is easy to see that the linear combinations of the generators a_{11}, a_{12}, a_{21}, a_{22}, b, c_1, c_2 give everywhere dense set of $M_2(C^*(F_2)) \otimes O_2$. The closure by norm gives our C^*-algebra. To prove that ψ is a surjection we point out the elements of the algebra $O_{4,\{\alpha_k=1/2\}_{k=1}^4}$ which give the generators of $M_2(C^*(F_2)) \otimes O_2$:

$$S_4S_4^* = a_{11}, \qquad S_4S_3^* = a_{12}, \qquad a_{21} = a_{12}^*, \qquad S_3S_3^* = a_{22},$$
$$S_4^*(S_1 + S_2) = b, \qquad S_4^2S_4^* + S_3S_4S_3^* = c_1, \qquad S_3^2S_3^* + S_4S_3S_4^* = c_2.$$ ■

The proof of Theorem 1 follows from Remark 1 and Lemmas 1, 2. Another proof follows from Remark 2 and Lemmas 1, 3.

The criterion of $*$-wildness of the algebra $O_{3,\{\alpha_k\}_{k=1}^3}$

For the algebras $\mathfrak{P}_3 = \mathfrak{P}(\alpha_1, \alpha_2, \alpha_3)$ all irreducible representations were described in [5]. The irreducible representations of these algebras exist only in the cases:

1) $\alpha_1 + \alpha_2 + \alpha_3 = 1$, $0 < \alpha_k < 1$, $k = 1, 2, 3$, $P_1 = P_3 = P_3 = I$;
2) $\alpha_i \in R\backslash\{1\}$, $\alpha_j + \alpha_k = 1$,$0 < \alpha_j < 1$, $0 < \alpha_k < 1$, here i, j, k are pairwise different integers from the set $\{1, 2, 3\}$, $P_j = P_k = I$; $P_i = I$ if $\alpha_i = 0$ and $P_i = 0$ otherwise;

3) $\alpha_1+\alpha_2+\alpha_3=2$, $0<\alpha_k<1$, $k=1,2,3$, P_1, P_2, P_3 are two-dimensional matrices:

$$\begin{gathered}P_1=\begin{pmatrix}1&0\\0&0\end{pmatrix},\\ P_2=\begin{pmatrix}\cos^2\phi&\cos\phi\sin\phi\\ \cos\phi\sin\phi&\sin^2\phi\end{pmatrix},\qquad P_3=\begin{pmatrix}\cos^2\theta&\cos\theta\sin\theta\\ \cos\theta\sin\theta&\sin^2\theta\end{pmatrix},\end{gathered}\tag{10}$$

here

$$\begin{gathered}\cos\phi=\sqrt{\frac{(1-\alpha_2)(\alpha_2+\alpha_3-1)}{\alpha_2(2-\alpha_2-\alpha_3)}},\qquad \sin\phi=\sqrt{\frac{1-\alpha_3}{\alpha_2(2-\alpha_2-\alpha_3)}},\\ \cos\theta=\sqrt{\frac{(1-\alpha_3)(\alpha_2+\alpha_3-1)}{\alpha_3(2-\alpha_2-\alpha_3)}},\qquad \sin\theta=-\sqrt{\frac{1-\alpha_2}{\alpha_3(2-\alpha_2-\alpha_3)}}.\end{gathered}\tag{11}$$

Theorem 2. *The C^*-algebra $O_{3,\{\alpha_1,\alpha_2,\alpha_3\}}$ is $*$-wild if one of the following conditions holds:*

1) $\alpha_1+\alpha_2+\alpha_3=1$, $0<\alpha_k<1$, $k=1,2,3$;

2) $\alpha_i=0$, $\alpha_j+\alpha_k=1$, $0<\alpha_j<1$, $0<\alpha_k<1$, here i, j, k are pairwise different integers from the set $\{1,2,3\}$;

3) $\alpha_j+\alpha_k=1$, $\alpha_i=\alpha_j$ or $\alpha_i=\alpha_k$, $0<\alpha_l<1$, $l=1,2,3$; here i, j, k are pairwise different integers from the set $\{1,2,3\}$.

Proof. One-dimensional representations of the algebra $\mathfrak{P}_3$ exist only when conditions 1, 2, 3 hold. In the first case we set the $*$-homomorphism $\psi: O_{3,\{\alpha_k\}_{k=1}^3}\to C^*(F_2)$ by the following way: $\psi(s_1)=e$, $\psi(s_2)=u$, $\psi(s_3)=v$ and $\psi(s_j)=u$, $\psi(s_k)=v$, $\psi(s_i)=e$ in the second case (here u, v are the generators of $C^*(F_2)$. It is easy to see that the map ψ is a surjection.

In the third case we restrict ourselves the case $\alpha_i=\alpha_j$. We give a $*$-homomorphism $\psi: O_{3,\{\alpha_k\}_{k=1}^3}\to M_2(C^*(F_2))\otimes O_2$ in such a way:

$$\begin{gathered}\psi(s_i)=S_i=\begin{pmatrix}e&0\\0&0\end{pmatrix}\otimes T_1+\begin{pmatrix}0&e\\0&0\end{pmatrix}\otimes T_2,\\ \psi(s_j)=S_j=\begin{pmatrix}0&0\\e&0\end{pmatrix}\otimes T_1+\begin{pmatrix}0&0\\0&e\end{pmatrix}\otimes T_2,\qquad \psi(s_k)=S_k=\begin{pmatrix}u&0\\0&v\end{pmatrix}\otimes I_0,\end{gathered}\tag{12}$$

here T_1, T_2 are the same as in (6). It is easy to check that the functor F induced by the $*$-homomorphism ψ is full. Therefore the algebra $O_{3,\{\alpha_1,\alpha_2,\alpha_3\}}$ majorizes $C^*(F_2)$ and is $*$-wild. ■

Theorem 3. *If $\alpha_1+\alpha_2+\alpha_3=2$, $0<\alpha_k<1$, $k=1,2,3$, then the C^*-algebra $O_{3,\{\alpha_1,\alpha_2,\alpha_3\}}$ is $*$-wild.*

Proof. We set the $*$-homomorphism $\psi: O_{3,\{\alpha_k\}_{k=1}^3}\to M_2(C^*(F_2))\otimes O_2$ in such a way:

$$\begin{gathered}\psi(s_1)=S_1=\begin{pmatrix}e&0\\0&0\end{pmatrix}\otimes T_1+\begin{pmatrix}0&e\\0&0\end{pmatrix}\otimes T_2,\\ \psi(s_2)=S_2=\begin{pmatrix}(\cos\phi)u&0\\(\sin\phi)e&0\end{pmatrix}\otimes T_1+\begin{pmatrix}0&(\cos\phi)v\\0&(\sin\phi)e\end{pmatrix}\otimes T_2,\\ \psi(s_3)=S_3=\begin{pmatrix}(\cos\theta)u&0\\(\sin\theta)e&0\end{pmatrix}\otimes T_1+\begin{pmatrix}0&(\cos\theta)v\\0&(\sin\theta)e\end{pmatrix}\otimes T_2,\end{gathered}\tag{13}$$

here T_1, T_2 are the same as in (6), $\cos\phi$, $\sin\phi$, $\cos\theta$, $\sin\theta$ are such as in (11).

It is easy to verify that the functor F generated by the $*$-homomorphism ψ is full. ■

Remark 3. One can see that these theorems together give also the needed conditions of $*$-wildness of $O_{3,\{\alpha_k\}_{k=1}^3}$ since either there are no representations of the corresponding algebra $\mathfrak{P}_3$ for other α_k, $k = 1, 2, 3$ (see [5]) or there are no representations of the algebra $O_{3,\{\alpha_k\}_{k=1}^3}$ (in the case $\alpha_i \neq 0$, if $\alpha_i \neq \alpha_j$ and $\alpha_i \neq \alpha_k$, here i, j, k are pairwise different integers from the set $\{1, 2, 3\}$).

The criterion of $*$-wildness of the C^*-algebra $O_{3,\{\alpha_k\}_{k=1}^3}$ follows from Theorems 2, 3 and Remark 3.

Theorem 4. *The algebra $O_{3,\{\alpha_k\}_{k=1}^3}$ is $*$-wild if and only if α_1, α_2, α_3 satisfy one of the following conditions:*

1) $\alpha_1 + \alpha_2 + \alpha_3 = 1$, $0 < \alpha_k < 1$, $k = 1, 2, 3$;
2) $\alpha_i = 0$, $\alpha_j + \alpha_k = 1$, $0 < \alpha_j < 1$, $0 < \alpha_k < 1$, *here* i, j, k *are pairwise different integers from the set* $\{1, 2, 3\}$;
3) $\alpha_j + \alpha_k = 1$, $\alpha_i = \alpha_j$ *or* $\alpha_i = \alpha_k$, $0 < \alpha_l < 1$, $l = 1, 2, 3$, *here* i, j, k *are pairwise different integers from the set* $\{1, 2, 3\}$;
4) $\alpha_1 + \alpha_2 + \alpha_3 = 2$, $0 < \alpha_k < 1$, $k = 1, 2, 3$.

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