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# Irreducible Representations of the Dirac Algebra for a System Constrained on a Manifold Diffeomorphic to $S^D$

*Y. OHNUKI*

*Nagoya Women's University, 1302 Takamiya Tempaku Nagoya 468-8507, Japan*
*E-mail: ohnuki@nagoya-wu.ac.jp*

The irreducible representations of the Dirac algebra for a particle constrained to move on $S^D$ are generalized to a system on a manifold diffeomorphic to $S^D$. It is shown that there exists a one-one correspondence between irreducible representations of two Dirac algebras given respectively on $S^D$ and on the manifold diffeomorphic to it. Among diffeomorphic mappings connecting $S^D$ to the manifold the area-preserving one plays a crucial role to derive out our main result. It is observed that the representation space of the Dirac algebra is kept unchanged through area-preserving mappings.

## 1 Dirac algebra

Let us consider a system constrained to move on a $D$-dimensional manifold embedded in the $(D+1)$-Euclidean space $R^{D+1}$ whose coordinates will be denoted as $x_1, x_2, \ldots, x_{D+1}$. The Hamiltonian in $R^{D+1}$ is assumed to be

$$H = \frac{1}{2}\sum_{\alpha=1}^{D+1} p_\alpha^2 + V(x) \tag{1.1}$$

and the $D$-dimensional smooth manifold on which the system is constrained will be written as

$$f(x) = 0 \tag{1.2}$$

with $f(x) \in C^\infty$. We further assume the manifold to be diffeomorphic to $S^D$.

Equation (1.2) is the so called primary constraint. According to the prescription by Dirac [1] the consistency of (1.2) under the time development leads us to the secondary constraint that can be written as

$$\{f_{,\alpha}(x), p_\alpha\} = 0, \tag{1.3}$$

where and in what follows $f_{,\alpha}(x) \equiv \partial_\alpha f(x)$, $f_{,\alpha\beta}(x) \equiv \partial_\alpha\partial_\beta f(x)$, $\{A, B\} \equiv AB + BA$ and repeated two Greek indices in a single term indicate a summation of such terms in which the pair of those indices run over 1 to $D+1$. The fundamental Dirac brackets [1] for canonical variables in classical mechanics are seen to be converted to

$$[x_\alpha, x_\beta] = 0, \tag{1.4}$$

$$[x_\alpha, p_\beta] = i\Lambda_{\alpha\beta}(x), \tag{1.5}$$

$$[p_\alpha, p_\beta] = -\frac{i}{2}\left\{\frac{1}{R^2(x)}\left(f_{,\alpha}(x) f_{,\beta\gamma}(x) - f_{,\beta}(x) f_{,\alpha\gamma}(x)\right), p_\gamma\right\}, \tag{1.6}$$

where

$$R^2(x) \equiv f_{,\alpha}(x) f_{,\alpha}(x) \tag{1.7}$$

and

$$\Lambda_{\alpha\beta}(x) \equiv \delta_{\alpha\beta} - \frac{f_{,\alpha}(x) f_{,\beta}(x)}{R^2(x)}. \tag{1.8}$$

With direct calculations one easily finds that Eqs. (1.4)~(1.6) are compatible with the constraints (1.2) and (1.3). The inner product of two wave functions $\chi(x)$ and $\varphi(x)$ is given by

$$\langle \chi | \varphi \rangle = \int d^{D+1} x \delta(f(x)) \chi^*(x) \varphi(x). \tag{1.9}$$

We call the algebra described by (1.2)~(1.6) the Dirac algebra on $f(x) = 0$.

## 2 Relation between two Dirac algebras

In order to examine the Dirac algebra on $f(x) = 0$ we introduce another manifold in $R^{D+1}$ which is also diffeomorphic to $S^D$. We denote it as

$$g(x) = 0. \tag{2.1}$$

Then we have the following Dirac algebra just corresponding to (1.3)~(1.8):

$$\{g_{,\alpha}(x), p_\alpha\} = 0, \tag{2.2}$$

$$[x_\alpha, x_\beta] = 0, \tag{2.3}$$

$$[x_\alpha, p_\beta] = i\Lambda'_{\alpha\beta}(x), \tag{2.4}$$

$$[p_\alpha, p_\beta] = -\frac{i}{2}\left\{ \frac{1}{R'^2(x)} \left( g_{,\alpha}(x) g_{,\beta\gamma}(x) - g_{,\beta}(x) g_{,\alpha\gamma}(x) \right), p_\gamma \right\}, \tag{2.5}$$

where

$$R'^2(x) = g_{,\alpha}(x) g_{,\alpha}(x) \quad \text{and} \quad \Lambda'_{\alpha\beta}(x) = \delta_{\alpha\beta} - \frac{g_{,\alpha}(x) g_{,\beta}(x)}{R'^2(x)}. \tag{2.6}$$

Since the manifold (1.2) is connected with (2.1) through a diffeomorphic mapping

$$x'_\alpha = x'_\alpha(x) \qquad (\text{or equivalently} \quad x_\alpha = x_\alpha(x')), \tag{2.7}$$

we may write the relation between them as

$$g(x') = f(x). \tag{2.8}$$

For the sake of simplicity by applying a scale transformation we will set up the following normalization condition for the volume of the manifold

$$\int d^{D+1} x \delta(f(x)) = \int d^{D+1} x \delta(g(x)) \tag{2.9}$$

without loss of generality[1].

[1]Applying the scale transformation $x_\alpha \to \rho x_\alpha$, $p_\alpha \to (1/\rho) p_\alpha$ with $\rho > 0$ and introducing $\bar{g}(x) \equiv g(\rho x)$ we find that (2.1)~(2.5) remain unchanged under the replacement $g(x) \to \bar{g}(x)$, and we have

$$\int d^{D+1} x \delta(g(x)) = \rho^{D+1} \int d^{D+1} \delta(\bar{g}(x)).$$

Now assuming that there exist operators $x_\alpha$ and $p_\alpha$ ($\alpha = 1, 2, \ldots, D+1$) that satisfy the Dirac algebra on $f(x) = 0$ we introduce a transformation such that

$$\begin{cases} x'_\alpha = x'_\alpha(x), \\ p'_\alpha = \dfrac{1}{2}\left\{(\Lambda'(x')[\partial x/\partial x'])_{\alpha\beta}, p_\beta\right\}, \end{cases} \tag{2.10}$$

where $\Lambda'(x')$ and $[\partial x/\partial x']$ stand for $(D+1)\times(D+1)$-matrices whose $(\alpha,\beta)$-elements are given by ${\Lambda'}_{\alpha\beta}(x')$ and $\partial x_\beta/\partial x'_\alpha$, respectively. Similarly the matrices $\Lambda(x)$ and $[\partial x'/\partial x]$ are defined by $(\Lambda(x))_{\alpha\beta} = \Lambda_{\alpha\beta}(x)$ and $[\partial x'/\partial x]_{\alpha\beta} = \partial x'_\beta/\partial x_\alpha$. The equations (2.10) provide us with a variable transformation $(x,p) \to (x',p')$ in the operator form. It must be written in the representation space of $x_\alpha$ and $p_\alpha$. Then there holds the following:

**Theorem.** *Given $x_\alpha$ and $p_\alpha$ that satisfy the Dirac algebra on $f(x) = 0$ then the operators $x'_\alpha$ and $p'_\alpha$ defined by (2.10) satisfy the Dirac algebra on $g(x) = 0$ for an arbitrary diffeomorphic mapping described with (2.7) and (2.8).*

Before entering a proof of the Theorem we remark the followings:

1. *Let $A$, $B$ and $C$ be operators. If $[[C,A],B] = 0$, then*

$$\{A, \{B, C\}\} = \{\{A,B\}, C, \}. \tag{2.11}$$

*Thus if further $[A,B] = 0$, we have*

$$\frac{1}{2}\{A, \{B, C\}\} = \{AB, C\}. \tag{2.12}$$

Proof: omitted.

2. *There holds true the identity*

$$\Lambda(x)[\partial x'/\partial x]\Lambda'(x') = \Lambda(x)[\partial x'/\partial x]. \tag{2.13}$$

Proof: Inserting ${\Lambda'}_{\gamma\beta}(x')$, which is defined by (2.6), into the left hand side (*l.h.s.*) of the above we find

$$\begin{aligned}(\alpha,\beta)\text{-element of } l.h.s. &= (\Lambda(x)[\partial x'/\partial x])_{\alpha\beta} - \frac{1}{R'^2(x')}\Lambda_{\alpha\rho}(x)\frac{\partial x'_\gamma}{\partial x_\rho}g_{,\gamma}(x')g_{,\beta}(x') \\ &= (\Lambda(x)[\partial x'/\partial x])_{\alpha\beta} - \frac{1}{R'^2(x')}\Lambda_{\alpha\rho}(x)f_{,\rho}(x)g_{,\beta}(x') = (\Lambda(x)[\partial x'/\partial x])_{\alpha\beta},\end{aligned}$$

where use has been made of (2.8) together with $\Lambda_{\alpha\rho}(x)f_{,\rho}(x) = 0$. (*q.e.d.*)

3. *We can uniquely solve the second equation of (2.10) with respect $p_\alpha$ to obtain*

$$p_\alpha = \frac{1}{2}\{(\Lambda(x)[\partial x'/\partial x])_{\alpha\beta}, p'_\beta\}. \tag{2.14}$$

Proof: Taking symmetrized products of $(\Lambda(x)[\partial x'/\partial x])_{\gamma\alpha}/2$ with the both sides of the second equation of (2.10) and making a sum over $\alpha$ we obtain with help of (2.11) and (2.13)

$$\begin{aligned}\frac{1}{2}\{(\Lambda(x)[\partial x'/\partial x])_{\gamma\alpha}, p'_\alpha\} &= \frac{1}{4}\left\{(\Lambda(x)[\partial x'/\partial x])_{\gamma\alpha}, \{(\Lambda'(x')[\partial x/\partial x'])_{\alpha\beta}, p_\beta\}\right\} \\ &= \frac{1}{2}\{(\Lambda(x)[\partial x'/\partial x]\Lambda'(x')[\partial x'/\partial x])_{\gamma\beta}, p_\beta\} \\ &= \frac{1}{2}\{(\Lambda(x)[\partial x'/\partial x][\partial x/\partial x'])_{\gamma\beta}, p_\beta\} = \frac{1}{2}\{\Lambda_{\gamma\beta}(x), p_\beta\}\end{aligned}$$

which reduces to

$$p_\gamma - \frac{1}{2}\left\{\frac{f_{,\gamma}(x)f_{,\beta}(x)}{R^2(x)}, p_\beta\right\} = p_\gamma - \frac{1}{4}\left\{\frac{f_{,\gamma}(x)}{R^2(x)}, \{f_{,\beta}(x), p_\beta\}\right\} = p_\gamma,$$

where we have used (1.8) together with (2.12) and (1.3). Thus we have proved (2.14).

With these preparations we will give a proof of the Theorem. To this end we first examine the constraint (2.2) starting with the Dirac algebra on $f(x) = 0$. Taking the anti-symmetrized products of $g_{,\alpha}(x')$ with the both sides of the second equation of (2.10) we find

$$\begin{aligned}\{g_{,\alpha}(x'), p'_\alpha\} &= \frac{1}{2}\left\{g_{,\alpha}(x'), \{(\Lambda'(x')[\partial x/\partial x'])_{\alpha\beta}, p_\beta\}\right\} \\ &= \left\{g_{,\alpha}(x'){\Lambda'}_{\alpha\gamma}(x')[\partial x/\partial x']_{\gamma\beta}, p_\beta\right\} = 0,\end{aligned}$$

where use has been made of (2.12) and the identity $g_{,\alpha}(x'){\Lambda'}_{\alpha\gamma}(x') = 0$. Thus the constraint (2.4) has been derived

Next to derive (2.6) we make a commutator of $x'_\alpha$ with $p'_\beta$. Then from (2.10) we obtain

$$\begin{aligned}[x'_\alpha, p'_\beta] &= \frac{1}{2}\left[x'_\alpha, \{(\Lambda'(x')[\partial x/\partial x'])_{\beta\gamma}, p_\gamma\}\right] \\ &= (\Lambda'(x')[\partial x/\partial x'])_{\beta\gamma}[x'_\alpha, p_\gamma] = i(\Lambda'(x')[\partial x/\partial x'])_{\beta\gamma}[\partial x'/\partial x]_{\rho\alpha}\Lambda_{\rho\gamma}(x) \\ &= i{\Lambda'}_{\alpha\beta}(x') - \frac{i}{R^2(x)}(\Lambda'(x')[\partial x/\partial x'])_{\beta\gamma}f_{,\gamma}(x)f_{,\rho}(x)[\partial x'/\partial x]_{\rho\alpha} \\ &= i{\Lambda'}_{\alpha\beta}(x') - \frac{i}{R^2(x)}{\Lambda'}_{\beta\gamma}(x')g_{,\gamma}(x')f_{,\rho}(x)[\partial x'/\partial x]_{\rho\alpha} = i{\Lambda'}_{\alpha\beta}(x'),\end{aligned}$$

which proves (2.4).

Finally we will derive (2.5). To avoid complications we will proceed in the following way: As seen from (1.5) and (1.6) the commutator $[p'_\alpha, p'_\beta]$ is linear in $p_\gamma$'s, thereby applying (2.14) we can write it as

$$[p'_\alpha, p'_\beta] = \frac{i}{2}\left\{c^{[\alpha\beta]}_\gamma(x'), p'_\gamma\right\} \tag{2.15}$$

with undetermined functions of $x'$, which have been denoted as $c^{[\alpha\beta]}_\gamma(x')$. Taking the commutators of $x'_\gamma$ with the both sides of (2.15) we obtain from the left hand side

$$\begin{aligned}[x'_\gamma, [p'_\alpha, p'_\beta]] &= [[x'_\gamma, p'_\alpha], p'_\beta] + [p'_\alpha, [x'_\gamma, p'_\beta]] \\ &= -i\left[\frac{g_{,\gamma}(x')g_{,\alpha}(x')}{R'^2(x')}, p'_\beta\right] + i\left[\frac{g_{,\gamma}(x')g_{,\beta}(x')}{R'^2(x')}, p'_\alpha\right]\end{aligned}$$

by virtue of (2.6), while from the right hand side

$$\left\{c^{[\alpha\beta]}_\rho(x'), [x'_\gamma, p'_\rho]\right\} = -{\Lambda'}_{\gamma\rho}(x')c^{[\alpha\beta]}_\rho(x') = -c^{[\alpha\beta]}_\gamma(x') + \frac{1}{R'^2(x')}g_{,\gamma}(x')g_{,\rho}(x')c^{[\alpha\beta]}_\rho(x').$$

Since the right hand sides of the above two equations are the same we find

$$c^{[\alpha\beta]}_\gamma(x') = \frac{1}{R'^2(x')}g_{,\gamma}(x')g_{,\rho}(x')c^{[\alpha\beta]}_\rho(x') + i\left[\frac{g_{,\gamma}(x')g_{,\alpha}(x')}{R'^2(x')}, p'_\beta\right] - i\left[\frac{g_{,\gamma}(x')g_{,\beta}(x')}{R'^2(x')}, p'_\alpha\right].$$

Then inserting this relation into the right hand side of (2.15) we find

$$[p'_\alpha, p'_\beta] = \frac{i}{2}\left\{\frac{1}{R'^2(x')}c_\rho^{[\alpha\beta]}(x')g_{,\rho}(x')g_{,\gamma}(x'), p'_\gamma\right\}$$
$$-\frac{1}{2}\left(\left\{\left[\frac{1}{R'^2(x')}g_{,\alpha}(x')g_{,\gamma}(x'), p'_\beta\right], p'_\gamma\right\} - (\alpha \leftrightarrow \beta)\right),$$

where the first term of the right hand side is found to vanish owing to (2.12) and (2.2). On the other hand, with the aid of (2.4) we have by direct calculation

$$\left[\frac{1}{R'^2(x')}g_{,\alpha}(x')g_{,\gamma}(x'), p'_\beta\right] = i\Lambda'_{\rho\beta}(x')\frac{\partial}{\partial x'_\rho}\left(\frac{1}{R'^2(x')}g_{,\alpha}(x')g_{,\gamma}(x')\right)$$
$$= \frac{i}{R'^2(x')}\left(g_{,\alpha\beta}(x')g_{,\gamma}(x') - g_{,\alpha}(x')g_{,\beta}(x')g_{,\rho}(x')g_{,\gamma}(x')\frac{\partial}{\partial x'_\rho}\left(\frac{1}{R'^2(x')}\right)\right.$$
$$\left.-\frac{g_{,\alpha}(x')g_{,\beta}(x')g_{,\rho}(x')g_{,\gamma\rho}(x')}{R'^2(x')}\right) + \frac{i}{R'^2(x')}g_{,\alpha}(x')g_{,\beta\gamma}(x')$$
$$-\frac{i}{R'^4(x')}\left(2g_{,\beta\rho}(x')g_{,\alpha}(x')g_{,\rho}(x') + g_{,\alpha\rho}(x')g_{,\rho}(x')g_{,\beta}(x')\right)g_{,\gamma}(x'),$$

which immediately leads to

$$\left[\frac{1}{R'^2(x')}g_{,\alpha}(x')g_{,\gamma}(x'), p'_\beta\right] - (\alpha \leftrightarrow \beta) = \frac{i}{R'^2(x')}\left(g_{,\alpha}(x')g_{,\beta\gamma}(x') - g_{,\beta}(x')g_{,\alpha\gamma}(x')\right)$$
$$-\frac{i}{R'^4(x')}\left(g_{,\alpha}(x')g_{,\beta\rho}(x')g_{,\rho}(x') - g_{,\beta}(x')g_{,\alpha\rho}(x')g_{,\rho}(x')\right)g_{,\gamma}(x').$$

Then taking the anti-commutators of $p'_\gamma$ with the both sides of the above equation we find the contribution from the second term of the right hand side turns zero due to (2.12) and (2.2), and finally obtain

$$[p'_\alpha, p'_\beta] = -\frac{i}{2}\left\{\frac{1}{R'^2(x')}\left(g_{,\alpha}(x')g_{,\beta\gamma}(x') - g_{,\beta}(x')g_{,\alpha\gamma}(x')\right), p'_\gamma\right\},$$

thereby proving (2.5). Thus we have completed the proof of the Theorem.

It is noted that among diffeomorphic mappings satisfying (2.9) there always exist [2] those which obey the condition

$$d^{D+1}x\delta(f(x)) = d^{D+1}x'\delta(g(x')). \tag{2.16}$$

We call them area-preserving mappings. Eq.(2.6) is of course equivalent to $\delta(f(x)) = \det[\partial x'/\partial x] \times\delta(f(x))$. After normalizing the constraints in a form of (2.9) we will apply this type of mapping[2] under which the transformation of the wave function $\varphi(x)$ is given by

$$\varphi'(x') = \varphi(x). \tag{2.17}$$

Then we are led to the invariance of the inner product of wave functions under the area-preserving mapping, i.e.,

$$\int d^{D+1}x\delta(f(x))\chi'^*(x)\varphi'(x) = \int d^{D+1}x\delta(g(x))\chi^*(x)\varphi(x). \tag{2.18}$$

[2]Physically the existence of the are-preserving mapping under the condition (2.9) could be understood by considering an incompressible fluid which uniformly covers the manifold.

Since, as was mentioned already, the transformation (2.10) has the inverse, the two descriptions based on the respective Dirac algebras on $f(x) = 0$ and $g(x) = 0$ are seen to be equivalent. Thus, if conversely starting with the canonical variables $x_\alpha$ and $p_\alpha$ that satisfy the Dirac algebra on $g(x) = 0$ we will then obtain those on $f(x) = 0$ by applying the inverse transformation of (2.10). It can be written as

$$\begin{cases} x'_\alpha = x'_\alpha(x), \\ p'_\alpha = \dfrac{1}{2}\left\{(\Lambda(x')[\partial x/\partial x'])_{\alpha\beta}, p_\beta\right\}, \end{cases} \tag{2.19}$$

where the first line stands for an area-preserving mapping from the manifold of $g(x) = 0$ to that of $f(x) = 0$ so that it satisfies $f(x') = g(x)$ together with (2.9). It is noted that as seen from the process of deriving (2.14) the transformation (2.19) is uniquely given by (2.10). Furthermore it is also remarkable that owing to (2.18) the irreducible representation space of $(x_\alpha, p_\alpha)$ is found to be the same as that of $(x'_\alpha, p'_\alpha)$, that is, in this case the irreducible representation space of the Dirac algebra is kept unchanged under a smooth deformation of the manifold.

Based on this fact we will determine, in the next section, all possible irreducible representations of the Dirac algebra on $f(x) = 0$. To this end we will use $S^D$ for the manifold $g(x) = 0$ in (2.19), since the irreducible representations of the Dirac algebra on $S^D$ have been known completely [3].

# 3 Irreducible representations and remarks

The operators $p_\beta$ in the irreducible representation space of the Dirac algebra on $S^D$ are given by [3]

$$\begin{cases} p_1 = -\dfrac{1}{2}\{x_2, L_{12}\} - \alpha x_2, \\ p_2 = \dfrac{1}{2}\{x_1, L_{12}\} + \alpha x_1 \end{cases} \qquad \text{for} \quad D = 1 \tag{3.1}$$

with $0 \le \alpha < 1$, and

$$p_\beta = \frac{1}{2}\{x_\rho, L_{\rho\beta}\} \qquad \text{for} \quad D \ge 2, \tag{3.2}$$

where, in $x$-diagonal representation, $L_{\alpha\beta}$ $(\alpha, \beta = 1, 2, \ldots, D+1)$ are defined by

$$L_{\alpha\beta} \equiv \frac{1}{i}\left(x_\alpha \frac{\partial}{\partial x_\beta} - x_\beta \frac{\partial}{\partial x_\alpha}\right). \tag{3.3}$$

In (3.1) and (3.2) we have assumed the radius of $S^D$ to be 1 for simplicity.

For $D = 1$ the irreducible representations are uniquely specified by $\alpha$, while for each of $D \ge 2$ we have one and only one irreducible representation. Furthermore it is known [3] that the irreducible representations of the Dirac algebra on $S^D$ are exhausted by the above. Thus inserting $p_\beta$ in (3.1) and (3.2) into the right hand side of (2.19) we can completely determine all possible irreducible representations of the Dirac algebra on $f(x) = 0$. They are expressed as

$$p'_\beta = \frac{1}{2}\{(\Lambda(x')[\partial x'/\partial x])_{\beta\gamma} x_\rho, L_{\rho\gamma}\} - \alpha(\Lambda(x')[\partial x'/\partial x])_{\beta\gamma} x_\rho \epsilon_{\rho\gamma} \qquad \text{for} \quad D = 1 \tag{3.4}$$

and

$$p'_\beta = \frac{1}{2}\{(\Lambda(x')[\partial x'/\partial x])_{\beta\gamma} x_\rho, L_{\rho\gamma}\} \qquad \text{for} \quad D \ge 2. \tag{3.5}$$

It is to be noted that for $D = 1$ there exist an infinite number of inequivalent irreducible representations corresponding to values of the parameter $\alpha$, while in the case of $D \geq 2$ the irreducible representation is uniquely given except for unitary equivalent representations.

Finally in concluding the present note we make a few remarks. It has been shown [3] that each of $p_\beta$'s in (3.1) and (3.2) is a self-adjoint operator. Hence it is obvious that the operators $p'_\beta$ given by (3.4) and (3.5) are all symmetric (hermitian) as easily seen from (2.18) and (2.19). Perhaps, however, they will be self-adjoint as well, although the proof has not yet been known. Moreover from the arguments made in this note it could be expected that if there exists a representation space of the Dirac algebra on a given manifold it is uniquely determined only by the topology of the manifold.

A detailed study on these problems would highly be desired.

## References

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[2] Omori H., Proc. of Symposia in Pure Mathematics, XV, 167, AMS, 1970.

[3] Ohnuki Y. and Kitakado S., *J. Math. Phys.*, 1993, V.34, 2827; especially see Section VI and Appendix.