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On $*$-Wild Algebras

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In this article, we consider discrete groups and study the complexity of their representations from the point of view of the theory of $*$-representations.

Let F_2 be the free group with two generators, $C^*(F_2)$ be the group C^*-algebra.

Definition (see [1]). *We call a discrete group F $*$-wild if there exist $n \in \mathbb{N}$ and an epimorphism $\varphi : C^*(F) \to M_n(C^*(F_2))$.*

In this note, we give two constructions that allow to construct examples of $*$-wild groups (other than the semi-direct products $F \rtimes G$, where F is a wild group). Note that the group $W = F \rtimes G$ is $*$-wild if F is a $*$-wild group.

1. Let G be a discrete group that has a faithful irreducible unitary representation in the algebra $M_n(\mathbb{C})$. In what follows, we assume that the group G is already an irreducible group of unitary operators in the algebra $M_n(\mathbb{C})$, i.e., such that if $[x, g] = 0$ for all $g \in G$ and some $x \in M_n(\mathbb{C})$, then $x = \lambda I_n$, $\lambda \in \mathbb{C}$, I_n is the identity operator in $M_n(\mathbb{C})$. Let F be a free group with generators $u_1, \ldots, u_m$ (the number of generators can be finite or infinite). Let also B_1, $\ldots$, B_m be unitary operators in the algebra $M_n(\mathbb{C})$. Consider the group $G_1 = \langle e \otimes g \mid g \in G$, e is the identity of the group $F\rangle \subset M_n(C^*(F))$. Note that $G \cong G_1$.

Theorem. *The group $G_2 = \langle G_1, u_1 \otimes B_1, \ldots, u_m \otimes B_m\rangle$ (the smallest subgroup of $M_n(C^*(F))$ containing the set of generating elements: elements of G_1, $u_1 \otimes B_1, \ldots, u_m \otimes B_m \in M_n(C^*(F))$) is $*$-wild.*

Proof. Let us construct a homomorphism $\phi\colon C^*(G_2) \to M_n(C^*(F))$ by extending the embedding $G_2 \subset M_n(C^*(F))$ to a homomorphism of the group C^*-algebra $C^*(G_2)$. Consider a unitary representation $\pi\colon F \to B(H)$ on a Hilbert space H. Denote by $\hat{\pi}$ the lift of the representation π, i.e., $\hat{\pi} = \pi \otimes I_n$. Define a functor $\Phi\colon \mathrm{Rep}\, F \to \mathrm{Rep}\, G_2$ as follows: $\Phi(\pi) = \hat{\pi} \circ \phi$. Let K be an intertwining operator for the representation π of the group F. Set $\Phi(K) = K \otimes I_n$.

Let us show that ϕ is an epimorphism. To prove that, it is sufficient to show (see [2]) that the functor $\Phi\colon \mathrm{Rep}\, F \to \mathrm{Rep}\, G_2$ is full (here $\mathrm{Rep}\, G$ is a category where the points are unitary representations of the group G, and morphisms are intertwining operators).

Recall that a functor Φ is called full, if it defines a one-to-one correspondence between intertwining operators for the corresponding representations of the groups F and G.

Let us show that the defined functor is full, i.e., there is a one-to-one correspondence between intertwining operators for the representation π of the group F and intertwining operators for the representation $\hat{\pi} \circ \phi$ of the group G_2. Let us consider an intertwining operator L for the representation $\hat{\pi} \circ \phi$ of the group G_2, i.e., the operator L commutes with all operators of the representation $\hat{\pi} \circ \phi$. Since the operator L commutes with elements of the group $\hat{\pi} \circ \phi(G_1)$, it follows that L must have the form $K \otimes I_n$, where $K \in B(H)$, I_n is the identity in the algebra $M_n(\mathbb{C})$. Since the operator L commutes with the elements $\pi(u_1) \otimes B_1, \ldots, \pi(u_m) \otimes B_m$, it follows that K commutes with all of the elements $\pi(u_1), \ldots, \pi(u_m)$, i.e., the functor Φ defines a one-to-one correspondence between intertwining operators. In virtue of paragraph 1.10 [1], the constructed functor $\Phi\colon \operatorname{Rep} F \to \operatorname{Rep} G_2$ is full and faithful, and since F is a $*$-wild group, there exists an epimorphism $\psi\colon C^*(F) \to M_n(C^*(F_2))$. Then the composition $\psi \circ \phi$ is an epimorphism and $\psi \circ \phi\colon C^*(G_2) \to M_n(C^*(F_2))$, hence G_2 is $*$-wild.

2. We give one more construction of $*$-wild groups that have the following form: they are extensions of a group F by a group G, where F is a $*$-wild group.

Consider the group $G = \langle g_1, \dots, g_p \rangle$, where the number of generators could be both finite or infinite. The conditions that are imposed on the group G are the same as in **1**. Consider the set $T = \{D$ is a discrete group, $D \subset U_n(\mathbb{C}) \mid$ so that there exists an extension of the group G by the group $D\}$. Consider the discrete groups $V = \langle w \otimes g \mid w \in F_m, g \in G \rangle$, $Z_1 = \langle e \otimes z \mid e$ is the identity in $F_m, z \in Z \rangle$. Note that $V \cong F_m \times G$. By construction, V is $*$-wild. Consider the group generated by all products of elements of the groups V and Z_1, and denote it by $Y = \langle V, Z_1 \rangle$. Note that Y is an extension of the group V by Z_1.

Proposition. *The group Y is $*$-wild.*

Proof. The proof is similar to the proof of Theorem. Construct a homomorphism $\phi\colon C^*(Y) \to M_n(C^*(V))$ by extending the embedding $Y \subset M_n(C^*(V))$ to a homomorphism of the group C^*-algebra $C^*(Y)$. We only prove that the correspondence for the intertwining operators for a representation π of the group V and the intertwining operators for the representation $\hat{\pi} \circ \phi$ of the group Y is one-to-one.

Consider an operator K that intertwines the representation $\hat{\pi} \circ \phi$ of the group Y. The operator K commutes with all operators of the group $\hat{\pi} \circ \phi(Y)$. Hence, K has the form $p \otimes I_n$, where $p \in B(H)$, H is the corresponding Hilbert space, and I_n is the identity of the algebra $M_n(\mathbb{C})$. Since K commutes with elements of the group F_m, it follows that p commutes with all elements $w \in F_m$, i.e., there exists an epimorphism $\phi\colon C^*(Y) \to M_n(C^*(V))$, i.e., Y is $*$-wild.

References

[1] Ostrovskyi V. and Samoĭlenko Yu., Introduction to the Theory of Representations of Finitely Presented $*$-Algebras. I. Representations by Bounded Operators, Amsterdam, The Gordon and Breach Publ., 1999.