Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 333–340.

Poincaré Parasuperalgebra with Central Charges and Parasupersymmetric Wess–Zumino Model

Alexander V. GALKIN

Institute of Mathematics of the National Academy of Sciences of Ukraine,
3 Tereshchenkivs'ka Str., Kyiv, Ukraine
E-mail: galkin@imath.kiev.ua

In the present paper we consider irreducible representations of Poincaré parasuperalgebra with central charges for even N and find the internal symmetry group for case $P_\mu P^\mu > 0$. The generalization of Wess–Zumino model for $N = 1$ and arbitrary p is also obtained.

1 Introduction

Supersymmetry (SUSY), introduced in theoretical physics and mathematics, has a lot of interesting applications [1]. One of them consists in mixing fermionic and bosonic states. It has important consequences in the quantum field theory, namely, this property provides a mechanism for cancellation of the ultraviolet divergences. Moreover, supersymmetric quantum field theory (SSQT) allows to unify the space symmetries of the Poincaré group with internal symmetries [2]. It allows to overcome the"no-go" theorem of Coleman and Mandula.

Supersymmetric quantum field theory (SSQFT) induced appearance of supersymmetric quantum mechanics (SSQM) [3]. SSQM stimulated deeper understanding of ordinary quantum mechanics and provided new ways for solving some problems [4].

SSQM has been generalized to the parasupersymmetric quantum mechanics (PSSQM) [5]. The latter deals with bosons and $p = 2$ parafermions having parastatistical properties. Here p is the so-called paraquantization order [6]. Soon an independent version of PSSQM yielding to positive defined Hamiltonians was proposed [7].

The crucial step in developing PSSQM was made by Beckers and Debergh [8] who required Poincaré invariance of the theory and formulated the group-theoretical foundations of the so-called parasupersymmetric quantum field theory (PSSQFT). This theory is a natural generalization of SSQFT, dealing with the Poincaré parasupergroup (or Poincaré parasuperalgebra (PPSA)) instead of the Poincaré supergroup (or Poincaré superalgebra (PSA)).

Recently IRs of the PPSA for $N = 1$ have been described [9] and then IRs for arbitrary N and internal symmetry group have been found [10, 11].

The present paper consists of two main parts. In the first part we consider the Poincaré parasuperalgebra with central charges. The second part includes the physical model, which is invariant under the Poincaré parasuperalgebra.

2 Extended Poincaré parasuperalgebra

Definition of the PPSA and the main Casimir operators. The Poincaré parasuperalgebra [8–11] is generated by ten generators P_μ, $J_{\mu\nu}$ of the Poincaré group, satisfying the commutation relations

$$
\begin{aligned}
&[P_\mu, P_\nu] = 0, \qquad [P_\mu, J_{\nu\sigma}] = i(g_{\mu\nu}P_\sigma - g_{\mu\sigma}P_\nu),\\
&[J_{\mu\nu}, J_{\rho\sigma}] = i(g_{\mu\sigma}J_{\nu\rho} + g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\rho}),\\
&J_{\mu\nu} = -J_{\nu\mu}, \qquad \mu,\nu = 0,1,2,3
\end{aligned}
\tag{2.1}
$$

and N parasupercharges Q^J_α, $(Q^J_\alpha)^\dagger$ ($\alpha = 1,2$, $J = 1,2,\ldots,N$), which satisfy the following double commutation relations

$$
\begin{aligned}
&[Q^I_\alpha, [Q^J_\beta, Q^K_\gamma]] = 4\varepsilon_{\alpha\beta}Z^{IJ}Q^K_\gamma - 4\varepsilon_{\alpha\gamma}Z^{IK}Q^J_\beta,\\
&[(Q^I_\alpha)^\dagger, [(Q^J_\beta)^\dagger, (Q^K_\gamma)^\dagger]] = 4\varepsilon_{\alpha\beta}Z^{*IJ}(Q^K_\beta)^\dagger - 4\varepsilon_{\alpha\gamma}Z^{*IK}(Q^J_\beta)^\dagger,\\
&[Q^I_\alpha, [Q^J_\beta, (Q^K_\gamma)^\dagger]] = 4\varepsilon_{\alpha\beta}Z^{IJ}(Q^K_\gamma)^\dagger - 4Q^J_\beta(\sigma_\mu)_{\alpha\gamma}P^\mu,\\
&[(Q^I_\alpha)^\dagger, [Q^J_\beta, t(Q^K_\gamma)^\dagger]] = 4(Q^K_\gamma)^\dagger(\sigma_\mu)_{\alpha\beta}P^\mu - 4\varepsilon_{\alpha\beta}Z^{*IK}Q^J_\beta,
\end{aligned}
\tag{2.2}
$$

where σ_ν are the Pauli matrices, $(\cdot)_{\alpha\gamma}$ stand for the matrix elements. Relations (2.1), (2.2) include operators Z^{IJ}, which we call the central charges. This definition is a direct generalization of Poincaré superalgebra with central charges.

In analogy with the Poincaré superalgebra the central charges must satisfy the relations $Z^*_{IJ} = Z^{IJ}$, $Z^{IJ} = -Z^{JI}$ and commute with generators of the PPSA. The spinor indices are risen and dropped using the universal spinor $\varepsilon^{\alpha\beta}$ ($\varepsilon^{11} = \varepsilon_{11} = \varepsilon^{22} = \varepsilon_{22} = 0$, $\varepsilon^{12} = \varepsilon_{21} = 1$, $\varepsilon^{21} = \varepsilon_{12} = -1$).

In addition, we have the following commutation relations between the generators of the Poincaré group and the parasupercharges:

$$
\begin{aligned}
&[J_{\mu\nu}, Q^J_\alpha] = -\frac{1}{2i}(\sigma_{\mu\nu})_{\alpha\beta}Q^J_\beta, \qquad [P_\mu, Q^J_\alpha] = 0,\\
&[J_{\mu\nu}, (Q^J_\alpha)^\dagger] = -\frac{1}{2i}(\sigma^*_{\mu\nu})_{\alpha\beta}(Q^J_\beta)^\dagger, \qquad [P_\mu, (Q^J_\alpha)^\dagger] = 0.
\end{aligned}
\tag{2.3}
$$

The PPSA, as well as PSA, can be extended by adding the generators Σ_l of the internal symmetry group, which satisfy the following relations:

$$
[Q^I_\alpha, \Sigma_l] = S^I_{lJ}Q^J_\alpha, \qquad [\Sigma_l, (Q^I_\alpha)^\dagger] = S^{*I}_{lJ}(Q^I_\alpha)^\dagger, \qquad [\Sigma_l, \Sigma_m] = f^k_{lm}\Sigma_k. \tag{2.4}
$$

By analogy with the PSA, P_σ and $J_{\mu\nu}$ are called even and Q^J_α, $(Q^J_\alpha)^\dagger$ are called odd elements of the PPSA.

In the papers [8–11] two main Casimir operators were found. They have the form

$$
C_1 = P_\mu P^\mu, \qquad C_2 = P_\mu P^\mu B_\nu B^\nu - (B_\mu P^\mu)^2, \tag{2.5}
$$

where

$$
B_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}J^{\nu\rho}P^\sigma + (\sigma_\mu)_{AB}\bar{Q}^I_A Q^I_B.
$$

The eigenvalues of C_1, C_2 are used for the classification of irreducible representations (IRs) of PPSA.

IRs with the central charges. Like the case of the ordinary Poincaré group, IRs of the PPSA can be divided into three main classes I. $P_\mu P^\mu > 0$, II. $P_\mu P^\mu = 0$, III. $P_\mu P^\mu < 0$. It is known [8–11] that for classes I and II there exists the additional Casimir operator $C_3 = P_0/|P_0|$, whose eigenvalues are $\varepsilon = \pm 1$. In other words, the classes I and II can be splitted into two subclasses corresponding to the fixed values of C_3, and we will mark them by I^+ and II^+ (for class $C_3 = 1$) and I^- , II^- (for class $C_3 = -1$).

The IRs for these main classes are described in [8–11]. Here we note that for the Poincaré superalgebra only classes I^+ and II^+ exist (since the relevant generators P_0 should be positive defined).

Now let us consider the IRs of PPSA with central charges (2.1)–(2.3). In the present paper we shall obtain the IRs for class I^+. The rest of IRs will be considered in our forthcoming papers.

Thus, for class I^+ we have $C_3 = 1$ and $P_\mu = (M, 0, 0, 0)$ in the rest frame. The central charges Z^{IJ} have to be equal to the unit matrix multiplied by the numeric coefficients which are elements of the $N \times N$ antisymmetric matrix Z.

By means of the unitary transformation $Z \longrightarrow \bar{Z} = UZU^\dagger$, any such matrix can be reduced to the quasidiagonal form

$$\tilde{Z}^{IJ} = U^I_L U^J_M Z^{LM}, \tag{2.6}$$

where

$$\tilde{Z}^{IJ} = \varepsilon \otimes D \quad (\text{even } N), \qquad \tilde{Z}^{IJ} = \begin{pmatrix} \varepsilon \otimes D & 0 \\ 0 & 0 \end{pmatrix} \quad (\text{odd } N), \tag{2.7}$$

D is a diagonal matrix with the positive real entries Z_m, and ε is the universal spinor. Relations (2.2) are invariant under the simultaneous transformation

$$Z^{IJ} \longrightarrow \tilde{Z}^{IJ} = U^{IK} U^{LJ} Z^{KJ}, \qquad Q^I_A \longrightarrow \tilde{Q}^I_A = U^{JK} Q^K_A, \tag{2.8}$$

where all nonzero Z^{IJ} are exhausted by the following ones $Z^{2m-1,2m} = -Z^{2m,2m-1} = Z^m$.

Choosing a new basis

$$\begin{aligned} Q^{2m-1}_1 &= \frac{1}{\sqrt{2}}(\hat{Q}^{2m-1}_1 + \hat{Q}^{2m}_1), \qquad & Q^{2m-1}_2 &= \frac{1}{\sqrt{2}}(\hat{\hat{Q}}^{2m}_2 - \hat{\hat{Q}}^{2m-1}_2), \\ Q^{2m}_1 &= \frac{1}{\sqrt{2}}(\hat{Q}^{2m-1}_2 + \hat{Q}^{2m}_2), \qquad & Q^{2m}_2 &= \frac{1}{\sqrt{2}}(\hat{\hat{Q}}^{2m}_1 - \hat{\hat{Q}}^{2m-1}_1), \end{aligned} \tag{2.9}$$

we reduce relations (2.2) in the rest frame $P = (M, 0, 0, 0)$ to the form

$$\begin{aligned} &[\hat{Q}^{2k-1}_A, [\hat{\hat{Q}}^{2m-1}_B, \hat{Q}^J_C]] = \delta_{AB}\delta_{km}(2M - Z_k)\hat{Q}^J_C, \\ &[\hat{Q}^{2k}_A, [\hat{\hat{Q}}^{2m}_B, \hat{Q}^J_C]] = \delta_{AB}\delta_{km}(2M + Z_m)\hat{Q}^J_C, \\ &[\hat{\hat{Q}}^{2k-1}_A, [\hat{Q}^{2m-1}_B, \hat{\hat{Q}}^J_C]] = \delta_{AB}\delta_{km}(2M - Z_m)\hat{\hat{Q}}^J_C, \\ &[\hat{\hat{Q}}^{2k}_A, [\hat{Q}^{2m}_B, \hat{\hat{Q}}^J_C]] = \delta_{AB}\delta_{km}(2M + Z_m)\hat{Q}^J_C, \end{aligned} \tag{2.10}$$

the remaining double commutators of the parasupercharges are equal to zero.

Below we shall restrict ourselves to the case, where all $Z_m < 2M$. Then, using the analogy of (2.10) with the formulas from [11], we easily find the general solution of (2.10):

$$\begin{aligned} \hat{Q}^{2m-1}_A &= (-1)^{A-1}\sqrt{2M - Z_m}(S_{4N+1,\,8m-11+4A} - iS_{4N+1,\,8m-10+4A}), \\ \hat{Q}^{2m}_A &= (-1)^{A-1}\sqrt{2M + Z_m}(S_{4N+1,\,8m-9+4A} - iS_{4N+1,\,8m-8+4A}), \end{aligned} \tag{2.11}$$

where S_{IJ} are generators of $SO(1, 4N+1)$ satisfying the commutation relations

$$[S_{kl}, S_{mn}] = -i(g_{km}S_{ln} + g_{ln}S_{km} - g_{kn}S_{lm} - g_{lm}S_{km}).$$

Substituting (2.11) into (2.9), we obtain parasupercharges in the rest frame

$$\begin{aligned} Q_A^{2m-1} = {} & \sqrt{M - Z_m/2}((-1)^{A-1} S_{4N+1,\,8m-11+4A} - S_{4N+1,\,8m-10+4A}) \\ & + \sqrt{M + Z_m/2}(S_{4N+1,\,8m-9+4A} + i(-1)^A S_{4N+1,\,8m-8+4A}), \\ Q_A^{2m} = {} & \sqrt{M - Z_m/2}(-S_{4N+1,\,8m-7+4A} + i(-1)^{A-1} S_{4N+1,\,8m-6+4A}) \\ & + \sqrt{M + Z_m/2}((-1)^A S_{4N+1,\,8m-5+4A} + i S_{4N+1,\,8m-4+4A}). \end{aligned} \tag{2.12}$$

In accordance with [11] the IRs of the class I^+ of the PPSA with central charges $Z_m < 2M$ are labeled by the following sets of numbers $(M, j, n_1, n_2, \ldots, n_{2N}, Z_1, Z_2, \ldots, Z_{\{\frac{n}{2}\}})$, here j label the IRs of $SO(3)$; $n_1 \geq n_2 \geq \ldots \geq n_{2N}$ label the IRs of $SO(1, 4N+1)$; n_1, n_2, $\ldots$, n_{2N} are either integer or half integer. Using the Lorentz transformation, we can find the basis elements P_μ, $J_{\mu\nu}$, Q_α^j, $\bar{Q}_\alpha^J$ in arbitrary frame.

Internal symmetry group. The internal symmetry group for PPSA without central charges was described in [11]. In paper [11] the authors showed that the internal symmetry group for IRs of classes I^+ and II^+ and arbitrary N is $SU(N)$.

Therefore, it is interesting to generalize these results to the case of nontrivial central charges. In this paper we shall obtain the IRs of class I^+, $Z_m < 2M$, $Z_m \neq 0$ and even N.

In analogy with SUSY it is easy to see that the internal symmetry group is smaller than in the case of central charges, equal to zero. Indeed, considering the first of relations (2.2) for $\alpha = \beta = 1$, $\gamma = 2$:

$$[Q_1^J, [Q_2^J, Q_1^K]] = 4Z^{IJ} Q_1^K \tag{2.13}$$

and evaluating the commutators of l.h.s. and r.h.s. of (2.13) with Σ_l and using (2.4) we come to the following condition

$$S_{lJ}^I Z^{JK} = S_{lJ}^K Z^{JI}. \tag{2.14}$$

In other words, the products of generators of the internal group with the matrix of central charges should be symmetric matrices.

Now let us present the explicit description of internal symmetry algebra for IRs of class I^+ and even N. In our case the internal symmetry algebra is isomorphic to $Sp(N)$. The basis elements have the following form

$$\begin{aligned} & A^{kk} = Z_k^{-1}(-S_{8k-7,\,8k-6} - S_{8k-5,\,8k-4} + S_{8k-3,\,8k-2} + S_{8k-1,\,8k}), \\ & B^{kk} = Z_k^{-1}(S_{8k-5,\,8k} - S_{8k-4,\,8k-1} + S_{8k-7,\,8k-2} - S_{8k-6,\,8k-3}) \\ & \qquad + i(S_{8k-5,\,8k-1} + S_{8k-4,\,8k} + S_{8k-7,\,8k-3} + S_{8k-6,\,8k-2}), \\ & C^{kk} = (B^{kk})^\dagger, \\ & A^{kn} = (f_{kn}^- + f_{nk}^-)\Sigma_{kn} + (f_{kn}^+ + f_{nk}^+)\Sigma_{k+2,\,n+2}, \\ & B^{kn} = f_{nk}^- \tilde{\Sigma}_{kn} + f_{kn}^- \Sigma_{kn}^\dagger + f_{nk}^+ \tilde{\Sigma}_{k+2,\,n+2} - f_{nk}^+ \tilde{\Sigma}_{k+2,\,n+2}^\dagger, \quad n > k, \\ & C^{kn} = (B^{kn})^\dagger, \quad n < k, \end{aligned} \tag{2.15}$$

where

$$f_{kn}^\pm = \frac{1}{Z_n}\sqrt{\frac{2M \pm Z_k}{2M \pm Z_n}}, \qquad f_{nk}^\pm = \frac{1}{Z_k}\sqrt{\frac{2M \pm Z_n}{2M \pm Z_k}},$$

$$
\begin{aligned}
\Sigma_{kn} &= S_{8k-7,\,8n-6} - S_{8k-6,\,8n-7} - S_{8k-3,\,8n-2} + S_{8k-2,\,8n-3} \\
&\quad - i(S_{8k-7,\,8n-7} + S_{8k-6,\,8n-6} + S_{8k-3,\,8n-3} + S_{8k-2,\,8n-2}), \\
\tilde{\Sigma}_{kn} &= -S_{8k-7,\,8n-2} + S_{8k-6,\,8n-3} + S_{8k-3,\,8n-6} - S_{8k-2,\,8n-7} \\
&\quad - i(S_{8k-7,\,8n-3} + S_{8k-6,\,8n-2} + S_{8k-3,\,8n-7} + S_{8k-2,\,8n+6}), \\
n &\neq k, \quad k, n = 1, 2, ..., N/2.
\end{aligned}
$$

Matrices (2.18) commute with the generators of Poincaré group and satisfy the following relations

$$
\begin{aligned}
&[A^{kk}, Q^J_A] = Z_k^{-1}(\delta_{J,\,2k-1} - \delta_{J,\,2k})Q^J_A, \\
&[B^{kk}, Q^J_A] = 2Z_k^{-1}\delta_{J,\,2k-1}Q^J_{2k}, \\
&[C^{kk}, Q^J_A] = 2Z_k^{-1}\delta_{J,\,2k}Q^{2k-1}_A, \\
&[A^{kn}, Q^J_A] = \delta_{J,\,2k-1}Z_k^{-1}Q^{2k-1}_A - \delta_{J,\,2n-1}Z_n^{-1}Q^{2k-1}_A + \delta_{J,\,2k}Z_K^{-1}Q^{2n}_A - \delta_{J,\,2k}Z_n^{-1}Q^{2k}_A, \\
&[B^{kn}, Q^J_A] = \delta_{j,\,2k-1}Z_k^{-1}Q^{2n}_A + \delta_{J,\,2n-1}Z_n^{-1}Q^{2k}_A, \\
&[C^{kn}, Q^J_A] = \delta_{J,\,2k}Z_k^{-1}Q^{2n-1}_A + \delta_{J,\,2n}Z_n^{-1}Q^{2k-1}_A, \\
&[A^{mn}, A^{kl}] = Z_k^{-1}\delta^{kn}A^{ml} - Z_m^{-1}\delta^{ml}A^{nk}, \\
&[A^{mn}, B^{kl}] = Z_n^{-1}(\delta^{nk}B^{ml} + \delta^{nl}B^{mk}), \\
&[A^{mn}, C^{kl}] = [C^{mn}, C^{kl}] = 0, \\
&[B^{mn}, C^{kl}] = Z_k^{-1}(\delta^{nk}A^{ml} + \delta^{mk}A^{nl}) + Z_k^{-1}(\delta^{nl}A^{mk} + \delta^{ml}A^{nk}).
\end{aligned}
$$

Thus the internal symmery group for IRs of class I^+, $Z_m < 2M$, $Z_m \neq 0$ and even N is $Sp(N)$.

3 Parasupersymmetric Wess–Zumino model

Now let us consider the PPSA without the central charges in the terms of the paragrassmannian variables for arbitrary p. Then the generators of PPSA will be of the form [8]:

$$
\begin{aligned}
&P_\mu = p_\mu = i\frac{\partial}{\partial x^\mu}, \\
&J_{12} = x_1p_2 - x_2p_1 + \frac{1}{4}(\theta^1Q_2 - Q_2\theta^1 - \theta^2Q_1 + Q_1\theta^2), \\
&J_{13} = x_1p_3 - x_3p_1 + \frac{i}{4}(\theta^1Q_1 - Q_1\theta^1 - \theta^2Q_2 + Q_2\theta^2), \\
&J_{23} = x_2p_3 - x_3p_2 + \frac{1}{4}(\theta^1Q_1 - Q_1\theta^1 - \theta^2Q_2 + Q_2\theta^2), \\
&J_{01} = x_0p_1 - x_1p_0 + \frac{i}{4}(Q_1\theta^1 - \theta^1Q_1 - \theta^2Q_2 + Q_2\theta^2), \\
&J_{02} = x_0p_2 - x_2p_0 + \frac{1}{4}(Q_2\theta^1 - \theta^1Q_2 + \theta^2Q_1 - Q_1\theta^2), \\
&J_{03} = x_0p_3 - x_3p_0 + \frac{i}{4}(\theta^2Q_1 - Q_1\theta^2 - \theta^1Q_2 + Q_2\theta^1), \\
&Q_1 = \frac{\partial}{\partial\theta^1}, \qquad (Q_1)^\dagger = -2((p_3 - p_0)\theta^1 + (p_1 + ip_2)\theta^2), \\
&Q_2 = \frac{\partial}{\partial\theta^2}, \qquad (Q_2)^\dagger = 2((p_3 + p_0)\theta^2 - (p_1 - ip_2)\theta^1),
\end{aligned}
$$

where θ_α are paragrassmanian variables defined by the Green anzats:

$$\theta_\alpha = \sum_{i=1}^{p} \theta_\alpha^{(i)}, \qquad \frac{\partial}{\partial\theta_\alpha} = \sum_{i=1}^{p} \frac{\partial}{\partial\theta_\alpha^{(i)}}, \qquad \left[\theta_\alpha^{(i)}, \theta_\beta^{(i)}\right]_+ = 0, \qquad \left[\theta_\alpha^{(i)}, \theta_\beta^{(j)}\right] = 0,$$
$$\left[\frac{\partial}{\partial\theta_\alpha^{(i)}}, \frac{\partial}{\partial\theta_\beta^{(i)}}\right]_+ = 0, \quad \left[\theta_\alpha^{(i)}, \frac{\partial}{\partial\theta_\beta^{(i)}}\right]_+ = \delta_{\alpha\beta}, \quad \left[\theta_\alpha^{(i)}, \frac{\partial}{\partial\theta_\beta^{(j)}}\right]_+ = 0, \quad \left[\theta_\alpha^{(i)}, \frac{\partial}{\partial\theta_\beta^{(j)}}\right] = 0, \tag{3.1}$$

p is paraquantization order, $i \neq j$. It should be noted that θ_α are Majorana spinors.

There exists the realization of $N = 1$ PPSA in terms of four paragrassmanian variables θ^1, θ^2, $(\theta^1)^\dagger$, $(\theta^2)^\dagger$. In this case we have

$$P_\mu = p_\mu = i\frac{\partial}{\partial x_\mu},$$
$$J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu - \frac{1}{2}\left((\sigma_{\mu\nu})_{\alpha\beta}\left[\theta^\alpha, \frac{\partial}{\partial\theta_\beta}\right] + (\sigma^\dagger{}_{\mu\nu})_{\alpha\beta}\left[(\theta^\alpha)^\dagger, \frac{\partial}{\partial(\theta^\beta)^\dagger}\right]\right), \tag{3.2}$$
$$Q_\alpha = -i\frac{\partial}{\partial\theta^\alpha} - i(\sigma_\mu)_{\alpha\beta}(\theta^\beta)^\dagger P^\mu, \qquad (Q_\alpha)^\dagger = i\frac{\partial}{\partial(\theta^\alpha)^\dagger} + i(\theta^\beta)^\dagger(\sigma_\mu)_{\beta\alpha}P^\mu.$$

Then we can define the covariant derivatives

$$(D_\alpha)^\dagger = \frac{\partial}{\partial\theta^\alpha} - (\sigma_\mu)_{\alpha\beta}(\theta^\beta)^\dagger P^\mu, \qquad (D_\alpha)^\dagger = -\frac{\partial}{\partial(\theta^\alpha)^\dagger} + \theta^\beta(\sigma_\mu)_{\beta\alpha}P^\mu. \tag{3.3}$$

Derivatives (3.3) have important property which will be used below, namely, the operators $L = D_\alpha D^\alpha = [D_1, D_2]$ and $(L)^\dagger = (D_\alpha)^\dagger(D^\alpha)^\dagger = [(D_1)^\dagger, (D_2)^\dagger]$ commute with the PPSA generators in representation (3.2).

We notice that the representation of PPSA in the terms of paragrassmannian variables can be found, using the covariant representation for the PPSA [11]. Then we put

$$\sqrt{2M}(S_{51} - iS_{52}) \longrightarrow -i\frac{\partial}{\partial\theta^1} - i(\theta^1)^\dagger M, \qquad \sqrt{2M}(S_{53} - iS_{54}) \longrightarrow -i\frac{\partial}{\partial\theta^2} - i(\theta^2)^\dagger M,$$
$$\sqrt{\frac{2}{M}}(S_{51} + iS_{52}) \longrightarrow -i\frac{\partial}{\partial(\theta^2)^\dagger} - i\theta^1 M, \qquad \sqrt{\frac{2}{M}}(S_{53} + iS_{54}) \longrightarrow -i\frac{\partial}{\partial(\theta^1)^\dagger} - i\theta^2 M.$$

Operators (3.2)–(3.3) act on the space of functions $\Phi(x, \theta, (\theta)^\dagger)$ which depend on space coordinates x_μ and paragrassmanian variables θ, $(\theta)^\dagger$. We shall call such functions parasuperfields. They form the linear space. In general case this space is reducible. In other words, the expansion $\Phi(x, \theta, (\theta)^\dagger)$ by powers of θ and $\theta^\dagger$ has superficial components. We can eliminate these components, if we impose the covariant constraints on Φ. But these constraints should not lead to the differential consequences in the x-space which restrict the dependence of the parasuperfields on x_μ. For the case $N = 1$ and arbitrary p these constraints have the following form

$$D_\alpha\Phi(x, \theta, (\theta)^\dagger) = 0 \qquad \text{or} \qquad (D_\alpha)^\dagger\Phi(x, \theta, (\theta)^\dagger) = 0. \tag{3.4}$$

Constraints (3.4) pick out the invariant subspaces, containing smaller number of component fields, from the space of the parasuperfields. The fields satisfying conditions (3.4) are called chiral.

In order to investigate equations (3.4) it is convenient to choose a new representation for the parasuperfields

$$\Phi^\pm = \exp(\mp G)\Phi, \qquad \text{where} \qquad G = \frac{1}{2}(\sigma_\mu)_{\alpha\beta}[(\theta^\alpha)^\dagger, \theta^\beta]P^\mu \tag{3.5}$$

(these representations will be called "+" and "−" representations). The operators A^+ and A^- in "+" and "−" representations are connected with the operator A in the initial representation by the formula

$$A^{\pm} = \exp(\mp G) A \exp(\pm G) \tag{3.6}$$

Using (3.5), we can show that Φ^+ doesn't depend on $(\theta)^\dagger$ and Φ^+ doesn't depend on θ.

In the case $N = 1$ and $p = 2$ the invariant spaces, picked out by equations (3.4), will contain 6 independent fields: 3 fields with spin 0, 2 filds with spin $\frac{1}{2}$ and 1 field with spin 1 (see [8]).

Using the fact that the operator L defined above commutes with the generators Poincaré parasuperalgebra, we can write down the equation for $\Phi^+(x,\theta)$ (without interaction), which will be invariant under the Poincaré parasuperalgebra:

$$(L^+)^\dagger \exp(-2G)\Phi^*_+(x,\theta) = 0, \tag{3.7}$$

where $(L^+)^\dagger = [(D_1^+)^\dagger, (D_2^+)^\dagger]$, $(D^+)^\dagger$ is the covariant derivative in "+" representation. For $p = 2$ we find, taking into account (3.3), (3.5), that the equations for the component fields A, ϕ_α, $\psi_{\alpha\beta}$, λ_α, B (below we omit the indices"+") are

$$(p_0 + \vec{S}\vec{p})\vec{\Lambda} = 0, \qquad \operatorname{div}\vec{\Lambda} = 0, \tag{3.8}$$

$$(p_0 + \vec{\sigma}\vec{p})\varphi = 0, \tag{3.9}$$

$$\Box A = 0, \tag{3.10}$$

$$\chi(x) = B(x) = \lambda_1 = \lambda_2 = 0, \tag{3.11}$$

where $\vec{\Lambda} = (\psi_{22} - \psi_{11},\ -i(\psi_{22} + \psi_{11}),\ \psi_{12} + \psi_{21})$, $\chi(x) = \psi_{12} - \psi_{21}$, the matrices S_a have the form

$$S_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \qquad S_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is obvious that (3.8) is the system of Maxwell's equations for the massless field with spin 1. System (3.9) is the system of the equations for the massless field with spin $\frac{1}{2}$. Equation (3.10) is the equation for the massless field with spin 0. In addition, it is obvious from (3.11) that two scalar fields and spinor field are equal zero.

Now let us consider the equation for $\Phi^+(x,\theta)$ including the interaction (parasupersymmetric Wess–Zumino model). In this case parasupersymmetric equation for $\Phi^+(x,\theta)$ has the form

$$((L^+)^\dagger)^p \exp(-2G)\Phi^*_+(x,\theta) = g\Phi^2_+(x,\theta), \tag{3.12}$$

where g is interaction constant. For $p = 1$ we recover supersymmetric Wess–Zumino model. For $p = 2$ it yields the model described in [8].

The Lagrangian which corresponds to equation (3.12) has the form

$$\mathcal{L} = (\Phi^*_+ \exp(2G)\Phi_+)_{(\theta^1)^p(\theta^2)^p((\theta^1)^\dagger)^p((\theta^2)^\dagger)^p} + \frac{1}{3}(g\Phi^3_+)_{(\theta^1)^p(\theta^2)^p} + (\text{c.c.}),$$

(c.c. is complex conjugation). In conclusion of this paper let us note that the Wess–Zumino model for $p > 1$ is incompatible with the description of massive particles (see [8]).

Acknowledgements

The author thanks Professor A.G. Nikitin for fruitful discussions.

References

[1] Gol'fand Yu.A. and Likhtman E.P., *Lett. JETP*, 1971, V.13, 452;
Ramond P., *Phys.Rev. D*, 1971, V.3, 2415;
Neveu A. and Schwartz J., *Nucl. Phys. B*, 1971, V.31, 86;
Volkov D.V. and Akulov V.P., *Lett. JETP*, 1972, V.16, 621; *Phys.Lett. B*, 1973, V.46, 109;
West P., Introduction to Supersymmetry and Supergravity, World Scientific, 1986.

[2] Haag R., Lopuszanski J.T. and Sohnius M.F., *Nucl. Phys. B*, 1975, V.88, 61.

[3] Witten E., *Nucl. Phys. B*, 1981, V.188, 513; 1982, V.202, 253.

[4] Piette B. and Vinet L., *Phys. Lett. A*, 1989, V.4, 2515;
D'Hoker E., Kostelecky V.A. and Vinet L., Spectrum generating superalgebras in dynamical groups and spectrum generating algebras, eds. A. Barut, A. Bohm and J. Ne'eman, Singapore, Word Scientific, 1989.

[5] Rubakov V.A. and Spiridonov V.P., *Mod. Phys. Lett. A*, 1988, V.3.

[6] Ohnuki Y. and Kamefuchi S., Quantum Field Theory and Parastatistics, Tokyo, University of Tokyo, 1982.

[7] Beckers J. and Debergh N., *Nucl. Phys. B*, 1990, V.340, 767; *J. Math. Phys.*, 1991, V.32, 1808, 1815.

[8] Beckers J. and Debergh N., *J. Mod. Phys. A*, 1993, V.8, 5041.

[9] Nikitin A.G. and Tretynyk V.V., *J. Phys. A*, 1995, V.28, 1665.

[10] Nikitin A.G., in 5th Wigner Symposium, eds. P. Kasperovitz and D. Gram, World Scientific, 1998, 227.

[11] Niederle J. and Nikitin A.G., *J. Phys. A*, 1999, V.32, 5141.