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Operators on $(A, *)$-Spaces and Linear Classification Problems

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In the paper we consider the problem about the canonical form of linear operators on vector spaces that are gradable by a partially ordered set with involution. In a natural and important (from the practical point of view) case we establish a connection between this problem and the problem of description representations of partially ordered sets with involution.

The problem of about the canonical form of linear operators on a finite-dimensional S-space over a field k, where $S = (A, *)$ is a partially ordered set with involution, was introduced by the author in [1] (see also [2–4]). In present paper we reduce a natural and important (from the practical point of view) case of this problem to the problem of description of representations of partially ordered sets with involution.

1. Main concepts. Through out the paper all vector spaces are finite-dimensional and all partially ordered sets (posets) are finite. A poset A with trivial involution $*$ is identified with A. Under consideration of linear maps, morphisms, functors, etc. we use the right-side notations (in particular, vector spaces are right).

For a poset with involution $S = (A, *)$ and a field k we denote by $\mathrm{mod}_S k$ the category with objects the vector k-spaces $U = \oplus_{x\in A} U_x$, where $U_{x^*} = U_x$ for all $x \in A$ (such k-spaces are called S-spaces over k), and with morphisms $\delta : U \to U'$ those linear maps $\delta \in \mathrm{Hom}_k(U, U')$ for which $\delta_{x^*x^*} = \delta_{xx}$ for all $x \in A$ and $\delta_{xy} = 0$ if $x \not\leq y$ (such maps are called S-maps) [1]; here δ_{xy} denotes (as usual in analogous situations) the linear map of U_x into U'_y, induced by the map δ). If $|A| = 1$, the category $\mathrm{mod}_A k$ coincide with the category $\mathrm{mod}\, k$ of all (finite-dimensional) vector k-spaces.

The set of all S-maps of U into U' (U and U' are S-spaces) is denoted by $\mathrm{Hom}_{S.k}(U, U')$. If U is a S-space and $C \subset A$, U_C denotes the subspace $\oplus_{x\in C} U_x \subset U$; if, moreover, V is a k-space and $\gamma \in \mathrm{Hom}_k(V, U)$, γ_C denotes the map of V into U_C induced by γ; if γ is a map of a S-space U into a S-space U', $\gamma_{C,D}$ denotes the map of U_C into U'_D induced by γ.

Let $S = (A, *)$ be a poset with involution and $f = f(t)$ be a polynomial over k. We denote by $\Lambda_{S,k}$ the category whose objects are the linear operators on S-spaces, i.e. the pairs (U, φ) formed a S-space U and a map $\varphi \in \mathrm{Hom}_k(U, U)$. A morphism from (U, φ) to (U', φ') is determined by a S-map $\delta : U \to U'$ such that $\varphi\delta = \delta\varphi'$. By $\Lambda_{S,k,f}$ we denote the full subcategory of $\Lambda_{S,k}$ consisting of all objects (U, φ) such that $f(\varphi) = 0$.

2. Formulation of the main result. Let $f = f(t)$ be a polynomial over k. If x is a root of $f(t)$, $r(x)$ denotes its multiplicity. We assume that each root of $f(t)$ belongs to k and has multiplicity < 3. For $f = f(t)$, define a poset with involution $\widehat{P}_f = (P_f, *_f)$ in the following way:

1) P_f consists of the triples (x, p, i) formed by a root x of $f(t)$, and integer numbers p and i such that $1 \leq i \leq p \leq r(x)$;

2) $(x, p, i) \leq (y, q, j)$ if and only if $x = y$, and $p \geq q$, $i \leq j$ or $p \leq q$, $j - i \geq q - p$;

3) $\overline{x}^{*_f} = \overline{y}$ for unequal $\overline{x} = (x, p, i)$ and $\overline{y} = (y, q, j)$ if and only if $x = y$ and $p = q$.

Let $S=(A,*)$ be a poset with involution. A representation of S is (in our terms) a triple (V,U,γ) formed by vector k-space $V\in \operatorname{mod} k, U\in \operatorname{mod}_S k$ and a linear map $\gamma\in \operatorname{Hom}_k(V,U)$; a morphism of representations $(V,U,\gamma)\to(V',U',\gamma')$ is given by a pair (μ,ν) of linear maps $\mu\in\operatorname{Hom}_k(V,V')$ and $\nu\in\operatorname{Hom}_{S,k}(U,U')$ such that $\gamma\nu=\mu\gamma'$ (see [5]). Thus defined category is denoted by $\mathcal{R}_k(S)$.

Denote by $\overline{\mathcal{R}}_k(S\coprod\widehat{P}_f)$, where $S\coprod\widehat{P}_f$ is the direct sum of S and $\widehat{P}_f$ (i.e. $S\coprod\widehat{P}_f=(A\cup P_f,*)$, $A\cap P_f=\varnothing$ and $*$ on $A\cup P_f$ is induced by $*$ on A and $*_f$ on P_f), the full subcategory of $\mathcal{R}_k(S\coprod\widehat{P}_f)$ consisting of all objects (V,U,γ) with $\gamma_A: V\to U_A$ and $\gamma_{P_f}: V\to U_{P_f}$ being isomorphisms $\operatorname{mod} k$.

In this paper we shall prove the following statement.

Theorem A. *Let $S=(A,*)$ be a poset with involution, and $f=f(t)$ be a polynomial (over k) such that each its root belong to k and has multiplicity <3. Then the categories $\Lambda_{S,k,f}$ and $\overline{\mathcal{R}}_k(S\coprod\widehat{P}_f)$ are equivalent.*

3. Proof of Theorem A. Recall that a functor $F:\Phi\to\Psi$ is called faithful (respectively, full) if, for an arbitrary $X,Y\in\operatorname{Ob}\Phi$, the map $F:\operatorname{Hom}_\Phi(X,Y)\to\operatorname{Hom}_\Psi(X,Y)$ is injective (respectively, surjective); a functor F is called dense if each $Y\in\operatorname{Ob}\Psi$ is isomorphic to some XF (a special case of a dense functor is a surjective on objects functor, i.e. such one that the map $F:\operatorname{Ob}\Phi\to\operatorname{Ob}\Psi$ is surjective). According to the well-known theorem a functor F is equivalence of categories if and only if it is full, faithful and dense.

For a $\widehat{P}_f$-space U, we denote by $[U]$ a linear operator on U with the following (Jordan) matrix $([U]_{\overline{x}\overline{y}})$, where $\overline{x}=(x,p,i)$ and $\overline{y}=(y,q,j)$ run through the set P_f:

$$[U]_{\overline{x}\,\overline{x}}=x\mathbf{1}\quad\text{for every } \overline{x};$$
$$[U]_{\overline{x}\,\overline{y}}=\mathbf{1}\quad\text{if } x=y,\ p=q,\ i=j-1;$$
$$[U]_{\overline{x}\,\overline{y}}=O\quad\text{in the remaining cases,}$$

where $\mathbf{1}=\mathbf{1}_{U(\overline{x})}$.

Define a functor $F:\overline{\mathcal{R}}_k(S\coprod\widehat{P}_f)\to\Lambda_{S,k,f}$ as follows:

$$(V,U,\gamma)F=(U_A,\gamma_A^{-1}\gamma_{P_f}[U_{P_f}]\gamma_{P_f}^{-1}\gamma_A)\quad\text{for an object } (V,U,\gamma);$$
$$(\mu,\nu)F=\nu_{A,A}\quad\text{for a morphism } (\mu,\nu):(V,U,\gamma)\to(V',U',\gamma').$$

Here S-map $\nu_{A,A}$ is a morphism in $\Lambda_{S,k,f}$ because the equality $\varphi\nu_{A,A}=\nu_{A,A}\varphi'$, where $\varphi=\gamma_A^{-1}\gamma_{P_f}[U_{P_f}]\gamma_{P_f}^{-1}\gamma_A$ and $\varphi'=(\gamma'_A)^{-1}\gamma'_{P_f}[U'_{P_f}](\gamma'_{P_f})^{-1}\gamma'_A$, easily reduces to the obviously equality $[U_{P_f}]\nu_{P_f,P_f}=\nu_{P_f,P_f}[U'_{P_f}]$ (using the equalities $\gamma_A\nu_{A,A}=\mu\gamma'_A$ and $\gamma_{P_f}\nu_{P_f,P_f}=\mu\gamma'_{P_f}$ which are equivalent to the equality $\gamma\nu=\mu\gamma'$)

It follows from the equalities $\gamma_A\nu_{A,A}=\mu\gamma'_A$ and $\gamma_{P_f}\nu_{P_f,P_f}=\mu\gamma'_{P_f}$ (taking into account the invertibility of the maps γ_A and γ_{P_f}) that $\mu=0$ and $\nu=0$ whenever $(\mu,\nu)F=0$. Hence the functor F is faithful.

The functor F is full — for a morphism $\alpha:(V,U,\gamma)F\to(V',U',\gamma')F$, we can take as a morphism $(\mu,\nu):(V,U,\gamma)\to(V',U',\gamma')$ such that $(\mu,\nu)F=\alpha$ the following morphism:

$$\mu=\gamma_A\alpha(\gamma'_A)^{-1},\quad \nu_{A,A}=\alpha,\quad \nu_{P_f,P_f}=\gamma_{P_f}^{-1}\gamma_A\alpha(\gamma'_A)^{-1}\gamma'_{P_f}\quad(\nu_{A,P_f}=0,\nu_{P_f,A=0}).$$

Finally, let us verify that the functor F is surjective on objects. If $(W,\varphi)\in\Lambda_{S,k,f}$, denote by W^0 the $\widehat{P}_f$-space $\oplus_{\overline{x}=(x,p,i)\in P_f}W_{\overline{x}}$, where $W_{\overline{x}}=\operatorname{Ker}(\varphi-x\mathbf{1}_W)$, and fix a map $\lambda\in\operatorname{Hom}_k(W,W^0)$ such that $\varphi=\lambda[W^0]\lambda^{-1}$. Then we can take as the objects $(V,U,\gamma)\in\overline{\mathcal{R}}_k(S\coprod\widehat{P}_f)$ such that $(V,U,\gamma)F=(W,\varphi)$ the following object: $V=W$, $U_A=W$, $U_{P_f}=W^0$ and $\gamma_A=\mathbf{1}_{U_A}$, $\gamma_{P_f}=\lambda$.

Thus the functor F is full, faithful and dense. Theorem A is proved.

4. Generalization of Theorem A. Our theorem can be generalized to the case of an arbitrary polynomial $f(t)$. Here we consider the case when each root of $f(t)$ belong to k and have an arbitrary multiplicity r ($1 \le r \le \deg f$).

In this situation we shall also need representations of so-called completed posets [6]. A completed poset consists of a poset B and an equivalence relation $\sim$ on $B^{\le} = \{(x,y) \in B \times B |\, x \le y\}$. These data are subjected to the condition that $x \le z \le y$ and $(x,y) \sim (x',y')$ imply the existence of a unique z' satisfying $x' \le z' \le y'$, $(x,z) \sim (x',z')$ and $(z,y) \sim (z',y')$. In case $(x,x) \sim (x',x')$ we shall write $x \sim x'$; therefore it is possible to describe restriction of the relation $\sim$ on B. A completed poset is called weakly completed if $(x,y) \sim (x',y')$ implies $x = y$ and $x' = y'$.

Let $T = (B, \sim)$ be a completed poset. A T-space (over k) is a B-space $U = \oplus_{b \in B} U_b$ such that $U_x = U_y$ if $x \sim y$. A T-map of U into U' (U and U' are T-spaces) is a B-map $\varphi : U \to U'$ such that $\varphi_{xy} = \varphi_{zt}$ if $(x,y) \sim (z,t)$; $\mathrm{Hom}_{T.k}(U,U')$ denotes the set of all T-maps of U into U'. The category of T-spaces over k (whose objects and morphisms are, respectively, the T-spaces and T-maps) is denoted by $\mathrm{mod}_T k$.

Representations of a completed poset $T = (B, \sim)$ are defined in a way analogous to that for a poset with involution $S = (A, *)$. A representation of $T = (B, \sim)$ is (in our terms) a triple (V, U, γ) formed by vector k-spaces $V \in \mathrm{mod}\, k$, $U \in \mathrm{mod}_T k$ and a linear map $\gamma \in \mathrm{Hom}_k(V, U)$. A morphism of representations $(V, U, \gamma) \to (V', U', \gamma')$ is given by a pair (μ, ν) of linear maps $\mu \in \mathrm{Hom}_k(V, V')$ and $\nu \in \mathrm{Hom}_{T,k}(U, U')$ such that $\gamma\nu = \mu\gamma'$. The category of all representations of $T = (B, \sim)$ is denoted by $\mathcal{R}_k(T)$.

For $f = f(t)$, we shall consider (instead of the poset with involution $\widehat{P}_f = (P_f, *_f)$ which was considered above) the completed posets $(\widetilde{P}_f, \sim_f)$, where P_f is defined by the conditions 1) and 2), and the relations $\sim_f$ on $P_f^{\le}$, defined in the following way: $((x,p,i,),(x,q,j)) \sim_f ((x',p',i'),(x',q',j'))$ if, and only if, $x = x'$, $p = p'$, $q = q'$ and $i - i' = j - j'$. This obviously implies that $(x,p,i) \sim_f (x',p',i'))$ if and only if $x = x'$ and $p = p'$.

Let $S = (A, *)$ be a poset with involution. We identify S with the weakly completed poset $(A, \sim_*)$, where $x \sim_* x'$ for $x \ne x'$ if, and only if, $x^* = x'\,(x, x' \in A)$. Consider the direct sum of S and $\widetilde{P}_f$: $S \coprod \widetilde{P}_f = (A, \sim_*) \coprod (P_f, \sim_f)$ .As in the case above denote by $\overline{\mathcal{R}}_k(S \coprod \widetilde{P}_f)$ the full subcategory of $\mathcal{R}_k(S \coprod \widetilde{P}_f)$ consisting of all objects (V, U, γ) with $\gamma_A : V \to U_A$ and $\gamma_{P_f} : V \to U_{P_f}$ being isomorphisms in $\mathrm{mod}\, k$.

We have the following generalization of Theorem A.

Theorem B. *Let $S = (A, *)$ be a poset with involution, and $f = f(t)$ be a polynomial (over k) with roots belonging to k. Then the categories $\Lambda_{S,k,f}$ and $\overline{\mathcal{R}}_k(S \coprod \widetilde{P}_f)$ are equivalent.*

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