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Discrete Subgroups of the Poincaré Group

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The framework of point form relativistic quantum mechanics is used to construct interacting four-momentum operators in terms of creation and annihilation operators of underlying constituents. It is shown how to write the creation and annihilation operators in terms of discrete momenta, arising from discrete subgroups of the Lorentz group, in such a way that the Poincaré commutation relations are preserved. For discrete momenta the bosonic creation and annihilation operators can be written as multiplication and differentiation operators acting on a holomorphic Fock space. It is shown that with such operators matrix elements of the relativistic Schrödinger equation become an infinite coupled set of first order partial differential equations.

1 Introduction

In nonrelativistic quantum mechanics one often puts a system of interest in a box, in order to deal with discrete momenta and avoid delta functions. This is equivalent to looking at representations of discrete subgroups of the Euclidean group, the group consisting of rotations and Galilei boosts, and itself a subgroup of the full Galilei group. It is of interest to see if the same thing can be done for relativistic systems, and in particular to see what happens to relativistic spin. The point of this paper is to show how to construct discrete subgroups of the Lorentz group and then embed these discrete subgroups in a quasidiscrete Poincaré subgroup, in order to construct interacting four-momentum operators.

The context for this work is point form relativistic quantum mechanics [1], wherein all interactions are put in the four-momentum operators, and the Lorentz generators are free of interactions. The Lorentz generators are readily exponentiated to give global Lorentz transformations; this is important since discrete subgroups do not have an associated Lie algebra. The point form is to be contrasted with the more usual instant form of dynamics, where interactions are present in the Hamiltonian and boost generators, and the momentum and angular momentum generators are free of interactions (for a discussion of the various forms of dynamics, see for example [2]).

One of the main reasons for introducing discrete subgroups is that the creation and annihilation operators that are used to build the interacting four-momentum operators then have discrete momenta in their arguments, and for bosonic creation and annihilation operators, can be realized as multiplication and differentiation operators acting on a holomorphic Hilbert space. In this representation matrix elements of the relativistic Schrödinger equation then take on the form of an infinite coupled set of first order partial differential equations.

2 Point form quantum mechanics

In order to have a relativistic theory it is necessary to satisfy the commutation relations of the Poincaré algebra. In quantum field theory this is done by integrating the stress-energy tensor – made up of polynomials of field operators – over a time constant surface (see for example [3]). It is however also possible to integrate over the forward hyperboloid, in which case the interactions will all be in the four-momentum operators, which must commute with one another, $[P^\mu, P^\nu] = 0$, where μ and ν run between zero and three. Here $P^\mu = P^\mu_{fr} + P^\mu_I$, the sum of free and interacting four-momentum operators. Since Lorentz generators do not contain interactions, it is more convenient to deal with representations of global Lorentz transformations, written as U_Λ, where Λ is a Lorentz transformation and U_Λ the unitary operator representing the Lorentz transformation. The Poincaré relations are then

$$[P^\mu, P^\nu] = 0, \tag{1}$$

$$U_\Lambda P^\mu U_\Lambda^{-1} = (\Lambda^\mu_\nu)^{-1} P^\nu. \tag{2}$$

Since P^μ are the generators for space-time translations, the relativistic Schrödinger equation can be written as

$$i\hbar\partial/\partial x^\mu |\Psi_x\rangle = P_\mu |\Psi_x\rangle, \tag{3}$$

where $|\Psi_x\rangle$ is an element of the Fock space and x $(= x^\mu)$ is a space-time point. The space-time independent Schrödinger equation is then

$$P^\mu |\Psi\rangle = p^\mu |\Psi\rangle, \tag{4}$$

where p^μ is an eigenvalue of P^μ. The mass operator is $M := \sqrt{P^\mu P_\mu}$ and must have a spectrum bounded from below.

For particles of mass m $(m > 0)$ and spin j, it is well known that the irreducible representations of the Poincaré group can be written as

$$U_\Lambda |p, \sigma\rangle = \sum_{\sigma'=-j}^{j} |\Lambda p, \sigma'\rangle D^j_{\sigma',\sigma}(R_W), \tag{5}$$

$$U_a |p, \sigma\rangle = e^{-ip^\mu a_\mu} |p, \sigma\rangle, \tag{6}$$

$$P^\mu |p, \sigma\rangle = p^\mu |p, \sigma\rangle. \tag{7}$$

Here p is a four-momentum vector satisfying $p^\mu p_\mu = m^2$, σ is a spin projection variable, and $|p, \sigma\rangle$ is a (nonnormalizable) state vector. R_W is a Wigner rotation (see, for example [4] and references cited therein) and $D^j_{\sigma',\sigma}(\ )$ an $SU(2)$ D function. For infinitesimal space-time elements a, equation (7) follows from equation (6).

To get a many-body theory, creation and annihilation operators are introduced which take on the transformation properties of the single particle states, equations (5), (6):

$$[a(p, \sigma), a^\dagger(p', \sigma')]_\pm = 2E\delta^3(p - p')\delta_{\sigma,\sigma'}, \tag{8}$$

$$U_\Lambda a^\dagger(p, \sigma) U_\Lambda^{-1} = \sum_{\sigma'=-j}^{j} a^\dagger(\Lambda p, \sigma') D^j_{\sigma',\sigma}(R_W), \tag{9}$$

$$U_a a^\dagger(p, \sigma) U_a^{-1} = e^{-ip^\mu a_\mu} a^\dagger(p, \sigma), \tag{10}$$

where $\pm$ means commutator or anticommutator. Then a free four-momentum operator can be written as

$$P^\mu_{fr} = \sum_\sigma \int dp(p^\mu) a^\dagger(p, \sigma) a(p, \sigma) \tag{11}$$

and satisfies the Poincaré commutation relations, equations (1), (2). $dp := d^3p/2\sqrt{m^2+p^2}$ is the Lorentz invariant measure.

The full interacting four-momentum operator, $P^\mu = P^\mu_{fr} + P^\mu_I$, must also satisfy equation(1):

$$\begin{aligned}[P^\mu, P^\nu] &= 0 \\ &= [P^\mu_{fr} + P^\mu_I, P^\nu_{fr} + P^\nu_I] \\ &= [P^\mu_{fr}, P^\nu_I] + [P^\mu_I, P^\nu_{fr}] + [P^\mu_I, P^\nu_I].\end{aligned} \tag{12}$$

Equation(12) can be satisfied if $[P^\mu_I, P^\nu_I] = 0$ and $[P^\mu_{fr}, P^\nu_I] = [P^\nu_{fr}, P^\mu_I]$.

A natural way to construct an interacting four-momentum operator that satisfies these equations is with an interaction Lagrangian built out of local fields. While the long range goal is to use discrete subgroups for pion-nucleon interactions, in this paper we will consider the simpler case of a charged scalar meson interacting with a neutral meson. If $a(p)$ and $b(p)$ denote the annihilation operators for the positively and negatively charged mesons, while $c(k)$ denotes the annihilation operator for the neutral meson, then local fields $\phi(x)$ and $\phi^\dagger(x)$ for the charged mesons and $\chi(x)$ for the neutral meson are defined by

$$\begin{aligned}&\phi(x) = \int dp \left(e^{-ip^\mu x_\mu} a(p) + e^{ip^\mu x_\mu} b^\dagger(p)\right), \quad \phi^\dagger(x) = \int dp \left(e^{-ip^\mu x_\mu} b(p) + e^{ip^\mu x_\mu} a^\dagger(p)\right), \\ &\chi(x) = \int dk \left(e^{-ik^\mu x_\mu} c(k) + e^{ik^\mu x_\mu} c^\dagger(k)\right),\end{aligned} \tag{13}$$

where $p^\mu p_\mu = m^2$ and $k^\mu k_\mu = m_\pi^2$.

An interacting four-momentum operator can be built out of these local fields by integrating (say a trilinear coupling) over the forward hyperboloid:

$$P^\mu_I = \lambda_0 \int d^4x \delta\left(x^\nu x_\nu - \tau^2\right) \theta(x_0) x^\mu \phi^\dagger(x)\phi(x)\chi(x); \tag{14}$$

if derivative couplings, say of the form $\partial\phi^\dagger(x)/\partial x^\alpha \partial\phi(x)/\partial x_\alpha \chi(x)$, and of all higher order derivatives are also added on to equation (14), then an interacting four-momentum operator with an arbitrary potential will result. That is, each differentiation of a field brings down a power of momentum, and if these are all added together, and the integration over space-time carried out, the interacting four-momentum operator will have the form,

$$P^\mu_I(1) = \int dp\, dp'\, dk \Delta^\mu(p - p' + k) v(p^\alpha p'_\alpha, -p^\alpha k_\alpha, p'_\alpha k^\alpha) a^\dagger(p) a(p') c^\dagger(k), \tag{15}$$

plus seven other terms of the same form involving different creation and annihilation operators. λ_0 is a coupling constant and the coupling constants for all the higher derivatives times the powers of momenta combine to give the potential function $v(\ )$. For example, there is a term of the form $a(p)b(p')c^\dagger(k)$ in which the potential function has the argument $v(-p^\alpha p'_\alpha, p'_\alpha k^\alpha, p^\alpha k_\alpha)$. Thus, $P^\mu_I = \sum_{i=1}^{8} P^\mu(i)$, and by construction satisfies equation (12). $\Delta^\mu(p) := \int d^4x \delta\left(x^\alpha x_\alpha - \tau^2\right) \theta(x_0) x^\mu e^{ip^\alpha x_\alpha}$ and comes from the exponentials in the field operators. The full four-momentum operator, a sum of free four-momentum operators for the three types of particles called a, b, and c and the interacting four-momentum operator, of the form given in equation (15), satisfies the point form equations, equations (1) and (2) (this is shown in reference [5]), and provides the starting point for the discrete subgroups of the Lorentz group.

3 Discrete subgroups of the Lorentz group

In order for the arguments in the creation and annihilation operators introduced in section (2) to become discrete, the Lorentz transformations which boost a particle from its rest frame must be discrete:

$$p_n = \Lambda(n) p(\text{rest}), \tag{16}$$

where $p(\text{rest})$ is the rest frame four-momentum, $p(\text{rest}) = (m, 0, 0, 0)$. To construct discrete subgroups of the Lorentz group, it is convenient to start with Lorentz transformations along the z direction, in which case the discrete elements have the form

$$\Lambda_z(n) = \begin{pmatrix} \operatorname{ch}\beta n & 0 & 0 & \operatorname{sh}\beta n \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \operatorname{sh}\beta n & 0 & 0 & \operatorname{ch}\beta n \end{pmatrix}. \tag{17}$$

For a fixed constant β, the set of elements with n ranging over all positive and negative integers forms a discrete subgroup of the group of all z axis Lorentz transformations. The minimum momentum is $p_1 = m \operatorname{sh}\beta$ and in the limit where β goes to zero, n to infinity, such that $\beta n = \alpha$, one recovers the continuum limit.

Next consider the finite rotation subgroups of the full rotation group, $SO(3)$, and in particular the crystallographic subgroups denoted by $SO(3)_D$ (see for example [6], chapters two and four). Now arbitrary Lorentz transformations can be written as

$$\Lambda = R\Lambda_z R', \tag{18}$$

where R, R' are elements of $SO(3)$. Then discrete subgroups $SO(1,3)_D$ of the continuous Lorentz group $SO(1,3)$ have elements of the form

$$\Lambda_D = R_D \Lambda_z(n) R'_D, \tag{19}$$

where R_D, R'_D are elements of $SO(3)_D$. Thus the relevant discrete subgroups of the Lorentz group are indexed by the finite subgroups of the rotation group.

The discrete Lorentz transformations can now be adjoined to space-time translations to give groups with a Poincaré-like structure. Because of the semidirect product nature of the Poincaré group, the Lorentz transformations act on space-time translations, but not the other way around. Hence it is possible to adjoin discrete Lorentz transformations to continuous space-time translations and still get a group structure. These groups will be called quasi-discrete Poincaré groups and denoted by $\mathcal{P}_D$, with a group law given by

$$(\Lambda_D, a)(\Lambda'_D, a') = (\Lambda_D \Lambda'_D, \Lambda_D a + a'), \tag{20}$$

with Λ_D in $SO(1,3)_D$ and a in $\mathcal{R}^4$.

The representations of $\mathcal{P}_D$ are obtained in exactly the same way as the ordinary Poincaré group representations, namely as induced representations [4]. For positive mass representations the little group is now $SO(3)_D$ and the action of a discrete Lorentz transformation on a state with discrete four-momentum p_D and spin projection σ is given by

$$U_{\Lambda_D} |p_D, \sigma\rangle = \sum_{\sigma'=-j}^{j} |\Lambda_D p_D, \sigma'\rangle D^j_{\sigma',\sigma}(R_W), \tag{21}$$

where now R_W is a discrete Wigner rotation given by

$$R_W = B^{-1}(\Lambda_D p_D)\Lambda_D B(p_D), \tag{22}$$

with the discrete momenta given by $p_D = B(p_D)p(\text{rest})$, and $B(p_D)$ a boost (coset) representative of $SO(1,3)_D$ with respect to $SO(3)_D$. It should be noted that since the irreducible representations of all the finite subgroups of $SO(3)$ are bounded in their spin values [6], the irreducible representation label j in equation (21) can only take on low lying spin values. Thus relativistic spin is well-defined for the discrete subgroups of the Lorentz group, but bounded in its possible values.

Given the representations of $\mathcal{P}_D$, creation and annihilation operators with discrete arguments can be defined in the usual way:

$$\begin{aligned} &|p_D, \sigma\rangle = a^\dagger(p_D, \sigma)|0\rangle, \\ &[a(p_D, \sigma), a^\dagger(p'_D, \sigma')]_\pm = \delta_{p_D, p'_D}\delta_{\sigma,\sigma'}, \qquad P^\mu_{fr} = \sum_{\sigma, p_D} p^\mu_D a^\dagger(p_D, \sigma)a(p_D, \sigma). \end{aligned} \tag{23}$$

A problem however arises when attempting to define local fields. As an example take the charged scalar field, equation (13) with

$$\begin{aligned} &\phi_D(x) := \sum_{p_D} e^{-ip^\mu_D x_\mu} a(p_D) + e^{ip^\mu_D x_\mu} b^\dagger(p_D), \\ &U_{\Lambda_D}\phi_D(x)U^{-1}_{\Lambda_D} = \phi_D(\Lambda_D x), \qquad U_a\phi_D(x)U_a^{-1} = \phi_D(x+a). \end{aligned} \tag{24}$$

But

$$[\phi(x), \phi^\dagger(y)] = \sum_{p_D} e^{-ip^\mu_D(x_\mu - y_\mu)} - e^{ip^\mu_D(x_\mu - y_\mu)} := \Delta_D(x-y). \tag{25}$$

Then $\Delta_D(\Lambda_D x) = \Delta_D(x)$ and $\Delta_D(-x) = -\Delta_D(x)$. However, for x spacelike, there is in general no Λ_D such that $\Lambda_D x = -x$; rather there is only a restricted set of space-time points x satisfying this equation. Therefore in general $\Delta_D(x) \neq 0$ for x spacelike and $\phi_D(x)$ is not local.

It is nevertheless still possible to define the discrete analogue of the interacting four-momentum operator, $P^\mu_{I_D}$ so that the sum $P^\mu_{I_D} = \sum_{i=1}^{8} P^\mu_{I_D}(i)$, as in equation (15), satisfies the point form equations, equation (1), with now the Lorentz transformations replaced by discrete Lorentz transformations (proofs of these statements are given in reference [7]).

4 Matrix elements of the point form relativistic Schrödinger equation

To conclude we show how to convert the point form equations, equations (1), (2), and (4) into a set of coupled first order partial differential equations. To keep the formalism as simple as possible we consider one space and one time dimension only. Then the bosonic creation and annihilation operators, $c^\dagger(k_n)$ and $c(k_n)$, can be replaced, respectively, by z_n and $\partial/\partial z_n$, with the bosonic Fock space now a holomorphic Hilbert space:

$$\mathcal{F}(\pi) = \{f(Z) | \|f\|^2 < \infty,\ f \text{ holomorphic}\}, \tag{26}$$

with a differentiation inner product given by $(f, f') = f^\star(\partial/\partial Z)f'(Z)|_{Z=0}$, where Z denotes the set of complex variables $\{z_n\}$ with n running over all the integers (see for example [8] and references therein).

Typical terms in the full four-momentum operator have the following form:

$$P^{\mu}_{fr}(\pi) = \sum_{n} k^{\mu}_{n} z_n \partial/\partial z_n, \qquad P^{\mu}_{I}(1) = \sum_{n,n',m} v^{\mu}(n,n',m) a^{\dagger}(n) a(n') z_m, \tag{27}$$

where now $v^{\mu}(n,n',m)$ includes both the Δ^{μ} and the potential $v()$ terms of the four-momentum operator, equation (15).

The goal is now to convert the relativistic Schrödinger equation, equation (4), into a coupled set of first order partial differential equations; this is possible because in all the terms in the full four-momentum operator, the meson annihilation operator, $\partial/\partial z_n$, never appears as a product with itself, but only as a product with other creation and annihilation operators. Therefore matrix elements of the relativistic Schrödinger equation will give a set of coupled first order differential equations.

The a, b creation and annihilation operators always occur in pairs that conserve baryon number, and hence it is easily shown that P^{μ} commutes with the baryon number operator, defined by

$$\hat{B} = \sum_{n} a^{\dagger}(n) a(n) + b^{\dagger}(n) b(n), \tag{28}$$

with positive or negative integers as eigenvalues. The Fock space thus decomposes into baryon number sectors, and there will be a coupled set of differential equations for each baryon number sector.

As an example consider the $B = 0$ sector, the physical vacuum, one pion, ..., sector, for which the wave function can be written as

$$|\Psi_{B=0}\rangle = f^{(0)}(Z)|0\rangle + \sum_{n,n'} f^{(2)}(Z,n,n')|n,n'\rangle + \cdots, \tag{29}$$

that is, a superposition of $0, 1, 2, \ldots$ particle-antiparticle pairs, where in each term there is a function of Z representing the meson cloud. $|0\rangle$ in equation (29) designates the a, b type particle bare vacuum only, since the bare meson vacuum is given by $f(Z) = 1$.

Taking $B = 0$ sector matrix elements of the relativistic Schrödinger equation with the wave function given in equation (29),

$$\begin{aligned} \langle 0|P^{\mu}|\Psi_{B=0}\rangle &= (m_{\pi}, 0)^{\mu}\langle 0|\Psi_{B=0}\rangle \\ \langle m,m'|P^{\mu}|\Psi_{B=0}\rangle &= (m_{\pi}, 0)^{\mu}\langle m,m'|\Psi_{B=0}\rangle \\ \vdots &= \vdots \end{aligned} \tag{30}$$

then gives a set of coupled partial differential equations in the 'meson cloud' amplitudes, $f^{(0)}(Z)$ for mesons with no particle-antiparticle pairs, $f^{(2)}(Z,n,n')$ for mesons with one particle-antiparticle pair having discrete momenta n and n' respectively (actually momentum $p = m\,\mathrm{sh}\,\beta n$ and $p' = m\,\mathrm{sh}\,\beta n'$ respectively).

The first and second of the equations given in equation (30) can be written more explicitly as

$$\begin{aligned} & m_{\pi}\sum_{n} k^{\mu}_{n} z_n \partial/\partial z_n f^{(0)}(Z) + \sum_{m,m',n} v^{\mu}(m,m',n)(z_n + \partial/\partial z_n) f^{(2)}(Z,m,m') \\ & = (m_{\pi}(v), 0)^{\mu} f^{(0)}(Z), \end{aligned}$$

$$\sum_n v^\mu(m,m',n)(z_n+\partial/\partial z_n)f^{(0)}(Z)+m(p^\mu_m+p^\mu_{m'})f^{(2)}(Z,m,m')$$
$$+m_\pi\sum_n k^\mu_n z_n\partial/\partial z_n f^{(2)}(Z,m,m')+\sum_{n,n'}v^\mu(m,n',n)(z_n+\partial/\partial z_n)f^{(2)}(Z,n',m')$$
$$+4\sum_{r,r',n}v^\mu(r,r',n)(z_n+\partial/\partial z_n)f^{(4)}(Z,m,m',r,r')=(m_\pi(v),0)^\mu f^{(2)}(Z,p,p),$$

where $m_\pi(v)$ is the eigenvalue for the physical (renormalized) pion, relative to the potential v^μ. If $m_\pi(v)$ is set equal to zero, the resulting eigenfuction is the eigenfunction for the physical vacuum. In all cases the eigenfunctions are those for a system at rest. Since the Lorentz boost generators are kinematic, these eigenfunctions can always be boosted to an arbitrary momentum. In terms of a column of unknown functions $f^{(0)}, f^{(2)}, f^{(4)}, \ldots$, one gets a tridiagonal matrix of first order partial differential equations. Possible methods of solution include truncating the f's at some order, but that is the subject of another paper.

In conclusion we have shown how quasidiscrete subgroups of the Poincaré group can be used to write the total four-momentum operator in terms of discrete momenta. This allows one to replace the mesonic creation and annihilation operators by multiplication and differentiation operators acting on a holomorphic Hilbert space. By taking matrix elements of the relativistic Schrödinger equation in this representation, one gets a set of coupled first order partial differential equations, in which the unknown functions are related to meson cloud amplitudes. Aside from the specific application discussed in this paper, one can think of the quasidiscrete subgroups of the Poincaré group as supplying a method for putting relativistic particles in a box.

It remains to find (at least approximate) solutions to the partial differential equations. Moreover this last section dealt only with systems in one spatial dimension, and to include spin, it is necessary to go to three dimensions. Finally it would be very interesting to see if clothing transformations [9] (see also [3], chapter 12), unitary transformations on the interacting four momentum operator, could be found that give an interacting four-momentum operator in terms of physical particles.

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