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On Casimir Elements of q-Algebras $U_q'(\mathrm{so}_n)$ and Their Eigenvalues in Representations

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The nonstandard q-deformed algebras $U_q'(\mathrm{so}_n)$ are known to possess q-analogues of Gel'fand–Tsetlin type representations. For these q-algebras, all the Casimir elements (corresponding to basis set of Casimir elements of so_n) are found, and their eigenvalues within irreducible representations are given explicitly.

1 Introduction

The nonstandard deformation $U_q'(\mathrm{so}_n)$, see [1], of the Lie algebra so_n admits, in contrast to standard deformation [2] of Drinfeld and Jimbo, an explicit construction of irreducible representations [1, 3] corresponding to those of Lie algebra so_n in Gel'fand–Tsetlin formalism. Besides, as it was shown in [4], $U_q'(\mathrm{so}_n)$ is the proper dual for the standard q-algebra $U_q(\mathrm{sl}_2)$ in the q-analogue of dual pair $(\mathrm{so}_n, \mathrm{sl}_2)$.

Let us mention that the algebra $U_q'(\mathrm{so}_3)$ appeared earlier in the papers [5]. As a matter of interest, this algebra arose naturally as the algebra of observables [6] in $2+1$ quantum gravity with 2D space fixed as torus. At $n > 3$, the algebras $U_q'(\mathrm{so}_n)$ are no less important, serving as intermediate algebras in deriving the algebra of observables in 2+1 quantum gravity with 2D space of genus $g > 1$, so that n depends on g, $n = 2g + 2$ [7, 8]. In order to obtain the algebra of observables, the q-deformed algebra $U_q'(\mathrm{so}_{2g+2})$ should be quotiented by some ideal generated by (combinations of) Casimir elements of this algebra. This fact, along with others, motivates the study of Casimir elements of $U_q'(\mathrm{so}_n)$.

2 The q-deformed agebras $U_q'(\mathrm{so}_n)$

According to [1], the nonstandard q-deformation $U_q'(\mathrm{so}_n)$ of the Lie algebra so_n is given as a complex associative algebra with $n-1$ generating elements $I_{21}, I_{32}, \ldots, I_{n,n-1}$ obeying the defining relations (denote $q + q^{-1} \equiv [2]_q$)

$$
\begin{aligned}
&I_{j,j-1}^2 I_{j-1,j-2} + I_{j-1,j-2} I_{j,j-1}^2 - [2]_q\, I_{j,j-1} I_{j-1,j-2} I_{j,j-1} = -I_{j-1,j-2},\\
&I_{j-1,j-2}^2 I_{j,j-1} + I_{j,j-1} I_{j-1,j-2}^2 - [2]_q\, I_{j-1,j-2} I_{j,j-1} I_{j-1,j-2} = -I_{j,j-1},\\
&[I_{i,i-1}, I_{j,j-1}] = 0 \qquad \text{if} \quad | i-j | > 1.
\end{aligned}
\tag{1}
$$

Along with definition in terms of trilinear relations, we also give a *'bilinear' presentation*. To this end, one introduces the generators (here $k > l+1$, $1 \le k, l \le n$)

$$I_{k,l}^{\pm} \equiv [I_{l+1,l}, I_{k,l+1}^{\pm}]_{q^{\pm 1}} \equiv q^{\pm 1/2} I_{l+1,l} I_{k,l+1}^{\pm} - q^{\mp 1/2} I_{k,l+1}^{\pm} I_{l+1,l}$$

together with $I_{k+1,k} \equiv I^+_{k+1,k} \equiv I^-_{k+1,k}$. Then (1) imply

$$\begin{aligned}
&[I^+_{lm}, I^+_{kl}]_q = I^+_{km}, \qquad [I^+_{kl}, I^+_{km}]_q = I^+_{lm}, \qquad [I^+_{km}, I^+_{lm}]_q = I^+_{kl} \qquad \text{if} \quad k > l > m,\\
&[I^+_{kl}, I^+_{mp}] = 0 \qquad \text{if} \quad k > l > m > p \quad \text{or} \quad k > m > p > l;\\
&[I^+_{kl}, I^+_{mp}] = (q - q^{-1})(I^+_{lp} I^+_{km} - I^+_{kp} I^+_{ml}) \qquad \text{if} \quad k > m > l > p.
\end{aligned} \tag{2}$$

Analogous set of relations exists involving I^-_{kl} along with $q \to q^{-1}$ (let us denote this "dual" set by $(2')$). If $q \to 1$ ('classical' limit), both (2) and $(2')$ reduce to those of so_n.

Let us give explicitly the two examples, namely $n = 3$ (the Odesskii–Fairlie algebra [5]) and $n = 4$, using the definition $[X, Y]_q \equiv q^{1/2} XY - q^{-1/2} YX$:

$$U_q'(\mathrm{so}_3): \quad [I_{21}, I_{32}]_q = I^+_{31}, \qquad [I_{32}, I^+_{31}]_q = I_{21}, \qquad [I^+_{31}, I_{21}]_q = I_{32}. \tag{3}$$

$$U_q'(\mathrm{so}_4): \quad \begin{cases}
[I_{32}, I_{43}]_q = I^+_{42}, \quad [I^+_{31}, I_{43}]_q = I^+_{41}, \quad [I_{21}, I^+_{42}]_q = I^+_{41},\\
[I_{43}, I^+_{42}]_q = I_{32}, \quad [I_{43}, I^+_{41}]_q = I^+_{31}, \quad [I^+_{42}, I^+_{41}]_q = I_{21}, \qquad (4)\\
[I^+_{42}, I_{32}]_q = I_{43}, \quad [I^+_{41}, I^+_{31}]_q = I_{43}, \quad [I^+_{41}, I_{21}]_q = I^+_{42},\\
[I_{43}, I_{21}] = 0, \quad [I_{32}, I^+_{41}] = 0, \quad [I^+_{42}, I^+_{31}] = (q - q^{-1})(I_{21} I_{43} - I_{32} I^+_{41}). \qquad (5)
\end{cases}$$

The first relation in (3) can be viewed as the definition for third generator needed to give the algebra in terms of q-commutators. Dual copy of the algebra $U_q'(\mathrm{so}_3)$ involves the generator $I^-_{31} = [I_{21}, I_{32}]_{q^{-1}}$ and the other two relations similar to (3), but with $q \to q^{-1}$.

In order to describe the basis of $U_q'(\mathrm{so}_n)$ we introduce a lexicographical ordering for the elements $I^+_{k,l}$ of $U_q'(\mathrm{so}_n)$ with respect to their indices, i.e., we suppose that $I^+_{k,l} \prec I^+_{m,n}$ if either $k < m$, or both $k = m$ and $l < n$. We define *an ordered monomial* as the product of non-decreasing sequence of elements $I^+_{k,l}$ with different k, l such that $1 \le l < k \le n$. The following proposition describes the Poincaré–Birkhoff–Witt basis for the algebra $U_q'(\mathrm{so}_n)$.

Proposition. *The set of all ordered monomials is a basis of* $U_q'(\mathrm{so}_n)$.

3 Casimir elements of $U_q'(\mathrm{so}_n)$

As it is well-known, tensor operators of Lie algebras so_n are very useful in construction of invariants of these algebras. With this in mind, let us introduce q-analogues of tensor operators for the algebras $U_q'(\mathrm{so}_n)$ as follows:

$$J^{\pm}_{k_1,k_2,\ldots,k_{2r}} = q^{\mp\frac{r(r-1)}{2}} \sum_{s \in S_{2r}}{}' \varepsilon_{q^{\pm 1}}(s) I^{\pm}_{k_{s(2)},k_{s(1)}} I^{\pm}_{k_{s(4)},k_{s(3)}} \cdots I^{\pm}_{k_{s(2r)},k_{s(2r-1)}}. \tag{6}$$

Here $1 \le k_1 < k_2 < \cdots < k_{2r} \le n$, and the summation runs over all the permutations s of indices $k_1, k_2, \ldots, k_{2r}$ such that

$$k_{s(2)} > k_{s(1)}, \qquad k_{s(4)} > k_{s(3)}, \quad \ldots, \quad k_{s(2r)} > k_{s(2r-1)}, \qquad k_{s(2)} < k_{s(4)} < \cdots < k_{s(2r)}$$

(the last chain of inequalities means that the sum includes only ordered monomials). Symbol $\varepsilon_{q^{\pm 1}}(s) \equiv (-q^{\pm 1})^{\ell(s)}$ stands for a q-analogue of the Levi–Chivita antisymmetric tensor, $\ell(s)$ means the length of permutation s. (If $q \to 1$, both sets in (6) reduce to the set of components of rank $2r$ antisymmetric tensor operator of the Lie algebra so_n.)

Using q-tensor operators given by (6) we obtain the Casimir elements of $U_q'(\mathrm{so}_n)$.

Theorem 1. *The elements*

$$C_n^{(2r)} = \sum_{1\le k_1<k_2<\ldots<k_{2r}\le n} q^{k_1+k_2+\cdots+k_{2r}-r(n+1)} J^+_{k_1,k_2,\ldots,k_{2r}} J^-_{k_1,k_2,\ldots,k_{2r}}, \tag{7}$$

where $r = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$ ($\lfloor x \rfloor$ *means the integer part of* x), *are Casimir elements of* $U_q'(\mathrm{so}_n)$, *i.e., they belong to the center of this algebra.*

In fact, for even n, *not only the product (which constitutes* $C_n^{(n)}$) *of elements* $C_n^{(n)+} \equiv J^+_{1,2,\ldots,n}$ *and* $C_n^{(n)-} \equiv J^-_{1,2,\ldots,n}$ *belongs to the center, but also each of them.*

We conjecture that, in the case of q being not a root of 1, the set of Casimir elements $C_n^{(2r)}$, $r = 1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor$, and the Casimir element $C_n^{(n)+}$ (for even n) generates the center of $U_q'(\mathrm{so}_n)$, i.e., any element of the algebra $U_q'(\mathrm{so}_n)$ which commutes with all other elements can be presented as a polynomial of elements from this set of Casimir elements.

Let us give explicitly some of Casimir elements. For $U_q'(\mathrm{so}_3)$ and $U_q'(\mathrm{so}_4)$ we have

$$C_3^{(2)} = q^{-1} I_{21}^2 + I_{31}^+ I_{31}^- + q I_{32}^2 = q I_{21}^2 + I_{31}^- I_{31}^+ + q^{-1} I_{32}^2,$$
$$C_4^{(2)} = q^{-2} I_{21}^2 + I_{32}^2 + q^2 I_{43}^2 + q^{-1} I_{31}^+ I_{31}^- + q I_{42}^+ I_{42}^- + I_{41}^+ I_{41}^-,$$
$$C_4^{(4)+} = q^{-1} I_{21} I_{43} - I_{31}^+ I_{42}^+ + q I_{32} I_{41}^+ = q I_{21} I_{43} - I_{31}^- I_{42}^- + q^{-1} I_{32} I_{41}^- = C_4^{(4)-}.$$

For $U_q'(\mathrm{so}_5)$ the fourth order Casimir element is

$$C_5^{(4)} = q^{-2} J^+_{1,2,3,4} J^-_{1,2,3,4} + q^{-1} J^+_{1,2,3,5} J^-_{1,2,3,5}$$
$$+ J^+_{1,2,4,5} J^-_{1,2,4,5} + q J^+_{1,3,4,5} J^-_{1,3,4,5} + q^2 J^+_{2,3,4,5} J^-_{2,3,4,5},$$

where $J^+_{i,j,k,l} = q^{-1} I^+_{ji} I^+_{lk} - I^+_{ki} I^+_{lj} + q I^+_{kj} I^+_{li}$ and $J^-_{i,j,k,l} = q I^-_{ji} I^-_{lk} - I^-_{ki} I^-_{lj} + q^{-1} I^-_{kj} I^-_{li}$. For $U_q'(\mathrm{so}_6)$, we present only the highest order Casimir element:

$$C_6^{(6)+} = q^{-3} I_{21} I_{43} I_{65} - q^{-2} I_{31}^+ I_{42}^+ I_{65} + q^{-1} I_{32} I_{41}^+ I_{65} - q^{-2} I_{21} I_{53}^+ I_{64}^+ + q^{-1} I_{31}^+ I_{52}^+ I_{64}^+$$
$$- I_{32} I_{51}^+ I_{64}^+ + q^{-1} I_{21} I_{54} I_{63}^+ - I_{41}^+ I_{52}^+ I_{63}^+ + q I_{42}^+ I_{51}^+ I_{63}^+ - I_{31}^+ I_{54} I_{62}^+ + I_{41}^+ I_{53}^+ I_{62}^+$$
$$- q^2 I_{43} I_{51}^+ I_{62}^+ + q I_{32} I_{54} I_{61}^+ - q^2 I_{42}^+ I_{53}^+ I_{61}^+ + q^3 I_{43} I_{52}^+ I_{61}^+.$$

Finally, let us give explicitly the quadratic Casimir element of $U_q'(\mathrm{so}_n)$,

$$C_n^{(2)} = \sum_{1\le i<j\le n} q^{i+j-n-1} I_{ji}^+ I_{ji}^-.$$

This formula coincides with that given in [4], and is a particular case of (7).

4 Irreducible representations of $U_q'(\mathrm{so}_n)$

Let us give a brief description of irreducible representations (irreps) of $U_q'(\mathrm{so}_n)$. More detailed description of these irreps can be found in [1, 3].

As in the case of Lie algebra so_n, finite-dimensional irreps T of the algebra $U_q'(\mathrm{so}_n)$ are characterized by the set $\mathbf{m}_n \equiv (m_{1,n}, m_{2,n}, \ldots, m_{\lfloor \frac{n}{2} \rfloor,n})$ (here $\lfloor x \rfloor$ means the integer part of x) of numbers, which are either all integers or all half-integers, and satisfy the well-known dominance conditions

$$m_{1,n} \geq m_{2,n} \geq \cdots \geq m_{\frac{n}{2}-1,n} \geq |m_{\frac{n}{2},n}| \qquad \text{if } n \text{ is even},$$

$$m_{1,n} \geq m_{2,n} \geq \cdots \geq m_{\frac{n-1}{2},n} \geq 0 \qquad \text{if } n \text{ is odd}.$$

To give the representations in Gel'fand–Tsetlin basis we denote, as in the case of Lie algebra so_n, the basis vectors $|\alpha\rangle$ of representation spaces by Gel'fand–Tsetlin patterns α. The representation operators $T_{\mathbf{m}_n}(I_{2p+1,2p})$ and $T_{\mathbf{m}_n}(I_{2p,2p-1})$ act on $|\alpha\rangle$ by the formulae

$$T_{\mathbf{m}_n}(I_{2p+1,2p})|\alpha\rangle = \sum_{r=1}^{p}\Big(A^r_{2p}(\alpha)|m^{+r}_{2p}\rangle - A^r_{2p}(m^{-r}_{2p})|m^{-r}_{2p}\rangle\Big),$$

$$T_{\mathbf{m}_n}(I_{2p,2p-1})|\alpha\rangle = \sum_{r=1}^{p-1}\Big(B^r_{2p-1}(\alpha)|m^{+r}_{2p-1}\rangle - B^r_{2p-1}(m^{-r}_{2p-1})|m^{-r}_{2p-1}\rangle\Big) + \mathrm{i}C_{2p-1}|\alpha\rangle.$$

Here the matrix elements A^r_{2p},B^r_{2p-1}, C_{2p-1} are obtained from the classical (non-deformed) ones by replacing each factor (x) with its respective q-number $[x] \equiv (q^x - q^{-x})/(q - q^{-1})$; besides, the coefficient $\frac{1}{2}$ in the 'classical' A^r_{2p} is replaced with the $l_{r,2p}$-dependent expression $(([l_{r,2p}][l_{r,2p}+1])/([2l_{r,2p}][2l_{r,2p}+2]))^{1/2}$, where $l_{r,2p} = m_{r,2p} + p - r$.

5 Casimir operators and their eigenvalues

The Casimir operators (the operators which correspond to the Casimir elements), within irreducible finite-dimensional representations of $U'_q(\mathrm{so}_n)$ take diagonal form. To give them explicitly, we employ the so-called *generalized factorial elementary symmetric polynomials* (see [9]). Fix an arbitrary sequence of complex numbers $\mathbf{a} = (a_1, a_2, \ldots)$. Then, for each $r = 0, 1, 2, \ldots, N$, introduce the polynomials of N variables z_1, $z_2, \ldots,$ z_N as follows:

$$e_r(z_1, z_2, \ldots, z_N|\mathbf{a}) = \sum_{1\leq p_1<p_2<\cdots<p_r\leq N}(z_{p_1} - a_{p_1})(z_{p_2} - a_{p_2-1})\ldots(z_{p_r} - a_{p_r-r+1}). \qquad (8)$$

The Casimir operators in the irreducible finite-dimensional representations characterized by the set $(m_{1,n},\ m_{2,n}, \ldots,\ m_{N,n})$, $N = \lfloor \frac{n}{2} \rfloor$, by the Schur Lemma, are presentable as (here $\mathbf{1}$ denotes the unit operator):

$$T_{\mathbf{m}_n}(C^{(2r)}_n) = \chi^{(2r)}_{\mathbf{m}_n}\mathbf{1}.$$

Theorem 2. *The eigenvalue of the operator* $T_{\mathbf{m}_n}\left(C^{(2r)}_n\right)$ *is*

$$\chi^{(2r)}_{\mathbf{m}_n} = (-1)^r e_r([l_{1,n}]^2, [l_{2,n}]^2, \ldots, [l_{N,n}]^2|\mathbf{a})$$

where $\mathbf{a} = ([\epsilon]^2, [\epsilon+1]^2, [\epsilon+2]^2, \ldots)$, $l_{k,n} = m_{k,n} + N - k + \epsilon$. *Here* $\epsilon = 0$ *for* $n = 2N$ *and* $\epsilon = \frac{1}{2}$ *for* $n = 2N+1$.

In the case of even n, *i.e.,* $n = 2N$,

$$T_{\mathbf{m}_n}\left(C^{(n)+}_n\right) = T_{\mathbf{m}_n}\left(C^{(n)-}_n\right) = \left(\sqrt{-1}\right)^N [l_{1,n}][l_{2,n}]\ldots[l_{N,n}]\mathbf{1}.$$

The eigenvalues of Casimir operators are important for physical applications. Let us quote some of Casimir operators together with their eigenvalues. For $U'_q(\mathrm{so}_3)$,

$$T_{(m_{13})}\left(C^{(2)}_3\right) = -[m_{13}][m_{13}+1]\mathbf{1}.$$

For $U'_q(\mathrm{so}_4)$ we have

$$T_{(m_{14},m_{24})}\left(C_4^{(2)}\right) = -\left([m_{14}+1]^2 + [m_{24}]^2 - 1\right)\mathbf{1},$$

$$T_{(m_{14},m_{24})}\left(C_4^{(4)+}\right) = T_{(m_{14},m_{24})}\left(C_4^{(4)-}\right) = -[m_{14}+1][m_{24}]\mathbf{1}.$$

Finally, for $U'_q(\mathrm{so}_5)$ the Casimir operators are

$$T_{(m_{15},m_{25})}\left(C_5^{(2)}\right) = -\left([m_{15}+3/2]^2 + [m_{25}+1/2]^2 - [1/2]^2 - [3/2]^2\right)\mathbf{1},$$

$$T_{(m_{15},m_{25})}\left(C_5^{(4)}\right) = \left([m_{15}+3/2]^2 - [1/2]^2\right)\left([m_{25}+1/2]^2 - [1/2]^2\right)\mathbf{1}.$$

6 Concluding remarks

In this note, for the nonstandard q-algebras $U'_q(\mathrm{so}_n)$ we have presented explicit formulae for all the Casimir operators corresponding to basis set of Casimirs of so_n. Their eigenvalues in irreducible finite-dimensional representations are also given. We believe that the described Casimir elements generate the whole center of the algebra $U'_q(\mathrm{so}_n)$ (of course, for q being not a root of unity).

As mentioned, the algebras $U'_q(\mathrm{so}_n)$ for $n > 4$ are of importance in the *construction of algebra of observables for* 2+1 *quantum gravity* (with 2D space of genus $g > 1$) serving as certain intermediate algebras. For that reason, the results concerning Casimir operators and their eigenvalues will be useful in the process of construction of the desired algebra of independent quantum observables for the case of higher genus surfaces, in the important and interesting case of anti-De Sitter gravity (corresponding to negative cosmological constant).

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