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# Use of Quantum Algebras in Quantum Gravity

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After brief survey of appearance of quantum algebras in diverse contexts of quantum gravity, we demonstrate that the particular deformed algebras, which arise within the approach of J. Nelson and T. Regge to $(2+1)$ anti-de Sitter quantum gravity (for space surface of genus $g$) and which should generate algebras of independent quantum observables, are in fact isomorphic to nonstandard $q$-deformed analogues $U'_q(so_n)$ (introduced in 1991) of Lie algebras of the orthogonal groups $SO(n)$, $n$ being related to $g$ as $n = 2g + 2$.

## 1 Introduction

Quantum or $q$-deformed algebras may appear in quantum (or $q$-versions of) gravity in various situations. Let us mention some of them.

• Case of $n$ spacetime dimensions ($n \geq 2$), straightforward approach to construct $q$-gravity (this is accomplished, e.g., in [1]). Basic steps are:

– Start with some version of quantum/$q$-deformed algebra $iso_q(n)$ (in [1] it is projected out from the standard quantum algebras $U_q(B_r)$, $U_q(D_r)$ of Drinfeld and Jimbo [2]). In the particular Poincare algebra $iso_q(3,1)$ exploited by Castellani, only those commutation relations which involve momenta do depend on the parameter $q$, while the Lorentz subalgebra remains non-deformed;

– Develop necessary bicovariant differential calculus;

– A $q$-gravity is constructed by "gauging" the $q$-analogue of Poincaré algebra. The resulting Lagrangian turns out to be a generalization [1] (see also [3]) of the usual Einstein or Einstein–Cartan one.

It is worth to emphasize that in this approach the obtained results, including physical implications, unambiguously depend on the specific features of chosen the $q$-algebra.

• Two-dimensional quantum Liouville gravity [4], within particular framework of quantization, leads to the appearance [5] of quantum algebras such as $U_q(sl(2,\mathbf{C}))$.

• Case of 3-dimensional (Euclidean) gravity. The simpl approach developed by Ponzano and Regge [6] employs irreducible representations of the algebra $su(2)$ labelled by spins $j$ and assigned to edges of tetrahedra in triangulation, the main ingredient being $6j$-symbols of $su(2)$. Within natural generalization of this approach by Turaev and Viro [7], see also [8], the underlying symmetry of the action (which can be related to Chern–Simons theory) is that of the quantum algebra $su_q(2)$, and basic objects are $q - 6j$ symbols. Due to this, physical quantities become expressible through topological (knot or link) invariants. The parameter $q$ takes into account the cosmological constant and, on the other hand, is connected with the (quantized) Chern–Simons coupling constant $k$ as $q = \exp \frac{2i\pi}{k+2}$.

• $(2+1)$-dimensional gravity with or without cosmological constant $\Lambda$ is known to possess important peculiar features [9, 10]. Within the approach to quantization developed by J. Nelson and T. Regge, specific deformed algebras arise [11, 12] for the situation with $\Lambda < 0$, and just this fact will be of our main concern here.

## 2 Nonstandard $q$-deformed algebras $U'_q(\mathrm{so}_n)$, their advantages

As defined in [13], the nonstandard $q$-deformation $U'_q(\mathrm{so}_n)$ of the Lie algebra $\mathrm{so}_n$ is given as a complex associative algebra with $n-1$ generating elements $I_{21}$, $I_{32}, \ldots$, $I_{n,n-1}$ obeying the defining relations (denote $q+q^{-1} \equiv [2]_q$)

$$\begin{aligned} &I^2_{j,j-1} I_{j-1,j-2} + I_{j-1,j-2} I^2_{j,j-1} - [2]_q \; I_{j,j-1} I_{j-1,j-2} I_{j,j-1} = -I_{j-1,j-2}, \\ &I^2_{j-1,j-2} I_{j,j-1} + I_{j,j-1} I^2_{j-1,j-2} - [2]_q \; I_{j-1,j-2} I_{j,j-1} I_{j-1,j-2} = -I_{j,j-1}, \\ &[I_{i,i-1}, I_{j,j-1}] = 0 \qquad \text{if} \quad | \, i-j \, | > 1. \end{aligned} \tag{1}$$

At $q \to 1$, $[2]_q \to 2$ (non-deformed or classical limit), these go over into the defining relations of the $so(n)$ Lie algebras.

Among the *advantages of these nonstandard $q$-deformed algebras* with regards to the Drinfeld–Jimbo quantum deformations, the following should be pointed out.

(**i**) Existence of the canonical chain of embedded subalgebras (from now on, we omit the prime in the symbol)

$$U_q(so_n) \supset U_q(so_{n-1}) \supset \cdots \supset U_q(so_4) \supset U_q(so_3)$$

in the case of $U_q(so_n)$ and, due to this, implementability of the $q$-analogue of Gelfand–Tsetlin formalism enabling one to construct finite dimensional representations [13, 14].

(**ii**) Existence, for all the real forms known in the nondeformed case $q=1$, of their respective $q$-analogues – the "compact" $U_q(so_n)$ and the "noncompact" $U_q(so_{p,s})$ (with $p+s=n$) real forms. Moreover, each such form exists along with the corresponding chain of embeddings. For instance, in the $n$-dimensional $q$-Lorentz case we have

$$U_q(so_{n-1,1}) \supset U_q(so_{n-1}) \supset U_q(so_{n-2}) \supset \cdots \supset U_q(so_3).$$

This fact enables one to develop the construction and analysis of infinite-dimensional representations of $U_q(so_{n-1,1})$, see [13, 15].

(**iii**) Existence of embedding $U_q(so_3) \subset U_q(sl_3)$ generalizable [16] to the embedding of higher $q$-algebras such that $U_q(so_n) \subset U_q(sl_n)$, – the fact which enables construction of the proper quantum analogue [16] of symmetric coset space $SL(n)/SO(n)$.

(**iv**) If one attempts to get a $q$-analogue of the Capelli identity known to hold for the dual pair $sl_2 \leftrightarrow so_n$, nothing but this nonstandard $q$-algebra $U_q(so_n)$ inevitably arises [17]. As a result, the relation Casimir$\{U_q(sl_2)\}$ = Casimir$\{U_q(so_n)\}$ is valid [17, 18] within particular representation.

(**v**) Natural appearance, as will be discussed in Sec.4, of these $q$-algebras within the Nelson–Regge approach to $2+1$ quantum gravity.

As a drawback let us mention the fact that Hopf algebra structure is not known for $U_q(so_n)$, although for the situation (**iii**) the nonstandard $q$-algebra $U_q(so_n)$ was shown to be a coideal [16] in the Hopf algebra $U_q(sl_n)$.

Recall that it was (**i**), (**ii**) which motivated introducing in [13] this class of $q$-algebras.

## 3 Bilinear formulation of $U_q(so_n)$

Along with the definition in terms of trilinear relations (1) above, a *'bilinear' formulation* of $U_q(so_n)$ can as well be provided. To this end, one introduces the generators (set $k > l+1$, $1 \le k,l \le n$)

$$I^{\pm}_{k,l} \equiv [I_{l+1,l}, I^{\pm}_{k,l+1}]_{q^{\pm 1}} \equiv q^{\pm 1/2} I_{l+1,l} I^{\pm}_{k,l+1} - q^{\mp 1/2} I^{\pm}_{k,l+1} I_{l+1,l}$$

together with $I_{k+1,k} \equiv I^+_{k+1,k} \equiv I^-_{k+1,k}$. Then (1) imply

$$\begin{aligned}
&[I^+_{lm}, I^+_{kl}]_q = I^+_{km}, \qquad [I^+_{kl}, I^+_{km}]_q = I^+_{lm}, \qquad [I^+_{km}, I^+_{lm}]_q = I^+_{kl} \qquad \text{if} \quad k > l > m,\\
&[I^+_{kl}, I^+_{mp}] = 0 \qquad \text{if} \quad k > l > m > p \quad \text{or if} \quad k > m > p > l;\\
&[I^+_{kl}, I^+_{mp}] = (q - q^{-1})(I^+_{lp} I^+_{km} - I^+_{kp} I^+_{ml}) \qquad \text{if} \quad k > m > l > p.
\end{aligned} \tag{2}$$

Analogous set of relations exists which involves $I^-_{kl}$ along with $q \to q^{-1}$ (denote this "dual" set by ($2'$)). In the 'classical' limit $q \to 1$ , both (2) and ($2'$) reduce to those of $\mathrm{so}_n$.

To illustrate, we give the examples of $n = 3$, isomorphic to Fairlie–Odesskii algebra [19], and $n = 4$ (recall that the $q$-commutator is defined as $[X, Y]_q \equiv q^{1/2} XY - q^{-1/2} YX$):

$$U_q(\mathrm{so}_3): \quad [I_{21}, I_{32}]_q = I^+_{31}, \qquad [I_{32}, I^+_{31}]_q = I_{21}, \qquad [I^+_{31}, I_{21}]_q = I_{32}. \tag{3}$$

$$U_q(\mathrm{so}_4): \begin{cases}
[I_{32}, I_{43}]_q = I^+_{42}, \quad [I^+_{31}, I_{43}]_q = I^+_{41}, \quad [I_{21}, I^+_{42}]_q = I^+_{41},\\
[I_{43}, I^+_{42}]_q = I_{32}, \quad [I_{43}, I^+_{41}]_q = I^+_{31}, \quad [I^+_{42}, I^+_{41}]_q = I_{21},\\
[I^+_{42}, I_{32}]_q = I_{43}, \quad [I^+_{41}, I^+_{31}]_q = I_{43}, \quad [I^+_{41}, I_{21}]_q = I^+_{42},
\end{cases} \tag{4}$$

$$[I_{43}, I_{21}] = 0, \quad [I_{32}, I^+_{41}] = 0, \quad [I^+_{42}, I^+_{31}] = (q - q^{-1})(I_{21} I_{43} - I_{32} I^+_{41}). \tag{5}$$

The first relation in (3) is viewed as definition for the third generator $I^+_{31}$; with this, the algebra is given in terms of $q$-commutators. Dual copy of $U_q(\mathrm{so}_3)$ involves the generator $I^-_{31} = [I_{21}, I_{32}]_{q^{-1}}$ which enters the relations same as (3), but with $q \to q^{-1}$. Similar remarks concern the generators $I^+_{42}$, $I^+_{41}$, as well as (dual copy of) the whole algebra $U_q(\mathrm{so}_4)$.

## 4 The deformed algebras $A(n)$ of Nelson and Regge

For $(2+1)$-dimensional gravity with cosmological constant $\Lambda < 0$, the Lagrangian involves spin connection $\omega_{ab}$ and dreibein $e^a$, $a, b = 0, 1, 2$, combined in the $SO(2, 2)$-valued (anti-de Sitter) spin connection $\omega_{AB}$ of the form

$$\omega_{AB} = \begin{pmatrix} \omega_{ab} & \frac{1}{\alpha} e^a \\ -\frac{1}{\alpha} e^b & 0 \end{pmatrix},$$

and is given in the Chern–Simons (CS) form [10]

$$\frac{\alpha}{8} \left( \mathrm{d}\omega^{AB} - \frac{2}{3} \omega^A_{\ F} \wedge \omega^{FB} \right) \wedge \omega^{CD} \epsilon_{ABCD}.$$

Here $A, B = 0, 1, 2, 3$, the metric is $\eta_{AB} = (-1, 1, 1, -1)$, and the CS coupling constant is connected with $\Lambda$, so that $\Lambda = -\frac{1}{3\alpha^2}$. The action is invariant under $SO(2, 2)$, leads to Poisson brackets and field equations. Their solutions (infinitesimal connections) describe space-time which is locally anti-de Sitter.

To describe global features of space-time, of principal importance are the *integrated connections* which provide a mapping $S : \pi_1(\Sigma) \to G$ of the homotopy group for a surface $\Sigma$ into the group $G = SL_+(2, R) \otimes SL_-(2, R)$ (spinorial covering of $SO(2, 2)$) and thoroughly studied in [11]. To generate the algebra of observables, one takes the traces

$$c^\pm(a) = c^\pm(a^{-1}) = \frac{1}{2} \mathrm{tr}[S^\pm(a)], \qquad a \in \pi_1, \qquad S^\pm \in SL_\pm(2, R).$$

For $g = 1$ (torus) surface $\Sigma$, the algebra of (independent) quantum observables was derived [11], which turned out to be isomorphic to the cyclically symmetric Fairlie–Odesskii algebra [19]. This latter algebra, however, is known to coincide [15] with the special $n = 3$ case of $U_q(so_n)$. So, natural question arises whether for surfaces of higher genera $g \geq 2$, the nonstandard $q$-algebras $U_q(so_n)$ also play a role.

Below, the positive answer to this question is given.

For the topology of spacetime $\Sigma \times \mathbf{R}$ (fixed-time formulation; $\Sigma$ is genus-$g$ surface), the homotopy group $\pi_1(\Sigma)$ is most efficiently described in terms of $2g + 2 = n$ generators $t_1, t_2, \ldots, t_{2g+2}$ introduced in [12] and such that

$$t_1 t_3 \cdots t_{2g+1} = 1, \qquad t_2 t_4, ..., t_{2g+2} = 1, \qquad \text{and} \qquad \prod_{i=1}^{2g+2} t_i = 1.$$

Classical gauge invariant trace elements ($n(n-1)/2$ in total) defined as

$$\alpha_{ij} = \frac{1}{2}\mathrm{Tr}(S(t_i t_{i+1} \cdots t_{j-1})), \qquad S \in SL(2, R), \tag{6}$$

generate concrete algebra with Poisson brackets, explicitly found in [12]. At the quantum level, to the algebra with generators (6) there corresponds quantum commutator algebra $A(n)$ specific for $2+1$ quantum gravity with negative $\Lambda$. For each quadruple of indices $\{j, l, k, m\}$, $j, l, k, m = 1, \ldots, n$, obeying (see [12]) 'anticlockwise ordering'

$$\begin{tikzcd} & j \arrow[dl] & \\ l \arrow[dr] & & m \arrow[ul] \\ & k \arrow[ur] & \end{tikzcd} \tag{7}$$

the quantum algebra $A(n)$ reads [12]:

$$\begin{aligned}
&[a_{mk}, a_{jl}] = [a_{mj}, a_{kl}] = 0, \\
&[a_{jk}, a_{kl}] = \left(1 - \frac{1}{K}\right)(a_{jl} - a_{kl}a_{jk}), \\
&[a_{jk}, a_{km}] = \left(\frac{1}{K} - 1\right)(a_{jm} - a_{jk}a_{km}), \\
&[a_{jk}, a_{lm}] = \left(K - \frac{1}{K}\right)(a_{jl}a_{km} - a_{kl}a_{jm}).
\end{aligned} \tag{8}$$

Here the parameter $K$ of deformation involves both $\alpha$ and Planck's constant, namely

$$K = \frac{4\alpha - ih}{4\alpha + ih}, \qquad \alpha^2 = -\frac{1}{3\Lambda}, \qquad \Lambda < 0. \tag{9}$$

Note that in (6) only one copy of the two $SL_{\pm}(2, R)$ is indicated. In conjunction with this, besides the deformed algebra $A(n)$ derived with, say, $SL_+(2, R)$ taken in (6) and given by (8), another identical copy of $A(n)$ (with the only replacement $K \to K^{-1}$) can also be obtained starting from $SL_-(2, R)$ taken in place of $SL(2, R)$ in (6). This another copy is independent from the original one: their generators mutually commute.

# 5 Isomorphism of the algebras $A(n)$ and $U_q(so_n)$

To establish isomorphism between the algebra $A(n)$ from (8) and the nonstandard $q$-deformed algebra $U_q(so_n)$ one has to make the following two steps.

$$\begin{aligned} &\text{Redefine:} \qquad \left\{K^{1/2}(K-1)^{-1}\right\} a_{ik} \longrightarrow A_{ik}, \\ &\text{Identify:} \qquad A_{ik} \longrightarrow I_{ik}, \qquad K \longrightarrow q. \end{aligned}$$

Then, the Nelson–Regge algebra $A(n)$ is seen to translate exactly into the nonstandard $q$-deformed algebra $U'_q(\mathrm{so}_n)$ described above, see (2). We conclude that these two deformed algebras are isomorphic to each other (of course, for $K \neq 1$). Recall that $n$ is linked to the genus $g$ as $n = 2g + 2$, while $K = (4\alpha - ih)/(4\alpha + ih)$ with $\alpha^2 = -\frac{1}{\Lambda}$.

Let us remark that it is the bilinear presentation (2) of the $q$-algebra $U_q(so_n)$ which makes possible establishing of this isomorphism. It should be stressed also that the algebra $A(n)$ plays the role of "intermediate" one: starting with it and reducing it appropriately, the algebra of quantum observables (gauge invariant global characteristics) is to be finally constructed. The role of Casimir operators in this process, as seen in [12], is of great importance. In this respect let us mention that the quadratic and higher Casimir elements of the $q$-algebra $U_q(so_n)$, for $q$ being not a root of 1, are known in explicit form [18, 20] along with eigenvalues of their corresponding (representation) operators [20].

As shown in detail in [11], the deformed algebra for the case of genus $g = 1$ surfaces (tori) reduces to the desired algebra of three independent quantum observables which coincides with $A(3)$, the latter being isomorphic to the Fairlie–Odesskii algebra $U_q(so_3)$. The case of $g = 2$ is significantly more involved: here one has to derive, starting with the 15-generator algebra $A(6)$, the necessary algebra of 6 (independent) quantum observables. J. Nelson and T. Regge have succeeded [21] in constructing such an algebra. Their construction however is highly nonunique and, what is more essential, is not seen to be extendable to general situation of $g \geq 3$.

# 6 Outlook

Our goal in this note was to attract attention to the isomorphism of the deformed algebras $A(n)$ from [12] and the nonstandard $q$-deformed algebras $U'_q(\mathrm{so}_n)$ introduced in [13]). The hope is that, taking into account a significant amount of the already existing results concerning diverse aspects of $U'_q(\mathrm{so}_n)$ (the obtained various classes of irreducible representations, knowledge of Casimir operators and their eigenvalues depending on representations, etc.) we may expect for a further progress concerning construction of the desired algebras of quantum observables for space surfaces of genera $g > 2$.

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