Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 2, 298–303.

Some Problems of Quantum Group Gauge Field Theory

I.M. BURBAN

Bogolyubov Institute for Theoretical Physics of the National Academy of Sciences of Ukraine, 14b Metrologichna Str., Kyiv 143, Ukraine
E-mail: mmtpitp@gluk.apc.org

Some difficulties of the construction of quantum group gauge field theory on the classical and quantum spacetime are clarified. The classical geometric interpretation of the ghost field is generalized to case of the quantum group gauge field theory.

1 Introduction

The notation of a generalized Lie group as a noncommutative and noncocommutative Hopf algebra was done by Drinfeld [1], Jimbo [2], Woronowicz [3]. The first step in the construction of noncommutative dynamics was undertaken by I.G. Biedenharn [4] and McFarlane [5] in their study of the quantum noncommutative harmonic oscillator. From this period attempts were undertaken to construct deformed dynamical theories [6, 7, 8, 9], in particular, the deformed gauge theory named the quantum group gauge field theory with the quantum group playing the role of the gauge group. The conceptual problems concerning of the definition of the gauge field theory where the quantum group is considered as object of the gauge symmetry, i.e. the quantum group gauge field theory are not settled. Such theories investigated by Bernard [10], Aref'eva and Volovich [11], Hietarinta [12], Isaev and Popowicz [13], Bernard [10], Watamura [15], Brzezinski, Majid [16, 17], Hajac [18], Sudbery [19]. The deformed gauge field theory is interesting from various points of view. The enlargement of the rigid frameworks of the gauge theory would help to solve the fundamental theoretical problems of the spontaneos symmetry breaking and the quark confinement. In particular, in the quantized deformed gravity theory the spacetime becomes noncommutative and could possible provide the regularization mechanism. In the quantized gauge theory the deformation could be interpreted as a kind of the symmetry breaking, which does not reduce the symmetry but deforms it. This mechanism could give the masses to some vector bosons without the necessity to consider Higgs fields. There are two approaches in the construction of q-deformed dynamical field theory. The spacetime in one of them is assumed to be the usual manifold and it deforms only the structure of the dynamical variables. In the second approach the spacetime becomes the quantum (noncommutative) manifold.

2 The quantum group gauge field theory on the classical spaces

2.1. The classical gauge field theory. Let T^a, $a = 1, 2, \ldots, N$ be generators of the Lie algebra of some compact Lie group G satisfying the relations

$$[T_a, T_b] = f^{abc}T^c, \tag{1}$$

where f^{abc} are structure constants of this algebra. The basic objects of nonabelian gauge theory are the gauge fields – the Yang–Mills potentials. These are the set of vector fields $A^a_\mu(x)$, $a = 1, 2, \dots, N$, $\mu = 0, 1, 2, 3$. The matrix gauge potentials

$$A_\mu(x) = T^a A^a_\mu(x) \tag{2}$$

define the matrix strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu], \tag{3}$$

or in component form

$$F_{\mu\nu} = T^a F^a_{\mu\nu}, \qquad F_{\mu\nu} = \partial_\mu A^a_\nu - \partial^a_\nu A_\mu + f^{abc} A^b_\mu A^c_\nu. \tag{4}$$

The Lagrangian density of the theory

$$\mathcal{L} = -1/4 F^a_{\mu\nu} F^{a\mu\nu} \tag{5}$$

is invariant under the gauge transformation

$$A_\mu \to g(x)^{-1} A_\mu g(x) + g(x)^{-1} \partial_\mu g(x), \tag{6}$$

where $g(x) = \exp\{\varepsilon^a(x) T^a\}$, and $\varepsilon^a(x)$ are real functions. The transformation (6) can be written in the infinitesimal form as

$$\delta A^a_\mu = f^{abc} \varepsilon^b(x) A^c_\mu(x) - \partial_\mu \varepsilon^a(x), \qquad \delta F^a(x) = f^{abc} \varepsilon^b F^c(x). \tag{7}$$

In the following we shall review the several approaches in the construction of the q-deformed gauge field theory.

2.2. The construction based on the differential extension of the quantum group G_q [13, 11]. The main efforts in this approach were directed to keep the classical form of the gauge transformation for the gauge potentials. The problem is in the following. Let A be an element of some extension of the quantum group G_q. What differential calculus on this group should be considered and from what extension of this group should be taken the potential A to guarantee that the gauge transformed element A' also belongs to that extension. In some cases this problem were solved [13, 11].

2.3. The construction based on the bicovariant differential calculus on the quantum group [14]. There are many methods to deform a Lie algebra. The one of them is the method of the bicovariant differential calculus on the quantum groups.

Definition 2.1. *A bicovariant bimodule over Hopf algebra $\mathcal{A}$ is a triplet* $(\Gamma, \Delta_L, \Delta_R)$ *bimodule* Γ *over $\mathcal{A}$ and of linear mappings*

$$\Delta_L : \Gamma \to \mathcal{A} \otimes \Gamma, \qquad \Delta_R : \Gamma \to \Gamma \otimes \mathcal{A}$$

such that diagrams

1. $$\begin{array}{ccc} \Gamma & \xrightarrow{\Delta_L} & \mathcal{A} \otimes \Gamma \\ \Delta_L \downarrow & & \downarrow 1 \otimes \Delta_L \\ \mathcal{A} \otimes \Gamma & \xrightarrow{\Delta \otimes 1} & \mathcal{A} \otimes \mathcal{A} \otimes \Gamma \end{array}, \qquad \begin{array}{ccc} \Gamma & \xrightarrow{\Delta_L} & \mathcal{A} \otimes \Gamma \\ & \searrow & \downarrow \varepsilon \otimes id \\ & & k \otimes \Gamma \end{array}$$

2. $$\begin{array}{ccc} \Gamma & \xrightarrow{\Delta_R} & \Gamma \otimes \mathcal{A} \\ \Delta_R \downarrow & & \downarrow 1 \otimes \Delta_R \\ \Gamma \otimes \mathcal{A} & \xrightarrow{id \otimes \Delta} & \Gamma \otimes \mathcal{A} \otimes \mathcal{A} \end{array}, \qquad \begin{array}{ccc} \Gamma & \xrightarrow{\Delta_R} & \Gamma \otimes \mathcal{A} \\ & \searrow & \downarrow id \otimes \varepsilon \\ & & \Gamma \otimes k \end{array}$$

$$
3. \quad \begin{array}{ccc}
\Gamma & \xrightarrow{\Delta_R} & \Gamma \otimes \mathcal{A} \\
\Delta_L \downarrow & & \downarrow \Delta_L \otimes id \\
\mathcal{A} \otimes \Gamma & \xrightarrow{id \otimes \Delta_R} & \mathcal{A} \otimes \Gamma \otimes \mathcal{A}
\end{array}
$$

commute and

$$
4. \quad \Delta_L(a\omega b) = \Delta_L(a)\Delta\omega\Delta_L(b), \qquad \Delta_R(a\omega b) = \Delta_R(a)\Delta\omega\Delta_R(b).
$$

Definition 2.2. *A first order differential calculus over Hopf algebra $\mathcal{A}$ is a pair (Γ, d), where Γ is bimodule over $\mathcal{A}$, and the liner mapping $d : \mathcal{A} \to \Gamma$ such that $d(ab) = dab + adb$ and $\Gamma = \{adb : a, b \in \mathcal{A}\}$.*

Definition 2.3. *A first order differential calculus is called bicovariant differential calculus on the quantum group if $(\Gamma, \Delta_L, \Delta_R)$ is a bicovariant bimodule and $\Delta_L(da) = (id \otimes d)\Delta(a)$, $\Delta_R(da) = (d \otimes id)\Delta(a)$.*

Definition 2.4. *A first order differential calculus is called universal if $\Gamma = \ker m$, $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is multiplication map in algebra $\mathcal{A}$, and $d = 1 \otimes a - a \otimes 1$.*

It easy to see that $d : \mathcal{A} \to \Gamma$ is linear map satisfying the Leibnitz rule, Γ has the bimodule structure $c\left(\sum\limits_k a_k \otimes b_k\right) = \sum\limits_k ca_k \otimes b_k$, $\left(\sum\limits_k a_k \otimes b_k\right) c = \sum\limits_k a_k \otimes b_k c$ and every element of Γ has the form $\sum\limits_k a_k db_k$.

Definition 2.5. *A Z_2-graded complex differential algebra (Ω, d) is Z_2-graded complex algebra $\Omega = \Omega^+ + \Omega^-$,equipped with the graded derivation d which is odd and of square zero $d\Omega^{+,-} \subset \Omega^{-,+}$, $d^2 = 0$.*

Every first order differential calculus (Γ, d) generate Z_2-graded complex differential algebra $(\Omega(\mathcal{A}), d)$. A complex differential algebra generated by universal calculus is called *a differential envelop* of $\mathcal{A}$ and is denoted as $(\Omega\mathcal{A}, d)$. The space dual to the left-invariant subspace Γ_{inv} can be introduced as a linear subspace of A' whose basis elements $\chi_i \in A'$ are defined by

$$
da = \chi_i * a\omega^i \qquad \text{for all } a \in \mathcal{A}. \tag{8}
$$

The analogue of the ordinary permutation operator is a bimodule automorphism Λ in $\Gamma \otimes \Gamma$ defined by

$$
\Lambda\left(\omega^i \otimes \eta^j\right) = \eta^j \otimes \omega^i, \tag{9}
$$

i.e. $\Lambda(a\tau) = a\Lambda(\tau)$, $\Lambda(\tau b) = \Lambda(\tau)b$, where $a \in \mathcal{A}$, $\tau \in \Gamma \otimes \Gamma$. With the help of braiding operator Λ the exterior product of the elements $\rho, \rho' \in \Gamma$ is given

$$
\rho \bigwedge \rho' = \rho \otimes \rho' - \Lambda(\rho \otimes \rho'), \tag{10}
$$

$$
\omega^i \bigwedge \omega^j = \omega^i \otimes \omega^j - \Lambda^{ij}_{kl}\left(\omega^k \otimes \omega^l\right). \tag{11}
$$

The exterior product of two left invariant forms satisfies the relation

$$
\omega^i \bigwedge \omega^j = \frac{1}{q^2 + q^{-2}}\left[\Lambda^{ij}_{kl} + \left(\Lambda^{-1}\right)^{ij}_{kl}\right] \omega^k \bigwedge \omega^l. \tag{12}
$$

There exists an *adjoint representation* M^i_j of the quantum group defined by the right action on the left invariant ω^i

$$\Delta_R\left(\omega^i\right) = \omega^i \otimes M^i_j, \qquad M^i_j \in \mathcal{A}. \tag{13}$$

The bicovariant calculus on a G_q is characterized by the functionals χ_i and f^i_j on A satisfying

$$\chi_i\chi_j - \Lambda^{kl}_{ij}\chi_k\chi_l = \mathbf{C}^k_{ij}\chi_k, \tag{14}$$

$$\Lambda^{nm}_{ij} f^i_p f^i_q = f^n_i f^m_j \Lambda^{ij}_{pq}, \tag{15}$$

$$\mathbf{C}^i_{mn} f^m_j f^n_k = \Lambda^{pq}_{jk}\chi_p f^i_q + \mathbf{C}^l_{jk} f^i_p, \tag{16}$$

$$\chi^n_k f^n_l = \Lambda^{ij}_{kl} f^n_i \chi_j, \tag{17}$$

where $\Lambda^{ij}_{kl} = f^i_l(M^j_k)$, $\mathbf{C}^i_{jk} = \chi_k(M^i_j)$. In adjoint representation these conditions have the form

$$\mathbf{C}^n_{ri}\mathbf{C}^s_{nj} - R^{kl}_{ij}\mathbf{C}^n_{rk}\mathbf{C}^s_{nl} = \mathbf{C}^k_{ij}\mathbf{C}^s_{rk}, \tag{18}$$

$$\Lambda^{nm}_{ij}\Lambda^{ik}_{rp}\Lambda^{js}_{kq} = \Lambda^{nk}_{ri}\Lambda^{ms}_{kj}\Lambda^{ij}_{pq}, \tag{19}$$

$$\mathbf{C}^i_{mn}\Lambda^m_i\Lambda^i_j\mathbf{C}^s_{lk} = \Lambda^{nm}_{jk}\Lambda^{il}_{rq}\mathbf{C}^s_{lp} + \mathbf{C}^m_{jk}\Lambda^{is}_{rm}, \tag{20}$$

$$\mathbf{C}^m_{rk}\Lambda^{ns}_{ml} = \Lambda^{ij}_{kl}\Lambda^{nm}_{ri}\mathbf{C}^s_{mj}. \tag{21}$$

In this case the Lie algebra (1) of the gauge group of the classical gauge theory, taking into account of (14), is replaced by the quantum Lie algebra

$$T_aT_b - \Lambda^{cd}_{ab}T_cT_d = \mathbf{C}^c_{ab}T_c. \tag{22}$$

As in the classical case (2) the gauge field is defined by the same formula $A_\mu = A^a_\mu T_a$, but now the gauge potentials are noncommutative and satisfy the commutation relations

$$A^a_{[\mu}A^b_{\nu]} = -\frac{1}{q^2+q^{-2}}\left(\Lambda + \Lambda^{-1}\right)^{ab}_{cd} A^c_{[\mu}A^d_{\nu]}. \tag{23}$$

The field strength can be represented in the form

$$F^a_{\mu\nu} = \partial_{[\mu}A^a_{\nu]} + P^{kl}_{A\ mn} C^a_{kl} A^m_{[\mu}A^n_{\nu]}, \tag{24}$$

where $\mathbf{C}^n_{kl} = C^n_{kl} - \Lambda^{ij}_{kl}C^n_{ij}$. The deformed gauge transformations are assumed to have the form

$$\delta A = -d\varepsilon - A\varepsilon + \varepsilon A, \qquad \varepsilon = \varepsilon^a T_a. \tag{25}$$

The gauge parameters ε are q-numbers and are assumed to have the following commutation relations

$$\varepsilon^a A_a = \Lambda^{ab}_{mn}\varepsilon^n. \tag{26}$$

Then F is transformed as $\delta F = \varepsilon F - F\varepsilon$ and the deformed Lagrangian density $\mathcal{L} = F^a_{\mu\nu}F^b_{\mu\nu}g_{ab}$ is invariant under transformations (25) if

$$\Lambda^{nb}_{rs}C^a_{mn}g_{ab} + C^b_{rs}g_{mb} = 0. \tag{27}$$

2.4. The construction based on the quantum deformation of the BRST algebra [6, 15, 9]. One of the alternative formulations of the gauge field theory is the BRST method. In this approach the BRST transformation s is defined and parameter $\varepsilon(x)$ is replaced by the ghost field $C(x)$. If we restrict ourselves by the pure Yang–Mills field theory, then BRST transformations are reduced to the form

$$sA^a_\mu = \partial C^a + f^a_{bc} A^b_\mu C^C, \qquad sC^a = -\frac{1}{2} f^a_{bc} C^b C^c. \tag{28}$$

The gauge field strength F^a is transformed covariantly $sF^a = f^a_{bc} F^b C^c$. As was noted in [17] if we use the gauge symmetry of the Hopf algebra then it is necessary to formulate all theory in the algebraic frameworks. The gauge transformations should be represented in the abstract language. As we saw at (26) it is not known to what algebra belongs the set of the parameters. The idea of [6] is as follows: replace local gauge parameters of the theory by the ghost fields which now placed at same level as the gauge and matter fields. The formulation of all theory is algebraic.

3 The quantum group gauge field theory on the quantum spaces

3.1. The geometrical meaning of the gauge field potentials [21]. Let P and M be smooth manifolds, a Lie group G smooth acting on P and the differentiable principal fiber bundle $P(M,G)$ over M with the group G. A global (local) cross-section of a principal fiber bundle is a map σ from the base space (neighbourhood U_α) to the bundle space P such that $\pi(\sigma(x)) = x$, $\forall\, x \in M$ ($\pi\sigma_\alpha(x) = x$, $\forall\, x \in U_\alpha$) Let ω_α be a 1-form in U_α. It can be written in terms of its components (Lie-algebra valued functions) $A^\mu_\alpha(x)$

$$\omega = \sum_\mu A^\mu_\alpha(x) dx_\mu. \tag{29}$$

Suppose we transform σ_α into σ'_α by the action of some $g \in G$. If $\sigma'_\alpha(x) = \sigma_\alpha(x) g(x)$, then $\omega'_\alpha = \sigma'^*_\alpha \omega = A'^\mu dx_\mu$, where

$$A'_\mu = g^{-1} A_\mu g + g^{-1} \partial_\mu g. \tag{30}$$

This reproduces the gauge transformation formula for gauge potentials (6). The connection form ω describes at the same time both the Yang–Mills potential and ghost fields. It is split into two components of the gauge field ϕ which is horisontal and the ghost field χ which is normal to the section σ. From the Cartan–Maurer theorem the equations follow:

$$s\chi + 1/2[\chi,\chi] = 0, \qquad s\phi + B\chi = 0, \tag{31}$$

which are the same as BRST transformation (28).

3.2. The construction based on the quantum group generalization of the fiber bundle [17, 18]. The quantum group gauge field theory is constructed also in the framework of the fiber bundle with the quantum structure group [17]. The Cartan–Maurer equation obtained by the universal bicovariant differential calculus on quantum group is the same as BRST transformation ghost fields of the quantum group gauge field theory. But for general quantum fiber bundle this problem is open.

Acknowledgements

This research was supported by the Ukrainian State Foundation of Fundamental Researches Grant No. 1.4/206.

References

[1] Drinfeld V.G., Quantum groups, in Proceedings of the International Congress of Mathematicians, 1986, V.1, 798.

[2] Jimbo M., *Lett. Math. Phys.*, 1986, V.10, 63.

[3] Woronowicz S. L., *Commun. Math. Phys.*, 1987, V.111, 613.

[4] Biedenharn L.C., *J. Phys. A*, 1989, V.22, L837.

[5] Macfarlane A.J., *J. Phys.*, 1998, V.22, 4581.

[6] Bernard D. and LeClair A., *Phys. Let. B*, 1989, V.227, 417.

[7] Matsyzaki T. and Suzuki T., *Phys. Lett. B*, 1992, V.33, 296.

[8] Burban I.M., *J. Nonlin. Math. Phys.*, 1995, V.2, 12.

[9] Burban I.M., *Ukr. J. Phys.*, 1998, V.43, 787.

[10] Bernard D., *Prog. Theor. Phys. Suppl.*, 1990, V.102, 49.

[11] Arefeva I.Ya. and Volovich I.V., *Mod. Phys. Lett. A*, 1991, V.6, 893.

[12] Hirayama M., *Prog. Theor. Phys.*, 1992. V.88, 111.

[13] Isaev A.P. and Popowicz Z., *Phys. Lett. B*, 1992, V.281, 271.

[14] Castelani L., *Phys. Lett. B*, 1992, V.292, 93; $U_q(N)$ gauge theories, Preprint, DFTT-74/92, 1992.

[15] Watamura S., *Commun. Math. Phys.*, 1993, V.158, 67.

[16] Brzezinski T. and Majid S., *Phys. Lett. B*, 1993, 298, 339.

[17] Brzezinski T. and Majid S., *Commun. Math. Phys. B*, 1993, V.298, 591.

[18] Hajac P.M., hep-th/9406129.

[19] Sudbery A., hep-th/9601033.

[20] Gurevitch D.I., *Sov. Math. Dokl.*, 1986, V.33, 758.

[21] Thierry-Mieg J., *Nuovo Cimento A*, 1980, V.56, 396.