

C_λ -Extended Oscillator Algebras: Theory and Applications to (Variants of) Supersymmetric Quantum Mechanics

C. QUESNE[†] and *N. VANSTEENKISTE*[‡]

Physique Nucléaire Théorique et Physique Mathématique, Université Libre de Bruxelles, Campus de la Plaine CP229, Boulevard du Triomphe, B-1050 Brussels, Belgium

[†] *Directeur de recherches FNRS; E-mail: cquesne@ulb.ac.be*

[‡] *E-mail: nvsteen@ulb.ac.be*

C_λ -extended oscillator algebras, where C_λ is the cyclic group of order λ , are introduced and realized as generalized deformed oscillator algebras. For $\lambda = 2$, they reduce to the well-known Calogero–Vasiliev algebra. For higher λ values, they are shown to provide in their bosonic Fock space representation some interesting applications to supersymmetric quantum mechanics and some variants thereof: an algebraic realization of supersymmetric quantum mechanics for cyclic shape invariant potentials of period λ , a bosonization of para-supersymmetric quantum mechanics of order $p = \lambda - 1$, and, for $\lambda = 3$, a bosonization of pseudosupersymmetric quantum mechanics and orthosupersymmetric quantum mechanics of order two.

1 Introduction

Deformations and extensions of the oscillator algebra have found a lot of applications to physical problems, such as the description of systems with non-standard statistics, the construction of integrable lattice models, the investigation of nonlinearities in quantum optics, as well as the algebraic treatment of quantum exactly solvable models and of n -particle integrable systems.

The generalized deformed oscillator algebras (GDOAs) (see e.g. Ref. [1] and references quoted therein) arose from successive generalizations of the Arik–Coon [2] and Biedenharn–Macfarlane [3] q -oscillators. Such algebras, denoted by $\mathcal{A}_q(G(N))$, are generated by the unit, creation, annihilation, and number operators I, a^\dagger, a, N , satisfying the Hermiticity conditions $(a^\dagger)^\dagger = a, N^\dagger = N$, and the commutation relations

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad [a, a^\dagger]_q \equiv aa^\dagger - qa^\dagger a = G(N), \quad (1.1)$$

where q is some real number and $G(N)$ is some Hermitian, analytic function.

On the other hand, \mathcal{G} -extended oscillator algebras, where \mathcal{G} is some finite group, appeared in connection with n -particle integrable models. For the Calogero model [4], for instance, \mathcal{G} is the symmetric group S_n [5].

For two particles, the S_2 -extended oscillator algebra $\mathcal{A}_\kappa^{(2)}$, where $S_2 = \{I, K \mid K^2 = I\}$, is generated by the operators I, a^\dagger, a, N, K , subject to the Hermiticity conditions $(a^\dagger)^\dagger = a, N^\dagger = N, K^\dagger = K^{-1}$, and the relations

$$\begin{aligned} [N, a^\dagger] &= a^\dagger, & [N, K] &= 0, & K^2 &= I, \\ [a, a^\dagger] &= I + \kappa K & (\kappa \in \mathbb{R}), & & a^\dagger K &= -Ka^\dagger, \end{aligned} \quad (1.2)$$

together with their Hermitian conjugates.

When the S_2 generator K is realized in terms of the Klein operator $(-1)^N$, $\mathcal{A}_\kappa^{(2)}$ becomes a GDOA characterized by $q = 1$ and $G(N) = I + \kappa(-1)^N$, and known as the Calogero–Vasiliev [6] or modified [7] oscillator algebra.

The operator K may be alternatively considered as the generator of the cyclic group C_2 of order two, since the latter is isomorphic to S_2 . By replacing C_2 by the cyclic group of order λ , $C_\lambda = \{I, T, T^2, \dots, T^{\lambda-1} \mid T^\lambda = I\}$, one then gets a new class of \mathcal{G} -extended oscillator algebras [8], generalizing that describing the two-particle Calogero model. In the present communication, we will define the C_λ -extended oscillator algebras, study some of their properties, and show that they have some interesting applications to supersymmetric quantum mechanics (SSQM) [9] and some of its variants.

2 Definition and properties of C_λ -extended oscillator algebras

Let us consider the algebras generated by the operators I, a^\dagger, a, N, T , satisfying the Hermiticity conditions $(a^\dagger)^\dagger = a, N^\dagger = N, T^\dagger = T^{-1}$, and the relations

$$\begin{aligned}
 [N, a^\dagger] &= a^\dagger, & [N, T] &= 0, & T^\lambda &= I, \\
 [a, a^\dagger] &= I + \sum_{\mu=1}^{\lambda-1} \kappa_\mu T^\mu, & a^\dagger T &= e^{-i2\pi/\lambda} T a^\dagger,
 \end{aligned}
 \tag{2.1}$$

together with their Hermitian conjugates [8]. Here T is the generator of (a unitary representation of) the cyclic group C_λ (where $\lambda \in \{2, 3, 4, \dots\}$), and $\kappa_\mu, \mu = 1, 2, \dots, \lambda - 1$, are some complex parameters restricted by the conditions $\kappa_\mu^* = \kappa_{\lambda-\mu}$ (so that there remain altogether $\lambda - 1$ independent real parameters).

C_λ has λ inequivalent, one-dimensional matrix unitary irreducible representations (unirreps) $\Gamma^\mu, \mu = 0, 1, \dots, \lambda - 1$, which are such that $\Gamma^\mu(T^\nu) = \exp(i2\pi\mu\nu/\lambda)$ for any $\nu = 0, 1, \dots, \lambda - 1$. The projection operator on the carrier space of Γ^μ may be written as

$$P_\mu = \frac{1}{\lambda} \sum_{\nu=0}^{\lambda-1} e^{-i2\pi\mu\nu/\lambda} T^\nu,
 \tag{2.2}$$

and conversely $T^\nu, \nu = 0, 1, \dots, \lambda - 1$, may be expressed in terms of the P_μ 's as

$$T^\nu = \sum_{\mu=0}^{\lambda-1} e^{i2\pi\mu\nu/\lambda} P_\mu.
 \tag{2.3}$$

The algebra defining relations (2.1) may therefore be rewritten in terms of I, a^\dagger, a, N , and $P_\mu = P_\mu^\dagger, \mu = 0, 1, \dots, \lambda - 1$, as

$$\begin{aligned}
 [N, a^\dagger] &= a^\dagger, & [N, P_\mu] &= 0, & \sum_{\mu=0}^{\lambda-1} P_\mu &= I, \\
 [a, a^\dagger] &= I + \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu, & a^\dagger P_\mu &= P_{\mu+1} a^\dagger, & P_\mu P_\nu &= \delta_{\mu,\nu} P_\mu,
 \end{aligned}
 \tag{2.4}$$

where we use the convention $P_{\mu'} = P_\mu$ if $\mu' - \mu = 0 \pmod{\lambda}$ (and similarly for other operators or parameters indexed by μ, μ'). Equation (2.4) depends upon λ real parameters

$\alpha_\mu = \sum_{\nu=1}^{\lambda-1} \exp(i2\pi\mu\nu/\lambda)\kappa_\nu$, $\mu = 0, 1, \dots, \lambda - 1$, restricted by the condition $\sum_{\mu=0}^{\lambda-1} \alpha_\mu = 0$. Hence, we may eliminate one of them, for instance $\alpha_{\lambda-1}$, and denote C_λ -extended oscillator algebras by $\mathcal{A}_{\alpha_0\alpha_1\dots\alpha_{\lambda-2}}^{(\lambda)}$.

The cyclic group generator T and the projection operators P_μ can be realized in terms of N as

$$T = e^{i2\pi N/\lambda}, \quad P_\mu = \frac{1}{\lambda} \sum_{\nu=0}^{\lambda-1} e^{i2\pi\nu(N-\mu)/\lambda}, \quad \mu = 0, 1, \dots, \lambda - 1, \tag{2.5}$$

respectively. With such a choice, $\mathcal{A}_{\alpha_0\alpha_1\dots\alpha_{\lambda-2}}^{(\lambda)}$ becomes a GDOA, $\mathcal{A}^{(\lambda)}(G(N))$, characterized by $q = 1$ and $G(N) = I + \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu$, where P_μ is given in Eq.(2.5).

For any GDOA $\mathcal{A}_q(G(N))$, one may define a so-called structure function $F(N)$, which is the solution of the difference equation $F(N + 1) - qF(N) = G(N)$, such that $F(0) = 0$ [1]. For $\mathcal{A}^{(\lambda)}(G(N))$, we find

$$F(N) = N + \sum_{\mu=0}^{\lambda-1} \beta_\mu P_\mu, \quad \beta_0 \equiv 0, \quad \beta_\mu \equiv \sum_{\nu=0}^{\mu-1} \alpha_\nu \quad (\mu = 1, 2, \dots, \lambda - 1). \tag{2.6}$$

At this point, it is worth noting that for $\lambda = 2$, we obtain $T = K$, $P_0 = (I + K)/2$, $P_1 = (I - K)/2$, and $\kappa_1 = \kappa_1^* = \alpha_0 = -\alpha_1 = \kappa$, so that $\mathcal{A}_{\alpha_0}^{(2)}$ coincides with the S_2 -extended oscillator algebra $\mathcal{A}_\kappa^{(2)}$ and $\mathcal{A}^{(2)}(G(N))$ with the Calogero–Vasiliev algebra.

In Ref. [10], we showed that $\mathcal{A}^{(\lambda)}(G(N))$ (and more generally $\mathcal{A}_{\alpha_0\alpha_1\dots\alpha_{\lambda-2}}^{(\lambda)}$) has only two different types of unirreps: infinite-dimensional bounded from below unirreps and finite-dimensional ones. Among the former, there is the so-called bosonic Fock space representation, wherein $a^\dagger a = F(N)$ and $aa^\dagger = F(N + 1)$. Its carrier space \mathcal{F} is spanned by the eigenvectors $|n\rangle$ of the number operator N , corresponding to the eigenvalues $n = 0, 1, 2, \dots$, where $|0\rangle$ is a vacuum state, i.e., $a|0\rangle = N|0\rangle = 0$ and $P_\mu|0\rangle = \delta_{\mu,0}|0\rangle$. The eigenvectors can be written as

$$|n\rangle = \mathcal{N}_n^{-1/2} (a^\dagger)^n |0\rangle, \quad n = 0, 1, 2, \dots, \tag{2.7}$$

where $\mathcal{N}_n = \prod_{i=1}^n F(i)$. The creation and annihilation operators act upon $|n\rangle$ in the usual way, i.e.,

$$a^\dagger |n\rangle = \sqrt{F(n+1)} |n+1\rangle, \quad a |n\rangle = \sqrt{F(n)} |n-1\rangle, \tag{2.8}$$

while P_μ projects on the μ th component $\mathcal{F}_\mu \equiv \{|k\lambda + \mu\rangle \mid k = 0, 1, 2, \dots\}$ of the Z_λ -graded Fock space $\mathcal{F} = \sum_{\mu=0}^{\lambda-1} \oplus \mathcal{F}_\mu$. It is obvious that such a bosonic Fock space representation exists if and only if $F(\mu) > 0$ for $\mu = 1, 2, \dots, \lambda - 1$. This gives the following restrictions on the algebra parameters α_μ ,

$$\sum_{\nu=0}^{\mu-1} \alpha_\nu > -\mu, \quad \mu = 1, 2, \dots, \lambda - 1. \tag{2.9}$$

In the bosonic Fock space representation, we may consider the bosonic oscillator Hamiltonian, defined as usual by

$$H_0 \equiv \frac{1}{2} \{a, a^\dagger\}. \tag{2.10}$$

It can be rewritten as

$$H_0 = a^\dagger a + \frac{1}{2} \left(I + \sum_{\mu=0}^{\lambda-1} \alpha_\mu P_\mu \right) = N + \frac{1}{2} I + \sum_{\mu=0}^{\lambda-1} \gamma_\mu P_\mu, \tag{2.11}$$

where $\gamma_0 \equiv \frac{1}{2}\alpha_0$ and $\gamma_\mu \equiv \sum_{\nu=0}^{\mu-1} \alpha_\nu + \frac{1}{2}\alpha_\mu$ for $\mu = 1, 2, \dots, \lambda - 1$.

The eigenvectors of H_0 are the states $|n\rangle = |k\lambda + \mu\rangle$, defined in Eq.(2.7), and their eigenvalues are given by

$$E_{k\lambda+\mu} = k\lambda + \mu + \gamma_\mu + \frac{1}{2}, \quad k = 0, 1, 2, \dots, \quad \mu = 0, 1, \dots, \lambda - 1. \tag{2.12}$$

In each \mathcal{F}_μ subspace of the Z_λ -graded Fock space \mathcal{F} , the spectrum of H_0 is therefore harmonic, but the λ infinite sets of equally spaced energy levels, corresponding to $\mu = 0, 1, \dots, \lambda - 1$, may be shifted with respect to each other by some amounts depending upon the algebra parameters $\alpha_0, \alpha_1, \dots, \alpha_{\lambda-2}$, through their linear combinations $\gamma_\mu, \mu = 0, 1, \dots, \lambda - 1$.

For the Calogero–Vasiliev oscillator, i.e., for $\lambda = 2$, the relation $\gamma_0 = \gamma_1 = \kappa/2$ implies that the spectrum is very simple and coincides with that of a shifted harmonic oscillator. For $\lambda \geq 3$, however, it has a much richer structure. According to the parameter values, it may be nondegenerate, or may exhibit some $(\nu + 1)$ -fold degeneracies above some energy eigenvalue, where ν may take any value in the set $\{1, 2, \dots, \lambda - 1\}$. In Ref. [11], we obtained for $\lambda = 3$ the complete classification of nondegenerate, twofold and threefold degenerate spectra in terms of α_0 and α_1 .

In the remaining part of this communication, we will show that the bosonic Fock space representation of $\mathcal{A}^{(\lambda)}(G(N))$ and the corresponding bosonic oscillator Hamiltonian H_0 have some useful applications to SSQM and some of its variants.

3 Application to supersymmetric quantum mechanics with cyclic shape invariant potentials

In SSQM with two supercharges, the supersymmetric Hamiltonian \mathcal{H} and the supercharges $Q^\dagger, Q = (Q^\dagger)^\dagger$, satisfy the sqm(2) superalgebra, defined by the relations

$$Q^2 = 0, \quad [\mathcal{H}, Q] = 0, \quad \{Q, Q^\dagger\} = \mathcal{H}, \tag{3.1}$$

together with their Hermitian conjugates [9]. In such a context, shape invariance [12] provides an integrability condition, yielding all the bound state energy eigenvalues and eigenfunctions, as well as the scattering matrix.

Recently, Sukhatme, Rasinariu, and Khare [13] introduced cyclic shape invariant potentials of period p in SSQM. They are characterized by the fact that the supersymmetric partner Hamiltonians correspond to a series of shape invariant potentials, which repeats after a cycle of p iterations. In other words, one may define p sets of operators $\{\mathcal{H}_\mu, Q_\mu^\dagger, Q_\mu\}, \mu = 0, 1, \dots, p - 1$, each satisfying the sqm(2) defining relations (3.1). The operators may be written as

$$\mathcal{H}_\mu = \begin{pmatrix} \mathcal{H}^{(\mu)} - \mathcal{E}_0^{(\mu)} I & 0 \\ 0 & \mathcal{H}^{(\mu+1)} - \mathcal{E}_0^{(\mu)} I \end{pmatrix}, \quad Q_\mu^\dagger = \begin{pmatrix} 0 & A_\mu^\dagger \\ 0 & 0 \end{pmatrix}, \quad Q_\mu = \begin{pmatrix} 0 & 0 \\ A_\mu & 0 \end{pmatrix}, \tag{3.2}$$

where

$$\begin{aligned} \mathcal{H}^{(0)} &= A_0^\dagger A_0, \\ \mathcal{H}^{(\mu)} &= A_{\mu-1} A_{\mu-1}^\dagger + \mathcal{E}_0^{(\mu-1)} I = A_\mu^\dagger A_\mu + \mathcal{E}_0^{(\mu)} I, \quad \mu = 1, 2, \dots, p, \\ A_\mu &= \frac{d}{dx} + W(x, b_\mu), \quad A_\mu^\dagger = -\frac{d}{dx} + W(x, b_\mu), \quad \mu = 0, 1, \dots, p, \end{aligned} \tag{3.3}$$

and $\mathcal{E}_0^{(\mu)}$ denotes the ground state energy of $\mathcal{H}^{(\mu)}$ (with $\mathcal{E}_0^{(0)} = 0$). Here the superpotentials $W(x, b_\mu)$ depend upon some parameters b_μ , such that $b_{\mu+p} = b_\mu$, and they satisfy p shape invariance conditions

$$W^2(x, b_\mu) + W'(x, b_\mu) = W^2(x, b_{\mu+1}) - W'(x, b_{\mu+1}) + \omega_\mu, \quad \mu = 0, 1, \dots, p-1, \tag{3.4}$$

where ω_μ , $\mu = 0, 1, \dots, p-1$, are some real constants.

From the solution of Eq.(3.4), one may then construct the potentials corresponding to the supersymmetric partners $\mathcal{H}^{(\mu)}$, $\mathcal{H}^{(\mu+1)}$ in the usual way, i.e., $V^{(\mu)} = W^2(x, b_\mu) - W'(x, b_\mu) + \mathcal{E}_0^{(\mu)}$, $V^{(\mu+1)} = W^2(x, b_\mu) + W'(x, b_\mu) + \mathcal{E}_0^{(\mu)}$. For $p = 2$, Gangopadhyaya and Sukhatme [14] obtained such potentials as superpositions of a Calogero potential and a δ -function singularity. For $p \geq 3$, however, only numerical solutions of the shape invariance conditions (3.4) have been obtained [13], so that no analytical form of $V^{(\mu)}$ is known. In spite of this, the spectrum is easily derived and consists of p infinite sets of equally spaced energy levels, shifted with respect to each other by the energies $\omega_0, \omega_1, \dots, \omega_{p-1}$.

Since for some special choices of parameters, spectra of a similar type may be obtained with the bosonic oscillator Hamiltonian (2.10) acting in the bosonic Fock space representation of $\mathcal{A}^{(p)}(G(N))$, one may try to establish a relation between the class of algebras $\mathcal{A}^{(p)}(G(N))$ and SSQM with cyclic shape invariant potentials of period p .

In Ref. [11], we proved that the operators $\mathcal{H}^{(\mu)}$, A_μ^\dagger , and A_μ of Eqs.(3.2) and (3.3) can be realized in terms of the generators of p algebras $\mathcal{A}^{(p)}(G^{(\mu)}(N))$, $\mu = 0, 1, \dots, p-1$, belonging to the class $\{\mathcal{A}^{(p)}(G(N))\}$. The parameters of such algebras are obtained by cyclic permutations from a starting set $\{\alpha_0, \alpha_1, \dots, \alpha_{p-1}\}$ corresponding to $\mathcal{A}^{(p)}(G^{(0)}(N)) = \mathcal{A}^{(p)}(G(N))$. Denoting by N , a_μ^\dagger , a_μ the number, creation, and annihilation operators corresponding to the μ th algebra $\mathcal{A}^{(p)}(G^{(\mu)}(N))$, where $a_0^\dagger = a^\dagger$, and $a_0 = a$, we may write the fourth relation in the algebra defining relations (2.4) as

$$\left[a_\mu, a_\mu^\dagger \right] = I + \sum_{\nu=0}^{p-1} \alpha_\nu^{(\mu)} P_\nu, \quad \alpha_\nu^{(\mu)} \equiv \alpha_{\nu+\mu}, \quad \mu = 0, 1, \dots, p-1, \tag{3.5}$$

while the remaining relations keep the same form.

The realization of $\mathcal{H}^{(\mu)}$, A_μ^\dagger , A_μ , $\mu = 0, 1, \dots, p-1$, is then given by

$$\begin{aligned} \mathcal{H}^{(\mu)} &= F(N + \mu) = N + \mu I + \sum_{\nu=0}^{p-1} \beta_{\nu+\mu} P_\nu = H_0^{(\mu)} - \frac{1}{2} \sum_{\nu=0}^{p-1} \left(1 + \alpha_\nu^{(\mu)} \right) P_\nu + \mathcal{E}_0^{(\mu)} I, \\ A_\mu^\dagger &= a_\mu^\dagger, \quad A_\mu = a_\mu, \end{aligned} \tag{3.6}$$

where $H_0^{(\mu)} \equiv \frac{1}{2} \{ a_\mu, a_\mu^\dagger \}$ is the bosonic oscillator Hamiltonian associated with $\mathcal{A}^{(p)}(G^{(\mu)}(N))$, $\mathcal{E}_0^{(\mu)} = \sum_{\nu=0}^{\mu-1} \omega_\nu$, and the level spacings are $\omega_\mu = 1 + \alpha_\mu$. For this result to be meaningful,

the conditions $\omega_\mu > 0$, $\mu = 0, 1, \dots, p - 1$, have to be fulfilled. When combined with the restrictions (2.9), the latter imply that the parameters of the starting algebra $\mathcal{A}^{(p)}(G(N))$ must be such that $-1 < \alpha_0 < \lambda - 1$, $-1 < \alpha_\mu < \lambda - \mu - 1 - \sum_{\nu=0}^{\mu-1} \alpha_\nu$ if $\mu = 1, 2, \dots, \lambda - 2$, and

$$\alpha_{\lambda-1} = - \sum_{\nu=0}^{\lambda-2} \alpha_\nu.$$

4 Application to parasupersymmetric quantum mechanics of order p

The $\text{sqm}(2)$ superalgebra (3.1) is most often realized in terms of mutually commuting boson and fermion operators. Plyushchay [15], however, showed that it can alternatively be realized in terms of only boson-like operators, namely the generators of the Calogero–Vasiliev algebra $\mathcal{A}^{(2)}(G(N))$ (see also Ref. [16]). Such an SSQM bosonization can be performed in two different ways, by choosing either $Q = a^\dagger P_1$ (so that $\mathcal{H} = H_0 - \frac{1}{2}(K + \kappa)$) or $Q = a^\dagger P_0$ (so that $\mathcal{H} = H_0 + \frac{1}{2}(K + \kappa)$). The first choice corresponds to unbroken SSQM (all the excited states are twofold degenerate while the ground state is nondegenerate and at vanishing energy), and the second choice describes broken SSQM (all the states are twofold degenerate and at positive energy).

SSQM was generalized to parasupersymmetric quantum mechanics (PSSQM) of order two by Rubakov and Spiridonov [17], and later on to PSSQM of arbitrary order p by Khare [18]. In the latter case, Eq. (3.1) is replaced by

$$\begin{aligned} Q^{p+1} &= 0 \quad (\text{with } Q^p \neq 0), \\ [\mathcal{H}, Q] &= 0, \\ Q^p Q^\dagger + Q^{p-1} Q^\dagger Q + \dots + Q Q^\dagger Q^{p-1} + Q^\dagger Q^p &= 2p Q^{p-1} \mathcal{H}, \end{aligned} \tag{4.1}$$

and is retrieved in the case where $p = 1$. The parasupercharges Q, Q^\dagger , and the parasupersymmetric Hamiltonian \mathcal{H} are usually realized in terms of mutually commuting boson and parafermion operators.

A property of PSSQM of order p is that the spectrum of \mathcal{H} is $(p + 1)$ -fold degenerate above the $(p - 1)$ th energy level. This fact and Plyushchay’s results for $p = 1$ hint at a possibility of representing \mathcal{H} as a linear combination of the bosonic oscillator Hamiltonian H_0 associated with $\mathcal{A}^{(p+1)}(G(N))$ and some projection operators, as in Eq.(3.6).

In Ref. [10] (see also Refs. [8, 19]), we proved that PSSQM of order p can indeed be bosonized in terms of the generators of $\mathcal{A}^{(p+1)}(G(N))$ for any allowed (i.e., satisfying Eq.(2.9)) values of the algebra parameters $\alpha_0, \alpha_1, \dots, \alpha_{p-1}$. For such a purpose, we started from ansätze of the type

$$Q = \sum_{\nu=0}^p \sigma_\nu a^\dagger P_\nu, \quad \mathcal{H} = H_0 + \frac{1}{2} \sum_{\nu=0}^p r_\nu P_\nu, \tag{4.2}$$

where σ_ν and r_ν are some complex and real constants, respectively, to be determined in such a way that Eq.(4.1) is fulfilled. We found that there are $p + 1$ families of solutions, which may be distinguished by an index $\mu \in \{0, 1, \dots, p\}$ and from which we may choose the following

representative solutions

$$Q_\mu = \sqrt{2} \sum_{\nu=1}^p a^\dagger P_{\mu+\nu},$$

$$\mathcal{H}_\mu = N + \frac{1}{2}(2\gamma_{\mu+2} + r_{\mu+2} - 2p + 3)I + \sum_{\nu=1}^p (p + 1 - \nu)P_{\mu+\nu},$$
(4.3)

where

$$r_{\mu+2} = \frac{1}{p} \left[(p-2)\alpha_{\mu+2} + 2 \sum_{\nu=3}^p (p-\nu+1)\alpha_{\mu+\nu} + p(p-2) \right].$$
(4.4)

The eigenvectors of \mathcal{H}_μ are the states (2.7) and the corresponding eigenvalues are easily found. All the energy levels are equally spaced. For $\mu = 0$, PSSQM is unbroken, otherwise it is broken with a $(\mu + 1)$ -fold degenerate ground state. All the excited states are $(p + 1)$ -fold degenerate. For $\mu = 0, 1, \dots, p - 2$, the ground state energy may be positive, null, or negative depending on the parameters, whereas for $\mu = p - 1$ or p , it is always positive.

Khare [18] showed that in PSSQM of order p , \mathcal{H} has in fact $2p$ (and not only two) conserved parasupercharges, as well as p bosonic constants. In other words, there exist p independent operators Q_r , $r = 1, 2, \dots, p$, satisfying with \mathcal{H} the set of equations (4.1), and p other independent operators I_t , $t = 2, 3, \dots, p + 1$, commuting with \mathcal{H} , as well as among themselves. In Ref. [10], we obtained a realization of all such operators in terms of the $\mathcal{A}^{(p+1)}(G(N))$ generators.

As a final point, let us note that there exists an alternative approach to PSSQM of order p , which was proposed by Beckers and Debergh [20], and wherein the multilinear relation in Eq.(4.1) is replaced by the cubic equation

$$\left[Q, \left[Q^\dagger, Q \right] \right] = 2Q\mathcal{H}.$$
(4.5)

In Ref. [8], we proved that for $p = 2$, this PSSQM algebra can only be realized by those $\mathcal{A}^{(3)}(G(N))$ algebras that simultaneously bosonize Rubakov–Spiridonov–Khare PSSQM algebra.

5 Application to pseudosupersymmetric quantum mechanics

Pseudosupersymmetric quantum mechanics (pseudoSSQM) was introduced by Beckers, Debergh and Nikitin [21] in a study of relativistic vector mesons interacting with an external constant magnetic field. In the nonrelativistic limit, their theory leads to a pseudosupersymmetric oscillator Hamiltonian, which can be realized in terms of mutually commuting boson and pseudofermion operators, where the latter are intermediate between standard fermion and $p = 2$ parafermion operators.

It is then possible to formulate a pseudoSSQM [21], characterized by a pseudosupersymmetric Hamiltonian \mathcal{H} and pseudosupercharge operators Q, Q^\dagger , satisfying the relations

$$Q^2 = 0, \quad [\mathcal{H}, Q] = 0, \quad QQ^\dagger Q = 4c^2 Q\mathcal{H},$$
(5.1)

and their Hermitian conjugates, where c is some real constant. The first two relations in Eq.(5.1) are the same as those occurring in SSQM, whereas the third one is similar to the multilinear relation valid in PSSQM of order two. Actually, for $c = 1$ or $1/2$, it is compatible with Eq.(4.1) or (4.5), respectively.

In Ref. [10], we proved that pseudoSSQM can be bosonized in two different ways in terms of the generators of $\mathcal{A}^{(3)}(G(N))$ for any allowed values of the parameters α_0, α_1 . This time, we started from the ansätze

$$Q = \sum_{\nu=0}^2 (\xi_\nu a + \eta_\nu a^\dagger) P_\nu, \quad \mathcal{H} = H_0 + \frac{1}{2} \sum_{\nu=0}^2 r_\nu P_\nu, \tag{5.2}$$

and determined the complex constants ξ_ν, η_ν , and the real ones r_ν in such a way that Eq.(5.1) is fulfilled.

The first type of bosonization corresponds to three families of two-parameter solutions, labelled by an index $\mu \in \{0, 1, 2\}$,

$$Q_\mu(\eta_{\mu+2}, \varphi) = \left(\eta_{\mu+2} a^\dagger + e^{i\varphi} \sqrt{4c^2 - \eta_{\mu+2}^2} a \right) P_{\mu+2},$$

$$\mathcal{H}_\mu(\eta_{\mu+2}) = N + \frac{1}{2}(2\gamma_{\mu+2} + r_{\mu+2} - 1)I + 2P_{\mu+1} + P_{\mu+2}, \tag{5.3}$$

where $0 < \eta_{\mu+2} < 2|c|$, $0 \leq \varphi < 2\pi$, and

$$r_{\mu+2} = \frac{1}{2c^2}(1 + \alpha_{\mu+2}) (|\eta_{\mu+2}|^2 - 2c^2). \tag{5.4}$$

Choosing for instance $\eta_{\mu+2} = \sqrt{2}|c|$, and $\varphi = 0$, hence $r_{\mu+2} = 0$ (producing an overall shift of the spectrum), we obtain

$$Q_\mu = c\sqrt{2} (a^\dagger + a) P_{\mu+2},$$

$$\mathcal{H}_\mu = N + \frac{1}{2}(2\gamma_{\mu+2} - 1)I + 2P_{\mu+1} + P_{\mu+2}. \tag{5.5}$$

A comparison between Eq.(5.3) or (5.5) and Eq.(4.3) shows that the pseudosupersymmetric and $p = 2$ parasupersymmetric Hamiltonians coincide, but that the corresponding charges are of course different. The conclusions relative to the spectrum and the ground state energy are therefore the same as in Sec. 4.

The second type of bosonization corresponds to three families of one-parameter solutions, again labelled by an index $\mu \in \{0, 1, 2\}$,

$$Q_\mu = 2|c|aP_{\mu+2},$$

$$\mathcal{H}_\mu(r_\mu) = N + \frac{1}{2}(2\gamma_{\mu+2} - \alpha_{\mu+2})I + \frac{1}{2}(1 - \alpha_{\mu+1} + \alpha_{\mu+2} + r_\mu)P_\mu + P_{\mu+1}, \tag{5.6}$$

where $r_\mu \in \mathbb{R}$ changes the Hamiltonian spectrum in a significant way. We indeed find that the levels are equally spaced if and only if $r_\mu = (\alpha_{\mu+1} - \alpha_{\mu+2} + 3) \bmod 6$. If r_μ is small enough, the ground state is nondegenerate, and its energy is negative for $\mu = 1$, or may have any sign for $\mu = 0$ or 2 . On the contrary, if r_μ is large enough, the ground state remains nondegenerate with a vanishing energy in the former case, while it becomes twofold degenerate with a positive energy in the latter. For some intermediate r_μ value, one gets a two or threefold degenerate ground state with a vanishing or positive energy, respectively.

6 Application to orthosupersymmetric quantum mechanics of order two

Mishra and Rajasekaran [22] introduced order- p orthofermion operators by replacing the Pauli exclusion principle by a more stringent one: an orbital state shall not contain more than one particle, whatever be the spin direction. The wave function is thus antisymmetric in spatial indices alone with the order of the spin indices frozen.

Khare, Mishra, and Rajasekaran [23] then developed orthosupersymmetric quantum mechanics (OSSQM) of arbitrary order p by combining boson operators with orthofermion ones, for which the spatial indices are ignored. OSSQM is formulated in terms of an orthosupersymmetric Hamiltonian \mathcal{H} , and $2p$ orthosupercharge operators $Q_r, Q_r^\dagger, r = 1, 2, \dots, p$, satisfying the relations

$$Q_r Q_s = 0, \quad [\mathcal{H}, Q_r] = 0, \quad Q_r Q_s^\dagger + \delta_{r,s} \sum_{t=1}^p Q_t^\dagger Q_t = 2\delta_{r,s} \mathcal{H}, \quad (6.1)$$

and their Hermitian conjugates, where r and s run over $1, 2, \dots, p$.

In Ref. [10], we proved that OSSQM of order two can be bosonized in terms of the generators of some well-chosen $\mathcal{A}^{(3)}(G(N))$ algebras. As ansätze, we used the expressions

$$Q_1 = \sum_{\nu=0}^2 (\xi_\nu a + \eta_\nu a^\dagger) P_\nu, \quad Q_2 = \sum_{\nu=0}^2 (\zeta_\nu a + \rho_\nu a^\dagger) P_\nu, \quad \mathcal{H} = H_0 + \frac{1}{2} \sum_{\nu=0}^2 r_\nu P_\nu, \quad (6.2)$$

and determined the complex constants $\xi_\nu, \eta_\nu, \zeta_\nu, \rho_\nu$, and the real ones r_ν in such a way that Eq.(6.1) is fulfilled. We found two families of two-parameter solutions, labelled by $\mu \in \{0, 1\}$,

$$\begin{aligned} Q_{1,\mu}(\xi_{\mu+2}, \varphi) &= \xi_{\mu+2} a P_{\mu+2} + e^{i\varphi} \sqrt{2 - \xi_{\mu+2}^2} a^\dagger P_\mu, \\ Q_{2,\mu}(\xi_{\mu+2}, \varphi) &= -e^{-i\varphi} \sqrt{2 - \xi_{\mu+2}^2} a P_{\mu+2} + \xi_{\mu+2} a^\dagger P_\mu, \\ \mathcal{H}_\mu &= N + \frac{1}{2} (2\gamma_{\mu+1} - 1) I + 2P_\mu + P_{\mu+1}, \end{aligned} \quad (6.3)$$

where $0 < \xi_{\mu+2} \leq \sqrt{2}$ and $0 \leq \varphi < 2\pi$, provided the algebra parameter $\alpha_{\mu+1}$ is taken as $\alpha_{\mu+1} = -1$. As a matter of fact, the absence of a third family of solutions corresponding to $\mu = 2$ comes from the incompatibility of this condition (i.e., $\alpha_0 = -1$) with conditions (2.9).

The orthosupersymmetric Hamiltonian \mathcal{H} in Eq.(6.3) is independent of the parameters $\xi_{\mu+2}, \varphi$. All the levels of its spectrum are equally spaced. For $\mu = 0$, OSSQM is broken: the levels are threefold degenerate, and the ground state energy is positive. On the contrary, for $\mu = 1$, OSSQM is unbroken: only the excited states are threefold degenerate, while the nondegenerate ground state has a vanishing energy. Such results agree with the general conclusions of Ref. [23].

For p values greater than two, the OSSQM algebra (6.1) becomes rather complicated because the number of equations to be fulfilled increases considerably. A glance at the 18 independent conditions for $p = 3$ led us to the conclusion that the $\mathcal{A}^{(4)}(G(N))$ algebra is not rich enough to contain operators satisfying Eq.(6.1). Contrary to what happens for PSSQM, for OSSQM the $p = 2$ case is therefore not representative of the general one.

7 Conclusion

In this communication, we showed that the S_2 -extended oscillator algebra, which was introduced in connection with the two-particle Calogero model, can be extended to the whole class of C_λ -extended oscillator algebras $\mathcal{A}_{\alpha_0\alpha_1\dots\alpha_{\lambda-2}}^{(\lambda)}$, where $\lambda \in \{2, 3, \dots\}$, and $\alpha_0, \alpha_1, \dots, \alpha_{\lambda-2}$ are some real parameters. In the same way, the GDOA realization of the former, known as the Calogero–Vasiliev algebra, is generalized to a class of GDOAs $\mathcal{A}^{(\lambda)}(G(N))$, where $\lambda \in \{2, 3, \dots\}$, for which one can define a bosonic oscillator Hamiltonian H_0 , acting in the bosonic Fock space representation.

For $\lambda \geq 3$, the spectrum of H_0 has a very rich structure in terms of the algebra parameters $\alpha_0, \alpha_1, \dots, \alpha_{\lambda-2}$. This can be exploited to provide an algebraic realization of SSQM with cyclic shape invariant potentials of period λ , a bosonization of PSSQM of order $p = \lambda - 1$, and, for $\lambda = 3$, a bosonization of pseudoSSQM and OSSQM of order two.

References

- [1] Quesne C. and Vansteenkiste N., *J. Phys. A*, 1995, V.28, 7019; *Helv. Phys. Acta*, 1996, V.69, 141; *Czech. J. Phys.*, 1997, V.47, 115.
- [2] Arik M. and Coon D.D., *J. Math. Phys.*, 1976, V.17, 524; Kuryshkin V., *Ann. Fond. L. de Broglie*, 1980, V.5, 111.
- [3] Biedenharn L.C., *J. Phys. A*, 1989, V.22, L873; Macfarlane A.J., *J. Phys. A*, 1989, V.22, 4581.
- [4] Calogero F., *J. Math. Phys.*, 1969, V.10, 2191, 2197; 1971, V.12, 419.
- [5] Polychronakos A.P., *Phys. Rev. Lett.*, 1992, V.69, 703; Brink L., Hansson T.H. and Vasiliev M.A., *Phys. Lett. B*, 1992, V.286, 109.
- [6] Vasiliev M.A., *Int. J. Mod. Phys. A*, 1991, V.6, 1115.
- [7] Brzeziński T., Egusquiza I.L. and Macfarlane A.J., *Phys. Lett. B*, 1993, V.311, 202.
- [8] Quesne C. and Vansteenkiste N., *Phys. Lett. A*, 1998, V.240, 21.
- [9] Witten E., *Nucl. Phys. B*, 1981, V.185, 513.
- [10] Quesne C. and Vansteenkiste N., C_λ -extended oscillator algebras and some of their deformations and applications to quantum mechanics, preprint ULB/229/CQ/99/1, submitted to *J. Math. Phys.*
- [11] Quesne C. and Vansteenkiste N., *Helv. Phys. Acta*, 1999, V.72, 71.
- [12] Gendenshtein L.E., *JETP Lett.*, 1983, V.38, 356.
- [13] Sukhatme U.P., Rasinariu C. and Khare A., *Phys. Lett. A*, 1997, V.234, 401.
- [14] Gangopadhyaya A. and Sukhatme U.P., *Phys. Lett. A*, 1996, V.224, 5.
- [15] Plyushchay M.S., *Ann. Phys. (N.Y.)*, 1996, V.245, 339.
- [16] Beckers J., Debergh N. and Nikitin A.G., *Int. J. Theor. Phys.*, 1997, V.36, 1991.
- [17] Rubakov V.A. and Spiridonov V.P., *Mod. Phys. Lett. A*, 1988, V.3, 1337.
- [18] Khare A., *J. Math. Phys.*, 1993, V.34, 1277.
- [19] Quesne C. and Vansteenkiste N., *Czech. J. Phys.*, 1998, V.48, 1477.
- [20] Beckers J. and Debergh N., *Nucl. Phys. B*, 1990, V.340, 767.
- [21] Beckers J., Debergh N. and Nikitin A.G., *Fortschr. Phys.*, 1995, V.43, 67, 81; Beckers J. and Debergh N., *Int. J. Mod. Phys. A*, 1995, V.10, 2783; in Second International Workshop on Harmonic Oscillators, Cocoyoc, Morelos, Mexico, March 23–25, 1994, edited by D. Han and K.B. Wolf, NASA Conference Publication 3286 (NASA Goddard Space Center, Greenbelt, MD, 1995), p. 313.
- [22] Mishra A.K. and Rajasekaran G., *Pramana – J. Phys.*, 1991, V.36, 537; 1991, V.37, 455(E).
- [23] Khare A., Mishra A.K. and Rajasekaran G., *Int. J. Mod. Phys. A*, 1993, V.8, 1245.