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# On Nonlinear Deformations of Lie Algebras and their Applications in Quantum Physics

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The $sl(2,R)$-Lie algebra is the one of the simplest Lie algebras dealing with particularly important concepts in *quantum* physics, i.e. the angular momentum theory. Taken as an example, we then study some of its specific *polynomial* deformations leading to quadratic and cubic nonlinearities appearing inside symmetry algebras of recent interest in conformal field theory and quantum optics. The determination of their finite-dimensional representations in terms of *differential* operators is then discussed and their interest in connection with multi-boson Hamiltonians is pointed out.

## 1 Introduction

Already introduced in the proceedings of the *second* conference [1], the role of the linear simple $sl(2,R)$-Lie algebra is very well understood by physicists and mathematicians due mainly to its interest in connection with the famous angular momentum theory [2, 3] when *quantum* aspects of physics are considered. There, I have reported on some new results already published elsewhere [4–7] obtained in the characterization of irreducible representations of finite dimensions *but* for the so-called "*nonlinear*" $sl(2,R)$-algebras with a particular emphasis on the Higgs algebra [8, 9] which is frequently mentioned as a *cubic* deformation of $sl(2,R)$.

Here I also want to insist on another approach of such finite-dimensional irreducible representations characterizing these "*nonlinear*" $sl(2,R)$-algebras by coming on already published [10] and not yet published [11] results dealing more particularly with *differential* realizations of the generators. These polynomial deformations of $sl(2,R)$, in prolongation of well known results obtained in the linear context by Turbiner [12, 13] or (and) Ushveridze [12, 14] in particular, will be of special interest for the study of multi-boson Hamiltonians introduced in quantum *optical* models [15], for example. In fact, these nonlinear structures can play the role of "spectrum generating algebras" for such Hamiltonian descriptions and their irreducible representations can give us a lot of nice and meaningful contexts.

In *Section 2*, we recall a few interesting relations and information on well known results but go relatively quickly to *Section 3* for characterizing the differential forms of special interest for the generators of the structures we are visiting. In *Section 4*, the connection with optical models is proposed and the discussion of the multi-boson Hamiltonians is considered: it finally leads to conclusions on constructive developments associated with the Higgs algebra. Some considerations on supersymmetric properties are also pointed out by taking care of Witten's proposal [16] of supersymmetric quantum mechanics when two supercharges enter the game.

## 2 A brief survey of the "nonlinear" context

Our *"nonlinear"* $sl(2,R)$-algebras [4] are characterized by the typical commutation relation

$$[J_+, J_-] = f(J_3) = \sum_{p=0}^{N} \beta_p (2J_3)^{2p+1} \tag{1}$$

instead of the following one

$$[J_+, J_-] = 2J_3 \tag{2}$$

referring to the *linear* context, each of these relations being evidently supplemented by the usual commutators

$$[J_3, J_\pm] = \pm J_\pm. \tag{3}$$

In the latest context, the raising ($J_+$), lowering ($J_-$) and diagonal ($J_3$) operators act on vectors belonging to the well known orthogonal basis $\{\mid j, m\rangle\}$ [2, 3] in the following way

$$J_\pm \mid j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)} \mid j, m \pm 1\rangle, \tag{4}$$

$$J_3 \mid j, m\rangle = m \mid j, m\rangle, \tag{5}$$

where $j$ refers to the Casimir eigenvalues

$$C \mid j, m\rangle \equiv \left(\frac{1}{2}(J_+J_- + J_-J_+) + J_3^2\right) \mid j, m\rangle = j(j+1) \mid j, m\rangle \tag{6}$$

and takes the values $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ while $m$, in eq.(5), runs from $-j$ to $j$ giving the dimensions $(2j+1)$ to the irreducible representations of the linear $sl(2,R)$-context.

If the relation (1) is substituted to eq.(2), we then get [4, 5] the irreducible representations characterized by the following relations

$$J_+ \mid j, m\rangle = \sqrt{g(m)} \mid j, m + c\rangle, \tag{7}$$

$$J_- \mid j, m\rangle = \sqrt{g(m-c)} \mid j, m - c\rangle, \tag{8}$$

$$J_3 \mid j, m\rangle = \left(\frac{m}{c} + \gamma\right) \mid j, m\rangle, \tag{9}$$

where $c$ is a nonnegative and nonvanishing integer, $\gamma$ is a real scalar parameter while the function $g$ is $\gamma$- and $c$-dependent [4, 5].

A "nonlinear" typical context is the one corresponding to the (cubic) Higgs algebra [8] given in (1) by $N = 1$, $p = 0, 1$, $\beta_0 = 1$ and $\beta_1 = 8\beta$, $\beta$ being a real continuous parameter so that we have

$$f(J_3) = 2J_3 + 8\beta J_3^3. \tag{10}$$

All its finite-dimensional irreducible representations can be obtained by exploiting the corresponding actions (7), (8) and (9). In that way, we recover old well known results [8, 9, 17] but also find new ones in these angular momentum basis developments [4–7].

## 3 On polynomial deformations and differential realizations

If we search for *finite-dimensional* representations, the operators $J_+$, $J_-$ and $J_3$ have to act, for example, on the $(n+1)$-dimensional vector spaces $P(n) \equiv \{1, x, x^2, \ldots, x^n\}$ when *differential* realizations are prescribed. Such a point of view has already been adopted in the *linear* context [12–14] since the late eighties. More recently Fradkin [18] has proposed a nice way for discussing such purposes and we have extended his method to the *nonlinear* context [10].

By coming back on the example of the Higgs algebra characterized by the structure relations (1) and (3) but with the expression (10), we can realize the generators in the following way:

$$J_+ = x^N F(D), \qquad J_- = G(D)\frac{d^N}{dx^N}, \qquad J_0 = \frac{1}{N}\,(D+\alpha), \qquad N = 1, 2, 3, \ldots, \tag{11}$$

where $\alpha$ is a constant and

$$D \equiv x\frac{d}{dx}$$

is the dilatation operator which, due to the Heisenberg commutation relation

$$\left[\frac{d}{dx}, x\right] = 1$$

satisfies

$$\left[D, x^N\right] = Nx^N, \qquad \left[\frac{d^N}{dx^N}, D\right] = N\frac{d^N}{dx^N}.$$

Let us introduce also the Fradkin notations [18]

$$\frac{d^N}{dx^N}x^N = \prod_{k=1}^{N}(D+k) = \frac{(D+N)!}{D!}, \qquad x^N\frac{d^N}{dx^N} = \prod_{k=0}^{N-1}(D-k) = \frac{D!}{(D-N)!} \tag{12}$$

and notice that the relations (1), (10) and (11) imply the constraint

$$F(D-N)G(D-N)\frac{D!}{(D-N)!} - F(D)G(D)\frac{(D+N)!}{D!} = \frac{2}{N}(D+\alpha) + \frac{8\beta}{N^3}(D+\alpha)^3.$$

With the simplifying choice $G(D) = 1$, we get in the *cubic* context

$$F(D) = -f\frac{D!}{(D+N)!}(D+\lambda_1)(D+\lambda_2)(D+\lambda_3)(D+\lambda_4),$$

where $f = 2\beta N^{-4}$ and where the four $\lambda$'s have to satisfy the system

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 4\alpha + 2N,$$

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 = N^2 + 6\alpha N + 6\alpha^2 + \frac{N^2}{2\beta},$$

$$\lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_3\lambda_4 + \lambda_2\lambda_3\lambda_4 = 2\alpha N^2 + 6\alpha^2 N + 4\alpha^3 + \frac{\alpha N^2}{\beta} + \frac{N^3}{2\beta}.$$

*Nonsingular* realizations (look at the definitions (12)) only appear when $N = 1, 2, 3, 4$ and finite-dimensional representations are obtained only for the $N =$ 1- and 2-cases. These results are in perfect agreement with those obtained in previous developments [5, 6] but in the angular momentum basis rather than, here, in the $P(n)$-basis. In particular, when $N = 2$ and $\alpha = -\frac{n}{2}$ we recover specific families already quoted elsewhere [5–7].

# 4 Differential realizations and quantum optical Hamiltonians

Lie algebras being strongly related to (kinematical as well as dynamical) symmetries as everybody knows, it is interesting to learn about new symmetries from "nonlinear" Lie algebras dealing with physical models. This is the aim of this section by visiting more particularly *quantum optical models* subtended by typical multi-photon Hamiltonians already put in evidence for describing some scattering processes. We refer more particularly to Karassiov–Klimov proposals [15] which, in 2-dimensional flat spaces, considered the superposition of two harmonic oscillators. By taking care of $\omega_1 = \omega_2 = \omega$ at the level of their angular frequencies and of a real coupling constant $g$, the corresponding Hamiltonian can be written on the form with integers $m$ and $n$ $(0 \leq m \leq n)$:

$$H = \omega \left( a_1^\dagger a_1 + a_2^\dagger a_2 \right) + g \left( a_1^\dagger \right)^n a_2^m + \left( a_2^\dagger \right)^m a_1^n, \tag{13}$$

where the characteristics of the two harmonic oscillators are immediately fixed through the commutation relations

$$[a_j, a_k^\dagger] = \delta_{jk} I, \qquad [a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0, \qquad j, k = 1, 2.$$

An interesting result due to Debergh [5] is that the Higgs algebra can play the role of the *"spectrum generating algebra"* for the quantum optical model subtended by the Hamiltonian (13) iff $m = n = 2$, the raising and lowering operators being second powers of the linear ones and the diagonal $J_3$ being half of the linear one. In such a context, the deformation parameter is fixed by

$$\beta = -\frac{2}{2j^2 + 2j - 1}, \qquad j = 0, \frac{1}{2}, 1, \ldots$$

and the specific actions of the generators $J_+$, $J_-$ and $J_3$ become, in correspondence with eqs. (7)–(9) when the angular momentum basis is considered:

$$J_+ \mid j, m\rangle = ((j - m)(j + m + 1)(j - m - 1)(j + m + 2))^{\frac{1}{2}} \mid j, m + 2\rangle,$$

$$J_- \mid j, m\rangle = ((j + m)(j + m - 1)(j - m + 1)(j - m + 2))^{\frac{1}{2}} \mid j, m - 2\rangle,$$

$$J_3 \mid j, m\rangle = \frac{m}{2} \mid j, m\rangle.$$

For the whole set of $j$-values, we thus have the ($c = 2$ and $\gamma = 0$)-family of representations pointed out by Debergh [5] and simply related to meaningful physical models. Let us also mention that another interesting result, once again due to Debergh [6], is the *twofold* degeneracy of all the energy eigenvalues of the Hamiltonian (13) inside a Schrödinger-type (stationary) equation with the above characteristics of the Higgs algebra seen as the spectrum generating algebra of a quantum optical model. These degeneracies have moreover been interpreted as a property of supersymmetry in quantum mechanics [16] as it can be shown [6] through the construction of (two) supercharges generating with the Hamiltonian the *graded* Lie algebra $sqm(2)$.

In order to show such an interesting supersymmetric property, we have also considered [11] the *differential* realizations of the generators $J_\pm$ and $J_3$ and their introduction in the Hamiltonian operator. So, coming back to the $(n + 1)$-dimensional vector spaces $P(n)$ of polynomials of degree at most $n$ in the variable $x$, the Hamiltonian with arbitrary $N$ is found on the form

$$H_n^{(N)} = \omega n + g \left( \frac{d^N}{dx^N} + x^N (D - n)(D - n + 1) \ldots (D - n + N - 1) \right).$$

In our previous $N = 2$-context this gives

$$H_n^{(2)} = \omega n + g\left(\left(1 + x^4\right)\frac{d^2}{dx^2} + 2(1-n)x^3\frac{d}{dx} + n(n-1)x^2\right).$$

It is easy to see that these Hamiltonians preserve the spaces $P(n)$ and act invariantly on the subspaces $\epsilon(n) \equiv \{e_a(x)\}$ and $O(n) \equiv \{o_a(x)\}$ of even $(e_a)$ and odd $(o_a)$ polynomials of $P(n) = \epsilon(n) \oplus O(n)$.

It is remarkable that we get the following properties

$$H_n^{(2)} e_a(x) = E_a e_a(x) \qquad \text{and} \qquad H_n^{(2)} o_a(x) = E_a o_a(x)$$

with positive eigenvalues $E_a = \lambda_a^2$ pointing out immediately the double degeneracies. The existence of two supercharges $Q$ and $\bar{Q}$ becomes evident if we require

$$Qe_a = 0, \qquad Qo_a = \lambda_a e_a \qquad \text{and} \qquad \bar{Q}e_a = \lambda_a o_a, \qquad \bar{Q}o_a = 0.$$

Specific realizations of such supercharges have been proposed elsewhere [11] as well as some contexts for different even values of $N$. Supersymmetry is always present in these applications so that we have some hope that, as in *nuclear physics* [19] or in *atomic physics* [20], supersymmetry can also reveal its presence in some models of *quantum optics*.

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