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On Some New Classes of Separable Fokker–Planck Equations

Alexander ZHALIJ

Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivska Str., Kyiv, Ukraine
E-mail: zhaliy@imath.kiev.ua

We communicate some recent results on variable separation in the (1+3)-dimensional Fokker–Planck equations with a constant diagonal diffusion matrix.

The principal object of the study is a problem of separation of variables in the Fokker–Planck equation (FPE) with a constant diagonal diffusion matrix

$$u_t + \Delta u + (B_a(\vec{x})u)_{x_a} = 0, \tag{1}$$

where $\vec{B}(\vec{x}) = (B_1(\vec{x}), B_2(\vec{x}), B_3(\vec{x}))$ is the drift velocity vector. Here $u = u(t,\vec{x})$ and $B_i(\vec{x})$, $i = 1,2,3$ are smooth real-valued functions. Hereafter, summation over the repeated Latin indices from 1 to 3 is understood.

FPE (1) is a basic equation in the theory of continuous Markov processes. Therefore, it is widely used in different fields of physics, chemistry and biology [1], where stochastic methods are utilized.

We solve the problem of variable separation in FPE (1) into second-order ordinary differential equations in a sense that we obtain possible forms of the drift coefficients $B_1(\vec{x})$, $B_2(\vec{x})$, $B_3(\vec{x})$ providing separability of (1). Furthermore, we construct inequivalent coordinate systems enabling to separate variables in the corresponding FPEs.

Our analysis is based on the direct approach to variable separation in linear PDEs suggested in [3, 4]. It has been successfully applied to solving variable separation problem the Schrödinger equations [3, 4, 5] with variable coefficients.

For an alternative (symmetry) approach to separation of variables in FPE, see [2].

We say that FPE (1) is *separable* in a coordinate system t, $\omega_a = \omega_a(t,\vec{x})$, $a = 1,2,3$ if the separation Ansatz

$$u(t,\vec{x}) = \varphi_0(t) \prod_{a=1}^{3} \varphi_a\left(\omega_a(t,\vec{x}), \vec{\lambda}\right) \tag{2}$$

reduces PDE (1) to four ordinary differential equations for the functions φ_μ, $(\mu = 0,1,2,3)$

$$\varphi_0' = U_0\left(t, \varphi_0; \vec{\lambda}\right), \qquad \varphi_a'' = U_a\left(\omega_a, \varphi_a, \varphi_a'; \vec{\lambda}\right). \tag{3}$$

Here $U_0, \ldots, U_3$ are some smooth functions of the indicated variables, $\vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \Lambda = \{\text{an open domain in } \mathbf{R}^3\}$ are separation constants (spectral parameters, eigenvalues) and, what is more,

$$\operatorname{rank} \left\| \frac{\partial U_\mu}{\partial \lambda_a} \right\|_{\mu=0}^{3} {}_{a=1}^{3} = 3. \tag{4}$$

For more details, see our paper [5].

Next, we introduce an equivalence relation $\mathcal{E}$ on the set of all coordinate systems providing separability of FPE. We say that two coordinate systems t, ω_1, ω_2, ω_3 and $\tilde{t}$, $\tilde{\omega}_1$, $\tilde{\omega}_2$, $\tilde{\omega}_3$ are *equivalent* if the corresponding Ansatzes (2) are transformed one into another by the invertible transformations of the form

$$t \to \tilde{t} = f_0(t), \qquad \omega_i \to \tilde{\omega}_i = f_i(\omega_i), \tag{5}$$

where $f_0, \ldots, f_3$ are some smooth functions and $i = 1, 2, 3$. These equivalent coordinate systems give rise to the same solution with separated variables, therefore we shall not distinguish between them. The equivalence relation (5) splits the set of all possible coordinate systems into equivalence classes. In a sequel, when presenting the lists of coordinate systems enabling us to separate variables in FPE we will give only one representative for each equivalence class.

Following [5] we choose the reduced equations (3) to be

$$\varphi_0' = (T_0(t) - T_i(t)\lambda_i)\,\varphi_0, \qquad \varphi_a'' = (F_{a0}(\omega_a) + F_{ai}(\omega_a)\lambda_i)\,\varphi_a, \tag{6}$$

where T_0, T_i, F_{a0}, F_{ai} are some smooth functions of the indicated variables, $a = 1, 2, 3$. With this remark the system of nonlinear PDEs for unknown functions ω_1, ω_2, ω_3 takes the form

$$\frac{\partial \omega_i}{\partial x_a}\frac{\partial \omega_j}{\partial x_a} = 0, \qquad i \neq j, \quad i, j = 1, 2, 3; \tag{7}$$

$$\sum_{i=1}^{3} F_{ia}(\omega_i)\frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_i}{\partial x_j} = T_a(t), \qquad a = 1, 2, 3; \tag{8}$$

$$B_j\frac{\partial \omega_a}{\partial x_j} + \frac{\partial \omega_a}{\partial t} + \Delta\omega_a = 0, \qquad a = 1, 2, 3; \tag{9}$$

$$\sum_{i=1}^{3} F_{i0}(\omega_i)\frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_i}{\partial x_j} + T_0(t) + \frac{\partial B_a}{\partial x_a} = 0. \tag{10}$$

The system of equations (7), (8) has been integrated in [5]. Its general solution $\vec{\omega} = \vec{\omega}(t, \vec{x})$ is given implicitly by the following formulae:

$$\vec{x} = \mathcal{T}(t)H(t)\,\vec{z}(\vec{\omega}) + \vec{w}(t). \tag{11}$$

Here $\mathcal{T}(t)$ is the time-dependent 3×3 orthogonal matrix:

$$\mathcal{T}(t) = \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta\cos\gamma & -\cos\alpha\sin\beta - \sin\alpha\cos\beta\cos\gamma & \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta\cos\gamma & -\sin\alpha\sin\beta + \cos\alpha\cos\beta\cos\gamma & -\cos\alpha\sin\gamma \\ \sin\beta\sin\gamma & \cos\beta\sin\gamma & \cos\gamma \end{pmatrix}, \tag{12}$$

α, β, γ being arbitrary smooth functions of t; $\vec{z} = \vec{z}(\vec{\omega})$ is given by one of the eleven formulae

1. Cartesian coordinate system,

$$z_1 = \omega_1, \qquad z_2 = \omega_2, \qquad z_3 = \omega_3, \qquad \omega_1, \omega_2, \omega_3 \in \mathbf{R}.$$

2. Cylindrical coordinate system,

$$z_1 = e^{\omega_1}\cos\omega_2, \qquad z_2 = e^{\omega_1}\sin\omega_2, \qquad z_3 = \omega_3, \qquad 0 \le \omega_2 < 2\pi, \qquad \omega_1, \omega_3 \in \mathbf{R}.$$

3. Parabolic cylindrical coordinate system,

$$z_1 = (\omega_1^2 - \omega_2^2)/2, \qquad z_2 = \omega_1\omega_2, \qquad z_3 = \omega_3, \qquad \omega_1 > 0, \quad \omega_2, \omega_3 \in \mathbf{R}.$$

4. Elliptic cylindrical coordinate system,

$$z_1 = a\cosh\omega_1\cos\omega_2, \qquad z_2 = a\sinh\omega_1\sin\omega_2, \qquad z_3 = \omega_3,$$
$$\omega_1 > 0, \qquad -\pi < \omega_2 \le \pi, \qquad \omega_3 \in \mathbf{R}, \qquad a > 0.$$

5. Spherical coordinate system,

$$z_1 = \omega_1^{-1}\operatorname{sech}\omega_2\cos\omega_3, \qquad z_2 = \omega_1^{-1}\operatorname{sech}\omega_2\sin\omega_3, \qquad z_3 = \omega_1^{-1}\tanh\omega_2,$$
$$\omega_1 > 0, \qquad \omega_2 \in \mathbf{R}, \qquad 0 \le \omega_3 < 2\pi.$$

6. Prolate spheroidal coordinate system,

$$z_1 = a\operatorname{csch}\omega_1\operatorname{sech}\omega_2\cos\omega_3, \quad a > 0, \qquad z_2 = a\operatorname{csch}\omega_1\operatorname{sech}\omega_2\sin\omega_3,$$
$$z_3 = a\coth\omega_1\tanh\omega_2, \qquad \omega_1 > 0, \qquad \omega_2 \in \mathbf{R}, \qquad 0 \le \omega_3 < 2\pi. \tag{13}$$

7. Oblate spheroidal coordinate system,

$$z_1 = a\csc\omega_1\operatorname{sech}\omega_2\cos\omega_3, \quad a > 0, \qquad z_2 = a\csc\omega_1\operatorname{sech}\omega_2\sin\omega_3,$$
$$z_3 = a\cot\omega_1\tanh\omega_2, \qquad 0 < \omega_1 < \pi/2, \qquad \omega_2 \in \mathbf{R}, \qquad 0 \le \omega_3 < 2\pi.$$

8. Parabolic coordinate system,

$$z_1 = e^{\omega_1+\omega_2}\cos\omega_3, \qquad z_2 = e^{\omega_1+\omega_2}\sin\omega_3, \qquad z_3 = (e^{2\omega_1} - e^{2\omega_2})/2,$$
$$\omega_1, \omega_2 \in \mathbf{R}, \qquad 0 \le \omega_3 \le 2\pi.$$

9. Paraboloidal coordinate system,

$$z_1 = 2a\cosh\omega_1\cos\omega_2\sinh\omega_3, \quad a > 0, \qquad z_2 = 2a\sinh\omega_1\sin\omega_2\cosh\omega_3,$$
$$z_3 = a(\cosh 2\omega_1 + \cos 2\omega_2 - \cosh 2\omega_3)/2, \qquad \omega_1, \omega_3 \in \mathbf{R}, \qquad 0 \le \omega_2 < \pi.$$

10. Ellipsoidal coordinate system,

$$z_1 = a\,\frac{1}{\operatorname{sn}(\omega_1,\, k)}\operatorname{dn}(\omega_2,\, k')\operatorname{sn}(\omega_3,\, k), \quad a > 0, \quad k^2 + k'^2 = 1,$$
$$z_2 = a\,\frac{\operatorname{dn}(\omega_1,\, k)}{\operatorname{sn}(\omega_1,\, k)}\operatorname{cn}(\omega_2,\, k')\operatorname{cn}(\omega_3,\, k), \quad 0 < k, k' < 1,$$
$$z_3 = a\,\frac{\operatorname{cn}(\omega_1,\, k)}{\operatorname{sn}(\omega_1,\, k)}\operatorname{sn}(\omega_2,\, k')\operatorname{dn}(\omega_3,\, k),$$
$$0 < \omega_1 < K, \qquad -K' \le \omega_2 \le K', \qquad 0 \le \omega_3 \le 4K.$$

11. Conical coordinate system,

$$z_1 = \omega_1^{-1}\operatorname{dn}(\omega_2,\, k')\operatorname{sn}(\omega_3,\, k), \quad k^2 + k'^2 = 1, \quad 0 < k, k' < 1,$$
$$z_2 = \omega_1^{-1}\operatorname{cn}(\omega_2,\, k')\operatorname{cn}(\omega_3,\, k), \qquad z_3 = \omega_1^{-1}\operatorname{sn}(\omega_2,\, k')\operatorname{dn}(\omega_3,\, k),$$
$$\omega_1 > 0, \qquad -K' \le \omega_2 \le K', \qquad 0 \le \omega_3 \le 4K.$$

$H(t)$ is the 3×3 diagonal matrix

$$H(t) = \begin{pmatrix} h_1(t) & 0 & 0 \\ 0 & h_2(t) & 0 \\ 0 & 0 & h_3(t) \end{pmatrix}, \tag{14}$$

where

(a) $h_1(t)$, $h_2(t)$, $h_3(t)$ are arbitrary smooth functions for the completely split coordinate system (case 1 from (13)),

(b) $h_1(t) = h_2(t)$, $h_1(t)$, $h_3(t)$ being arbitrary smooth functions, for the partially split coordinate systems (cases 2–4 from (13)),

(c) $h_1(t) = h_2(t) = h_3(t)$, $h_1(t)$ being an arbitrary smooth function, for non-split coordinate systems (cases 5–11 from (13))

and $\vec{w}(t)$ stands for the vector-column whose entries $w_1(t)$, $w_2(t)$, $w_3(t)$ are arbitrary smooth functions of t.

Note that we have chosen the coordinate systems ω_1, ω_2, ω_3 with the use of the equivalence relation $\mathcal{E}$ (5) in such a way that the relations

$$\Delta\omega_a = 0, \qquad a = 1, 2, 3 \tag{15}$$

hold for all the cases 1–11 in (13). Solving (9) with respect to $B_j(\vec{x})$, $i = 1, 2, 3$ we get (see, also [5])

$$\vec{B}(\vec{x}) = \mathcal{M}(t)(\vec{x} - \vec{w}) + \dot{\vec{w}}. \tag{16}$$

Here we use the designation

$$\mathcal{M}(t) = \dot{\mathcal{T}}(t)\mathcal{T}^{-1}(t) + \mathcal{T}(t)\dot{H}(t)H^{-1}(t)\mathcal{T}^{-1}(t), \tag{17}$$

where $\mathcal{T}(t)$, $H(t)$ are variable 3×3 matrices defined by formulae (12) and (14), correspondingly, $\vec{w} = (w_1(t), w_2(t), w_3(t))^T$ and the dot over a symbol means differentiation with respect to t.

As the functions B_1, B_2, B_3 are independent of t, it follows from (16) that

$$\vec{B}(\vec{x}) = \mathcal{M}\vec{x} + \vec{v}, \qquad \vec{v} = \mathrm{const}, \tag{18}$$

$$\mathcal{M} = \mathrm{const}, \tag{19}$$

$$\dot{\vec{w}} = \mathcal{M}\vec{w} + \vec{v}. \tag{20}$$

Taking into account that $\dot{\mathcal{T}}\mathcal{T}^{-1}$ is antisymmetric and $\mathcal{T}\dot{H}H^{-1}\mathcal{T}^{-1}$ is symmetric part of $\mathcal{M}$ (17), correspondingly, we get from (19)

$$\dot{\mathcal{T}}(t)\mathcal{T}^{-1}(t) = \mathrm{const}, \tag{21}$$

$$\mathcal{T}(t)\dot{H}(t)H^{-1}(t)\mathcal{T}^{-1}(t) = \mathrm{const}. \tag{22}$$

Relation (21) yields the system of three ordinary differential equations for the functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$

$$\begin{aligned} &\dot{\alpha} + \dot{\beta}\cos\gamma = C_1,\\ &\dot{\beta}\cos\alpha\sin\gamma - \dot{\gamma}\sin\alpha = C_2,\\ &\dot{\beta}\sin\alpha\sin\gamma + \dot{\gamma}\cos\alpha = C_3, \end{aligned} \tag{23}$$

where C_1, C_2, C_3 are arbitrary real constants. Integrating the above system we obtain the following form of the matrix $\mathcal{T}(t)$:

$$\mathcal{T}(t) = \mathcal{C}_1\tilde{\mathcal{T}}\mathcal{C}_2, \tag{24}$$

where $\mathcal{C}_1$, $\mathcal{C}_2$ are arbitrary constant 3×3 orthogonal matrices and

$$\tilde{\mathcal{T}} = \begin{pmatrix} -\cos s\cos bt & \sin s & \cos s\sin bt\\ \sin bt & 0 & \cos bt\\ \sin s\cos bt & \cos s & -\sin s\sin bt \end{pmatrix} \tag{25}$$

with arbitrary constants b and s.

The substitution of equality (24) into (22) with subsequent differentiation of the obtained equation with respect to t yields

$$\mathcal{C}_2^{-1}\tilde{\mathcal{T}}^{-1}\dot{\tilde{\mathcal{T}}}\,\mathcal{C}_2 L + \dot{L} + L\,\mathcal{C}_2^{-1}(\dot{\tilde{\mathcal{T}}^{-1}})\tilde{\mathcal{T}}\,\mathcal{C}_2 = 0, \tag{26}$$

where $L = \dot{H}H^{-1}$, i.e. $l_i = \dot{h}_i/h_i$, $i = 1, 2, 3$. From (26) we have

$$\begin{aligned}
&l_i = \text{const}, \qquad i = 1, 2, 3;\\
&b\,(l_1 - l_2)\cos\alpha_2\,\sin\gamma_2 = 0,\\
&b\,(l_1 - l_3)(-\sin\alpha_2\,\sin\beta_2 + \cos\alpha_2\,\cos\beta_2\cos\gamma_2) = 0,\\
&b\,(l_2 - l_3)(\sin\alpha_2\,\cos\beta_2 + \cos\alpha_2\,\sin\beta_2\cos\gamma_2) = 0,
\end{aligned} \tag{27}$$

where α_2, β_2, γ_2 are the Euler angles for the orthogonal matrix $\mathcal{C}_2$. Thus we obtain the following forms of h_i:

$$h_i = c_i\exp(l_i t), \qquad c_i = \text{const}, \quad l_i = \text{const}, \quad i = 1, 2, 3. \tag{28}$$

From (27) we get the possible forms of b, l_i and $\mathcal{C}_2$:

$$\begin{aligned}
&(i) \quad b = 0, \quad l_1, l_2, l_3 \text{ are arbitrary constants,}\\
&\qquad \mathcal{C}_2 \text{ is an arbitrary constant orthogonal matrix;}\\
&(ii) \quad b \neq 0, \quad l_1 = l_2 = l_3,\\
&\qquad \mathcal{C}_2 \text{ is an arbitrary constant orthogonal matrix;}\\
&(iii) \quad b \neq 0, \quad l_1 = l_2 \neq l_3, \quad \mathcal{C}_2 = \begin{pmatrix} \varepsilon_1\cos\theta & -\varepsilon_1\sin\theta & 0\\ 0 & 0 & -\varepsilon_1\varepsilon_2\\ \varepsilon_2\sin\theta & \varepsilon_2\cos\theta & 0\end{pmatrix},
\end{aligned} \tag{29}$$

where ε_1, $\varepsilon_2 = \pm 1$, and θ is arbitrary constant. We do not adduce cases $b \neq 0$, $l_1 \neq l_2 = l_3$ and $b \neq 0$, $l_2 \neq l_1 = l_3$ because they are equivalent to case (iii).

Finally, we give a list of the drift velocity vectors $\vec{B}(\vec{x})$ providing separability of the corresponding FPEs. They have the following form:

$$\vec{B}(\vec{x}) = \mathcal{M}\vec{x} + \vec{v},$$

where $\vec{v}$ is arbitrary constant vector and $\mathcal{M}$ is constant matrix given by one of the following formulae:

1. $\mathcal{M} = \mathcal{T}L\mathcal{T}^{-1}$, where $L = \begin{pmatrix} l_1 & 0 & 0\\ 0 & l_2 & 0\\ 0 & 0 & l_3\end{pmatrix}$, l_1, l_2, l_3 are constants and $\mathcal{T}$ is an arbitrary constant 3×3 orthogonal matrix, i.e. $\mathcal{M}$ is a real symmetric matrix with eigenvalues l_1, l_2, l_3.

   (a) l_1, l_2, l_3 are all distinct. The new coordinates ω_1, ω_2, ω_3 are given implicitly by formula

   $$\vec{x} = \mathcal{T}H(t)\,\vec{z}(\vec{\omega}) + \vec{w}(t), \tag{30}$$

   where $\vec{z}(\vec{\omega})$ is given by formula 1 from (13), $\vec{w}(t)$ is solution of system of ordinary differential equations (20) and

   $$H(t) = \begin{pmatrix} c_1e^{l_1t} & 0 & 0\\ 0 & c_2e^{l_2t} & 0\\ 0 & 0 & c_3e^{l_3t}\end{pmatrix} \tag{31}$$

   with arbitrary constants c_1, c_2, c_3.

(b) $l_1 = l_2 \neq l_3$. The new coordinates ω_1, ω_2, ω_3 are given implicitly by (30), where $\vec{z}(\vec{\omega})$ is given by one of the formulae 1–4 from (13) and $H(t)$ is given by (31) with arbitrary constant c_1, c_2, c_3 satisfying the condition $c_1 = c_2$ for the partially split coordinates 2–4 from (13).

(c) $l_1 = l_2 = l_3$, i.e. $M = l_1 I$, where I is unit matrix. The new coordinates ω_1, ω_2, ω_3 are given implicitly by formula (30), where $\vec{z}(\vec{\omega})$ is given by one of the eleven formulae (13) and $H(t)$ is given by (31) with arbitrary constants c_1, c_2, c_3 satisfying the condition $c_1 = c_2$ for the partially split coordinates 2–4 from (13) and the condition $c_1 = c_2 = c_3$ for the non-split coordinates 5–11 from (13).

2. $M = b\,\mathcal{C}_1 \begin{pmatrix} 0 & \cos s & 0 \\ -\cos s & 0 & \sin s \\ 0 & -\sin s & 0 \end{pmatrix} \mathcal{C}_1^{-1} + l_1 I$, where I is the unit matrix and $\mathcal{C}_1$ is an arbitrary constant 3×3 orthogonal matrix, b, s, l_1 are arbitrary constants and $b \neq 0$. The new coordinates ω_1, ω_2, ω_3 are given implicitly by formula (11), where $\vec{z}(\vec{\omega})$ is given by one of the eleven formulae (13), $\mathcal{T}(t)$ is given by (24)–(25), $\vec{w}(t)$ is solution of system of ordinary differential equations (20) and $H(t) = \exp(l_1 t) \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}$ with arbitrary constants c_1, c_2, c_3 satisfying the condition $c_1 = c_2$ for the partially split coordinates 2–4 from (13) and the condition $c_1 = c_2 = c_3$ for non-split coordinates 5–11 from (13).

3. $M = \mathcal{C}_1 \begin{pmatrix} \frac{1}{2}(l_1 + l_3 + (l_1 - l_3)\cos 2s) & b\cos s & \frac{1}{2}(l_3 - l_1)\sin 2s \\ -b\cos s & l_1 & b\sin s \\ \frac{1}{2}(l_3 - l_1)\sin 2s & -b\sin s & \frac{1}{2}(l_1 + l_3 - (l_1 - l_3)\cos 2s) \end{pmatrix} \mathcal{C}_1^{-1}$, where $\mathcal{C}_1$ is an arbitrary constant 3×3 orthogonal matrix, b, s, l_1, l_2 are arbitrary constants, $l_1 \neq l_3$ and $b \neq 0$. The new coordinates ω_1, ω_2, ω_3 are given implicitly by formula (11), where $\vec{z}(\vec{\omega})$ is given by one of the formulae 1–4 from (13), $\mathcal{T}(t)$ is given by (24), (25) and (iii) from (29), $\vec{w}(t)$ is solution of system of ordinary differential equations (20) and

$$H(t) = \begin{pmatrix} c_1 e^{l_1 t} & 0 & 0 \\ 0 & c_2 e^{l_1 t} & 0 \\ 0 & 0 & c_3 e^{l_3 t} \end{pmatrix}$$

with arbitrary constants c_1, c_2, c_3 satisfying the condition $c_1 = c_2$ for the partially split coordinates 2–4 from (13).

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