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On Symmetry of a Class of First Order PDEs Equations

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The problem of group classiffication for the class of first-order scalar PDEs invariant under the Euclid algebra $E(n)$ in considered. We found new nonlinear equations of the form $u_a u_a = F(u_t)$ with wide symmetry properties.

In this paper we study group classification of a class of nonlinear first-order multidimensional equations

$$u_t = \Phi(u, u_a u_a). \tag{1}$$

$u_a u_a$ is a designation for the sum

$$\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 + \ldots + \left(\frac{\partial u}{\partial x_n}\right)^2, \qquad u_t = \frac{\partial u}{\partial t}.$$

u is a scalar function of time t and n spatial variables $(x_1, x_2, \ldots, x_n)$. The class (1) includes many well-known equations with wide symmetry properties.

We will not consider cases when $n < 3$. It will be more convenient to investigate the class (1) in the form

$$u_a u_a = F(u, u_t). \tag{2}$$

The function F is assumed to be sufficiently smooth.

Why is it interesting to study symmetries for this particular class of equations? First, it is the general class of first order PDEs, that includes many physically interesting equations. It is interesting to find new equations invariant under known symmetry algebras and new symmetry algebras. Invariant first-order equations can be used for study of conditional symmetry of higher order PDEs. First order PDEs may also have interesting generalizations.

The class of equations (2) includes such well-known equations with wide symmetries as the eikonal equation, the Hamilton–Jacobi and the Hamilton equations.

The Hamilton–Jacobi equation

$$u_t + u_a u_a = 0 \tag{3}$$

is invariant under the Galilei group. Its maximal Lie invariance algebra was studied in [4] and can be described by the following basis operators:

$$P_0 = \partial_t, \qquad P_a = \partial_a, \qquad P_u = \partial_u, \qquad J_{ab} = x_a \partial_b - x_b \partial_a, \qquad G_a^{(1)} = t\partial_a + \frac{1}{2} x_a \partial_u,$$

$$D^{(1)} = t\partial_0 + \frac{1}{2} x_a \partial_a, \qquad A^{(1)} = t^2 \partial_0 + t x_a \partial_a + \frac{1}{4} x_a x_a \partial_u, \qquad G_a^{(2)} = u\partial_a + \frac{x_a}{2} \partial_t,$$

$$D_a^{(2)} = u\partial_u + \frac{1}{2}x_a\partial_a, \qquad A^{(2)} = u^2\partial_u + ux_a\partial_a + \frac{1}{4}x^2\partial_t,$$
$$K_a = 2x_a\left(D^{(1)} + D^{(2)}\right) + \left(\frac{1}{4}tu - x^2\right)\partial_a \qquad \left(x^2 \equiv x_a x_a\right).$$

The equation (3) is also invariant under a discrete transformation $u \to t$, $t \to u$.

Symmetry of the relativistic Hamilton equation

$$u_\alpha u_\alpha = 1 \tag{4}$$

was studied in [1, 5]. Here

$$u_\alpha u_\alpha \equiv u_0^2 - u_1^2 - \ldots - u_n^2,$$

$u_0 \equiv u_t$.

The maximal Lie invariance group of the equation (4) is the conformal group $C(1, n+1)$.

Basis elements for the corresponding Lie algebra can be written as follows:

$$\partial_A = ig_{AB}\frac{\partial}{\partial x_B}, \qquad g_{AB} = \mathrm{diag}\,(1, -1, \ldots, -1),$$
$$J_{AB} = x_A\partial_B - x_B\partial_A, \qquad D = x_A\partial_A, \qquad K_A = 2x_A D - x_B x_B\partial_A,$$

where $A, B = 0, 1, 2, \ldots, n+1$; $x_{n+1} \equiv u$, summation over the repeated indices is as follows:

$$x_A x_A = x_0^2 - x_1^2 - x_2^2 - \cdots - x_{n+1}^2.$$

The eikonal equation

$$u_\alpha u_\alpha = 0, \tag{5}$$

$\alpha = 0, 1, \ldots, n$; is invariant [1, 5] under an infinite-dimensional algebra, defined by operators

$$X = (b^{\mu\nu}x_\nu + a^\mu)\partial_\mu + \eta\partial_u,$$

where $b^{\mu\nu} = -b^{\nu\mu}$, a^μ, η are arbitrary differentiable functions on u;

$$\partial_\alpha = ig_{\alpha\beta}\frac{\partial}{\partial x_B}, \qquad g_{\alpha\beta} = \mathrm{diag}\,(1, -1, \ldots, -1).$$

The class of equations we consider will be a natural generalization of equations (3)–(5).

We look for a Lie symmetry operator of the equation (2) in the form

$$X = \xi^t(t, x_a, u)\partial_t + \xi^a(t, x_b, u)\partial_{xa} + \eta(t, x_a, u)\partial_u. \tag{6}$$

The general Lie invariance condition is

$$\overset{1}{X}(u_a u_a - F(u, u_t))\Big|_{u_a u_a = F(u, u_t)} = 0, \tag{7}$$

where $\overset{1}{X}$ is the first Lie prolongation for the operator X.

The condition (7) gives the the following determining equations for operators of invarianse algebra of the equation (2):

$$\xi_b^a + \xi_a^b = 0, \quad b \neq a; \qquad \xi_a^a = \xi_b^b \tag{8}$$

(we will designate $\xi^a_a = d(x_a, t, u)$);

$$2(\eta_a - \xi^t_a u_t - \xi^a_u F) + F_{u_t}(\xi^a_t + \xi^a_u u_t) = 0; \tag{9}$$

$$2F(\eta_u - d - \xi^t_u u_t) = \eta F_u + F_{u_t}(\eta_t + (\eta_u - \xi^t_t)u_t - \xi^t_u u_t^2). \tag{10}$$

Lower indices always designate corresponding derivatives.

Determining equations (8) are fulfilled for all equations from the class (2). From (8) we get the following form for coefficients ξ^a of the operator X (6):

$$\xi^a = c_a + \tilde{d}x_a + \lambda_{ab}x_b + 2k_b x_b x_a - k_a x_b x_b, \tag{11}$$

where $\lambda_{ab} = -\lambda_{ba}$, c_a, $\tilde{d}$, k_a are functions on u and t.

The following operators are symmetry operators for all equations from the class (2) irrespective of the form of the function $F(u, u_t)$:

$$P_t = \partial_t, \qquad P_a = \partial_a, \qquad J_{ab} = x_a\partial_b - x_b\partial_a, \tag{12}$$

that form the basis of the Euclid algebra $E(n)$ in the space of n variables $x_1, \ldots, x_n$, plus the translation operator by time variable.

Now we look for equations from the class (2) admitting wider symmetry than the algebra (12). We need to find functions F for which the conditions (9), (10) are fulfilled with some coefficients being non-zero.

From the determining equation (9) we conclude that there are two options:

$$I.\ \eta_a = \xi^t_a = \xi^a_u = \xi^a_t = 0, \tag{13}$$

and $F = F(u, u_t)$ is determined by the equation (10).

$$II.\ F = r(u)u_t^2 + s(u)u_t + q(u). \tag{14}$$

The class (2) with F having the form (14) includes all well-known equations (3)–(5).

Let us consider the first option in detail. It follows from the conditions (13) that

$$\eta = \eta(t, u), \qquad \xi^t = \xi^t(t, u), \qquad \xi^a = \xi^a(x_1, \ldots, x_n).$$

The equation (10) takes the form

$$2F(\eta_u - d - \xi^t_u u_t) = \eta F_u + F_{u_t}\left(\eta_t + \eta_u u_t - \xi^t_t u_t - \xi^t_u u_t^2\right). \tag{15}$$

As $d_u = 0$, we conclude from (15) that $d = \text{const}$, and the expression for the coefficients ξ^a can only take the form

$$\xi^a = c^a + dx_a + \lambda_{ab}x_b,$$

where $\lambda_{ab} = -\lambda_{ba}$, d, c_a are constants.

There will be no conformal or projective symmetry operators in this case.

We adduce some new equations with additional symmetry to (12). For example, if we put $\eta_u = \xi^t_t$, then in the case $\xi^t_u \cdot \eta_t < 0$ we get the function F of the form

$$F = \left(1 + u_t^2\right)\exp(\lambda \operatorname{arctg} u_t), \qquad \lambda = \text{const}. \tag{16}$$

In the case $\xi^t_u \cdot \eta_t > 0$ we get $F = (a + bu_t)^2$ (a, b are constants) from the class (14).

The equation

$$u_a u_a = \left(1 + u_t^2\right)\exp(\lambda \operatorname{arctg} u_t) \tag{17}$$

has three additional symmetry operators of the form

$$\partial_u, \qquad -u\partial_t + t\partial_u - \frac{\lambda}{2}x_a\partial_a, \qquad u\partial_u + t\partial_t + x_a\partial_a.$$

It is interesting to note that the change $u \to t$, $t \to u$ leaves the equation (17) invariant up to the change of λ. In this aspect this equation is similar to the Hamilton–Jacobi equation (3).

There are other examples of equations of the form

$$u_a u_a = F(u_t)$$

with additional to (12) symmetry operators:

1. $u_a u_a = u_t^k$.

If $k \neq 0$, $k \neq 1$, $k \neq 2$, we get three additional operators:

$$t\partial_t + u\partial_u + x_a\partial_a, \ \partial_u, \qquad kx_a\partial_a + 2t\partial_t.$$

2. $u_a u_a = \exp u_t$.

We get two additional symmetry operator

$$\partial_u, \qquad 2t\partial_u - x_a\partial_a.$$

Summary. We studied the problem of the group classification for the equation (2). Determining equations for the function F were found, and some partial solutions for these equations constructed. Further research will be required for description of all nonequivalent equations of the form (2) that have additional invariance operators compared to space rotations and space and time translations. Other research opportunities in this respect include investigation of higher order PDEs invariant under the some algebras, of conditional symmetry of second-order PDEs with new equations as additional conditions.

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