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Symmetry Groups and Conservation Laws in Structural Mechanics

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Recent results concerning the application of Lie transformation group methods to structural mechanics are presented. Focus is placed on the point Lie symmetries and conservation laws inherent to the Bernoulli–Euler and Timoshenko beam theories as well as to the Marguerre-von Kármán equations describing the large deflection of thin elastic shallow shells within the framework of the nonlinear Donnell–Mushtari–Vlasov theory.

1 Introduction

The present paper is concerned with the invariance properties (point Lie symmetries) of three classes of self-adjoint partial differential equations arising in structural mechanics – the dynamic beam equations of Bernouli–Euler and Timoshenko type governing vibration of beams on a variable elastic foundation and dynamic stability of fluid conveying pipes, and Marguerre-von Kármán equations describing the large deflection of thin isotropic elastic shallow shells subjected to an external transverse load and a nonuniform heating.

Once the invariance properties of a given differential equation are established, several important applications are available. First, it is possible to obtain classes of group-invariant solutions. For a self-adjoint equation another application of its symmetries arises since it is the Euler–Lagrange equation of a certain functional. If a symmetry group of such an equation turned out to be its variational symmetry as well, that is a symmetry of the associated functional, then Noether's theorem guarantees the existence of a conservation law for the solutions of this equation. Needless to recall the fundamental role of the conserved quantities and conservation laws (or the corresponding balance laws) for the natural sciences, however it is worthy to point out that the available conservation (balance) laws should not be overlooked in the examination of discontinuous solutions (acceleration waves, shock waves, etc.) or in the numerical analysis (when constructing finite difference schemes or verifying numerical results, for instance) of any system of differential equations of physical interest. It should be remarked also that the path-independent integrals (such as the well known J-, L- and M-integrals) related to the conservation laws are basic tools in fracture analysis of solids and structures.

Throughout this paper: Greek (Latin) indices have the range 1, 2 (1, 2, 3), unless explicitly stated otherwise, and the usual summation convention over a repeated index is employed. The k-th order partial derivatives of a dependent variable, say w, that is $\partial^k w/\partial x^{\alpha_1}\partial x^{\alpha_2}\ldots\partial x^{\alpha_k}$ $(k,\alpha_1,\alpha_2,\ldots,\alpha_k = 1,2,\ldots)$, are denoted either by $w_{\alpha_1\alpha_2\ldots\alpha_k}$ or $w_{x^{\alpha_1}x^{\alpha_2}\ldots x^{\alpha_k}}$, where $x^1, x^2, \ldots$ are the independent variables. A similar notation is used for the partial derivatives of any other function, say f, of the independent variables but, in this case, the indices indicating the differentiation are preceded by a coma. D_α $(\alpha = 1,2,\ldots)$ denote the total derivative operators. For the basic notions and statements used in the group analysis of differential equations and variational problems see [1] or [2].

2 Symmetries and conservation laws of beam equations

Bernoulli–Euler beams. Consider the class of self-adjoint partial differential equations

$$\gamma w_{1111} + \chi^{\alpha\beta} w_{\alpha\beta} + \kappa(x) w = 0, \tag{1}$$

in two independent variables $x = (x^1, x^2)$ and one dependent variable $w(x)$, where $\gamma = \text{const} \neq 0$, $\chi^{\alpha\beta}$ are arbitrary constants, and $\kappa(x)$ is an arbitrary smooth function. Equations of this special type are used to study problems concerning dynamics and stability of both elastic beams resting on elastic foundations and pipes conveying fluid. In these cases, x^1 is associated with the spatial variable along the rod axis, x^2 – with the time, and w represents the transversal displacement field.

In [3] the point Lie symmetries of (1) are examined and the solution of the corresponding group-classification problem with respect to the arbitrary element $\{\gamma, \chi^{\alpha\beta}, \kappa(x)\}$ is given. Evidently, each equation of form (1) is invariant under the point Lie groups generated by the vector fields $X_0 = w\partial/\partial w$ and $X_u = u(x)\,\partial/\partial w$, where $u(x)$ is any smooth solution of the respective equation. The results of the group-classification are summarized in Table 1 below, where the equations invariant under larger groups are given through their coefficients together with the generators of the associated symmetry groups.

Table 1

#	Coefficients	Generators
1	$\kappa(x) = f(\beta^2 x^1 - \beta^1 x^2)$	$\beta^1 X_1 + \beta^2 X_2$
2	$\chi^{22} \neq 0,\ \det(\chi^{\alpha\beta}) = 0,\ \kappa(x) = \left(\beta^2 + x^2\right)^{-2} f(y),$ $y = \left(\beta^2 + x^2\right)^{-1/2} \left\{\beta^1 + x^1 - \chi x^2\right\}$	$\{\beta^1 + 2\chi\beta^2\}X_1$ $+2\beta^2 X_2 + X_3$
3	$\chi^{22} = 0,\ \det(\chi^{\alpha\beta}) \neq 0,\ \kappa(x) = \left(\beta^2 + x^2\right)^{-4/3} f(y),$ $y = \left(\beta^2 + x^2\right)^{-1/3} \left\{\beta^1 + 2x^1 - \left(\chi^{11}/\chi^{12}\right) x^2\right\}$	$\{\beta^1 + 3(\chi^{11}/\chi^{12})\beta^2\}X_1$ $+6\beta^2 X_2 + 2\tilde{X}_3$
4	$\chi^{22} \neq 0,\ \det(\chi^{\alpha\beta}) = 0,\ \kappa(x) = \kappa_0 \left(\beta + x^2\right)^{-2},$	$X_1,\ 2\beta X_2 + X_3$
5	$\chi^{22} = 0,\ \det(\chi^{\alpha\beta}) \neq 0,\ \kappa(x) = \kappa_0 \left(\beta + x^2\right)^{-4/3}$	$X_1,\ 3\beta X_2 + \tilde{X}_3$
6	$\chi^{22} \neq 0,\ \det(\chi^{\alpha\beta}) = 0,$ $\kappa(x) = \kappa_0 \left(\beta + x^1 - \chi x^2\right)^{-4}$	$\beta X_1 + X_3,$ $\chi X_1 + X_2$
7	$\chi^{22} = 0,\ \det(\chi^{\alpha\beta}) \neq 0,$ $\kappa(x) = \kappa_0 \left(\beta + 2x^1 - \left(\chi^{11}/\chi^{12}\right) x^2\right)^{-4}$	$\beta X_1 + 2\tilde{X}_3,$ $(\chi^{11}/\chi^{12})X_1 + 2X_2$
8	$\chi^{22} \det(\chi^{\alpha\beta}) \neq 0,\ \kappa(x) = \text{const}$	$X_1,\ X_2$
9	$\chi^{22} \det(\chi^{\alpha\beta}) = 0,\ \kappa(x) = \text{const} \neq 0$	$X_1,\ X_2$
10	$\chi^{22} \neq 0,\ \det(\chi^{\alpha\beta}) = 0,\ \kappa(x) = 0$	$X_1,\ X_2,\ X_3$
11	$\chi^{22} = 0,\ \det(\chi^{\alpha\beta}) \neq 0,\ \kappa(x) = 0$	$X_1,\ X_2,\ \tilde{X}_3$

Here f is an arbitrary function, β, β^1, β^2 are arbitrary real constants, $\chi = \chi^{12}/\chi^{22}$ and $X_\alpha = \partial/\partial x^\alpha$, $X_3 = \left(x^1 + \chi x^2\right)\partial/\partial x^1 + 2x^2\partial/\partial x^2$, $\tilde{X}_3 = \left(x^1 + \chi x^2\right)\partial/\partial x^1 + 3x^2\partial/\partial x^2$.

It is found [3] that all vector fields quoted under numbers 1, 3, 5, 7, 8, 9 and 11 generate variational symmetries of the respective equations of form (1), while in case # 2 variational symmetries are associated with $\{\beta^1 + 2\chi\beta^2\}X_1 + 2\beta^2 X_2 + X_3 + (1/2)X_0$, in case # 4 – with X_1

and $2\beta X_2 + X_3 + (1/2)X_0$, in case # 6 – with $\chi X_1 + X_2$ and $\beta X_1 + X_3 + (1/2)X_0$, and in case # 10 – with X_1, X_2 and $X_3 + (1/2)X_0$.

Once the variational symmetries are identified, we derive the corresponding conservation laws. They are listed in Table 2 in the same order as in Table 1 using the notation:

$$
\begin{aligned}
B^1_{(1)} &= -(1/2)\left\{\gamma(2w_1w_{111} - w_{11}^2) + \chi^{11}w_1^2 - \chi^{22}w_2^2 + \kappa w^2\right\} - (1/2)(\chi^{2\mu}ww_\mu)_{,2}\,,\\
B^2_{(1)} &= -\chi^{2\mu}w_1w_\mu + (1/2)(\chi^{2\mu}ww_\mu)_{,1}\,,\\
B^1_{(2)} &= -\chi^{1\mu}w_2w_\mu + \gamma(w_2w_{111} - w_{11}w_{12}) - (1/2)(\gamma w_1w_{11} - \chi^{1\mu}ww_\mu)_{,2}\,,\\
B^2_{(2)} &= -(1/2)\left\{\gamma w_{11}^2 + \chi^{22}w_2^2 - \chi^{11}w_1^2 + \kappa w^2\right\} + (1/2)(\gamma w_1w_{11} - \chi^{1\mu}ww_\mu)_{,1}\,,\\
B^\alpha_{(3)} &= \left\{x^1 + \chi x^2\right\}B^\alpha_{(1)} + 2x^2B^\alpha_{(2)} + \chi^{\alpha\mu}ww_\mu + (1/2)\gamma\delta^{1\alpha}(ww_{111} - w_1w_{11}),\\
\tilde{B}^\alpha_{(3)} &= \left\{x^1 + (\chi^{11}/\chi^{12})x^2\right\}B^\alpha_{(1)} + 3x^2B^\alpha_{(2)}\\
&\quad + (1/2)\left\{\chi^{\alpha\mu}ww_\mu + \delta^{1\alpha}(\chi^{11}ww_1 + 2\chi^{12}ww_2 - \gamma w_1w_{11})\right\}.
\end{aligned}
$$

Table 2

#	Conservation laws
1	$D_\alpha\left\{\beta^1B^\alpha_{(1)} + \beta^2B^\alpha_{(2)}\right\} = 0$
2	$D_\alpha\left\{\left(\beta^1 + 2\chi\beta^2\right)B^\alpha_{(1)} + 2\beta^2B^\alpha_{(2)} + B^\alpha_{(3)}\right\} = 0$
3	$D_\alpha\left\{\left(\beta^1 + 3\chi\beta^2\right)B^\alpha_{(1)} + 6\beta^2B^\alpha_{(2)} + 2\tilde{B}^\alpha_{(3)}\right\} = 0$
4	$D_\alpha B^\alpha_{(1)} = 0,\quad D_\alpha\left\{2\beta B^\alpha_{(2)} + B^\alpha_{(3)}\right\} = 0$
5	$D_\alpha B^\alpha_{(1)} = 0,\quad D_\alpha\left\{3\beta B^\alpha_{(2)} + \tilde{B}^\alpha_{(3)}\right\} = 0$
6	$D_\alpha\left\{\beta B^\alpha_{(1)} + B^\alpha_{(3)}\right\} = 0,\quad D_\alpha\left\{\chi B^\alpha_{(1)} + B^\alpha_{(2)}\right\} = 0$
7	$D_\alpha\left\{\beta B^\alpha_{(1)} + 2\tilde{B}^\alpha_{(3)}\right\} = 0,\quad D_\alpha\left\{(\chi^{11}/\chi^{12})B^\alpha_{(1)} + 2B^\alpha_{(2)}\right\} = 0$
8	$D_\alpha B^\alpha_{(1)} = 0,\quad D_\alpha B^\alpha_{(2)} = 0$
9	$D_\alpha B^\alpha_{(1)} = 0,\quad D_\alpha B^\alpha_{(2)} = 0$
10	$D_\alpha B^\alpha_{(1)} = 0,\quad D_\alpha B^\alpha_{(2)} = 0,\quad D_\alpha B^\alpha_{(3)} = 0$
11	$D_\alpha B^\alpha_{(1)} = 0,\quad D_\alpha B^\alpha_{(2)} = 0,\quad D_\alpha\tilde{B}^\alpha_{(3)} = 0$

In addition, each equation (1) admits conservation laws of form

$$D_\alpha\{\chi^{\alpha\mu}(uw_\mu - u_{,\mu}w) + \delta^{1\alpha}\gamma(uw_{111} + u_{,11}w_1 - u_{,111}w - u_{,1}w_{11})\} = 0,$$

$u(x)$ being any solution of the equation considered.

Timoshenko beams. The Timoshenko beam equations

$$\varrho J\varphi_{tt} = EJ\varphi_{xx} + nGA(w_x - \varphi), \qquad \varrho Aw_{tt} = nGA(w_{xx} - \varphi_x), \tag{2}$$

describe the motion of beams accounting for the buckling of the beam cross-section. They are two coupled second order linear partial differential equations in two independent variables – the time t and the coordinate along the beam axis x, the dependent variables being $w(x,t)$ and

$\varphi(x,t)$, associated with the transversal displacement of the beam axis and the rotation angle, respectively. In these equations ϱ is the density of the beam material, E – the modulus of elasticity, G – the shear modulus, J and A – the moment of inertia and the area of the beam cross-section, n – a coefficient related to the buckling of the cross-section.

The generator X_H of each one-parameter group H, admitted by (2), has the form

$$X_H = C_1X_1 + C_2X_2 + C_3X_3 + X_S$$

(see [4, 5]), where C_i $(i = 1, 2, 3)$ are real constants, $(\tilde{w}, \tilde{\varphi})$ is a solution of (2), and

$$X_1 = \partial/\partial x, \quad X_2 = \partial/\partial t, \quad X_3 = w\partial/\partial w + \varphi\partial/\partial\varphi, \quad X_S = \tilde{w}(x,t)\partial/\partial w + \tilde{\varphi}(x,t)\partial/\partial\varphi.$$

Denoting $r_* = \operatorname{rank}(C_1, C_2, C_3w + \tilde{w}(x,t),\ C_3\varphi + \tilde{\varphi}(x,t))$, $r_{**} = \operatorname{rank}(C_1, C_2)$, where $\operatorname{rank}(\cdot)$ is the rank of the matrix in parentheses, the necessary conditions for existence of solutions to Timoshenko beam equations invariant under the transformations of H (i.e. H-invariant solutions) are of the form

$$r_* \le 2, \qquad r_{**} = r_*. \tag{3}$$

The inequality (3) holds for every choice of C_i, $\tilde{w}(x,t)$ and $\tilde{\varphi}(x,t)$, because r_* is either 1 or 0. There exist only two opportunities to satisfy the equality (3). They are $C_1^2+C_2^2 > 0$, if $r_* = 1$ or $C_1^2 + C_2^2 = 0$, if $r_* = 0$. The only interesting alternative here is the first one, because if $r_* = 0$, the group H consists of the identity only. Thus, we proved the following.

Proposition 1 [4, 5]. *H-invariant solutions of the Timoshenko beam equations exist only if the group generator X_H incorporates at least one of the vector fields X_1 or X_2 associated with the translations along the independent variables.*

The invariant of the group H with generator X_H could be obtained, seeking for solutions to the equation $X_H(f) = 0$. Examining the cases $C_1 \ne 0$ and $C_1 = 0$, we found the most general form of the H-invariant solutions of (2) to be

$$w(x,t) = [\overline{w}(y) + W(x,t)]\Sigma, \qquad \varphi(x,t) = [\overline{\varphi}(y) + \Phi(x,t)]\Sigma, \tag{4}$$

where $y = C_2x - C_1t$ and the functions $W(x,t)$ and $\Phi(x,t)$ are solutions of the equations

$$C_1W_x + C_2W_t = \tilde{w}(x,t)\Sigma^{-1}, \qquad C_1\Phi_x + C_2\Phi_t = \tilde{\varphi}(x,t)\Sigma^{-1}. \tag{5}$$

In (4) and (5) we denote $\Sigma = \exp(C_3x/C_1)$ if $C_1 \ne 0$, otherwise $\Sigma = \exp(C_3t/C_2)$. Equations (5) are first-order linear partial differential equations, so it is a simple matter to obtain their solutions once C_i, $\tilde{w}(x,t)$ and $\tilde{\varphi}(x,t)$ are specified.

The following basic conservation laws of densities A^t and fluxes A^x, that is

$$\partial A^t/\partial t + \partial A^x/\partial x = 0,$$

are found to hold on the smooth solutions of the Timoshenko beam equations [4, 5].

Table 3

w - translations	**transversal linear momentum**
$X_w = \frac{\partial}{\partial w}$	$A^t_w = \rho A w_t$, $A^x_w = nGA(w_x - \varphi)$
x - translations	**wave momentum**
$X_1 = \frac{\partial}{\partial x}$	$A^t_1 = \rho A w_x w_t + \rho J \varphi_x \varphi_t$, $A^x_1 = -\mathcal{E} - nGA(w_x - \varphi)\varphi$
time - translations	**energy**
$X_2 = \frac{\partial}{\partial t}$	$A^t_2 = \mathcal{E} = (1/2)\left\{EJ\varphi_x^2 + nGA(w_x - \varphi)^2 + \rho A w_t^2 + \rho J \varphi_t^2\right\}$ $A^x_2 = -nGA(w_x - \varphi)w_t - EJ\varphi_x\varphi_t$
	reciprocity relation
$X_S = \tilde{w}\frac{\partial}{\partial w} + \tilde{\varphi}\frac{\partial}{\partial \varphi}$	$\tilde{A}^t = \rho A(w\tilde{w}_t - w_t\tilde{w}) + \rho J(\varphi\tilde{\varphi}_t - \varphi_t\tilde{\varphi})$ $\tilde{A}^x = EJ(\varphi_x\tilde{\varphi} - \varphi\tilde{\varphi}_x) + nGA\{(w_x - \varphi)\tilde{w} - w(\tilde{w}_x - \tilde{\varphi})\}$

3 Marguerre-von Kármán equations

Marguerre-von Kármán (MvK) equations (see e.g. [6, 7, 8]) describe the large deflection of thin isotropic elastic shallow shells. They can be written in the form [7, 8]:

$$\begin{aligned} &D\Delta^2 W - \varepsilon^{\alpha\mu}\varepsilon^{\beta\nu} W_{\alpha\beta}\Phi_{\mu\nu} = P, \\ &\frac{1}{Eh}\Delta^2\Phi + \frac{1}{2}\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu} W_{\alpha\beta}W_{\mu\nu} = Q. \end{aligned} \tag{6}$$

Here, the independent variables are the coordinates $x = (x^1, x^2)$ on the shell middle-surface F supposed to be given by the equation $z = f(x^1, x^2)$, $(x^1, x^2) \in \Omega \subset \mathbf{R}^2$, where (x^1, x^2, z) is a fixed right-handed rectangular Cartesian coordinate system in the 3-dimensional Euclidean space in which the middle-surface F of the shell is embedded, and $f : \mathbf{R}^2 \to \mathbf{R}$ is a smooth function on a certain domain of interest Ω. The dependent variables are Airy's stress function Φ, and $W = w + f$, where w is the transversal displacement function. In (6): $\varepsilon^{\alpha\beta}$ is the alternating tensor of F; E, h and $D = Eh^3/\left[12\left(1-\nu^2\right)\right]$ are Young's modulus, thickness and bending rigidity of the shell, respectively, ν being Poisson's ratio; Δ is the Laplace–Beltrami operator on F;

$$P = D\delta^{\alpha\beta}\delta^{\mu\nu} f_{,\alpha\beta\mu\nu} + p, \qquad Q = \frac{1}{2}\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu} f_{,\alpha\beta} f_{,\mu\nu} + q$$

(functions p and q appear when the shell is subjected to an external transversal load and nonuniform heating). System (6) includes as a special case, with $f_{,\alpha\beta} = 0$, the well-known von Kármán equations for large deflection of plates.

Actually (6) describe the state of equilibrium of the shell, but introducing, according to d'Alembert principle, the inertia force $-\rho w_{33} = -\rho W_{33}$ in the right-hand side of the first MvK equation, w_{33} being the second derivative of the displacement field with respect to the time $t \equiv x^3$ and ρ – the mass per unit area of the shell middle-surface, one can extend (6) to describe the dynamic behaviour of shells. In this case we will speak about the time-dependent MvK equations, otherwise (6) will be referred to as the time-independent MvK equations. In both cases, the moment tensor $M^{\alpha\beta}$, membrane stress tensor $N^{\alpha\beta}$, and shear-force vector Q^α are given in terms of W and Φ by the expressions

$$\begin{aligned} &M^{\alpha\beta} = D\left\{(1-\nu)\delta^{\alpha\mu}\delta^{\beta\nu} + \nu\delta^{\alpha\beta}\delta^{\mu\nu}\right\}\left\{W_{\mu\nu} - f_{,\mu\nu}\right\}, \\ &N^{\alpha\beta} = \varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}\Phi_{\mu\nu}, \qquad Q^\alpha = M^{\alpha\mu}_{,\mu} + N^{\alpha\mu}\left\{W_\mu - f_{,\mu}\right\}. \end{aligned}$$

Symmetry groups. The following is known [11] for the symmetry groups of the homogeneous time-independent and time-dependent MvK equations.

Proposition 2. *The homogeneous time-independent MvK equations admit the group $G_{(S)}$ generated by the basic vector fields (operators):*

$$\begin{aligned} &Y_1 = \partial/\partial W,\ Y_2 = \partial/\partial x^1,\ Y_3 = \partial/\partial x^2,\ Y_4 = x^2\partial/\partial x^1 - x^1\partial/\partial x^2,\ Y_5 = x^1\partial/\partial\Phi, \\ &Y_6 = x^2\partial/\partial\Phi,\ Y_7 = \partial/\partial\Phi,\ Y_8 = x^1\partial/\partial W,\ Y_9 = x^2\partial/\partial W,\ Y_{10} = x^1\partial/\partial x^1 + x^2\partial/\partial x^2. \end{aligned}$$

Proposition 3. *The homogeneous time-dependent MvK equations admit the group $G_{(D)}$ generated by the basic vector fields:*

$$\begin{aligned} &X_1 = Y_1,\ X_2 = Y_2,\ X_3 = Y_3,\ X_4 = \partial/\partial x^3,\ X_5 = x^1\partial/\partial x^1 + x^2\partial/\partial x^2 + 2x^3\partial/\partial x^3, \\ &X_6 = Y_4,\ X_7 = x^1\partial/\partial W,\ X_8 = x^2\partial/\partial W,\ X_9 = x^3\partial/\partial W,\ X_{10} = x^1x^3\partial/\partial W, \\ &X_{11} = x^2x^3\partial/\partial W,\ X_{12} = x^1 f(x^3)\partial/\partial\Phi,\ X_{13} = x^2 g(x^3)\partial/\partial\Phi,\ X_{14} = h(x^3)\partial/\partial\Phi, \end{aligned}$$

where f, g and h are arbitrary functions depending on the time only.

As for the symmetries of the nonhomogeneous MvK equations, we proved that:

Proposition 4. *A nonhomogeneous time-independent MvK system is invariant under a vector field* Y *iff* $Y = c^j Y_j$ $(j = 1, \dots, 10)$*, where* c^j *are real constants, and*

$$2P\xi^\mu_{,\mu} + \xi^\mu P_{,\mu} = 0, \qquad 2Q\xi^\mu_{,\mu} + \xi^\mu Q_{,\mu} = 0, \qquad \xi^\alpha = Y(x^\alpha), \tag{7}$$

Y *being regarded as an operator acting on the functions* $\zeta : \Omega \to \mathbf{R}$, $\Omega \subset \mathbf{R}^2$.

Proposition 5. *A nonhomogeneous time-dependent MvK system is invariant under a vector field* X *iff* $X = C^j X_j$ $(j = 1, \dots, 14)$*, where* C^j *are real constants, and*

$$P\xi^i_{,i} + \xi^i P_{,i} = 0, \qquad Q\xi^i_{,i} + \xi^i Q_{,i} = 0, \qquad \xi^i = X(x^i), \tag{8}$$

X *being regarded as an operator acting on the functions* $\chi : \Omega \times T \to \mathbf{R}$, $\Omega \subset \mathbf{R}^2$, $T \subset \mathbf{R}$.

The above Propositions imply the following group classification results.

Proposition 6. *The time-independent MvK equations admit a group* G *iff* G *is generated by a vector field* $Y = c^j Y_j$ $(j = 1, \dots, 10)$ *and the right-hand sides* P *and* Q *are invariants of* G *(when* $c^{10} = 0$*) or eigenfunctions (when* $c^{10} \neq 0$*) of its generator* Y.

Proposition 7. *The time-dependent MvK equations admit a group* G *iff* G *is generated by a vector field* $X = C^j X_j$ $(j = 1, \dots, 14)$ *and the right-hand sides* P *and* Q *are invariants of* G *(when* $C^5 = 0$*) or eigenfunctions (when* $C^5 \neq 0$*) of its generator* X.

Conservation laws. Both the time-independent and time-dependent MvK equations constitute self-adjoint systems and are the Euler–Lagrange equations for the functionals

$$I^{(S)}[W, \varPhi] = \int\!\!\int\!\!\int \Pi \, dx^1 dx^2 \quad \text{and} \quad I^{(D)}[W, \varPhi] = \int\!\!\int\!\!\int (\mathrm{T} - \Pi) \, dx^1 dx^2 dx^3,$$

$$\Pi = \frac{D}{2}\left\{(\Delta W)^2 - (1-\nu)\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu} W_{\alpha\beta} W_{\mu\nu}\right\} + \frac{1}{2}\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}\varPhi_{\alpha\beta} W_\mu W_\nu$$
$$-\frac{1}{2Eh}\left\{(\Delta\varPhi)^2 - (1+\nu)\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu}\varPhi_{\alpha\beta}\varPhi_{\mu\nu}\right\} - PW - Q\varPhi,$$
$$\mathrm{T} = \frac{\rho}{2}(W_3)^2,$$

Π and T being the strain and kinetic energies per unit area of the shell middle-surface.

In [10], the variational symmetries of the above functionals with $P = Q = 0$ are established and the associated conservation laws admitted by the smooth solutions of the homogeneous MvK equations are presented (see Appendices A and B). In particular, each such conservation law for the time-dependent MvK equations is a linear combination of the basic linearly independent conservation laws

$$\partial\varPsi_{(j)}/\partial x^3 + \partial P^\mu_{(j)}/\partial x^\mu = 0 \qquad (j = 1, 2, \dots, 14)$$

whose densities $\varPsi_{(j)}$ and fluxes $P^\mu_{(j)}$ are presented (together with the generators of the respective symmetries) on the Table 4 below in terms of Q^α, $M^{\alpha\beta}$, $G^{\alpha\beta}$ and F^α,

$$G^{\alpha\beta} = \frac{1}{Eh}\left\{(1+\nu)\delta^{\alpha\mu}\delta^{\beta\nu} - \nu\delta^{\alpha\beta}\delta^{\mu\nu}\right\}\varPhi_{\mu\nu} - \frac{1}{2}\varepsilon^{\alpha\mu}\varepsilon^{\beta\nu} w_\mu w_\nu, \qquad F^\alpha = G^{\alpha\nu}_{,\nu}.$$

Table 4

w - translations	**transversal linear momentum** (first MvK equation)
$X_1 = \frac{\partial}{\partial w}$	$P^\alpha_{(1)} = -Q^\alpha,\ \Psi_{(1)} = \rho\, w_3$
Φ - translations	**compatibility condition** (second MvK equation)
$X_{14} = \frac{\partial}{\partial \Phi}$	$P^\alpha_{(14)} = F^\alpha,\ \Psi_{(14)} = 0$
time - translations	**energy**
$X_4 = \frac{\partial}{\partial x^3}$	$P^\alpha_{(4)} = -w_3 Q^\alpha - \Phi_3 F^\alpha + w_{3\beta} M^{\alpha\beta} + \Phi_{3\beta} G^{\alpha\beta}$ $\Psi_{(4)} = \mathrm{T} + \Pi$
x^1 & x^2- translations	**wave momentum**
$X_2 = \frac{\partial}{\partial x^1}$	$P^\alpha_{(2)} = \delta^{\alpha 1}(\mathrm{T} - \Pi) + w_1 Q^\alpha + \Phi_1 F^\alpha - w_{1\beta} M^{\alpha\beta} - \Phi_{1\beta} G^{\alpha\beta}$ $\Psi_{(2)} = -\rho\, w_1 w_3$
$X_3 = \frac{\partial}{\partial x^2}$	$P^\alpha_{(3)} = \delta^{\alpha 2}(\mathrm{T} - \Pi) + w_2 Q^\alpha + \Phi_2 F^\alpha - w_{2\beta} M^{\alpha\beta} - \Phi_{2\beta} G^{\alpha\beta}$ $\Psi_{(3)} = -\rho\, w_2 w_3$
rotations	**moment of the wave momentum**
$X_6 = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2}$	$P^\alpha_{(6)} = x^2 P^\alpha_{(2)} - x^1 P^\alpha_{(3)} + \varepsilon_\nu^{\ \mu} w_\mu M^{\alpha\nu} + \varepsilon_\nu^{\ \mu} \Phi_\mu G^{\alpha\nu}$ $\Psi_{(6)} = x^2\, \Psi_{(2)} - x^1\, \Psi_{(3)}$
rigid body rotations	**angular momentum**
$X_7 = x^1 \frac{\partial}{\partial w}$	$P^\alpha_{(7)} = M^{\alpha 1} - x^1 Q^\alpha + w \varepsilon^{\alpha\nu} \Phi_{\nu 2},\ \Psi_{(7)} = \rho x^1 w_3$
$X_8 = x^2 \frac{\partial}{\partial w}$	$P^\alpha_{(8)} = M^{\alpha 2} - x^2 Q^\alpha + w \varepsilon^{\nu\alpha} \Phi_{\nu 1},\ \Psi_{(8)} = \rho x^2 w_3$
scaling	
$X_5 = x^\mu \frac{\partial}{\partial x^\mu} + 2x^3 \frac{\partial}{\partial x^3}$	$P^\alpha_{(5)} = x^1 P^\alpha_{(2)} + x^2 P^\alpha_{(3)} - 2x^3 P^\alpha_{(4)} - w_\beta M^{\alpha\beta} - \Phi_\beta G^{\alpha\beta}$ $\Psi_{(5)} = x^1\, \Psi_{(2)} + x^2\, \Psi_{(3)} - 2x^3\, \Psi_{(4)}$
Galilean boost	**center-of-mass theorem**
$X_9 = x^3 \frac{\partial}{\partial w}$	$P^\alpha_{(9)} = -x^3 Q^\alpha,\ \Psi_{(9)} = \rho\,(x^3 w_3 - w)$
$X_{10} = x^1 x^3 \frac{\partial}{\partial w}$	$P^\alpha_{(10)} = x^3 P^\alpha_{(7)},\ \Psi_{(10)} = x^1\, \Psi_{(9)}$
$X_{11} = x^2 x^3 \frac{\partial}{\partial w}$	$P^\alpha_{(11)} = x^3 P^\alpha_{(8)},\ \Psi_{(11)} = x^2\, \Psi_{(9)}$
$X_{12} = x^1 \frac{\partial}{\partial \Phi}$	$P^\alpha_{(12)} = x^1 F^\alpha - G^{\alpha 1},\ \Psi_{(12)} = 0$
$X_{13} = x^2 \frac{\partial}{\partial \Phi}$	$P^\alpha_{(13)} = x^2 F^\alpha - G^{\alpha 2},\ \Psi_{(13)} = 0$

The following statements [6] hold for the nonhomogeneous MvK equations.

Proposition 8. *A conservation law of flux* $A^\alpha_{(j)}$ *and characteristic* $\Lambda^\alpha_{(j)}$ $(j = 1, \ldots, 9)$ *admitted by the smooth solutions of the homogeneous time-independent MvK equations takes the form*

$$A^\mu_{(j),\mu} + S_{(j)} = 0, \qquad S_{(j)} = -\Lambda^1_{(j)} P - \Lambda^2_{(j)} Q, \tag{9}$$

on the smooth solutions of the non-homogeneous time-independent MvK equations;

$$S_{(j)} = \widetilde{A}^\mu_{(j),\mu},$$

iff (7) hold, and then (9) can be written as a divergence free expression (i.e. it becomes a proper conservation law), otherwise it has supply (production) $S_{(j)}$.

Proposition 9. *Each conservation law of density* $\Psi_{(i)}$*, flux* $P^{\alpha}_{(i)}$ *and characteristic* $\Lambda^{\alpha}_{(i)}$ $(i = 1, \ldots, 14)$ *admitted by the smooth solutions of the homogeneous time-dependent MvK equations takes the form*

$$\Psi_{(i),3} + P^{\mu}_{(i),\mu} + S_{(i)} = 0, \qquad S_{(i)} = -\Lambda^{1}_{(i)}P - \Lambda^{2}_{(i)}Q, \tag{10}$$

on the smooth solutions of the non-homogeneous time-dependent MvK equations;

$$S_{(i)} = \widetilde{\Psi}_{(i),3} + \widetilde{P}^{\mu}_{(i),\mu},$$

iff (8) hold, and hence (10) becomes a proper conservation law, otherwise it has supply (production) $S_{(i)}$.

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