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Symmetries and Reductions of Partial Differential Equations

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We consider different types of symmetries of partial differential equations. Using symmetry operators we construct corresponding ansatzes, reducing initial equations to the system with fewer independent variables.

It is well known that invariance of system of partial differential equations with respect to a Lie group of point transformations of independent and dependent variables is a sufficient condition of reduction of the system under study to a system of equations with fewer number of independent variables with help of a corresponding ansatz. This property is sucessfully exploited in constructing of exact solutions for many linear and nonlinear equations of mathematical physics [1]. By using the results of [2] we construct an ansatz for $\vec{D}$, $\vec{B}$, $\vec{E}$, $\vec{H}$, which reduces the nonlinear Maxwell equations

$$\begin{aligned}
&\frac{\partial \vec{D}}{\partial t} = \mathrm{rot}\vec{H}, \qquad && \frac{\partial \vec{B}}{\partial t} = -\mathrm{rot}\vec{E}, \\
&\mathrm{div}\vec{D} = 0, && \mathrm{div}\vec{B} = 0,
\end{aligned} \tag{1}$$

$$\vec{D} = M(I_1, I_2)\vec{E} + N(I_1, I_2)\vec{B}, \qquad \vec{H} = M(I_1, I_2)\vec{B} - N(I_1, I_2)\vec{E}, \tag{2}$$

where $M, N \in C^1(R^2, R^1)$, to the system of ordinary differential equations.

The ansatz invariant with respect to the 3-dimensional subalgebra $\langle -J_{01} - J_{13}, J_{03}, P_2\rangle$ of the Poincaré algebra has the form

$$\begin{aligned}
E^1 &= \frac{1}{2}\left(\frac{1}{\xi} + \xi\right)\widetilde{E}_1 + \frac{1}{2}\left(\frac{1}{\xi} - \xi\right)\widetilde{B}_2 - \frac{x_1}{\xi}\widetilde{E}_3 + \frac{x_1^2}{2\xi}\left(\widetilde{B}_2 - \widetilde{E}_1\right), \\
E^2 &= \frac{1}{2}\left(\frac{1}{\xi} + \xi\right)\widetilde{E}_2 - \frac{1}{2}\left(\frac{1}{\xi} - \xi\right)\widetilde{B}_1 + \frac{x_1}{\xi}\widetilde{B}_3 + \frac{x_1^2}{2\xi}\left(\widetilde{E}_2 - \widetilde{B}_1\right), \\
E^3 &= \widetilde{E}_3 - x_1\left(\widetilde{B}_2 - \widetilde{E}_1\right),
\end{aligned} \tag{3}$$

$$\begin{aligned}
B^1 &= \frac{1}{2}\left(\frac{1}{\xi} + \xi\right)\widetilde{B}_1 - \frac{1}{2}\left(\frac{1}{\xi} - \xi\right)\widetilde{E}_2 - \frac{x_1}{\xi}\widetilde{B}_3 - \frac{x_1^2}{2\xi}\left(\widetilde{E}_2 + \widetilde{B}_1\right), \\
B^2 &= \frac{1}{2}\left(\frac{1}{\xi} + \xi\right)\widetilde{B}_2 + \frac{1}{2}\left(\frac{1}{\xi} - \xi\right)\widetilde{E}_1 - \frac{x_1}{\xi}\widetilde{E}_3 + \frac{x_1^2}{2\xi}\left(\widetilde{B}_2 - \widetilde{E}_1\right), \\
B^3 &= \widetilde{B}_3 - x_1\left(\widetilde{E}_2 + \widetilde{B}_1\right),
\end{aligned} \tag{4}$$

$$\begin{aligned}
&D^1 = \frac{1}{2}\left(\frac{1}{\xi}+\xi\right)\widetilde{D}_1 + \frac{1}{2}\left(\frac{1}{\xi}-\xi\right)\widetilde{H}_2 - \frac{x_1}{\xi}\widetilde{D}_3 + \frac{x_1^2}{2\xi}\left(\widetilde{H}_2-\widetilde{D}_1\right),\\
&D^2 = \frac{1}{2}\left(\frac{1}{\xi}+\xi\right)\widetilde{D}_2 - \frac{1}{2}\left(\frac{1}{\xi}-\xi\right)\widetilde{H}_1 + \frac{x_1}{\xi}\widetilde{H}_3 + \frac{x_1^2}{2\xi}\left(\widetilde{D}_2-\widetilde{H}_1\right),\\
&D^3 = \widetilde{D}_3 - x_1\left(\widetilde{H}_2-\widetilde{D}_1\right),
\end{aligned}\tag{5}$$

$$\begin{aligned}
&H^1 = \frac{1}{2}\left(\frac{1}{\xi}+\xi\right)\widetilde{H}_1 - \frac{1}{2}\left(\frac{1}{\xi}-\xi\right)\widetilde{D}_2 - \frac{x_1}{\xi}\widetilde{H}_3 - \frac{x_1^2}{2\xi}\left(\widetilde{D}_2+\widetilde{H}_1\right),\\
&H^2 = \frac{1}{2}\left(\frac{1}{\xi}+\xi\right)\widetilde{H}_2 + \frac{1}{2}\left(\frac{1}{\xi}-\xi\right)\widetilde{D}_1 - \frac{x_1}{\xi}\widetilde{D}_3 + \frac{x_1^2}{2\xi}\left(\widetilde{H}_2-\widetilde{D}_1\right),\\
&H^3 = \widetilde{H}_3 - x_1\left(\widetilde{D}_2+\widetilde{H}_1\right),
\end{aligned}\tag{6}$$

where $\widetilde{E}_a$, $\widetilde{B}_a$, $\widetilde{D}_a$, $\widetilde{H}_a$ are unknown functions of the variable $\omega = x_0^2 - x_1^2 - x_3^2$, $\xi = x_0 - x_3$.

Substituting (3)–(6) in (1) we obtain the reduced system

$$\begin{aligned}
&\left(\widetilde{B}_1' + \widetilde{E}_2'\right)\omega + \widetilde{B}_1' - \widetilde{E}_2' + \widetilde{B}_1 + \widetilde{E}_2 = 0,\\
&\left(\widetilde{B}_2' - \widetilde{E}_1'\right)\omega + \widetilde{B}_2' + \widetilde{E}_1' + 2\left(\widetilde{B}_2 - \widetilde{E}_1\right) = 0,\\
&\widetilde{B}_3' = 0, \qquad \widetilde{B}_3 = 0,
\end{aligned}\tag{7}$$

$$\begin{aligned}
&\left(\widetilde{H}_1' + \widetilde{D}_2'\right)\omega - \widetilde{H}_1' + \widetilde{D}_2' + 2\left(\widetilde{H}_1 + \widetilde{D}_2\right) = 0,\\
&\left(\widetilde{H}_2' - \widetilde{D}_1'\right)\omega - \left(\widetilde{H}_2' + \widetilde{D}_1'\right) + \widetilde{H}_2 - \widetilde{D}_1 = 0,\\
&\widetilde{D}_3' = 0, \qquad \widetilde{D}_3 = 0,
\end{aligned}\tag{8}$$

$$\vec{\widetilde{D}} = M\vec{\widetilde{E}} + N\vec{\widetilde{B}}, \qquad \vec{\widetilde{H}} = M\vec{\widetilde{B}} - N\vec{\widetilde{E}},\tag{9}$$

where M, N are functions of $I_1 = \vec{\widetilde{E}}^2 - \vec{\widetilde{B}}^2$, $I_2 = \vec{\widetilde{B}}\vec{\widetilde{E}}$, "$'$" designates differentiation.

To construct invariant solutions it is necessary to the integrate nonlinear system of differential equations (7)–(9). We obtained a partial solution of the system, when $N = 0$, $M = M(I_1)$ in the form

$$\widetilde{E}_1 = C_1\left(\omega^{-1/2} - \omega^{-3/2}\right), \qquad \widetilde{E}_2 = C_1\left(\omega^{-1/2} + \omega^{-3/2}\right), \qquad \widetilde{E}_3 = 0,\tag{10}$$

$$\widetilde{B}_1 = -C_1\left(\omega^{-1/2} - \omega^{-3/2}\right), \qquad \widetilde{B}_2 = C_1\left(\omega^{-1/2} + \omega^{-3/2}\right), \qquad \widetilde{B}_3 = 0,\tag{11}$$

$$\widetilde{D}_1 = mC_1\left(\omega^{-1/2} - \omega^{-3/2}\right), \qquad \widetilde{D}_2 = mC_1\left(\omega^{-1/2} + \omega^{-3/2}\right), \qquad \widetilde{D}_3 = 0,\tag{12}$$

$$\widetilde{H}_1 = -mC_1\left(\omega^{-1/2} - \omega^{-3/2}\right), \qquad \widetilde{H}_2 = mC_1\left(\omega^{-1/2} + \omega^{-3/2}\right), \qquad \widetilde{H}_3 = 0,\tag{13}$$

where $m = M(0)$.

Substituting the solution in (3)–(6), we obtain an exact solution of the nonlinear Maxwell equations

$$E^1 = \frac{2C_1x_3}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}}, \qquad E^2 = \frac{2C_1x_0}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}}, \qquad E^3 = -\frac{2C_1x_1}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}},$$

$$B^1 = -\frac{2C_1x_3}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}}, \qquad B^2 = \frac{2C_1x_0}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}}, \qquad B^3 = \frac{2C_1x_1}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}},$$

$$D^1 = \frac{2C_1mx_3}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}}, \qquad D^2 = \frac{2C_1mx_0}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}}, \qquad D^3 = -\frac{2C_1mx_1}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}},$$

$$H^1 = -\frac{2C_1mx_3}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}}, \qquad H^2 = \frac{2C_1mx_0}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}}, \qquad H^3 = \frac{2C_1mx_1}{\left(x_0^2 - x_1^2 - x_3^2\right)^{3/2}}.$$

In analogous way we construct solutions invariant under the following subalgebras of Poincaré algebra: $\langle J_{03}, P_1, P_2\rangle$, $\langle J_{12}+\alpha P_0, P_1, P_2\rangle$, $\langle J_{03}+\alpha J_{12}, P_1, P_2\rangle$, $\langle P_0 - J_{01} - J_{13}, P_0+P_3, P_2\rangle$, where $\alpha = \text{const}$.

The existence of the operator of the classical symmetry is not a necessary condition for reduction of partial differential equations, as it is shown in [3, 4, 5]. It was proved in [6] that the conditional symmetry under involutive set of operators is the necessary and sufficient condition for reduction of partial differential equations by means of a corresponding ansatz.

Operators of nonpoint symmetry can be used to reduction of differential equations too. For simplicity we consider a second order equation of the form

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}, u_{x_1x_1}, u_{x_1x_2}, u_{x_2x_2}) = 0. \tag{14}$$

We search for a solution of (14) as a solution of system

$$\frac{\partial u}{\partial x_1} = v^1(x_1, x_2, u), \qquad \frac{\partial u}{\partial x_2} = v^2(x_1, x_2, u). \tag{15}$$

Denote $u \equiv x_3$ and consider v^1, v^2 as a functions of variables x_1, x_2, x_3; $v^1, v^2 \in C^1\left(R^3, R^1\right)$. Then the compability condition of the system (15) takes the form

$$v_2^1 + v_3^1 v^2 = v_1^2 + v_3^2 v^1. \tag{16}$$

Any solution of (15) satisfies (14), if the following equality holds

$$F\left(x_1, x_2, x_3, v_1^1 + v_3^1 v^1, v_2^1 + v_3^1 v^2, v_2^2 + v_3^2 v^2\right) = 0. \tag{17}$$

Thus the problem of construction of an ansatz of type (15) is reduced to the problem of finding of operators of classical and conditional symmetry of the system (16), (17).

Let us consider the infinitesimal operator of one–parametrical group of transfomations of independent and dependent variables

$$Q = \xi^1(x_1, x_2, u)\partial_{x_1} + \xi^2(x_1, x_2, u)\partial_{x_2} + \eta(x_1, x_2, u)\partial_u \tag{18}$$

and first prolongation of Q

$$\begin{aligned} \underset{1}{Q} = Q &+ \left(\eta_1 + \eta_u u_1 - u_1\left(\xi_1^1 + \xi_u^1 u_1\right) - u_2\left(\xi_1^2 + \xi_u^2 u_1\right)\right)\partial_{u_1} \\ &+ \left(\eta_2 + \eta_u u_2 - u_1\left(\xi_2^1 + \xi_u^1 u_2\right) - u_2\left(\xi_2^2 + \xi_u^2 u_2\right)\right)\partial_{u_2}, \end{aligned} \tag{19}$$

where lower indices designate differentiation of ξ^p, η $(p=1,2)$ with respect to the corresponding variables. We associate the operator Q'

$$\begin{aligned} Q' &= \xi^1(x_1,x_2,x_3)\partial_{x_1} + \xi^2(x_1,x_2,x_3)\partial_{x_2} + \xi^3(x_1,x_2,x_3)\partial_{x_3} \\ &\quad + \left(\xi^3_1 + \xi^3_3 v^1 - v^1\left(\xi^1_1 + \xi^1_3 v^1\right) - v^2\left(\xi^2_1 + \xi^2_3 v^1\right)\right)\partial_{v^1} \\ &\quad + \left(\xi^3_2 + \xi^3_3 v^2 - v^1\left(\xi^1_2 + \xi^1_3 v^2\right) - v^2\left(\xi^2_2 + \xi^2_3 v^2\right)\right)\partial_{v^2}, \\ \eta(x_1,x_2,x_3) &= \xi^3(x_1,x_2,x_3) \end{aligned} \tag{20}$$

with operator Q.

Theorem. *Let equation (14) be invariant with respect to one-parameter group generator Q (18). Then the operator Q' belongs to the invariance algebra of system (16), (17).*

Proof. Acting by the operator $\underset{1}{Q'}$ on the manifold (16), we obtain

$$\begin{aligned} &\xi^3_{12} + \xi^3_{32}v^1 - v^1\left(\xi^1_{12} + \xi^1_{32}v^1\right) - v^2\left(\xi^2_{12} + \xi^2_{32}v^1\right) + v^1_2\left(\xi^3_3 - \xi^1_1 - \xi^2_3 v^2 - 2v^1\xi^1_3\right) \\ &\quad -v^2_2\left(\xi^2_1 + \xi^2_3 v^1\right) - \xi^1_2 v^1_1 - \xi^2_2 v^1_2 - v^1_3\xi^3_2 + v^2\left(\xi^3_{13} + \xi^3_{33}v^1 - v^1\left(\xi^1_{13} + \xi^1_{33}v^1\right)\right. \\ &\quad -v^2\left(\xi^2_{13} + \xi^2_{33}v^1\right) + v^1_3\left(\xi^3_3 - \xi^1_1 - 2v^1\xi^1_3 - \xi^2_3 v^2\right) - v^2_3\left(\xi^2_1 + \xi^2_3 v^1\right) \\ &\quad \left.-v^1_1\xi^1_3 - v^1_2\xi^2_3 - v^1_3\xi^3_3\right) + \left(\xi^3_2 + \xi^3_3 v^2 - v^1\left(\xi^1_2 + \xi^1_3 v^2\right) - v^2\left(\xi^2_2 + \xi^2_3 v^2\right)\right)v^1_3 \\ &\quad = \xi^3_{21} + \xi^3_{31}v^2 - v^1\left(\xi^1_{21} + \xi^1_{31}v^2\right) - v^2\left(\xi^2_{21} + \xi^2_{31}v^2\right) + v^2_1\left(\xi^3_3 - \xi^2_2 - \xi^1_3 v^1 - 2v^2\xi^2_3\right) \\ &\quad -v^1_1\left(\xi^1_2 + \xi^1_3 v^2\right) - \xi^1_1 v^2_1 - \xi^2_1 v^2_2 - v^2_3\xi^3_1 + v^1\left(\xi^3_{23} + \xi^3_{33}v^2 - v^1\left(\xi^1_{23} + \xi^1_{33}v^2\right)\right. \\ &\quad -v^2\left(\xi^2_{23} + \xi^2_{33}v^2\right) + v^2_3\left(\xi^3_3 - \xi^2_2 - 2v^2\xi^2_3 - \xi^1_3 v^1\right) - v^1_3\left(\xi^1_2 + \xi^1_3 v^2\right) - v^2_1\xi^1_3 \\ &\quad \left.-v^2_2\xi^2_3 - v^2_3\xi^3_3\right) + \left(\xi^3_1 + \xi^3_3 v^1 - v^1\left(\xi^1_1 + \xi^1_3 v^1\right) - v^2\left(\xi^2_1 + \xi^2_3 v^1\right)\right)v^2_3. \end{aligned}$$

It is easy to verify, that this equality is fulfilled identically on the manifold (16). Thus we obtain

$$\underset{1}{Q'}\left(v^1_2 + v^1_3 v^2 - v^2_1 - v^2_3 v^1\right)\Big|_{v^1_2+v^1_3v^2=v^2_1+v^2_3v^1} \equiv 0. \tag{21}$$

It is necessary to prove that the equation (17) admits operator Q' to prove the theorem. One property of coordinates of prolonged operators $\underset{2}{Q}$, $\underset{1}{Q'}$ is used for this purpose, where

$$\underset{2}{Q} = \underset{1}{Q} + \varepsilon^{u_{11}}\partial_{u_{11}} + \varepsilon^{u_{12}}\partial_{u_{12}} + \varepsilon^{u_{22}}\partial_{u_{22}}, \tag{22}$$

$$\begin{aligned} \varepsilon^{u_{ab}} &= \eta_{ab} + u_b\eta_{au} + u_a\eta_{bu} + u_a u_b\eta_{uu} + u_{ab}\eta_u - u_c\left(\xi^c_{ab} + u_b\xi^c_{au}\right) \\ &\quad - u_a u_c\left(\xi^c_{bu} + u_b\xi^c_{uu}\right) - u_{ac}\left(\xi^c_b + u_b\xi^c_u\right) - u_{cb}\left(\xi^c_a + u_a\xi^c_u\right) - u_{ab}u_c\xi^c_u, \end{aligned} \tag{23}$$

$a,b,c = 1,2$, we mean summation over the index c.

Making the substitution $u = x_3$, $u_1 = v^1$, $u_2 = v^2$, $u_{11} = v^1_1 + v^1_3 v^1$, $u_{12} = v^1_2 + v^1_3 v^2$, $u_{22} = v^2_2 + v^2_3 v^2$ in (23), we obtain coefficients $\varepsilon'^{u_{12}}$, $\varepsilon'^{u_{22}}$ associated with $\varepsilon^{u_{11}}$, $\varepsilon^{u_{12}}$, $\varepsilon^{u_{22}}$. Then the following equalities

$$\varepsilon'^{u_{11}} = \underset{1}{Q'}\left(v^1_1 + v^1_3 v^1\right), \qquad \varepsilon'^{u_{12}} = \underset{1}{Q'}\left(v^1_2 + v^1_3 v^2\right), \qquad \varepsilon'^{u_{22}} = \underset{1}{Q'}\left(v^2_2 + v^2_3 v^2\right) \tag{24}$$

are fulfilled. The correctness of (24) is verified by direct calculations. Thus, derivatives u_{11}, u_{12}, u_{22} are transformed in the same way as the combinations $v^1_1 + v^1_3 v^1$, $v^1_2 + v^1_3 v^2$, $v^2_2 + v^2_3 v^2$ under the group transformations. From this it follows that the system (16), (17) is invariant under the group G'_1 provided G_1 is the invariance group of equation (14). ■

Thus, we conclude that the group of transformations admissible by equation (14) is not wider than the symmetry group of the system (16), (17). In the general case the symmetry group of system (16), (17) contains the invariance group of (14) as a subgroup. There is a possibility of expansion of this group by studying the symmetry properties of the system, as it is shown in [7, 8]. To obtain new solutions it is necessary to use the symmetry operators of system (16), (17), which are not prolongated operators of point symmetry of equation (14), as well as the operators of conditional symmetry of the system. By using this approach we constructed ansatzes reducing nonlinear equations to the system of ordinary differential equations. Integrating the reduced system we obtained new solutions of nonlinear heat and wave equations.

The method of conditional symmetry is generalised to the Lie–Bäcklund operators (see for example [9])

$$X = \eta(x, u, \ldots, \underset{r}{u})\partial_u. \tag{25}$$

Let us consider differential equations

$$U(x, u, \underset{1}{u}, \ldots, \underset{k}{u}) = 0, \tag{26}$$

where $u \in C^k\left(R^n, R^1\right)$, $x \in R^n$.

Definition. *Equation (26) is conditionaly invariant with respect to operator (25), if the following condition is satisfied*

$$\underset{k}{X} U\Big|_{[\eta=0]_\infty} = M, \tag{27}$$

where $M \neq 0$, $M\Big|_{[U=0]_r} = 0$, $[\eta = 0]_\infty$ *is a set of all differential consequences of the equation* $\eta = 0$, $[U = 0]_r$ *is a set of all differential consequences of* r*-th order of the equation* $U = 0$.

The corresponding ansatz, which is a solution of the equation

$$\eta(x, u, \ldots, \underset{r}{u}) = 0,$$

reduces equation (26) to the system of equations with smaller number of independent variables. Using this property we can construct exact solutions of partial differential equations.

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