Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 204–209.

One-Dimensional Fokker–Planck Equation Invariant under Four- and Six-Parametrical Group

Stanislav SPICHAK † *and Valerii STOGNII* ‡

† *Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivska Street, Kyiv, Ukraine*
E-mail: spichak@apmat.freenet.kiev.ua

‡ *National Technical University of Ukraine "Kyiv Polytechnic Institute",*
Peremohy Ave., 37, Kyiv 56, Ukraine
E-mail: valerii@apmat.freenet.kiev.ua

Symmetry properties of the one-dimensional Fokker–Planck equations with arbitrary coefficients of drift and diffusion are investigated. It is proved that the group symmetry of these equations can be one-, two-, four- or six-parametric and corresponding criteria are obtained. The changes of the variables reducing Fokker–Planck equations to the heat and Schrödinger equations with certain potential are determined.

1 Introduction

Fokker–Planck equation (FPE) is a basic equation in the theory of continuous Markovian processes. In an one-dimensional case FPE has the form [1, 2]

$$L = \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}[A(t,x)u] - \frac{1}{2}\frac{\partial^2}{\partial x^2}[B(t,x)u] = 0, \tag{1}$$

where $u = u(t,x)$ is the probability density, $A(t,x)$ and $B(t,x)$ are differentiable functions meaning coefficients of drift and diffusion correspondingly.

We investigated symmetry properties of the equation (1) under the infinitesimal basis operators [3–5]

$$X = \xi^0(t,x,u)\frac{\partial}{\partial t} + \xi^1(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{2}$$

The symmetry operators are defined from the invariance condition

$$\underset{2}{\hat{X}} L\Big|_{L=0} = 0, \tag{3}$$

where $\underset{2}{\hat{X}}$ is the second prolongation of the operator X, which is constructed according to the formulae [3–5]. From the condition of invariance (3), equating coefficients by a function u and its derivatives u_x, u_{tt}, u_{tx}, u_{xx} (u_t can be expressed from equation (1)) to zero it is possible to determine the following system of equations on functions ξ^0, ξ^1, η:

$$\begin{gathered}
\xi^0 = \xi^0(t), \qquad \xi^1 = \xi^1(t,x), \qquad \eta = \chi(t,x)u, \qquad 2\xi^1_x B - \xi^0_t B - \xi^1 B_x - \xi^0 B_t = 0, \\
\xi^0_t(A - B_x)\xi^1_t + \xi^0(A_t - B_{tx}) + \xi^1(A_x - B_{xx}) - \xi^1_x(A - B_x) + \frac{1}{2}B\xi^1_{xx} = B\chi_x, \\
\chi_t + \xi^0_t\left(A_x - \frac{1}{2}B_{xx}\right) + \xi^0\left(A_{tx} - \frac{1}{2}B_{txx}\right) \\
+\xi^1\left(A_{xx} - \frac{1}{2}B_{xxx}\right) + \chi_x(A - B_x) - \frac{1}{2}B\chi_{xx} = 0.
\end{gathered} \tag{4}$$

Here lower indexes t, x mean differentiation on corresponding variables. Let us also introduce the following notations $\frac{\partial}{\partial t} = \partial_t$, $\frac{\partial}{\partial x} = \partial_x$, $\frac{\partial}{\partial u} = \partial_u$.

2 Criterion of invariance FPE under four- and six-parametrical group of symmetry

In [6] following Theorem was proved:

Theorem 1. *If there is a symmetry operator (2) $Q \neq u\partial_u$ for FPE (1) then there exists a transformation of a form*

$$\tilde{t} = T(t), \qquad \tilde{x} = X(t,x), \qquad u = v(t,x)\tilde{u},$$

which reduces it to equation (1) with coefficients of drift and diffusion $\tilde{A} = A(\tilde{x})$, $\tilde{B} = B(\tilde{x})$. And, if $\xi^0 \not\equiv 0$ then

$$\tilde{t} = T(t), \qquad \tilde{x} = \omega, \qquad u = v(t,x)\tilde{u}, \tag{5}$$

where $T(t) = \int \frac{dt}{\xi^0(t)}$, and the functions $\omega = \omega(t,x)$, $v(t,x)$ satisfy the equations:

$$\xi^0\omega_t + \xi^1\omega_x = 0, \qquad \xi^0 v_t + \xi^1 v_x = \chi v, \tag{6}$$

where $\omega \neq \mathrm{const}$ is further meant as any fixed solution of the equation (6).

The consequence of this theorem is

Theorem 2. *The dimension of an invariance algebra of FPE (1) can be equal to 1, 2, 4, 6.*

Proof. If dimension of algebra more than 1 then equation (1) is reduced to the equation with $\tilde{A} = \tilde{A}(\tilde{x})$, $\tilde{B} = \tilde{B}(\tilde{x})$, but classification of such equations is known: dimension of their invariance algebra is either 2 or 4 or 6 [7].

In work [8] it is shown that any diffusion process with coefficient of drift $A(t,x)$ and diffusion $B(t,x)$ can be reduced to a process with appropriate coefficienct $\tilde{A}(t,x) = A(t,x)/B(t,x)$ and $\tilde{B}(t,x) = 1$ through random replacement of time $\tau(t)$. Using result of the theorem 1 we carry out symmetry classification of FPE for the coefficient $B(t,x) = 1$ and any $A(t,x)$ just as it was made in [7] for a case $A = A(x)$ (homogeneous process). So puting in the equations (4) $B = 1$ it is easy to show that

$$\begin{aligned}
&\xi^0 = \tau(t), \qquad \xi^1 = \frac{1}{2}x\tau' + \varphi(t),\\
&\frac{3}{2}\tau' M + \tau M_t + (\frac{1}{2}\tau' x + \varphi)M_x = \frac{1}{2}\tau' x + \varphi'',\\
&\chi = \frac{1}{2}\tau' x A(t,x) - \frac{1}{4}x^2\tau'' - \varphi' x + \varphi A(t,x) + \tau\int\limits_{x_0}^{x} \frac{\partial A(t,\xi)}{\partial t} d\xi + \theta(t),
\end{aligned} \tag{7}$$

where $M = A_t + \frac{1}{2}A_{xx} + AA_x$, x_0 and $\theta(t)$ are arbitrary point and function correspondently. Let us find a condition on M under wich there existes at least two the linearly independent solutions $\tau(t)$ of the equations (7). In this case from the Theorem 2 it is followed that there exists either 3 or 5 operators of symmetry (besides trivial $u\partial_u$). Let's assume that $M_{xx} \neq 0$. After differentiating twice on x both parts (7) we have:

$$\frac{5}{2}\tau' M_{xx} + \tau M_{txx} + \left(\frac{1}{2}\tau' x + \varphi\right) M_{xxx} = 0. \tag{8}$$

Now if we assume that $M_{xxx} = 0$, i.e. $M_{xx} = F(t)$, then the following condition takes place:

$$\frac{5}{2}\tau' F + \tau F' = 0. \tag{9}$$

For this equation there is only one linearly independent solution, therefore $M_{xxx} \neq 0$. Then from (8):

$$-\varphi(t) = \frac{5M_{xx} + xM_{xxx}}{2M_{xxx}}\tau' + \frac{M_{txx}}{M_{xxx}}\tau = h(t,x)\tau' + r(t,x)\tau.$$

So if (τ_1, φ_1), (τ_2, φ_2) are linearly independent then τ_1, τ_2 are linearly independent, and also $h_x\tau' + r_x\tau = 0$. Thus

$$h_x\tau_1' + r_x\tau_1 = 0, \qquad h_x\tau_1' + r_x\tau_1 = 0.$$

As Wronskian $\begin{vmatrix} \tau_1' & \tau_1 \\ \tau_2' & \tau_2 \end{vmatrix} \neq 0$, then from this system it is followed that $h_x \equiv 0$, $r_x \equiv 0$, i.e.

$$\frac{5M_{xx} + xM_{xxx}}{2M_{xxx}} = h(t), \qquad \frac{M_{xxt}}{M_{xxx}} = r(t). \tag{10}$$

From conditions (10) it is easy to deduce that

$$M = \lambda(x - H(t))^{-3} + F(t)x + G(t), \tag{11}$$

where $\lambda = \text{const} \neq 0$, H, F, G are arbitrary functions. Now notice that if $M_{xx} = 0$, M has form (11) with $\lambda = 0$. Thus the condition (11) is necessary for the invariance algebra to have dimension either 4 or 6. Substituting (11) in (8) and equating zero factors at $x - H$, $(x - H)^{-4}$ and 1 we obtain the following conditions:

$$\begin{gathered} 2\tau' F + \tau F' = \frac{1}{2}\tau''', \qquad \lambda\left(\tau H' - \frac{1}{2}\tau' H - \varphi\right) = 0, \\ \frac{3}{2}\tau'(FH + G) + \tau(F'H + G') + F\left(\frac{1}{2}\tau' H + \varphi\right) = \frac{1}{2}\tau''' H + \varphi'''. \end{gathered} \tag{12}$$

1) Let $\lambda \neq 0$. Then expressing from the second equation $\varphi(t) = \tau H' - \frac{1}{2}\tau' H$ and substituting it in the third equation we have

$$\frac{3}{2}\tau'(FH + G - H'') + \tau(FH + G - H'')' = 0.$$

Condition of existence of at least 2 independent solutions τ_1, τ_2 results in the equation $FH + G - H'' = 0$. In this case the number of the fundamental solutions of system (12) is three. Really, there are three linear independent solutions τ_1, τ_2, τ_3 of the first equation (12). From the second equation (12) φ_i is expressed through τ_i, $i = 1, 2, 3$.

2) If $\lambda = 0$ the system of the equations (12) has 5 linearly independent solution (τ_i, φ_i), $i = \overline{1, 5}$.

So the following theorem is proved.

Theorem 3. *1) The class FPE (1) with $B = 1$ admitting four-dimentional algebra of invariance is described by the condition*

$$A_t + \frac{1}{2}A_{xx} + AA_x = \lambda(x - H(t))^{-3} + F(t)x + G(t), \tag{13}$$

where $\lambda = \mathrm{const} \neq 0$, G *satisfies the condition*

$$G = H'' - FH, \tag{14}$$

$F(t)$, $H(t)$ *are arbitrary functions.*

2) The class FPE (1) with $B = 1$ *admitting six-dimensional invariance algebra invariance is described by condition (13) in which* $\lambda = 0$, F, G *are arbitrary functions.*

Remark. In particular, if the coefficient $A(t, x)$ satisfies the Burgers equation then FPE (1) is reduced to the heat equation (see [9]).

3 Transformation of the Fokker–Planck equations to homogeneous equations

1) It turns out that FPE (1) ($B = 1$), (13) at $\lambda = 0$ is reduced to the heat equation [9]. We find the appropriate transformation (5), (6). Let τ be any solution of system (12) and $\tau > 0$ (evidently that it is always possible to choose a solution $\tau(t) > 0$ on some interval). From the formulae (6), (7) it is easy to prove that $\omega(t,x) = \tau^{1/2}x - \int\limits_{t_0}^{t} \varphi(\xi)\tau^{-3/2}(\xi)dt$, where t_0 is arbitrary fixed point. Let us consider the transformation:

$$\begin{aligned}
&\tilde{t} = \frac{1}{2}\int \frac{dt}{\tau},\\
&\tilde{x} = \omega(t,x) = \tau^{-1/2}x - \int\limits_{t_0}^{t} \varphi(\xi)\tau^{-3/2}(\xi)d\xi,\\
&u(t,x) = v(t,x)\tilde{u}(\tilde{t},\tilde{x}).
\end{aligned} \tag{15}$$

Having substitued into (1), (13) the replacement variable (15) we come to the equation:

$$\begin{aligned}
\tilde{u}_{\tilde{t}} = {} & -2\tau\left(\frac{v_t}{v} + A_x + A\frac{v_x}{v} - \frac{1}{2}\frac{v_{xx}}{v}\right)\tilde{u}\\
& -2\left(-\frac{1}{2}\tau^{1/2}\tau' x - \varphi\tau^{-1/2} + A\tau^{1/2} - \frac{v_x}{v}\tau^{1/2}\right)\tilde{u}_{\tilde{x}} + \tilde{u}_{\tilde{x}\tilde{x}}.
\end{aligned} \tag{16}$$

Equating zero factor at $\tilde{u}_{\tilde{x}}$, we shall get:

$$v = \exp\left(-\frac{1}{4}\tau^{-1}\tau' x^2 - \tau^1\varphi x + \int\limits_{x_0}^{x} A(t,\xi)d\xi + h(t)\right), \tag{17}$$

where $h(t)$ is an arbitrary function, x_0 is some fixed point. Substituting (17) into the expression $\frac{v_t}{v} + A_x + A\frac{v_x}{v} - \frac{1}{2}\frac{v_{xx}}{v}$ (factor at $\tilde{u}$ in (16)) and equating its to zero we get:

$$h'(t) = \frac{1}{2}\left[\tau^{-2}\varphi^2 - \frac{1}{2}\tau^{-1}\tau^1 - A_x(t,x_0) - A^2(t,x_0)\right], \tag{18}$$

$$\frac{1}{2}\tau^{-1}\tau'' - \frac{1}{4}\tau^2(\tau')^2 = F, \qquad \tau^{-1}\varphi' - \frac{1}{2}\tau^2\tau'\varphi = G. \tag{19}$$

It is easy to prove that if ($\tau \neq 0$, φ) is some solution of system (19) then it satisfies to system (12) ($\lambda = 0$, $M = 0$). Then we have the transformation (15), where funcitons $v(t,x)$, $\tau(t)$, $\varphi(t)$ can be found from (17)–(19), resulting FPE (1), (13) ($\lambda = 0$) to the heat equation

$$\tilde{u}_{\tilde{t}} = \tilde{u}_{\tilde{x}\tilde{x}}. \tag{20}$$

Let us notice, that the system (19) is reduced to following:

$$2y' + y^2 = 4F, \qquad y = \frac{\tau'}{\tau}, \qquad \varphi = \tau^{1/2}\int\limits_{t_0}^{t} \tau^{1/2} G dt. \tag{21}$$

2) We consider now FPE (1), (13) with $\lambda \neq 0$. As in the case of 1) the transformation (15) reduces this equation to the equation (16). The conditions for (16) to be FPE are the following:

$$\begin{aligned} &\tilde{A} = \tilde{A}(\omega) = -\tau^{-1/2}\tau' x - 2\varphi\tau^{-1/2} + 2A\tau^{1/2} - 2\tau^{1/2}\frac{v_x}{v}, \\ &\tilde{A}_\omega = 2\tau\left(\frac{v_t}{v} + A_x + A\frac{v_x}{v} - \frac{1}{2}\frac{v_{xx}}{v}\right), \end{aligned} \tag{22}$$

where ω is given in (15). The first condition is equivalent to the equation

$$\partial_{\tilde{t}}\tilde{A} = \left[\tau\partial_t + \left(\frac{1}{2}\tau' x + \varphi\right)\partial_x\right]\left(-\tau^{-1/2}\tau' x - 2\varphi\tau^{1/2} + 2A\tau^{1/2} - 2\tau^{1/2}\frac{v_x}{v}\right) = 0. \tag{23}$$

Omitting intermediate calculations we give the general solution $v(t,x)$ of equation (23):

$$v(t,x) = \exp\left[\int\limits_{x_0}^{x} A(t,\xi)d\xi - \frac{1}{4}\tau^{-1}\tau' x^2 - \tau^{-1}\varphi x + k(\omega)\right], \tag{24}$$

where $k(\omega)$ is an arbitrary function, x_0 is some fixed point. Substituting (24) into the first equation (22) one can prove that $\tilde{A} = -k'(\omega)$ $\left(k'(\omega) = \frac{dk(\omega)}{d\omega}\right)$. Let us substitute $\tilde{A}(\omega) = -k'(\omega)$, $v(t,x)$ (24) in the second equation (22). Under chosen conditions

$$\tau^{1/2}\int\limits_{t_0}^{t} \varphi\tau^{-3/2}dt = H, \qquad \frac{1}{2}\tau^{-1}\tau'' - \frac{1}{4}\tau^{-2}\tau'^2 = F, \qquad \tau^{-1}\varphi' - \frac{1}{2}\tau^{-2}\tau'\varphi = G, \tag{25}$$

$$k'' - k'^2 = \lambda\omega^{-2}, \tag{26}$$

the second equaiton (22) is satisfied. It is possible to choose the condition (25) because, as it is easy to prove, any solution $\tau \neq 0$, φ of the given system is a particular solution of th system equations (12), (14) that it is enough for construction of the transformation (15). System (25) (taking into account (14)) is equivalent to:

$$2y' + y^2 = 4F, \qquad y = \frac{\tau'}{\tau}, \qquad \varphi = \tau^{3/2}(\tau^{-1/2}H)'. \tag{27}$$

Thus we have proved

Theorem 4. *FP equation (1), (13), (14) with $\lambda \neq 0$, invariant under four-parameter algebra of invariance, through transformations*

$$\tilde{t} = T(t), \qquad \tilde{x} = \tau^{-1/2}x - \tau^{-1/2}H(t), \qquad u = v(t,x)\tilde{u}(\tilde{t},\tilde{x}),$$

where $T = \dfrac{1}{2}\displaystyle\int \dfrac{dt}{\tau(t)}$*,* $v(t,x)$ *has the form (24),* $\tau \neq 0$ *is any solution of the first equation (27),* $k(\omega)$ *is a solution of the equation (26), is reduced to the equation*

$$\tilde{u}_{\tilde{t}} = 2k''(\omega)\tilde{u} + 2k'(\omega)\tilde{u}_\omega + \tilde{u}_{\omega\omega}.$$

Remark. Making the replacement in last equation

$$\bar{t} = \tilde{t}, \qquad \bar{x} = \omega, \qquad \tilde{u} = \exp(k(\omega))\bar{u},$$

and taking into account the condition (26), we can reduce this equation to the following Schrödinger equation:

$$\bar{u}_{\bar{t}} = \bar{u}_{\bar{x}\bar{x}} + \frac{\lambda}{\bar{x}^2}\bar{u}.$$

Thus in the case FPE with four-parametrical group of symmetry there exists an “initial” equation, to which they are reduced; though it is not FPE as it is in the case of the six-parametrical group.

Acknowlegments

Authors is grateful to Prof. R.Z. Zhdanov for valuable discusions.

References

[1] Gardiner K.V., Handbook of Stochastic Methods, Springer, Berlin, 1985.

[2] Risken H., The Fokker–Planck Equation, Springer, Berlin, 1989.

[3] Ovsiannikov L.V., Group Analysis of Differential Equations, Academic, New York, 1982.

[4] Olver P., Applications of Lie Groups of Differential Equations, Springer, New York, 1986.

[5] Fushchych W., Shtelen W. and Serov N., Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathematical, Kyiv, Naukova Dumka, 1989 (in Russian).

[6] Spichak S. and Stognii V., Symmetry classification and exact solutions of the one-dimensional Fokker–Planck equation with arbitrary coefficients of drift and diffusion, *J. Math. Phys.*, 1999, V.32, N 47, 8341–8353.

[7] Cicogna G. and Vitali D., Classification on the extended symmetries of Fokker–Planck equations, *J. Phys. A: Math. Gen.*, 1990, V.23, L85–L88.

[8] Dynkin E.B., Markov Processes, Moscow, Fizmatgiz, 1963.

[9] Zhdanov R.Z. and Lagno V.I., On separability criteria for a time-invariant Fokker–Planck equaiton, *Dokl. Akad. Nauk of Ukraine*, 1993, N 2, 18–21.