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On Local Time-Dependent Symmetries of Integrable Evolution Equations

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We consider scalar $(1+1)$-dimensional evolution equation of order $n \geq 2$, which possesses time-independent formal symmetry (i.e. it is integrable in the sense of symmetry approach), shared by all local generalized time-independent symmetries of this equation. We show that if such equation possesses the nontrivial canonical conserved density ρ_m, $m \in \{-1, 1, 2, \ldots\}$, then it has no polynomial in time local generalized symmetries (except time-independent ones) of order higher than $n+m+1$. Some generalizations of this result and related results are also presented. Using them, we have found all local generalized time-dependent symmetries of Harry Dym and mKdV equations.

Introduction

The scalar $(1+1)$-dimensional evolution equation, having the time-independent formal symmetry, is either linearizable or integrable via inverse scattering transform (see e.g. [1, 2, 3, 4] for the survey of known results and [5] for the generalization to $(2+1)$ dimensions).

It is natural to ask whether such equation may have local generalized *time-dependent* symmetries, different from time-independent ones, forming the integrable hierarchy of the equation considered, and how to find all of them (cf. Ch. V of [4]). To the best of our knowledge, there were no attempts to find general answer to this question, although long ago all local generalized symmetries of KdV [6, 7] and Burgers [6] equations were found.

In this paper we present the results, enabling one to answer this question for a large class of evolution equations. In particular, we prove that if $\rho_{-1} = (\partial F/\partial u_n)^{-1/n} \notin \operatorname{Im} D$, but $\nabla_F(\rho_{-1}) \in \operatorname{Im} D$, then the equation $u_t = F(x, u, \ldots, u_n)$, $n \geq 2$, has no local generalized time-dependent symmetries of order higher than n (see Section 1 and Theorem 1 in Section 2 for details).

Next, for the majority of nonlinear evolution equations one can prove the *polynomiality* in time of *all* their local generalized symmetries, using scaling or other arguments, so it is interesting to consider the conditions of existence of polynomial in time symmetries, especially for the equations with $\rho_{-1} \in \operatorname{Im} D$. To this end one can apply our Theorem 2, stating that if canonical conserved density $\rho_m \notin \operatorname{Im} D$ for some $m \in \mathbb{N}$, then the equation $u_t = F(x, u, \ldots, u_n)$, $n \geq 2$, possessing time-independent formal symmetry, has no polynomial in time local generalized symmetries (except time-independent ones) of order higher than the number $p_F = p_F(m)$, given by (20).

Finally, on the basis of Theorems 1 and 2 we suggest the scheme of finding all local generalized time-dependent symmetries of a given integrable evolution equation and apply it to Harry Dym and modified KdV equations. We also discuss in brief the generalization of our results to the systems of evolution equations.

1 Basic definitions and known results

We consider the scalar $(1+1)$-dimensional evolution equation

$$\partial u/\partial t = F(x, u, u_1, \ldots, u_n), \qquad n \geq 2, \quad \partial F/\partial u_n \neq 0, \tag{1}$$

where $u_l = \partial^l u/\partial x^l$, $l = 0, 1, 2, \ldots$, $u_0 \equiv u$, and the local generalized symmetries of this equation, i.e. the right hand sides G of evolution equations

$$\partial u/\partial \tau = G(x, t, u, u_1, \ldots, u_k), \tag{2}$$

compatible with equation (1) (following [3, 6] we identify the symmetries with their characteristics).

For any function $H = H(x, t, u, u_1, \ldots, u_q)$ the greatest number m such that $\partial H/\partial u_m \neq 0$ is called its order and is denoted as $m = \operatorname{ord} H$. For $H = H(x, t)$ we assume that $\operatorname{ord} H = 0$. We shall call the function f of $x, t, u, u_1, \ldots$ *local* [4], if it has finite order.

We shall denote by $S_F^{(k)}$ the space of local generalized symmetries of order not higher than k of Eq.(1). Let also $S_F = \bigcup\limits_{j=0}^{\infty} S_F^{(j)}$, $\Theta_F = \{H(x,t) | H(x,t) \in S_F\}$, $S_{F,k} = S_F^{(k)}/S_F^{(k-1)}$ for $k = 1, 2, \ldots$, $S_{F,0} = S_F^{(0)}/\Theta_F$.

Finally, let Ann_F be the set of all local *time-independent* generalized symmetries of Eq.(1). In what follows we shall always consider time-dependent local generalized symmetries of Eq.(1) as the elements of quotient space S_F/Ann_F. In other words, we shall consider time-dependent symmetries modulo time-independent ones (i.e. up to the addition of linear combinations of time-independent symmetries).

S_F is Lie algebra with respect to the so-called Lie bracket [2, 4]

$$\{h, r\} = r_*(h) - h_*(r) = \nabla_h(r) - \nabla_r(h),$$

where for any sufficiently smooth function f of $x, t, u, u_1, \ldots, u_s$ we have introduced the notation

$$f_* = \sum_{i=0}^{s} \partial h/\partial u_i D^i, \qquad \nabla_f = \sum_{i=0}^{\infty} D^i(f) \partial/\partial u_i.$$

Here $D = \partial/\partial x + \sum\limits_{i=0}^{\infty} u_{i+1} \partial/\partial u_i$ is the total derivative with respect to x. We shall denote by $\operatorname{Im} D$ the image of the space of local functions under the action of the operator D.

G is symmetry of Eq.(1) if and only if [4]

$$\partial G/\partial t = -\{F, G\}. \tag{3}$$

Let us note without proof (cf. Lemma 5.21 from [4]) that for any $G \in S_F$, $\operatorname{ord} G = k \geq n_0$, we have

$$\partial G/\partial u_k = c_k(t) \Phi^{k/n}, \tag{4}$$

where $c_k(t)$ is a function of t, $\Phi = \partial F/\partial u_n$,

$$n_0 = \begin{cases} \max(1-j, 0), \text{ if } F \text{ is such that } \partial F/\partial u_{n-i} = \phi_i(x), \quad i = 0, \ldots, j, \\ 2 \text{ otherwise.} \end{cases} \tag{5}$$

In what follows we shall assume without loss of generality that any symmetry $G \in S_{F,k}$, $k \geq n_0$ vanishes, provided the relevant function $c_k(t)$ is identically equal to zero.

We make also a blanket assumption that all the functions that appear below in this paper (the function F, symmetries G, etc.) are locally analytical functions of their arguments.

For any local functions P, Q the relation $R = \{P, Q\}$ implies [2]

$$R_* = \nabla_P(Q_*) - \nabla_Q(P_*) + [Q_*, P_*], \tag{6}$$

$$[\nabla_P, \nabla_Q] = \nabla_R. \tag{7}$$

Here $\nabla_P(Q_*) \equiv \sum\limits_{i,j=0}^{\infty} D^j(P)\frac{\partial^2 Q}{\partial u_j \partial u_i} D^i$ and likewise for $\nabla_Q(P_*)$; $[\cdot,\cdot]$ stands for the usual commutator of linear differential operators.

For $P = G$, $Q = F$, using Eq.(3), we obtain

$$\partial G_*/\partial t \equiv (\partial G/\partial t)_* = \nabla_G(F_*) - \nabla_F(G_*) + [F_*, G_*]. \tag{8}$$

Now let us remind some facts concerning the formal series in powers of D (see e.g. [1, 3, 5] for more information; in contrast with these references we let the coefficients of the series depend explicitly on time, but this obviously doesn't alter the results, listed below), i.e. the expressions of the form

$$\mathrm{H} = \sum_{j=-\infty}^{m} h_j(x, t, u, u_1, \ldots) D^j. \tag{9}$$

The greatest integer m such that $h_m \neq 0$ is called the degree of formal series H and is denoted by $\deg \mathrm{H}$. For any formal series H of degree $m \neq 0$ there exists unique [5] (up to the multiplication by m-th root of unity) formal series $\mathrm{H}^{1/m}$ of degree 1 (or -1 for $m < 0$) such that $(\mathrm{H}^{1/m})^m = \mathrm{H}$. The fractional powers of H are defined as $\mathrm{H}^{l/m} = (\mathrm{H}^{1/m})^l$ for all integer l.

Let us also define [3] the residue of the formal series H as the coefficient at D^{-1}, i.e. $\operatorname{res} \mathrm{H} = h_{-1}$, and the logarithmic residue as $\operatorname{res} \ln \mathrm{H} = h_{m-1}/h_m$.

The formal series R is called the *formal symmetry* (of infinite rank) of Eq.(1), if it satisfies the relation (cf. [3])

$$\partial \mathrm{R}/\partial t + \nabla_F(\mathrm{R}) - [F_*, \mathrm{R}] = 0. \tag{10}$$

Finally, let us introduce the important notion of master symmetry [8] for the particular case of local functions: the local function $B(x, u, u_1, \ldots)$ is called (time-independent local) *master symmetry* of Eq.(1), if for any $P \in \mathrm{Ann}_F$ we have $\{B, P\} \in \mathrm{Ann}_F$. If in addition $\{B, F\} \neq 0$, we shall call B *strong master symmetry*. Like for the time-dependent symmetries, we shall always consider master symmetries up to the addition of the terms, being the linear combinations of time-independent symmetries.

2 The no-go theorem

By Theorem 1 from [10] for any symmetry G of Eq.(1) of order $k > n + n_0 - 2$ we have

$$G_* = \sum_{j=k-n+1}^{k} c_j(t) F_*^{j/n} + \left(\frac{1}{n}\dot{c}_k(t) D^{-1}(\Phi^{-1/n}) - \frac{k}{n} c_k(t) D^{-1}(\nabla_F(\Phi^{-1/n}))\right) F_*^{\frac{k-n+1}{n}} + \mathrm{N}, \tag{11}$$

where $c_j(t)$ are some functions of t and N is some formal series, $\deg \mathrm{N} < k - n + 1$.

Analyzing the terms, standing under D^{-1}, we conclude that if $\Phi^{-1/n} \notin \operatorname{Im} D$, while $\nabla_F(\Phi^{-1/n}) \in \operatorname{Im} D$, then G_* (and hence G itself) becomes nonlocal, and nonlocal terms vanish only if $\dot{c}_k(t) = 0$.

Lemma 1 *If $G \in S_{F,k}$, $k \geq n_0$, and there exists a linear differential operator $L = \sum_{j=0}^{q} a_j \partial^j/\partial t^j$, $a_j \in \mathbb{C}$, such that $L(c_k(t)) = 0$, then $L(G) = 0$.*

Proof of the lemma. Let us assume that the statement of the lemma is wrong, i.e. $L(G) = \tilde{G} \neq 0$. It is obvious that $\tilde{G} \in S_F^{(k-1)}$ ($\tilde{G} \in \Theta_F$ for $k = 0$) and that the determining equations (3) for $\tilde{G}$ contain neither $c_k(t)$ nor its time derivatives. Since by assumption $G \in S_{F,k}$, G must vanish if $c_k(t)$ vanishes. On the other hand, this is impossible, because $\tilde{G}$ is independent of $c_k(t)$ and its derivatives and $\tilde{G} \neq 0$. This contradiction may be avoided only if $\tilde{G} = 0$, what proves the lemma.

Using Lemma 1 with $L = \partial/\partial t$, we conclude from the above that if $\Phi^{-1/n} \notin \operatorname{Im} D$ and $\nabla_F(\Phi^{-1/n}) \in \operatorname{Im} D$, then all the elements of $S_{F,q}$ for $q > n + n_0 - 2$ are time-independent, and hence, since we consider time-dependent symmetries modulo time-independent ones, Eq.(1) has no local time-dependent generalized symmetries of order higher than $n + n_0 - 2$.

Finally, from the definition (5) of n_0 it is clear that for $n_0 = 0, 1$ $\Phi^{-1/n} = \tilde{\phi}(x) \in \operatorname{Im} D$ and hence the case $\Phi^{-1/n} \notin \operatorname{Im} D$ is possible only for $n_0 = 2$. Thus, we have proved

Theorem 1 *If $(\partial F/\partial u_n)^{-1/n} \notin \operatorname{Im} D$, while $\nabla_F((\partial F/\partial u_n)^{-1/n}) \in \operatorname{Im} D$, then Eq.(1) has no local time-dependent generalized symmetries of order higher than n.*

Let B be strong master symmetry of Eq.(1) and $\operatorname{ord} B > n$. Then $Q = B + t\{B, F\}$ obviously is time-dependent symmetry of Eq.(1) of order higher than n, what contradicts to Theorem 1. This contradiction proves the following

Corollary 1 *If the conditions of Theorem 1 hold for Eq.(1), then it has no local time-independent strong master symmetries of order higher than n.*

As an example, let us consider Harry Dym equation

$$u_t = u^3 u_3.$$

It is straightforward to check that we have $(\partial F/\partial u_3)^{-1/3} = u^{-1} \notin \operatorname{Im} D$, but $\nabla_F(u^{-1}) \in \operatorname{Im} D$. Hence, the equation in question has no time-dependent symmetries of order higher than 3. Further computation of symmetries of orders $0, \ldots, 3$ shows that, apart from the infinite hierarchy of time-independent symmetries, Harry Dym equation has only two local time-dependent generalized symmetries: $u + 3tu^3u_3$ and $xu_1 + 3tu^3u_3$, and both of them are equivalent to Lie point symmetries.

3 Structure of linear in time symmetries

Consider polynomial in time t symmetries of Eq.(1) from the space $S_{F,q}$. Using Lemma 1 with $L = \partial^s/\partial t^s$, one may easily check that in order to possess polynomial in time symmetry from $S_{F,q}$ Eq.(1) must possess (at least one) linear in t symmetry $Q = K + tH \in S_{F,q}$, $\partial K/\partial t = \partial H/\partial t = 0$, $H \in S_{F,q}$. It is obvious that

$$\{F, H\} = 0. \tag{12}$$

Since $Q \in S_{F,q}$, it is clear that $k \equiv \operatorname{ord} K \leq q$. The substitution of $G = Q$ and $P = F$ into (3) and (8) yields

$$\{F, K\} = -H, \tag{13}$$

$$\nabla_K(F_*) - \nabla_F(K_*) + [F_*, K_*] = H_*. \tag{14}$$

Since for arbitrary F and K $\mathrm{ord}\{F,K\} \le k+n-1$, Eq.(13) implies that $k+n-1 \ge q$ and hence $k \ge q-n+1$.

Plugging the symmetry Q into (11) and setting $t=0$, we immediately obtain the following representation for K_*, provided $q > n+n_0-2$:

$$K_* = \sum_{j=q-n+1}^{k} \kappa_j F_*^{j/n} + \left(\frac{\gamma}{n} D^{-1}(\Phi^{-1/n}) - \delta_{k,q}\frac{k}{n}\kappa_k D^{-1}(\nabla_F(\Phi^{-1/n}))\right) F_*^{\frac{q-n+1}{n}} + \mathrm{N}, \qquad (15)$$

where $\kappa_j \in \mathbb{C}$, $\gamma = \Phi^{-q/n}\partial H/\partial u_q \in \mathbb{C}$, $\gamma \neq 0$; N is some formal series with time-independent coefficients, $\deg \mathrm{N} < q-n+1$; $\delta_{k,q} = 1$ if $k=q$ and 0 otherwise.

Let us mention that if $k=q$, $\kappa_k \neq 0$, we may consider the symmetry $Q' = Q - (\kappa_q/\gamma)H = tH + K' \in S_{F,q}$ instead of Q, and for Q' we have $\mathrm{ord}\, K' < q$. Thus, we can always assume that $\mathrm{ord}\, K < q$ and hence $\kappa_k \delta_{k,q} = 0$.

4 Polynomial in time symmetries of evolution equation having formal symmetry

From now on we shall consider the evolution equation (1), possessing a time-independent ($\partial \mathrm{L}/\partial t = 0$) formal symmetry L of nonzero degree p. By definition, L satisfies the equation

$$[\nabla_F - F_*, \mathrm{L}] = 0. \qquad (16)$$

It is clear that for any integer q $c\mathrm{L}^{q/p}$, $c = \mathrm{const}$ also is formal symmetry of Eq.(1) [3]. Therefore, without loss of generality we may assume in what follows that $\deg \mathrm{L} = 1$ and $\mathrm{L} = (\partial F/\partial u_n)^{1/n} D + \cdots$ [3].

It is known [2] that there exists at most one (up to the addition of linear combination of local generalized *time-independent* symmetries $Z = Z(x,u,u_1,\ldots)$ of Eq.(1), satisfying $[\nabla_Z - Z_*, \mathrm{L}] = 0$) such local generalized time-independent symmetry Y of Eq.(1) that[1]

$$[\nabla_Y - Y_*, \mathrm{L}] \neq 0.$$

Let us choose Y to be of minimal possible order r, adding to it, if necessary, the appropriate linear combination of the symmetries $Z \in \mathrm{Ann}_F$, which satisfy the condition $[\nabla_Z - Z_*, \mathrm{L}] = 0$.

From now on we shall assume (it is clear that this does not lead to the loss of generality) that for any local generalized *time-independent* symmetry $P = P(x,u,u_1,\ldots) \in S_F/S_F^{(r)}$

$$[\nabla_P - P_*, \mathrm{L}] = 0. \qquad (17)$$

Finally, let us assume that $(\partial F/\partial u_n)^{-1/n} \in \mathrm{Im}\, D$, i.e. the necessary condition of existence of time-dependent symmetries of order higher than n, given in Theorem 1, is satisfied.

Now let us consider again the symmetry $Q = K + tH \in S_{F,q}$, $\mathrm{ord}\, Q = q > \max(r, n+n_0-2)$. Using Jacobi identity, Eq.(17) for $P = H$, Eqs. (13), (16), and Eqs.(6), (7) for $P = F$, $Q = K$, we obtain that for all integer s

$$[\nabla_F - F_*, [\mathrm{L}^s, \nabla_K - K_*]] = -[\nabla_K - K_*, [\nabla_F - F_*, \mathrm{L}^s]] - [\mathrm{L}^s, [\nabla_K - K_*, \nabla_F - F_*]] = \\ -[\mathrm{L}^s, [\nabla_K - K_*, \nabla_F - F_*]] = [\mathrm{L}^s, \nabla_H - H_*] = 0.$$

[1]If, e.g., as in the majority of cases, all time-independent symmetries are generated by hereditary recursion operator R from one seed symmetry, leaving R invariant, the symmetry Y with such property does not exist.

Hence, by Lemma 8 from [2] $[\mathrm{L}^s, \nabla_K - K_*] = \sum_{j=-\infty}^{k_s} c_{j,s}\mathrm{L}^j, c_{j,s} \in \mathbb{C}$. Straightforward but lengthy check, which we omit here, shows that in fact $k_s = s + q - n$, and thus

$$[\mathrm{L}^s, \nabla_K - K_*] = \sum_{j=-\infty}^{s+q-n} c_{j,s}\mathrm{L}^j, c_{j,s} \in \mathbb{C}, c_{s+q-n,s} \neq 0. \tag{18}$$

Since $\operatorname{res} \nabla_G(\mathrm{L}^s) = 0$ for $s \leq -2$ and $\operatorname{res} \mathrm{L}^j = 0$ for $j < -1$, Eq.(18) for $s \leq -2$ yields

$$\operatorname{res}\,[\mathrm{L}^s, K_*] = -\sum_{j=-\infty}^{s+q-n} c_{j,s}\operatorname{res} \mathrm{L}^j = -\sum_{j=-1}^{s+q-n} c_{j,s}\operatorname{res} \mathrm{L}^j. \tag{19}$$

But the residue of the commutator of two formal series always lies in $\operatorname{Im} D$ [3]. On the other hand, $\rho_j = \operatorname{res} \mathrm{L}^j$, $j = -1, 1, 2, 3, \ldots$ (and $\rho_0 = \operatorname{res} \ln \mathrm{L}$) are nothing but the so-called *canonical conserved densities* for Eq.(1) [3], and hence $\nabla_F(\rho_j) \in \operatorname{Im} D$. The density ρ_j is called *nontrivial*, if $\rho_j \notin \operatorname{Im} D$, and *trivial* otherwise.

If the density ρ_{s+q-n} is nontrivial, while ρ_j, $j = -1, 1, \ldots, s+q-n-1$ are trivial, then Eq.(19) contains a contradiction. Namely, its l.h.s. lies in $\operatorname{Im} D$, while the nonzero term $c_{s+q-n,s}\rho_{s+q-n}$ on its r.h.s. does not belong to $\operatorname{Im} D$.

Since the density $\rho_{-1} = (\partial F/\partial u_n)^{-1/n}$ is trivial by assumption, let us restrict ourselves to the case $s + q - n \geq 1$. This inequality is compatible with the condition $s \leq -2$ for the non-empty range of values of s if and only if $q > n+2$. Therefore, the range of values of s, for which Eq.(19) may contain the contradiction, is $n - q + 1, \ldots, -2$.

Thus, if for $q > \max(n + 2, r)$ at least one of the densities $\rho_1, \ldots, \rho_{q-n-2}$ is nontrivial, then Eq.(18) (and hence Eq.(13) with $H \in S_{F,q}$ as well) has no local time-independent solutions K.

Let

$$p_F = \begin{cases} m + n + 1, & \text{if (17) is satisfied for all } P \in \operatorname{Ann}_F, \\ \max(r, m + n + 1) & \text{otherwise,} \end{cases} \tag{20}$$

where $m \in \mathbb{N}$ is the smallest number such that $\rho_m \notin \operatorname{Im} D$, while for $j = -1, 1, \ldots, m-1$, $j \neq 0$, $\rho_j \in \operatorname{Im} D$ ($\rho_{-1} \in \operatorname{Im} D$ by assumption).

It is clear from the above that Eq.(1) has no polynomial in time symmetries from $S_F/S_F^{(p_F)}$ (except time-independent ones), and we obtain (cf. Theorem 1 and Corollary 1)

Theorem 2 *If Eq.(1) has time-independent formal symmetry* L, $\deg \mathrm{L} \neq 0$, *of infinite rank, and for some* $m \in \mathbb{N}$ $\rho_m \notin \operatorname{Im} D$, *while for* $j = -1, 1, \ldots, m-1$, $j \neq 0$, $\rho_j \in \operatorname{Im} D$, *then Eq.(1) has no polynomial in time*[2] *local generalized symmetries from* $S_F/S_F^{(p_F)}$.

Corollary 2 *If the conditions of Theorem 2 hold for Eq.(1), then it has no local time-independent strong master symmetries of order higher than* p_F.

Let us mention that provided one can prove that all the symmetries from $S_F/S_F^{(p_F)}$ are polynomial in time, Theorem 2, exactly like Theorem 1, implies the absence of *any* time-dependent local generalized symmetries of order higher than p_F of Eq.(1).

It is also important to stress that the application of Theorem 2 does *not* require the check of triviality of the density ρ_0, as shows the well known example of Burgers equation

$$u_t = u_2 + uu_1.$$

[2] Of course, except time-independent ones.

This equation has time-independent formal symmetry of degree 1 and local generalized polynomial in time symmetries of all orders [6], and its canonical densities $\rho_{-1}, \rho_1, \rho_2, \ldots$ are trivial, while ρ_0 is nontrivial.

Let us mention without proof that our results may be partially generalized to the case of systems of evolution equations of the form (1), where u is s-component vector, provided $s \times s$ matrix $\Phi = \partial F/\partial u_n$ is nondegenerate ($\det \Phi \neq 0$), may be diagonalized by means of some similarity transformation $\Phi \to \Phi' = \Omega\Phi\Omega^{-1}$ and has s distinct eigenvalues λ_i. We shall call the systems (1) with such properties *nondegenerate weakly diagonalizable.* For such systems we have the following analogs of Theorems 1 and 2:

Theorem 3 *If for all eigenvalues λ_i of Φ we have $\lambda_i^{-1/n} \notin \operatorname{Im} D$, but $\nabla_F(\lambda_i^{-1/n}) \in \operatorname{Im} D$, then nondegenerate weakly diagonalizable system (1) has no local time-dependent generalized symmetries of order higher than n.*

Theorem 4 *If nondegenerate weakly diagonalizable system (1) has time-independent formal symmetry $\mathrm{L} = \eta D^q + \cdots$ of infinite rank, with $\det \eta \neq 0$ and $q = \deg \mathrm{L} \neq 0$, (17) is satisfied for all time-independent symmetries P of (1), and*[3] *for some $m \in \mathbb{N}$ $\rho_m^l \notin \operatorname{Im} D$ for all $l = 1, \ldots, s$, while for $j = -1, 1, \ldots, m-1$, $j \neq 0$, $\rho_j^a \in \operatorname{Im} D$ for all $a = 1, \ldots, s$, then (1) has no polynomial in time local generalized symmetries (except time-independent ones) from $S_F/S_F^{(m+n+1)}$.*

The modification of Corollaries 1 and 2 for the case of nondegenerate weakly diagonalizable systems is obvious, so we leave it to the reader.

Let us note that the requirements of Theorem 4 may be relaxed. Namely, if there exist nontrivial densities ρ_j^a with $j < m$, $j \neq 0$, but only for $j = m$ *all* the densities ρ_m^l, $l = 1, \ldots, s$, are nontrivial, and the nontrivial densities ρ_j^a with $j < m$, $j \neq 0$, are linearly independent of ρ_m^a with the same value of index a, then the statement of Theorem 4 remains true.

Thus, Theorems 1 – 4 reveal interesting duality between time-dependent symmetries and canonical conserved densities of integrable evolution equations, which is completely different from the one coming e.g. from the famous Noether's theorem. Namely, as one can conclude from Theorems 1 – 4, the *nontriviality* of these densities (except ρ_0) turns out to be an obstacle to existence of polynomial in time (or even any time-dependent) local generalized symmetries of sufficiently high order of such equations, provided they possess time-independent formal symmetry. This result appears to be rather unexpected in view of the well known fact that the existence of canonical conserved densities is the necessary condition of existence of high order time-independent symmetries of the evolution equations, see e.g. [1]. However, the apparent contradiction between these two results vanishes, if we consider nonlocal symmetries. Indeed, integrable evolution systems usually possess the infinite number of *nonlocal* polynomial in time symmetries, which form the so-called hereditary algebra, see e.g. [9], and the nonlocal variables that these symmetries depend on turn out to be nothing but the integrals of nontrivial conserved densities.

5 Applications

It is well known that the straightforward finding of all time-dependent local generalized symmetries of a given integrable evolution equation, and especially the proof of completeness of the obtained set of symmetries, is a highly nontrivial task (see e.g. [7] for the case of KdV equation), in contrast with time-independent symmetries, all of which usually can be obtained by the repeated application of the recursion operator to one seed symmetry.

[3]See [3] for the definition of densities ρ_k^l.

Fortunately, our results allow to suggest a very simple and efficient way to find *all* local generalized time-dependent symmetries of a given integrable evolution equation.

First of all, one should find the smallest $m \in \{-1, 1, 2, \ldots\}$ such that the canonical conserved density ρ_m is nontrivial. If $m \neq -1$, then one should evaluate the number p_F and check the polynomiality in time of all local generalized symmetries from the space $S_F/S_F^{(p_F)}$, using scaling arguments or e.g. the results of [11]. If $m = -1$ or the polynomiality really takes place, then by Theorem 1 or 2 there exist no time-dependent symmetries (of course, modulo the infinite hierarchy of time-independent ones) of order higher than n or p_F respectively. Finally, all time-dependent symmetries of orders $0, \ldots, n$ or $0, \ldots, p_F$ can be found by straightforward computation, using e.g. computer algebra.

The similar scheme, this time based on Theorems 3 and 4, works for integrable nondegenerate weakly diagonalizable systems of evolution equations as well, provided all $P \in \mathrm{Ann}_F$ satisfy (17) (for $m = -1$ this is not required).

Our method fails, if ρ_j are trivial for all $j = -1, 1, 2, \ldots$ or it is impossible (for $m \neq -1$) to prove that all the elements of the space $S_F/S_F^{(p_F)}$ are polynomial in time. However, such situations are typical for linearizable equations, while for genuinely nonlinear integrable equations one usually encounters no difficulties in the application of the above scheme.

Let us consider for instance the modified Korteweg-de Vries (mKdV) equation

$$u_t = u_3 + u^2 u_1.$$

It has the recursion operator (see e.g. [4]) $\mathrm{R} = D^2 + (2/3)u^2 - (2/3)u_1 D^{-1} u$ and $\mathrm{L} = \mathrm{R}^{1/2}$ is the formal symmetry of degree 1, which satisfies (17) for all $P \in \mathrm{Ann}_{\mathrm{mKdV}}$, since the operator R is hereditary. The density $\rho_1 = u^2$ is nontrivial, while $\rho_{-1} = 1 \in \mathrm{Im}\, D$, so we have $p_{\mathrm{mKdV}} = 5$. All local time-dependent generalized symmetries of mKdV equation are polynomial in time [11], so by Theorem 2 it has no local generalized time-dependent symmetries of order greater than 5. The computation of symmetries of orders $0, \ldots, 5$ shows that the only generalized symmetry of mKdV equation, that doesn't belong to the infinite hierarchy of time-independent symmetries, is the dilatation $xu_1 + u + 3t(u_3 + u^2 u_1)$, which is equivalent to Lie point symmetry.

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