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Lie Submodels of Rank 1 for MHD Equations

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The MHD equations describing flows of a viscous homogeneous incompressible fluid of finite electrical conductivity are reduced by means of Lie symmetries to partial differential equations in three independent variables. Symmetry properties of the reduced systems are investigated.

1. Introduction. The MHD equations (the MHDEs) describing flows of a viscous homogeneous incompressible fluid of finite electrical conductivity have the following form:

$$\begin{aligned} &\vec{u}_t + (\vec{u}\cdot\vec{\nabla})\vec{u} - \triangle\vec{u} + \vec{\nabla}p + \vec{H}\times\operatorname{rot}\vec{H} = \vec{0}, \qquad \operatorname{div}\vec{u} = 0,\\ &\vec{H}_t - \operatorname{rot}(\vec{u}\times\vec{H}) - \nu_m\triangle\vec{H} = \vec{0}, \qquad \operatorname{div}\vec{H} = 0. \end{aligned} \tag{1}$$

System (1) is very complicated and construction of its new exact solutions is a difficult problem. In [1, 2] the MHDEs (1) are reduced to ordinary differential equations and to partial differential equations in two independent variables. Following [3], in this paper we reduce the MHDEs (1) to partial differential equations in three independent variables by means of one-dimensional subalgebras of the maximal Lie invariance algebra of the MHDEs.

In (1) and below, $\vec{u} = \{u^a(t,\vec{x})\}$ denotes the velocity field of a fluid, $p = p(t,\vec{x})$ denotes the pressure, $\vec{H} = \{H^a(t,\vec{x})\}$ denotes the magnetic intensity, ν_m is the coefficient of magnetic viscosity, $\vec{x} = \{x_a\}$, $\partial_t = \partial/\partial t$, $\partial_a = \partial/\partial x_a$, $\vec{\nabla} = \{\partial_a\}$, $\triangle = \vec{\nabla}\cdot\vec{\nabla}$ is the Laplacian. The kinematic coefficient of viscosity and fluid density are set equal to unity, permeability is done $(4\pi)^{-1}$. Subscripts of functions denote differentiation with respect to the corresponding variables.

The maximal Lie invariance algebra of the MHDEs (1) is an infinite-dimensional algebra $A(\mathrm{MHD})$ with the basis elements (see [4])

$$\begin{aligned} &\partial_t, \quad D = t\partial_t + \tfrac{1}{2}x_a\partial_a - \tfrac{1}{2}u^a\partial_{u^a} - \tfrac{1}{2}H^a\partial_{H^a} - p\partial_p,\\ &J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial_{u^b} - u^b\partial_{u^a} + H^a\partial_{H^b} - H^b\partial_{H^a}, \ {\scriptstyle a<b},\\ &R(\vec{m}) = m^a\partial_a + m^a_t\partial_{u^a} - m^a_{tt}x_a\partial_p, \quad Z(\chi) = \chi\partial_p, \end{aligned} \tag{2}$$

where $m^a = m^a(t)$ and $\chi = \chi(t)$ are arbitrary smooth functions of t (for example, from $C^\infty((t_0,t_1),\mathbb{R})$). Summation is understood over repeated indices. The indices a, b take values $1,2,3$ and i, j takes respectivily values $1,2$. The algebra $A(\mathrm{MHD})$ is isomorphic to the maximal Lie invariance algebra $A(\mathrm{NS})$ of the Navier–Stokes equations [5, 6, 7].

In addition to continuous transformations generated by operators (2), the MHDEs admit discrete transformations I_b of the form

$$\begin{aligned} &\tilde{t} = t, \qquad x_b = -x_b, \qquad \tilde{x}_a = x_a,\\ &\tilde{p} = p, \qquad \tilde{u}^b = -u^b, \qquad \tilde{H}^b = -H^b, \qquad \tilde{u}^a = u^a, \qquad \tilde{H}^a = H^a, \quad a \neq b, \end{aligned}$$

where b is fixed.

2. Inequivalent one-dimensional subalgebras of $A(\mathrm{MHD})$. A complete set of $A(\mathrm{MHD})$-inequivalent one-dimensional subalgebras of $A(\mathrm{MHD})$ is exhausted by the following algebras:

1. $A^1_1(\varkappa) = \langle D + \varkappa J_{12}\rangle$, where $\varkappa \geq 0$.

2. $A_2^1(\varkappa) = \langle \partial_t + \varkappa J_{12} \rangle$, where $\varkappa \in \{0;1\}$.

3. $A_3^1(\eta,\chi) = \langle J_{12} + R(0,0,\eta(t)) + Z(\chi(t)) \rangle$ with smooth functions η and χ. Algebras $A_3^1(\eta,\chi)$ and $A_3^1(\tilde\eta,\tilde\chi)$ are equivalent if $\exists\,\varepsilon,\delta \in \mathbb{R}$, $\exists\,\varepsilon_1,\varepsilon_2 \in \{-1;1\}$, $\exists\,\lambda \in C^\infty((t_0,t_1),\mathbb{R})$:

$$\tilde\eta(\tilde t) = \varepsilon_1 e^{-\varepsilon}\eta(t), \quad \tilde\chi(\tilde t) = \varepsilon_2 e^{2\varepsilon}(\chi(t) + \lambda_{tt}(t)\eta(t) - \lambda(t)\eta_{tt}(t)), \tag{3}$$

where $\tilde t = te^{-2\varepsilon} + \delta$.

4. $A_4^1(\vec m,\chi) = \langle R(\vec m(t)) + Z(\chi(t)) \rangle$ with smooth functions $\vec m$ and χ: $(\vec m,\chi) \not\equiv (\vec 0, 0)$. Algebras $A_4^1(\vec m,\chi)$ and $A_4^1(\tilde{\vec m},\tilde\chi)$ are equivalent if $\exists\,\varepsilon,\delta \in \mathbb{R}$, $\exists\,\varepsilon_1 \in \{-1;1\}$, $\exists\,C \neq 0$, $\exists\,B \in O(3)$, $\exists\,\vec l \in C^\infty((t_0,t_1),\mathbb{R}^3)$:

$$\tilde{\vec m}(\tilde t) = Ce^{-\varepsilon}B\vec m(t), \quad \tilde\chi(\tilde t) = C\varepsilon_1 e^{2\varepsilon}(\chi(t) + \vec l_{tt}(t)\cdot\vec m(t) - \vec m_{tt}(t)\cdot\vec l(t)), \tag{4}$$

where $\tilde t = te^{-2\varepsilon} + \delta$.

3. Lie ansatzes of codimension one for the MHD field. Using of the algebras $A_1^1 - A_4^1$ (in the case when additional restrictions for parameters are satisfied), we can construct ansatzes of codimension one for the MHD field. Let us list these ansatzes.

1.
$$\begin{aligned} &\vec u = |t|^{-1/2}O(\tau)\vec v + \tfrac12 t^{-1}\vec x + \varkappa t^{-1}\vec e_3 \times \vec x, \\ &\vec H = |t|^{-1/2}O(\tau)\vec G, \\ &p = |t|^{-1}q + \tfrac18 t^{-2}|\vec x|^2 + \tfrac12\varkappa t^{-2}r^2, \end{aligned} \tag{5}$$

where $\vec y = |t|^{-1/2}O^T(\tau)\vec x$, $\tau = \varkappa\ln|t|$. Here and below

$$v^a = v^a(y_1,y_2,y_3), \quad G^a = G^a(y_1,y_2,y_3), \quad q = q(y_1,y_2,y_3),$$

$$O(\tau) = \begin{pmatrix} \cos\tau & -\sin\tau & 0 \\ \sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad r = (x_1^2 + x_2^2)^{1/2}, \quad \vec e_3 = (0,0,1).$$

2. $$\vec u = O(\tau)\vec v + \varkappa\vec e_3 \times \vec x, \quad \vec H = O(\tau)\vec G, \quad p = q + \tfrac12\varkappa r^2, \tag{6}$$

where $\vec y = O^T(\tau)\vec x$, $\tau = \varkappa t$.

3.
$$\begin{aligned} &u^1 = x_1 r^{-1}v^1 - x_2 r^{-2}v^2, \quad u^2 = x_2 r^{-1}v^1 + x_1 r^{-2}v^2, \\ &u^3 = v^3 + \eta(t)r^{-2}v^2 + \eta_t(t)\arctan x_2/x_1, \\ &H^1 = x_1 r^{-1}G^1 - x_2 r^{-2}G^2, \quad H^2 = x_2 r^{-1}G^1 + x_1 r^{-2}G^2, \quad H^3 = G^3 + \eta(t)r^{-2}G^2, \\ &p = q - \tfrac12\eta_{tt}(t)(\eta(t))^{-1}x_3^2 + \chi(t)\arctan x_2/x_1, \end{aligned} \tag{7}$$

where $y_1 = r$, $y_2 = x_3 - \eta(t)\arctan x_2/x_1$, $y_3 := \tau = t$.

Notion. The expression for the pressure p from ansatz (7) is indeterminate in the points $t \in (t_0,t_1)$ where $\eta(t) = 0$. If there are such points t, we will consider ansatz (7) on the intervals (t_0^n,t_1^n) that are contained in the interval (t_0,t_1) and that satisfy one of the conditions:

a) $\eta(t) \neq 0 \quad \forall\, t \in (t_0^n,t_1^n)$;

b) $\eta(t) = 0 \quad \forall\, t \in (t_0^n,t_1^n)$.

In the latter case we consider $\eta_{tt}/\eta := 0$.

With the algebra $A_4^1(\vec m,\chi)$, an ansatz can be constructed only for such t wherefore $\vec m(t)\neq\vec 0$. If this condition is satisfied, it follows from (4) that the algebra $A_4^1(\vec m,\chi)$ is equivalent to the algebra $A_5^1(\vec m,0)$. An ansatz constructed with the algebra $A_4^1(\vec m,0)$ has the following form:

$$\begin{aligned}
4.\quad &\vec u = v^i\vec n^i + v^3|\vec m|^{-2}\vec m + (\vec m\cdot\vec x)|\vec m|^{-2}\vec m_t - y_i|\vec m|^{-1}\vec n^i_t,\\
&\vec H = G^i\vec n^i + G^3|\vec m|^{-2}\vec m,\\
&p = |\vec m|q - \tfrac12|\vec H|^2 - |\vec m|^{-2}(\vec m_{tt}\cdot\vec x)(\vec m\cdot\vec x) + \tfrac12(\vec m_{tt}\cdot\vec m)|\vec m|^{-4}(\vec m\cdot\vec x)^2-\\
&\quad -\tfrac32|\vec m|^{-4}\big((\vec m_t\cdot\vec n^i)y_i\big)^2 + \big(\tfrac14|\vec m|_{tt}|\vec m|^{-2} - \tfrac38(|\vec m|_t)^2|\vec m|^{-3}\big)y_iy_i,
\end{aligned}\tag{8}$$

where $y_i=\vec n^i\cdot\vec x$, $y_3=\tau:=\int|\vec m|dt$, $\vec n^i$ are smooth vector-functions such that

$$\vec n^i\cdot\vec m=\vec n^1\cdot\vec n^2=\vec n^1_t\cdot\vec n^2=0,\quad |\vec n^i|=|\vec m|^{1/2}.\tag{9}$$

Notion. There exist vector-functions $\vec n^i$ which satisfy conditions (9). They can be constructed in the following way [3]: let us fix smooth vector-functions $\vec k^i=\vec k^i(t)$ such that $\vec k^i\cdot\vec m=\vec k^1\cdot\vec k^2=0$, $|\vec k^i|=|\vec m|^{1/2}$, and set

$$\vec n^1=\vec k^1\cos\psi(t)-\vec k^2\sin\psi(t),\qquad \vec n^2=\vec k^1\sin\psi(t)+\vec k^2\cos\psi(t).\tag{10}$$

Then $\quad \vec n^1_t\cdot\vec n^2=\vec k^1_t\cdot\vec k^2-\psi_t=0\quad$ if $\quad\psi=\int(\vec k^1_t\cdot\vec k^2)dt$.

4. Reduced systems in three independent variables. Substituting ansatzes (5) and (6) into the MHDEs (1), we obtain reduced systems of PDEs with the same general form

$$\begin{aligned}
&(\vec v\cdot\nabla)\vec v-\triangle\vec v+\nabla q+\vec G\times\operatorname{rot}\vec G+\gamma_1\vec e_3\times\vec v=\vec 0,\\
&(\vec v\cdot\nabla)\vec G-(\vec G\cdot\nabla)\vec v-\nu_m\triangle\vec G+\gamma_2\vec G=\vec 0,\\
&\operatorname{div}\vec v=\tfrac32\gamma_2,\quad \operatorname{div}\vec G=0.
\end{aligned}\tag{11}$$

Hereafter the functions v^a, G^a, and q are differentiated with respect to the variables y_1, y_2, and y_3. The constants γ_a take the values

1. $\gamma_1=2\varkappa\operatorname{sign}t,\quad \gamma_2=-\operatorname{sign}t;$
2. $\gamma_1=2\varkappa,\quad \gamma_2=0.$

For ansatzes (7) and (8) the reduced equations have the form

$$\begin{aligned}
3.\quad &\mathcal M^1+q_1+y_1^{-3}((G^3)^2-(v^3)^2-2\eta v^3_2)-v^1_1y_1^{-1}+v^1y_1^{-2}=0,\\
&\mathcal M^2+(1+\eta^2y_1^{-2})q_2+2\eta y_1^{-3}(G^1G^3-v^1v^3+v^3_1-\eta v^1_2-\\
&\quad -2v^3y_1^{-1})-y_1^{-1}v^2_1+2\eta_ty_1^{-2}v^3-\eta_{tt}\eta^{-1}y_2-\eta\chi y_1^{-2}=0,\\
&\mathcal M^3-\eta q_2+v^3_1y_1^{-1}+2\eta y_1^{-1}v^1_2+\chi=0,\\
&\mathcal N^1+\nu_m(-2\eta y_1^{-3}G^3_2+y_1^{-2}G^1-y_1^{-1}G^1_1)=0,\\
&\mathcal N^2+\nu_m(-2\eta^2y_1^{-3}G^1_2+2\eta y_1^{-3}G^3_1-y_1^{-1}G^2_1-4\eta G^3y_1^{-4})=0,\\
&\mathcal N^3+2y_1^{-1}(v^3G^1-v^1G^3)+2\nu_m\eta y_1^{-1}G^1_2+\nu_mG^3y_1^{-1}=0,\\
&v^i_i+v^1y_1^{-1}=0,\quad G^i_i+G^1y_1^{-1}=0,
\end{aligned}\tag{12}$$

where $\quad\mathcal M^a=v^a_\tau+v^jv^a_j-G^jG^a_j-v^a_{11}-(1+\eta^2y_1^{-2})v^a_{22},$

$\qquad\quad\mathcal N^a=G^a_t+v^iG^a_i-G^iv^a_i-\nu_mG^a_{11}-\nu_m(1+\eta^2y_1^{-2})G^a_{22}.$

$$
\begin{aligned}
4.\quad & v^i_\tau + v^j v^i_j - G^j G^i_j - v^i_{jj} + q_i + 2\beta^i\alpha^{-3}v^3 = 0,\\
& v^3_\tau + v^j v^3_j - G^j G^3_j - v^3_{jj} = 0,\\
& G^i_\tau + v^j G^i_j - G^j v^i_j - \nu_m G^i_{jj} + \alpha_\tau\alpha^{-1}G^i = 0,\\
& G^3_\tau + v^j G^3_j - G^j v^3_j - \nu_m G^3_{jj} - 2\beta^j G^j - 2\alpha_\tau\alpha^{-1}G^3 = 0,\\
& v^i_i = 0, \quad G^i_i = 0,
\end{aligned}
\tag{13}
$$

where $\alpha = \alpha(\tau) = |\vec{m}|$, $\beta^i = \beta^i(\tau) = (\vec{m}_\tau \cdot \vec{n}^i)$.

5. Symmetry of reduced systems. Let us study symmetry properties of systems (11), (12), and (13). All results of this subsection are obtained by means of the standard Lie algorithm [9, 8].

Symmetry properties of the systems (11). The maximal Lie invariance algebra of system (11) is the algebra

a) $\langle \partial_a,\ \partial_q,\ J^1_{12}\rangle$ if $\gamma_1 \neq 0$;

b) $\langle \partial_a,\ \partial_q,\ J^1_{ab}\rangle$ if $\gamma_1 = 0$, $\gamma_2 \neq 0$;

c) $\langle \partial_a,\ \partial_q,\ J^1_{ab},\ D^1_1\rangle$ if $\gamma_1 = \gamma_2 = 0$.

Here $J^1_{ab} = y_a\partial_{y_b} - y_b\partial_{y_a} + v^a\partial_{v^b} - v^b\partial_{v^a} + G^a\partial_{G^b} - G^b\partial_{G^a}$,

$D^1_1 = y_a\partial_{y_a} - v^a\partial_{v^a} - G^a\partial_{G^a} - 2q\partial_q$.

Note. All Lie symmetry operators of (11) are induced by operators from $A(\mathrm{MHD})$: The operators J^1_{ab} and D^1_1 are induced by J_{ab} and D. The operators $c_a\partial_a$ ($c_a = \mathrm{const}$) and ∂_q are induced by either

$$R(|t|^{1/2}(c_1\cos\tau - c_2\sin\tau, c_1\sin\tau + c_2\cos\tau, c_3)), \quad Z(|t|^{-1}),$$

where $\tau = \varkappa\ln|t|$, for ansatz (5) or

$$R(c_1\cos\varkappa t - c_2\sin\varkappa t, c_1\sin\varkappa t + c_2\cos\varkappa t, c_3), \quad Z(1)$$

for ansatz (6), respectively. Therefore, Lie reduction of system (11) gives only solutions that can be obtained by reducing the MHDEs with two- and three-dimensional subalgebras of $A(\mathrm{MHD})$.

Symmetry properties of the systems (12). Let $A^{\max}$ be the maximal Lie invariance algebra of system (12). Studying symmetry properties of (12), one has to consider the following cases:

A. $\eta, \chi \equiv 0$. Then

$$A^{\max} = \langle \partial_\tau,\ D^1_2,\ R_1(\zeta(\tau)),\ Z^1(\lambda(\tau))\rangle,$$

where $D^1_2 = 2\tau\partial_\tau + y_i\partial_{y_i} - v^i\partial_{v^i} - 2v^3\partial_{v^3} - G^i\partial_{G^i} - 2G^3\partial_{G^3} - 2q\partial_q$,

$R_1(\zeta(\tau)) = \zeta\partial_2 + \zeta_\tau\partial_{v^2} - \zeta_{\tau\tau}y_2\partial_q$, $\quad Z^1(\lambda(\tau)) = \lambda(\tau)\partial_q$.

Here and below $\zeta = \zeta(\tau)$ and $\lambda = \lambda(\tau)$ are arbitrary smooth functions of $\tau = t$.

B. $\eta \equiv 0$, $\chi \not\equiv 0$. In this case an extension of $A^{\max}$ exists for $\chi = (C_1\tau + C_2)^{-1}$, where $C_1, C_2 = \mathrm{const}$. Let $C_1 \neq 0$. We can make C_2 vanish by means of equivalence transformation (3), i.e., $\chi = C\tau^{-1}$, where $C = \mathrm{const}$. Then

$$A^{\max} = \langle D^1_2,\ R_1(\zeta(\tau)),\ Z^1(\lambda(\tau))\rangle.$$

If $C_1 = 0$, $\chi = C = \mathrm{const}$ and $A^{\max} = \langle \partial_\tau,\ R_1(\zeta(\tau)),\ Z^1(\lambda(\tau))\rangle$.

For other values of χ, i.e., when $\chi_{\tau\tau}\chi \neq \chi_\tau\chi_\tau$, $A^{\max} = \langle R_1(\zeta(\tau)),\ Z^1(\lambda(\tau))\rangle$.

C. $\eta \neq 0$. Using equivalence transformation (3) we always can make $\chi = 0$. In this case an extension of $A^{\max}$ exists for $\eta = \pm|C_1\tau + C_2|^{1/2}$, where $C_1, C_2 = \mathrm{const}$. Let $C_1 \neq 0$. We can annihilate C_2 by means of equivalence transformation (3), i.e., $\eta = C|\tau|^{1/2}$, where $C = \mathrm{const}$. Then

$$A^{\max} = \langle D^1_2,\ R_2(|\tau|^{1/2}),\ R_2(|\tau|^{1/2}\ln|\tau|),\ Z^1(\lambda(\tau))\rangle,$$

where $R_2(\zeta(\tau)) = \zeta\partial_{y_2} + \zeta_\tau\partial_{v^2}$. If $C_1 = 0$, i.e., $\eta = C = \mathrm{const}$,

$$A^{\max} = \langle \partial_\tau,\ \partial_{y_2},\ \tau\partial_{y_2} + \partial_{v^2},\ Z^1(\lambda(\tau))\rangle.$$

For other values of η, i.e., when $(\eta^2)_{\tau\tau} \neq 0$,

$$A^{\max} = \langle R_2(\eta(\tau)),\ R_2(\eta(\tau)\textstyle\int(\eta(\tau))^{-2}d\tau), Z^1(\lambda(\tau))\rangle.$$

Note. In all cases considered above the Lie symmetry operators of (12) are induced by operators from $A(\mathrm{MHD})$: The operators ∂_τ, D^1_2, and $Z^1(\lambda(\tau))$ are induced by ∂_t, D, and $Z(\lambda(t))$, respectively. The operator $R(0,0,\zeta(t))$ induces the operator $R_1(\zeta(\tau))$ for $\eta \equiv 0$ and the operator $R_2(\zeta(\tau))$ (if $\zeta_{\tau\tau}\eta - \zeta\eta_{\tau\tau} = 0$) for $\eta \neq 0$. Therefore, the Lie reduction of system (12) gives only the solutions that can be obtained by reducing the MHDEs with two- and three-dimensional subalgebras of $A(\mathrm{MHD})$.

Symmetry properties of the systems (13). Let us introduce the notations

$$S^1 = \partial_{v^3} - 2\beta^i\alpha^{-3}y_i\partial_q, \quad S^2 = (\alpha)^2\partial_{G^3}, \quad \tilde Z(\lambda(\tau)) = \lambda\partial_q,$$
$$\tilde R(\bar\psi(\tau)) = \psi^i\partial_{y_i} + \psi^i_\tau\partial_{v^i} - \psi^i_{\tau\tau}y_i\partial_q, \quad \bar\psi = (\psi^1, \psi^2),$$
$$\tilde D = \tau\partial_\tau + \tfrac12 y_i\partial_{y_i} - \tfrac12 v^i\partial_{v^i} - \tfrac12 G^i\partial_{G^i} - q\partial_q,\ \tilde I = v^3\partial_{v^3} + G^3\partial_{G^3},$$
$$\tilde J_{12} = y_1\partial_{y_2} - y_2\partial_{y_1} + v^1\partial_{v^2} - v^2\partial_{v^1} + G^1\partial_{G^2} - G^2\partial_{G^1}.$$

For arbitrary values of the parameter-functions α and β^i, the system (13) is invariant under the algebra

$$A^{\mathrm{all}} = \langle \tilde R(\bar\psi),\ S^1,\ S^2,\ \tilde Z(\lambda)\rangle.$$

Extensions of the maximal Lie invariance algebra of system (13) exist in the following cases (for each extension we write down its basis operators):

1. $\beta^i = 0$, $\alpha_\tau = 0$, $\nu_m = 1$: $\tilde D,\ \partial_\tau,\ \tilde J_{12},\ I,\ G^3\partial_{v^3} + v^3\partial_{G^3}$.
2. $\beta^i = 0$, $\alpha_\tau = 0$, $\nu_m \neq 1$: $\tilde D,\ \partial_\tau,\ \tilde J_{12},\ I$.
3. $\beta^i = 0$, $\alpha = a_2|\tau + a_0|^{a_1}$, $a_1a_2 \neq 0$: $\tilde D + a_0\partial_\tau,\ \tilde J_{12},\ I$.
4. $\beta^i = 0$, $\alpha = a_2e^{a_1\tau}$, $a_1a_2 \neq 0$: $\partial_\tau,\ \tilde J_{12},\ I$.
5. $\beta^i = 0$, $\alpha\alpha_\tau\alpha_{\tau\tau\tau} + (\alpha_\tau)^2\alpha_{\tau\tau} - 2\alpha(\alpha_{\tau\tau})^2 \neq 0$: $\tilde J_{12},\ I$.
6. $\beta^i \neq 0$, $\beta^i_\tau = 0$, $\alpha_\tau = 0$: $\tilde D - \frac32 I,\ \partial_\tau$.
7. $\beta^1 = \rho\cos\theta$, $\beta^2 = \rho\sin\theta$, $\alpha = a_2|\tau + a_0|^{a_1}$, where
   $\rho = b_1|\tau + a_0|^{a_1/2-1}$, $\theta = b_2\ln|\tau + a_0| + b_3$, and $(a_1b_1, b_2) \neq (0,0)$:
   $\tilde D + a_0\partial_\tau - b_2\tilde J_{12} + \frac12(a_1 - 1)I$.
8. $\beta^1 = \rho\cos\theta$, $\beta^2 = \rho\sin\theta$, $\alpha = a_2e^{a_1\tau}$, where
   $\rho = b_1e^{3a_1\tau/2}$, $\theta = b_2\tau + b_3$, and $(a_1b_1, b_2) \neq (0,0)$:
   $\partial_\tau - b_2\tilde J_{12} + \frac32 a_1 I$.

Note. The vector-functions $\vec{n}^i$ from Ansatz 4 are determined up to the transformation

$$\vec{n}^1 \longrightarrow \vec{n}^1 \cos\delta - \vec{n}^2 \sin\delta, \qquad \vec{n}^2 \longrightarrow \vec{n}^1 \sin\delta + \vec{n}^2 \cos\delta,$$

where $\delta = \mathrm{const}$. Therefore, δ can be chosen such that $b_3 = 0$.

Note. The operators $R(\bar{\psi}(t)) + C_1 S^1$, $\tilde{Z}(\lambda(\tau))$ from $A^{\rm all}$ are induced by the operators $R(\vec{l}(t))$, $Z(\chi(t))$, respectively. Here

$$\chi(t) = \lambda(\tau(t)), \quad \vec{l}(t) = \psi^i(\tau(t))\vec{n}^i(t) + \varphi(t)\vec{m}(t),$$
$$\text{where} \quad 2\psi^i(\tau(t))(\vec{n}^i_t(t)\cdot\vec{m}(t)) + \varphi(t)|\vec{m}(t)|^2 = C_1.$$

The operator S^2 is not induced by operators from $A(\mathrm{MHD})$. Therefore, Lie reduction of system (13) can give solutions that can not be obtained by reducing the MHDEs with two- and three-dimensional subalgebras of $A(\mathrm{MHD})$.

Consider inducing the operators from extension of $A^{\rm all}$. The operators I and $G^3\partial_{v^3} + v^3\partial_{G^3}$ are not induced by operators from $A(\mathrm{MHD})$.

The operator $\tilde{J}_{12}$ belongs to the maximal Lie invariance algebra of the system (13) if $\beta^i = 0$. In this case $\vec{m} = |\vec{m}|\vec{e}$, where $\vec{e} = \mathrm{const}$ and $|\vec{e}| = 1$. Then, the operator $\tilde{J}_{12}$ is induced by $e_1J_{23} + e_2J_{31} + e_3J_{12}$.

For $\vec{m} = e^{\sigma t}(c_2\cos\theta,\, c_2\sin\theta,\, c_1)$ with $c_1, c_2, \sigma, \varkappa, \delta = \mathrm{const}$ and $\theta = \varkappa t + \delta$, where $c_1^2 + c_2^2 = 1$, the operator $\partial_t + \varkappa J_{12}$ induces the operator $\partial_\tau - c_1\varkappa\tilde{J}_{12} + \sigma I$ if the following vector-functions $\vec{n}^i$ are chosen:

$$\vec{n}^1 = \vec{k}^1\cos c_1\theta + \vec{k}^2\sin c_1\theta, \quad \vec{n}^2 = -\vec{k}^1\sin c_1\theta + \vec{k}^2\cos c_1\theta, \tag{14}$$

where $\vec{k}^1 = (-\sin\theta, \cos\theta, 0)$ and $\vec{k}^2 = (c_1\cos\theta, c_1\sin\theta, -c_2)$.

For $\vec{m} = |t+\tilde{\delta}|^{\sigma+1/2}(c_2\cos\theta,\, c_2\sin\theta,\, c_1)$ with $\theta = \varkappa\ln|t+\tilde{\delta}| + \delta$ and $c_1, c_2, \sigma, \varkappa, \delta, \tilde{\delta} = \mathrm{const}$, where $c_1^2 + c_2^2 = 1$, the operator $D + \tilde{\delta}\partial_t + \varkappa J_{12}$ induces the operator $\tilde{D} + \tilde{\delta}\partial_\tau - c_1\varkappa\tilde{J}_{12} + \sigma I$, if the vector-functions $\vec{n}^i$ are chosen in form (14).

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