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On $SO(3)$-Partially Invariant Solutions of the Euler Equations

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$SO(3)$-partially invariant solutions having minimal defect are constructed for the Euler equations describing flows of an ideal incompressible fluid.

The concept of partially invariant solutions was introduced by Ovsiannikov [1] as a generalization of invariant solutions, which is possible for systems of partial differential equations (PDEs). The algorithm for finding partially invariant solutions is very difficult to apply. For this reason it is used more rarely than the classical Lie algorithm for constructing invariant solutions.

The Euler equations (EEs) describing flows of an ideal incompressible fluid have the following form:

$$\vec{u}_t + (\vec{u}\cdot\vec{\nabla})\vec{u} + \vec{\nabla}p = \vec{0}, \qquad \operatorname{div}\vec{u} = 0. \tag{1}$$

It is well known [2, 3] that the maximal Lie invariance algebra of EEs is the infinite dimensional algebra $A(E)$, generated by the following basis elements:

$$\begin{aligned}
&\partial_t, \quad J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial_{u^b} - u^b\partial_{u^a} \quad (a<b),\\
&D^t = t\partial_t - u^a\partial_{u^a} - 2p\partial_p, \quad D^x = x_a\partial_a + u^a\partial_{u^a} + 2p\partial_p,\\
&R(\vec{m}) = R(\vec{m}(t)) = m^a(t)\partial_a + m^a_t(t)\partial_{u^a} - m^a_{tt}(t)x_a\partial_p,\\
&Z(\chi) = Z(\chi(t)) = \chi(t)\partial_p.
\end{aligned} \tag{2}$$

In the following $\vec{u} = \{u^a(t,\vec{x})\}$ denotes the velocity of the fluid, $p = p(t,\vec{x})$ denotes the pressure, $\vec{x} = \{x_a\}$, $\partial_t = \partial/\partial t$, $\partial_a = \partial/\partial x_a$, $\vec{\nabla} = \{\partial_a\}$, $\triangle = \vec{\nabla}\cdot\vec{\nabla}$ is the Laplacian, $m^a = m^a(t)$ and $\chi = \chi(t)$ are arbitrary smooth functions of t (for example, from $C^\infty((t_0,t_1),\mathbb{R})$). The fluid density is set equal to unity. Summation over repeated indices is implied, and we have $a,b = 1,2,3$. Subscripts of functions denote differentiation with respect to the corresponding variables.

Let us note that the algebra $so(3)$ generated by the operators J_{ab} is a subalgebra of $A(E)$.

Invariant solutions of (1) have been already constructed. For example, in [4, 5] EEs are reduced to partial differential equations in two and three independent variables by means of the Lie algorithm. In this paper we obtain $SO(3)$-partially invariant solutions of the minimal defect that is equal to 1 for the given representation of $so(3)$.

A complete set of functionally independent invariants of the group $SO(3)$ in the space of the variables $(t,\vec{x},\vec{u},p)$ is exhausted by the functions t, $|\vec{x}|$, $\vec{x}\cdot\vec{u}$, $|\vec{u}|$, p, so $SO(3)$-partially invariant solution of the minimal defect has the form

$$\begin{aligned}
u^R &= v(t,R),\\
u^\theta &= w(t,R)\sin\psi(t,R,\theta,\varphi),\\
u^\varphi &= w(t,R)\cos\psi(t,R,\theta,\varphi),\\
p &= p(t,R).
\end{aligned} \tag{3}$$

Hereafter for convenience the spherical coordinates are used:

$$R = |\vec{x}|, \qquad \varphi = \arctan \frac{x_2}{x_1}, \qquad \theta = \arccos \frac{x_3}{|\vec{x}|},$$

$$u^R = \frac{\vec{x} \cdot \vec{u}}{|\vec{x}|}, \qquad u^\varphi = \frac{(\vec{e}_3 \times \vec{x}) \cdot \vec{u}}{|(\vec{e}_3 \times \vec{x})|}, \qquad u^\theta = \frac{((\vec{e}_3 \times \vec{x}) \times \vec{x}) \cdot \vec{u}}{|((\vec{e}_3 \times \vec{x}) \times \vec{x})|}, \qquad \vec{e}_3 := (0,0,1).$$

Substituting (3) into EEs (1), we obtain the system of PDEs for the functions v, w, ψ, p :

$$\begin{aligned} & v_t + v v_R - R^{-1} w^2 + p_r = 0, \\ & w_t + v w_R + R^{-1} v w = 0, \\ & w(\psi_t + v\psi_R + R^{-1} w \psi_\theta \sin\psi + R^{-1} w \cos\psi (\sin\theta)^{-1}(\psi_\varphi - \cos\theta)) = 0, \\ & R v_r + 2v + w\psi_\theta \cos\psi - (\sin\theta)^{-1} w \sin\psi (\psi_\varphi - \cos\theta) = 0. \end{aligned} \tag{4}$$

It follows from (4) if $w = 0$ that $v = \eta R^{-2}$, $p = \eta_t R^{-1} - \frac{1}{2}\eta^2 R^{-4} + \chi$, where η and χ are arbitrary smooth functions of t. The corresponding solution of EEs

$$u^R = \frac{\eta}{R^2}, \qquad u^\theta = u^\varphi = 0, \qquad p = \frac{\eta_t}{R} - \frac{\eta^2}{2R^4} + \chi \tag{5}$$

is invariant with respect to $SO(3)$. Note that flow (5) is a solution of the Navier-Stokes equations too, and it is the unique $SO(3)$-partially invariant solutions of the minimal defect for the Navier-Stokes equations.

Below $w \neq 0$. Then two last equations of (4) form an overdetermined system in the function ψ. This system can be rewritten as follows

$$\begin{aligned} & \psi_\theta + R w^{-1} \sin\psi (\psi_t + v\psi_R) = -G \cos\psi, \\ & \psi_\varphi + R w^{-1} \cos\psi (\psi_t + v\psi_R) \sin\theta = G \sin\psi \sin\theta + \cos\theta, \end{aligned} \tag{6}$$

where $G = G(t, R) := w^{-1}(R v_R + 2v)$. The Frobenius theorem gives the compatibility condition of (6):

$$G_t + v G_R = R^{-1} w (1 + G^2). \tag{7}$$

If condition (7) holds, system (6) can be integrated implicitely. Namely, its general solution has the form

$$F(\Omega_1, \Omega_2, \Omega_3) = 0, \tag{8}$$

where F is an arbitrary function of Ω_1, Ω_2, and Ω_3,

$$\Omega_1 = \frac{\sin\theta \sin\psi - G\cos\theta}{\sqrt{1 + G^2}}, \qquad \Omega_2 = \varphi + \arctan \frac{\cos\psi}{\cos\theta \sin\psi + G \sin\theta}, \qquad \Omega_3 = h(t, r),$$

$h = h(t, R)$ is a fixed solution of the equation $h_t + v h_R = 0$ such that $(h_t, h_R) \neq (0, 0)$. Equation (8) can be solved with respect to ψ in a number of cases, for example, if either $F_{\Omega_1} = 0$ or $F_{\Omega_2} = 0$.

Equation (7) and two first equations of (4) form the "reduced" system for the invariant functions v, w, and p. It can be represented as the union of the system

$$\begin{aligned} & R^2 f_{tR} + f f_{RR} - (f_R)^2 = g, \qquad && f := R^2 v \\ & R^2 g_t + f g_R = 0, && g := (Rw)^2, \end{aligned} \tag{9}$$

for the functions v and w (this system can be also considered a system for the functions f and g) and the equation

$$p_R = -v_t - vv_R - R^{-1}w^2 \tag{10}$$

which is one for the function p if v and w are known. Therefore, to construct solutions for EEs, we are to carry out the following chain of actions: 1) to solve the system (9); 2) to integrate (10) with respect to p; 3) to find the function ψ from (8).

Theorem. *The maximal Lie invariance algebra of (9) is the algebra*

$$\mathcal{A} = \langle\, \partial_t,\ D^R = R\partial_R + v\partial_v + w\partial_w,\ D^t = t\partial_t - v\partial_v - w\partial_w \,\rangle.$$

A complete set of $\mathcal{A}$-inequivalent one-dimensional subalgebras of $\mathcal{A}$ is exhausted by four algebras. Let us enumerate these algebras and the corresponding ansatzes for the functions v and w as well as the reduced systems arising after substituting the ansatzes into (9).

1. $\langle\, \partial_t \,\rangle:$ $\quad v = R^{-2}\varphi^1(\omega),\ w = R^{-1}\varphi^2(\omega),\ \omega = R, \quad \varphi^2 \neq 0;$
   $\varphi^1\varphi^1_{\omega\omega} - (\varphi^1_\omega)^2 = (\varphi^2)^2, \quad \varphi^1\varphi^2_\omega = 0.$
2. $\langle\, D^R \,\rangle:$ $\quad v = R\varphi^1(\omega),\ w = R/\varphi^2(\omega),\ \omega = t, \quad \varphi^1\varphi^2 \neq 0;$
   $3\varphi^1_\omega = 3(\varphi^1)^2 + (\varphi^2)^{-2}, \quad \varphi^2_\omega = 2\varphi^1\varphi^2.$
3. $\langle\, \partial_t + D^R \,\rangle:$ $\quad v = R\varphi^1(\omega),\ w = R\varphi^2(\omega),\ \omega = \ln R - t, \quad \varphi^1_\omega\varphi^2_\omega \neq 0;$
   $(\varphi^1 - 1)\varphi^1_{\omega\omega} - (\varphi^1_\omega)^2 - \varphi^1_\omega(\varphi^1 + 3) - 3(\varphi^1)^2 = (\varphi^2)^2,$
   $(\varphi^1 - 1)\varphi^2_\omega + 2\varphi^1\varphi^2 = 0.$
4. $\langle\, D^t + \kappa D^R \,\rangle:$ $\quad v = Rt^{-1}\varphi^1(\omega),\ w = Rt^{-1}\varphi^2(\omega),\ \omega = \ln R - \kappa\ln|t|, \quad \varphi^1\varphi^2 \neq 0;$
   $(\varphi^1 - \kappa)\varphi^1_{\omega\omega} - (\varphi^1_\omega)^2 - \varphi^1_\omega(\varphi^1 + 3\kappa + 1) - 3(\varphi^1)^2 - 3\varphi^1 = (\varphi^2)^2,$
   $(\varphi^1 - \kappa)\varphi^2_\omega + (2\varphi^1 - 1)\varphi^2 = 0.$

Two first reduced systems can be integrated completely. As a result we obtain the following expressions for the functions v, w, and p :

1. $v = \dfrac{C_3}{2R^2}\Big(e^{C_2R} + C_1^2 e^{-C_2R}\Big), \quad w = \dfrac{C_1C_2C_3}{R}, \quad C_1, C_2, C_3 = \mathrm{const}, \quad C_1C_2C_3 \neq 0,$

$$\Longrightarrow \quad p = -\frac{C_3^2}{8R^4}\Big(e^{2C_2R} + 2C_1^2 + C_1^4 e^{-2C_2R}\Big) - \frac{C_1^2C_2^2C_3^2}{2R^2} + \chi(t),$$

$$G = \frac{1}{2C_1}\Big(e^{C_2R} - C_1^2 e^{-C_2R}\Big), \quad h = t - \int \frac{dR}{v(R)}.$$

2. $v = \dfrac{\wp_t}{2\wp}R, \quad w = \dfrac{3C}{2\wp}R, \quad \Longrightarrow \quad p = C^2\Big(\dfrac{1}{\wp^2} - \wp\Big)R^2 + \chi(t), \quad G = \dfrac{\wp_t}{C}, \quad h = \dfrac{R^2}{\wp}.$

Here $C = \mathrm{const}$, $C \neq 0$, $\wp = \wp(Ct, 0, 1)$ is the Weierstrass function, χ is an arbitrary smooth function of t.

System (9) has solutions for which f and g are polynomial with respect to R. Thus, if $\deg(f, R) = 1$ and, therefore, $\deg(g, R) \leqslant 2$, then $f = C^2tR$, $g = C^2R^2 - C^4t^2$, where $C = \mathrm{const}$, $C \neq 0$, i.e.

$$v = \frac{C^2t}{R}, \quad w = \frac{C}{R}\sqrt{R^2 - C^2t^2} \quad \Longrightarrow \quad p = \chi(t), \quad G = \frac{Ct}{\sqrt{R^2 - C^2t^2}}, \quad h = \sqrt{R^2 - C^2t^2}.$$

The solution of system (9), given above, is invariant with respect to the algebra $\langle D^t + D^R \rangle$.

If $\deg(f, R) = 3$ and, therefore, $\deg(g, R) \leqslant 4$, we have two families of solutions:

a) $f = -\dfrac{R^3}{t} + C_1^2(t^3 + C_2)R, \quad g = 3C_1^2 t^2 R^2 - C_1^4(t^3 + C_2)^2, \quad C_1, C_2 = \text{const}, \ C_1 \neq 0,$ i.e.

$$v = -\frac{R}{t} + C_1^2\frac{t^3 + C_2}{R}, \quad w = \frac{C_1}{R}\sqrt{3t^2R^2 - C_1^2(t^3 + C_2)^2}, \quad \Longrightarrow \quad p = -\frac{R^2}{t^2} + \chi(t),$$

$$G = \frac{-3R^2 + C_1^2 t(t^3 + C_2)^2}{C_1 t\sqrt{3t^2R^2 - C_1^2(t^3 + C_2)^2}}, \quad h = \sqrt{3t^2R^2 - C_1^2(t^3 + C_2)^2}.$$

b) $f = \dfrac{\wp_t}{2\wp}R^3 + (C_1\cos\alpha + C_2\sin\alpha)R, \quad g = \left(\dfrac{3C_0}{2\wp}R^2 - C_1\sin\alpha + C_2\cos\alpha\right)^2 - C_1^2 - C_2^2,$

$$v = \frac{f}{R^2}, \quad w = \frac{\sqrt{g}}{R} \quad \Longrightarrow \quad p = C_0^2\left(\frac{1}{\wp^2} - \wp\right)R^2 + \chi(t), \quad G = \frac{f_R}{\sqrt{g}}, \quad h = \sqrt{g}.$$

Here $C_0, C_1, C_2 = \text{const}$, $C_0 \neq 0$, $\wp = \wp(C_0 t, 0, 1)$ is the Weierstrass function, $\alpha = \int 3C_0\wp^{-1}dt$, χ is an arbitrary smooth function of t. The last solution is a generalization of the invariant solution with respect to the algebra $\langle D^R \rangle$.

The solutions given above exhaust all the solutions of system (9), for which f and g are polynomial with respect to R.

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