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A Class of Non-Lie Solutions for a Non-linear d'Alembert Equation

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New class of exact solutions of multidimensional complex nonlinear d'Alembert equation is constructed. These solutions in principle can not be obtained within the frame work of the traditional Lie approach.

In paper [1] the symmetry reduction of multidimensional complex non-linear d'Alembert equation was carried out

$$\Box u = F(|u|)u \tag{1}$$

to ordinary differential equations. In (1) $\Box = \partial^2/\partial x_0^2 - \Delta$ is the d'Alembert operator, $u = u(x_0, x_1, x_2, x_3)$ is a complex twice continuously differentiable function, $F(|u|)$ is a continuous arbitrary function.

Amongst the obtained reduced equations there appear not only ordinary differential equations, but also purely algebraical ones.

For example, if we take the Ansatz

$$u(x) = \exp\left(-i(\sigma x_1 + \gamma x_2 + \sigma x_2(x_0 + x_3))\right)\varphi(x_0 + x_3),$$

where σ, γ are real constants, so that $|\gamma| + |\sigma| \neq 0$.

Substituting this Ansatz into (1), we obtain

$$\left(\sigma^2 + \gamma^2 + 2\gamma\sigma\omega + \sigma\omega^2\right)\varphi = F(|\varphi|)\varphi.$$

Hence, we find that $|\varphi|$ is given implicitly:

$$F(|\varphi|) = \sigma^2 + \gamma^2 + 2\gamma\sigma\omega + \sigma\omega^2.$$

So, this procedure gives a class of solutions of equation (1) with the only real function $\Phi(x_0 + x_3)$:

$$\varphi = F^{-1}\left(\sigma^2 + \gamma^2 + 2\gamma\sigma\omega + \sigma\omega^2\right)\exp(i\Phi(x_0 + x_3)).$$

Our aim is to describe all possible Ansätze of the type:

$$u(x) = \exp(ia(x))\varphi(\omega(x)), \tag{2}$$

which reduce equation (1) to algebraic one.

The full solution of this task is given by the following theorem.

Theorem. *Ansatz (2) reduces equation (1) to algebraic one if and only if*

$$\begin{gathered} 1)\quad A_\mu(\omega)x^\mu + B(\omega) = 0, \qquad A_\mu(\omega)A^\mu(\omega) = 0, \qquad a_{x_\mu}\omega_{x^\mu} = 0, \\ \Box a = 0, \qquad a_{x_\mu}a_{x^\mu} = -w_1^2(\omega), \qquad a(x) = \frac{w_1(\omega)}{\left(-\dot{A}_\nu \dot{A}^\nu\right)}\dot{A}_\mu x^\mu + w_2(\omega), \end{gathered} \tag{3}$$

where $\nu = 0, 1, 2, 3$*;* $A_\mu(\omega)$*,* $B(\omega)$*,* $w_1(\omega)$*,* $w_2(\omega)$ *are arbitrary functions; the point above the symbol means derivative by* ω*; by the repeated indexes ment summing up (raising and lowering of the index is carried out with the help of the metrical tensor of the Minkowski space* $g_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$*).*

$$\begin{aligned} 2)\quad & \omega(x) = w_0\left(\theta_\mu x^\mu\right), \\ & a(x) = w_1\left(\theta_\mu x^\mu\right) a_\nu x^\nu + w_2\left(\theta_\mu x^\mu\right) b_\nu x^\nu + w_3\left(\theta_\mu x^\mu\right), \end{aligned}$$

where w_0*,* w_1*,* w_2*,* w_3 *are arbitrary functions of their arguments;* θ_μ*,* a_μ*,* b_μ *are arbitrary real parameters, which satisfy the following orthogonal relations*

$$a_\mu a^\mu = b_\mu b^\mu = -1, \qquad a_\mu b^\mu = a_\mu \theta^\mu = b_\mu \theta^\mu = \theta_\mu \theta^\mu = 0.$$

We omit the proof, which bases on the results of papers [2–3].

The result of substituting Ansatz (2), where $a(x)$, $\omega(x)$ are given by formulas (3), into equation (1) will give the following algebraic equation:

$$w_1^2(\omega)\varphi = F(|\varphi|)\varphi.$$

Hence, we obtain the following class of solutions of the complex non-linear d'Alembert equation (1)

$$u(x) = \rho(\omega)\exp\{ia(x)\},$$

where $a(x)$, $\omega(x)$ are given by the formulas (3), and $\rho(\omega) > 0$ is determined in implicit way:

$$F(\rho(\omega)) = w_1^2(\omega).$$

Let us mention that the obtained class of solutions is non-Lie and thus can not be derived with the help of the method of symmetry reduction.

Consider the example, where the exact solution of d'Alembert equation with the cubical non-linearity

$$\Box u = \lambda |u|^2 u \tag{4}$$

is constructed in explicit way. Substituting $A_0 = 1$, $A_1 = \omega$, $A_2 = \sqrt{1-\omega^2}$, $A_3 = B = 0$ into (3) we obtain that

$$\omega = \left(x_1^2 + x_2^2\right)^{-1}\left(x_0 x_1 \pm x_2\sqrt{x_1^2 + x_2^2 - x_0^2}\right).$$

Hence, we find

$$a(x) = iw_1(\omega)x_3 + iw_2(\omega).$$

So, we obtain the following class of solutions

$$u(x) = \frac{1}{\sqrt{\lambda}} w_1(\omega)\exp\{iw_1(\omega)x_3 + iw_2(\omega)\}$$

of the equation (4). We emphasize that this solution has singularity at the point $\lambda = 0$, so it can not be obtained by the methods of the perturbation theory by a small parameter λ.

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References

[1] Panchak O.A., Symmetry reduction of non-linear wave equation fora complex field, in Symmetrical and Analytical Methods in Mathematical Physics, Kyiv, Institute of Mathematics of the National Academy of Sciences of Ukraine, 1998, V.19, 166–173.

[2] Fushchych W.I., Zhdanov R.Z. and Revenko I.V., General solutions of the nonlinear wave and eikonal equations, *Ukr. Math. J.*, 1991, V.43, N 11, 1364–1379.

[3] Zhdanov R.Z., Revenko I.V. and Fushchych W.I., On the general solution of the d'Alembert equation with a nonlinear eikonal constraint and its applications, *J. Math. Phys.*, 1995, V.36, N 12, 7109–7127.