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# Hamiltonian Formulation and Order Reduction for Nonlinear Splines in the Euclidean 3-Space

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The authors use the procedure developed in [9] to develop a Hamiltonian structure into the variational problem given by the integral of the squared curvature on the spatial curves. The solutions of that problem are the elasticae or nonlinear splines. The symmetry of the problem under rigid motions is then used to reduce the Euler–Lagrange equations to a first-order dynamical system.

## 1 Introduction

Elasticae, or nonlinear spline curves, are the extremal curves of the second order variational problem given by the functional $\int \kappa^2 \mathrm{d}s$ (*cf.* [2, 4, 5]), where $\kappa$ is the geodesic curvature of the path, and $\mathrm{d}s$ is the arc-length element. This is one of the simplest examples of a general type of variational problems called parameter-invariant problems (see [6]), as the action integral is invariant under arbitrary changes of parameter. As all the problems in this class, our problem is *singular*, that is, its Hessian vanishes. As a consequence, the standard Hamiltonian formalism (momenta, Hamilton equations) cannot be directly applied. Hence it is not clear how to reduce the order of the equations by using Noether invariants attached to the symmetries of the problem.

In [9], the authors devised a general procedure called *parameter elimination*, in order to pass from parameter-invariant Lagrangian to a nonparametric version of it. Roughly speaking, the parameter elimination consists in taking the parameter “time” apart and using one of the “spatial” coordinates as the new parameter. So, the Lagrangian projects onto a “deparametrized” Lagrangian, whose extremals, parametrized arbitrarily, are the extremals of the original problem. Furthermore, the projected Lagrangian gives rise in some cases (which can be suitably characterized) to a regular problem, and hence Hamiltonian formalism can be applied.

In this paper this general procedure is applied to the particular case of the nonlinear splines in the 3-dimensional space, thus introducing a natural Hamiltonian formulation to the problem. Within this setting, the Noether invariants associated to the rigid motions of the space are found (the rigid motions – translations and rotations – are symmetries of the variational problem, as both the curvature and the arc-length are invariant under isometries). This invariants are then used to reduce the order of the equations of extremals, from a fourth order system to a nonlinear system of the form

$$\left\{ \begin{array}{l} \dfrac{\mathrm{d}y'}{\mathrm{d}x} = F(y', z'), \\[2mm] \dfrac{\mathrm{d}z'}{\mathrm{d}x} = G(y', z'), \end{array} \right.$$

where $y' = \mathrm{d}y/\mathrm{d}x$ and $z' = \mathrm{d}z/\mathrm{d}x$.

The term *nonlinear splines* is used to distinguish the extremal curves of the squared curvature functional of the "usual" linear splines (or just splines), or piecewise cubic polynomials. The latter are the extremals of the linear approximation to the variational problem under consideration. Of course, we are just interested in the *exact* extremals, not in the approximations.

# 2 Hamiltonian formulation for elasticae

In the standard coordinates of $\mathbb{R}^3$, the expression of the nonlinear splines Lagrangian is (see [12]):

$$\mathcal{L} = \frac{(\dot{y}\ddot{z} - \ddot{y}\dot{z})^2 + (\dot{x}\ddot{z} - \ddot{x}\dot{z})^2 + (\dot{x}\ddot{y} - \ddot{x}\dot{y})^2}{(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{5/2}}.$$

The variational problem defined by this Lagrangian is not regular. In fact, as a simple computation shows, we have $\det\left(\partial^2\mathcal{L}/\partial\ddot{x}_i\partial\ddot{x}_j\right) \equiv 0$. Hence, Hamiltonian formalism cannot be applied. Nevertheless, this variational problem is parameter-invariant (see [6]); *i.e.*,

$$\int_a^b \mathcal{L}\left(j_t^2(\sigma\circ u)\right)\mathrm{d}t = \int_\alpha^\beta \mathcal{L}\left(j_u^2(\sigma)\right)\mathrm{d}u,$$

for every orientation-preserving diffeomorphism $u\colon [a,b] \to [\alpha,\beta]$. This fact can easily be checked, either geometrically, or making use of the Zermelo conditions (see [6]).

Using the procedure described in [9], we pass from the Lagrangian $\mathcal{L}$ given above to the deparametrized version $\bar{\mathcal{L}}$. Roughly speaking, this is done by eliminating the variable $t$, using $x$ as the new independent variable, and passing from the "dots" (which represent derivatives with respect to $t$), to the "primes" (standing for derivatives with respect to $x$). The following relations are used:

$$y' = \frac{\dot{y}}{\dot{x}}, \qquad z' = \frac{\dot{z}}{\dot{x}}, \qquad y'' = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}, \qquad z'' = \frac{\dot{x}\ddot{z} - \dot{z}\ddot{x}}{\dot{x}^3}. \tag{1}$$

It is important to notice that the process is applied to the Lagrangian density $\mathcal{L}\,\mathrm{d}t$ to convert it into a density $\bar{\mathcal{L}}\,\mathrm{d}x$ modulo a contact form $\mathrm{d}x - \dot{x}\,\mathrm{d}t$.

The following are some of the main results given in [9] (here $\mathcal{L}$ stands for a generic second-order Lagrangian):

(i) $\bar{\mathcal{L}}$ is regular if and only if the Hessian matrix of $\mathcal{L}$ has rank 2.

(ii) The extremals of $\mathcal{L}\,\mathrm{d}t$ are the extremals of $\bar{\mathcal{L}}\,\mathrm{d}x$, endowed with an arbitrary parametrization.

In the case of the squared curvature functional, the rank of the Hessian matrix is 2, so the non-parametric Lagrangian, which can be easily computed by using the formulas (1),

$$\bar{\mathcal{L}} = \frac{(y'z'' - y''z')^2 + y''^2 + z''^2}{(1 + y'^2 + z'^2)^{5/2}}, \tag{2}$$

is regular, and we can apply the Hamiltonian formalism to it.

## 2.1 Hamilton equations for elasticae

Using the standard Hamiltonian formalism for second order problems (see, [1, 3, 4, 7, 11]) we can write the Jacobi–Ostrogradski momenta associated to the non-parametric Lagrangian $\bar{\mathcal{L}}$,

$$\begin{aligned} p = {} & v^{-5}\left[-2\left(z'^2+1\right)y''' + 2y'z'z'''\right] + v^{-7}\left[5y'\left(z'^2+1\right)y''^2\right. \\ & \left. + 2z'\left(-2y'^2+3z'^2+3\right)y''z'' - y'\left(y'^2+6z'^2+1\right)z''^2\right], \end{aligned}$$

$$
\begin{aligned}
q &= v^{-5}\left[2y'z'y''' - 2\left(y'^2+1\right)z'''\right] + v^{-7}\left[-z'\left(6y'^2+z'^2+1\right)y''^2\right.\\
&\qquad \left. + 2y'\left(3y'^2-2z'^2+3\right)y''z'' + 5z'\left(y'^2+1\right)z''^2\right],\\
p' &= v^{-5}\left[2\left(z'^2+1\right)y'' - 2y'z'z''\right],\\
q' &= v^{-5}\left[-2y'z'y'' + 2\left(y'^2+1\right)z''\right],\\
v &= \left(1+y'^2+z'^2\right)^{1/2},
\end{aligned}
$$

and also the Hamiltonian $H = py' + qz' + \left(v^3/4\right)\left[(p'y'+q'z')^2 + (p')^2 + (q')^2\right]$, the Poincaré–Cartan form

$$
\begin{aligned}
\bar{\Theta} &= \bar{\mathcal{L}}\,\mathrm{d}x + p(\mathrm{d}y - y'\,\mathrm{d}x) + q(\mathrm{d}z - z'\,\mathrm{d}x) + p'(\mathrm{d}y' - y''\,\mathrm{d}x) + q'(\mathrm{d}z' - z''\,\mathrm{d}x)\\
&= -H\,\mathrm{d}x + p\,\mathrm{d}y + q\,\mathrm{d}z + p'\,\mathrm{d}y' + q'\,\mathrm{d}z',
\end{aligned}
\tag{3}
$$

and, finally, the Hamilton equations:

$$
\begin{aligned}
&\frac{\mathrm{d}y}{\mathrm{d}x} = y', \qquad \frac{\mathrm{d}z}{\mathrm{d}x} = z', \qquad \frac{\mathrm{d}p}{\mathrm{d}x} = 0, \qquad \frac{\mathrm{d}q}{\mathrm{d}x} = 0,\\
&\frac{\mathrm{d}y'}{\mathrm{d}x} = \frac{1}{2}v^3\left[\left(y'^2+1\right)p' + y'z'q'\right],\\
&\frac{\mathrm{d}z'}{\mathrm{d}x} = \frac{1}{2}v^3\left[y'z'p' + \left(z'^2+1\right)q'\right],\\
&\frac{\mathrm{d}p'}{\mathrm{d}x} = -p - \frac{1}{4}v\left[y'\left(5y'^2+2z'^2+5\right)(p')^2 + 2z'\left(4y'^2+z'^2+1\right)p'q' + 3y'\left(z'^2+1\right)(q')^2\right],\\
&\frac{\mathrm{d}q'}{\mathrm{d}x} = -q - \frac{1}{4}v\left[3z'\left(y'^2+1\right)(p')^2 + 2y'\left(y'^2+4z'^2+1\right)p'q' + z'\left(2y'^2+5z'^2+5\right)(q')^2\right].
\end{aligned}
\tag{4}
$$

Our aim shall be to reduce this system to a first-order ordinary differential system in the variables $y'$, $z'$. Also note that $p' = q' = 0$ if and only if the spline curve is a geodesic; *i.e.*, $y = \alpha_1 x + \beta_1$, $z = \alpha_2 x + \beta_2$, $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, 2$.

## 3 Generalized symmetries and reduction to first order

It is straightforward to see that the variational problem defined by the squared curvature functional is invariant under isometries, as the curvature and arc-length element are themselves invariant under isometries. Thus, the infinitesimal generators of the rigid motions of $\mathbb{R}^3$ are infinitesimal symmetries of $\mathcal{L}\,\mathrm{d}t$, that is, if $X \in \mathfrak{X}(\mathbb{R}^3)$ is the infinitesimal generator of a rigid motion, $\mathrm{L}_{X_{(2)}}(\mathcal{L}\,\mathrm{d}t) = 0$, where $X_{(r)}$ is the prolongation of the vector field $(0, X) \in \mathfrak{X}(\mathbb{R}\times\mathbb{R}^3)$ to $J^r(\mathbb{R}, \mathbb{R}^3)$ by means of infinitesimal contact transformations (*cf.* [8, 9]). For infinitesimal symmetries, Noether's Theorem (see [7]) states that the function $f_X = i_{X_{(3)}}\Theta(\mathcal{L}\,\mathrm{d}t)\colon J^{2r-1}(\mathbb{R}, M) \to \mathbb{R}$ is constant along each extremal of the variational problem defined by $\mathcal{L}\,\mathrm{d}t$. The function $f_X$ is called the Noether invariant associated to $X$.

Nevertheless, as it was stated in [9], if $(0, X) \in \mathfrak{X}(\mathbb{R}\times\mathbb{R}^3)$ is an infinitesimal symmetry of $\mathcal{L}\,\mathrm{d}t$, it does not need to be $X \in \mathfrak{X}(\mathbb{R}\times\mathbb{R}^2)$ an infinitesimal symmetry of $\bar{\mathcal{L}}\,\mathrm{d}t$ (in fact, it can even be not projectable onto $\mathbb{R}^2$). But it is a generalized infinitesimal symmetry of $\bar{\mathcal{L}}\,\mathrm{d}t$, *i.e.*, $\mathrm{L}_{X_{[2]}}(\bar{\mathcal{L}}\,\mathrm{d}t)$

is a contact form, that is, it vanishes on every 2-jet of curve on $M$ (*cf.* [10, Definition 5.25]). Here, $X_{[r]}$ denotes the prolongation of $X \in \mathfrak{X}(\mathbb{R}^3)$ to $\mathfrak{X}(J^r(\mathbb{R},\mathbb{R}^2))$ by means of infinitesimal contact transformations.

It can be shown that if $X \in \mathfrak{X}(\mathbb{R} \times M)$ is a generalized infinitesimal symmetry of $\bar{\mathcal{L}}\,\mathrm{d}x$, then $f_X$ is constant on the extremals of $\bar{\mathcal{L}}\,\mathrm{d}x$; *i.e.*, generalized symmetries also produce Noether invariants.

Now we shall use the (generalized) symmetries of the deparametrized squared curvature Lagrangian to reduce (by means of the associated Noether invariants) the order of the Hamilton equations for the nonlinear splines on $\mathbb{R}^3$. More precisely, in the general case we shall reduce the equations (4) to a system of the form:

$$\frac{\mathrm{d}z'}{\mathrm{d}y'} = F\left(y', z'\right), \qquad \frac{\mathrm{d}y'}{\mathrm{d}x} = G\left(y', z'\right).$$

Hence the problem is reduced to two ordinary differential equations in the plane as we can first solve $z'$ in terms of $y'$ from the first equation above and then substitute the obtained expression for $z'$ into the second equation thus leading us to an equation of the form $\mathrm{d}y'/\mathrm{d}x = \bar{G}(\mu, x, y')$, where $\mu$ is a constant.

As we have already said, the infinitesimal generators of the isometries of $\mathbb{R}^3$ are infinitesimal symmetries of the Lagrangian density $\mathcal{L}\,\mathrm{d}t$, and hence infinitesimal generalized symmetries of $\bar{\mathcal{L}}\,\mathrm{d}x$. The isometries of $\mathbb{R}^3$ are generated by the translations along the three axes, and the rotations around these axes. The infinitesimal generators of the translations are $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial z$, which are their own prolongations by infinitesimal contact transformations. As the Poincaré–Cartan form has the expression given in (3), the Noether invariants associated to the translations are $H = -\bar{\Theta}(\partial/\partial x)$, $p = \bar{\Theta}(\partial/\partial y)$ and $q = \bar{\Theta}(\partial/\partial z)$. The infinitesimal generators of the three rotations are

$$X = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \qquad Y = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}, \qquad Z = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

We obtain their prolongations $X_{[3]}, Y_{[3]}, Z_{[3]} \in \mathfrak{X}(J^3(\mathbb{R},\mathbb{R}^2))$ by imposing that $X_{[3]}$, $Y_{[3]}$ and $Z_{[3]}$ project onto $X$, $Y$ and $Z$ respectively, and leave invariant the differential system spanned by the contact forms $\mathrm{d}y - y'\,\mathrm{d}x$, $\mathrm{d}z - z'\,\mathrm{d}x$, $\mathrm{d}y' - y''\,\mathrm{d}x$, $\mathrm{d}z' - z''\,\mathrm{d}x$, $\mathrm{d}y'' - y'''\,\mathrm{d}x$ and $\mathrm{d}z'' - z'''\,\mathrm{d}x$:

$$X_{[3]} = X + z'\frac{\partial}{\partial y'} - y'\frac{\partial}{\partial z'} + z''\frac{\partial}{\partial y''} - y''\frac{\partial}{\partial z''} + z'''\frac{\partial}{\partial y'''} - y'''\frac{\partial}{\partial z'''},$$

$$\begin{aligned} Y_{[3]} &= Y + y'z'\frac{\partial}{\partial y'} + \left(z'^2 + 1\right)\frac{\partial}{\partial z'} + \left(y'z'' + 2y''z'\right)\frac{\partial}{\partial y''} + 3z'z''\frac{\partial}{\partial z''} \\ &\quad + \left(y'z''' + 3y''z'' + 3y'''z'\right)\frac{\partial}{\partial y'''} + \left(4z'z''' + 3z''^2\right)\frac{\partial}{\partial z'''}, \end{aligned}$$

$$\begin{aligned} Z_{[3]} &= Z + (y'^2 + 1)\frac{\partial}{\partial y'} + y'z'\frac{\partial}{\partial z'} + 3y'y''\frac{\partial}{\partial y''} + \left(2y'z'' + y''z'\right)\frac{\partial}{\partial z''} \\ &\quad + \left(4y'y''' + 3y''^2\right)\frac{\partial}{\partial y'''} + \left(y'''z' + 3y''z''' + 3y'z'''\right)\frac{\partial}{\partial z'''}. \end{aligned}$$

The associated Noether invariants are:

$$\begin{aligned} C_x &= pz - qy + p'z' - q'y', \\ C_y &= Hz + qx + y'z'p' + \left(z'^2 + 1\right)q', \\ C_z &= Hy + px + \left(y'^2 + 1\right)p' + y'z'q'. \end{aligned}$$

### 3.1 The level $H = 0$

The Hamiltonian $H$ has a unique critical point at the point $y' = z' = p = q = p' = q' = 0$, which is non-degenerate of signature $(4, 2)$ as follows from Morse's lemma taking into account that $H = py' + qz' + P^2 + Q^2$, where

$$P = \frac{1}{2} v^{3/2} \left( \left( {y'}^2 + 1 \right)^{1/2} p' + y'z' \left( {y'}^2 + 1 \right)^{-1/2} q' \right),$$

$$Q = \frac{1}{2} v^{5/2} \left( {y'}^2 + 1 \right)^{-1/2} q'.$$

If the Hamiltonian vanishes, then we can substitute the expressions of $C_y$ and $C_z$ into the third and fourth Hamilton equations (4) to obtain the following system of first-order ordinary differential equations:

$$\frac{\mathrm{d}y'}{\mathrm{d}x} = \frac{1}{2} v^3 (C_z - px), \qquad \frac{\mathrm{d}z'}{\mathrm{d}x} = \frac{1}{2} v^3 (C_y - qx).$$

### 3.2 $H \neq 0$

If $H$ is not zero, we use the formulas of $C_y$ and $C_z$ to eliminate $y$ and $z$, and the third and fourth Hamilton equations (4) to eliminate $p'$ and $q'$. Substitution in the expressions of $H$ and $C_x$ then yields, after defining $C$ as

$$2C = C_x - \frac{pC_y - qC_z}{H},$$

the following system of ordinary differential equations of the first order:

$$Cv^3 = \left( \frac{q}{H} + z' \right) \frac{\mathrm{d}y'}{\mathrm{d}x} - \left( \frac{p}{H} + y' \right) \frac{\mathrm{d}z'}{\mathrm{d}x}, \tag{5}$$

$$(H - py' - qz')v^5 = \left( {z'}^2 + 1 \right) \left( \frac{\mathrm{d}y'}{\mathrm{d}x} \right)^2 - 2y'z' \frac{\mathrm{d}y'}{\mathrm{d}x} \frac{\mathrm{d}z'}{\mathrm{d}x} + \left( {y'}^2 + 1 \right) \left( \frac{\mathrm{d}z'}{\mathrm{d}x} \right)^2. \tag{6}$$

#### 3.2.1 The case $C = 0$

If $C$ vanishes, then the equation (5) is just

$$(Hz' + q) \frac{\mathrm{d}y'}{\mathrm{d}x} - (Hy' + p) \frac{\mathrm{d}z'}{\mathrm{d}x} = 0,$$

or, equivalently,

$$\begin{pmatrix} \mathrm{d}y'/\mathrm{d}x \\ \mathrm{d}z'/\mathrm{d}x \end{pmatrix}^{\perp} = \begin{pmatrix} \mathrm{d}z'/\mathrm{d}x \\ -\mathrm{d}y'/\mathrm{d}x \end{pmatrix} \perp \left( H \begin{pmatrix} y' \\ z' \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix} \right).$$

Hence, for every $x$ there exists a $\lambda(x) \in \mathbb{R}$ such that

$$\frac{\mathrm{d}y'}{\mathrm{d}x} = \lambda \left( Hy' + p \right), \qquad \frac{\mathrm{d}z'}{\mathrm{d}x} = \lambda \left( Hz' + q \right).$$

Solving this pair of differential equations, we obtain

$$y'(x) = K_y e^{H\Lambda} - \frac{p}{H}, \qquad z'(x) = K_z e^{H\Lambda} - \frac{q}{H},$$

where $\Lambda(x) = \int \lambda(x)\mathrm{d}x$ and $K_y, K_z \in \mathbb{R}$. Substitution in equation (6) then yields

$$\left(H - (pK_y + qK_z)e^{H\Lambda} - \frac{p^2+q^2}{H}\right)\left(\left(K_y^2 + K_z^2\right)e^{2H\Lambda} - \frac{pK_y + qK_z}{H}e^{H\Lambda} + \frac{H^2+p^2+q^2}{H^2}\right)^{5/2}$$
$$= (\Lambda')^2\left(H^2(K_y^2+K_z^2) + (qK_y - pK_z)^2\right)e^{2H\Lambda},$$

which is a first-order ordinary differential equation on $\Lambda$.

### 3.2.2 The general case

Let us now assume that $H \neq 0$ and $C \neq 0$. From there, and (5), we know that $p + Hy'$ and $q + Hz'$ do not vanish simultaneously. Let us suppose that $p + Hy' \neq 0$. Then, from equation (5) we can write the derivative of $z'$ in terms of the derivative of $y'$ as

$$\frac{\mathrm{d}z'}{\mathrm{d}x} = \frac{q+Hz'}{p+Hy'}\frac{\mathrm{d}y'}{\mathrm{d}x} - \frac{CHv^3}{p+Hy'}, \tag{7}$$

and then we can write this equation as a differential equation for $z'(y')$:

$$\frac{\mathrm{d}z'}{\mathrm{d}y'} = \frac{q+Hz'}{p+Hy'} - \frac{CHv^3}{(\mathrm{d}y'/\mathrm{d}x)(p+Hy')}. \tag{8}$$

Substitution of (7) in (6) then yields

$$\left(\frac{\mathrm{d}y'}{\mathrm{d}x}\right)^2\left[(pz'-qy')^2 + (Hz'+q)^2 + (hy'+p)^2\right] + 2\frac{\mathrm{d}y'}{\mathrm{d}x}CHv^3\left[y'(pz'-qy') - (Hz'+q)\right]$$
$$+C^2H^2v^6(y'^2+1) - (H - py' - qz')(Hy'+p)v^5 = 0,$$

whose solutions are

$$\frac{\mathrm{d}y'}{\mathrm{d}x} = \frac{v^3\left(y'(pz'-qy') - (Hz'+q)\right) \pm \Delta^{1/2}}{A}, \tag{9}$$

where

$$\Delta = -C^2H^2v^8(Hy'+p)^2 + v^5(Hy'+p)(H-py'-qz')A,$$
$$A = (qy'-pz')^2 + (Hy'+p)^2 + (Hz'+q)^2.$$

We have thus reduced the Hamilton equations to a pair of first-order ordinary differential equations, (8) and (9). Also note that for $C \neq 0$ and $H \neq 0$, the above system is non-singular where it is defined so that the singularities of the system can only arise in the particular cases previously studied.

We remark that in order to find out where the solutions of (9) are defined, we have to study the behaviour of the discriminant $\Delta$. It is clear that where $H - py' - qz' < 0$ the discriminant is negative and hence no solution is defined in that region. On the half-plane $H - py' - qz' > 0$, $\Delta$ is positive if and only if

$$(H-py'-qz')^2\left((qy'-pz')^2 + (Hy'+p)^2 + (Hz'+q)^2\right)^2 > C^4H^4\left(1+y'^2+z'^2\right)^3.$$

The analysis of the sign of this expression has turned too long to be stated here, and we leave it for a future work.

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