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The Group Classification of Nonlinear Wave Equations Invariant under Two-Dimensional Lie Algebras

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The group classification of the one class of the nonlinear wave equations which are invariant under one- and two-dimensional Lie algebras is obtained.

The problem of group classification of differential equations is one of the central problems of the modern symmetry analysis of the differential equations. In this raper we consider the problem of the group classification of the equations of form

$$u_{tt} = u_{xx} + F(t, x, u, u_x), \tag{1}$$

where $u = u(t, x)$, F is an arbitrary nonlinear differentiable function of its variables. In (1) $F_{u_x} \neq 0$ is an arbitrary nonlinear smooth function, which depends on variables u or u_x. Also we denoted $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, $F_{u_x} = \frac{\partial F}{\partial u_x}$, $u_t = \frac{\partial u}{\partial t}$, $u_{tt} = \frac{\partial^2 u}{\partial t^2}$.

We note that the problem of group classification of nonlinear hyperbolic equations was studied in the works [1–5]. We describe the equations of the form (1), which are invariant under one- and two-dimensional Lie algebras.

At first we give a form of an infinitesimal operator of the symmetry group of the equation (1), and the group of equivalence transformation of this equation.

Theorem 1. *The infinitesimal operator of the symmetry group of the equation (1) has following form:*

$$X = (\lambda t + \lambda_1)\partial_t + (\lambda x + \lambda_2)\partial_x + (h(x)u + r(t, x))\partial_u, \tag{2}$$

where λ, λ_1, λ_2 are the arbitrary real constants, $h(x)$, $r(t, x)$ are the arbitrary functions, which satisfy the following condition:

$$\begin{gathered} r_{tt} - \frac{d^2 h}{dx^2} - r_{xx} + (h - 2\lambda)F - (\lambda t + \lambda_1)F_t - (\lambda x + \lambda_2)F_x \\ -(hu + r)F_u - 2u_x\frac{dh}{dx} - u_x(h - \lambda)F_{u_x} - \frac{dh}{dx}uF_{u_x} - r_xF_{u_x} = 0. \end{gathered} \tag{3}$$

The proof of the theorem is done by the Lie method [6]. Then by the way of direct calculations it is not difficult to show, that the group of the equvalence of the equation (1) is determined by the transformation:

$$\bar{t} = \gamma t + \gamma_1, \qquad \bar{x} = \epsilon\gamma x + \gamma_2, \qquad v = \rho(x)u + \theta(t, x), \tag{4}$$

$\gamma \neq 0$, $\rho \neq 0$, $\epsilon = \pm 1$.

The first step of the group classification is the study of nonequivalent realizations of one-dimensional Lie algebra in class of operators (2).

Theorem 2. *There are transformations (4) which reduce operator (2) to one of the following operators:*

1) $X = \lambda(t\partial_t + x\partial_x), \quad \lambda \neq 0, \quad \lambda = \text{const};$
2) $X = \partial_t + \beta\partial_x, \quad \beta > 0;$
3) $X = \partial_x;$
4) $X = \partial_t;$
5) $X = \partial_t + h(x)u\partial_u, \quad h(x) \neq 0;$
6) $X = h(x)u\partial_u, \quad h(x) \neq 0;$
7) $X = r(t,x)\partial_u.$

Proof. We distinguish two cases.

Case 1. $\lambda \neq 0$.

$$X = \gamma[\lambda t + \lambda_1]\partial_{\bar{t}} + (\lambda x + \lambda_2)\epsilon\gamma\partial_{\bar{x}} + [\theta_t(\lambda t + \lambda_1) + (\lambda x + \lambda_2)(\rho' u + \theta_x) + (hu + r)\rho]\partial_v. \quad (5)$$

Check that operator (5) which by means of transformations (4) one be reduced to

$$X = \lambda\bar{t}\partial_{\bar{t}} + \lambda\bar{x}\partial_{\bar{x}}. \quad (6)$$

According to (5) and (6) we have

$$\begin{aligned}
&\gamma(\lambda t + \lambda_1) = \lambda(\gamma t + \gamma_1),\\
&\epsilon\gamma(\lambda x + \lambda_2) = \lambda(\epsilon\gamma x + \gamma_1),\\
&\theta_t(\lambda t + \lambda_2) + (\lambda x + \lambda_2)\theta_x + r\rho = 0,\\
&(\lambda x + \lambda_2)\rho' + h\rho = 0.
\end{aligned} \quad (7)$$

We have $\gamma_1 = \lambda_1\gamma\lambda^{-1}$, $\gamma_2 = \lambda_2\epsilon\gamma\lambda^{-1}$, ρ, θ are solutions of the system (7).

With the help of trasformations (4) the operator (2) reduces to form

$$X = \lambda(t\partial_t + x\partial_x).$$

Case 2. $\lambda = 0$ is treated the same way. ■

In accordance with Theorem 2 there are seven nonequivalent one-dimensional algebras:

$$\begin{aligned}
&A_1^1 = \langle t\partial_t + x\partial_x\rangle;\\
&A_1^2 = \langle \partial_t + \beta\partial_x\rangle, \quad \beta > 0;\\
&A_1^3 = \langle \partial_x\rangle;\\
&A_1^4 = \langle \partial_t\rangle;\\
&A_1^5 = \langle \partial_t + h(x)u\partial_u\rangle, \quad h(x) \neq 0;\\
&A_1^6 = \langle h(x)u\partial_u\rangle, \quad h(x) \neq 0;\\
&A_1^7 = \langle r(t,x)\partial_u\rangle.
\end{aligned}$$

Below we give the list of corresponding values of function F in the equation (1), when those one-dimensional algebras will be the algebras of invariance.

1) $A_1^1:\quad F = u_x^2 G(u, \omega_1, \omega_2), \quad \omega_1 = tx^{-1}, \quad \omega_2 = xu_x;$

2) $A_1^2:\quad F = G(\omega, u, u_x), \quad \omega = x - \beta t;$

3) $A_1^3:\quad F = G(t, u, u_x);$

4) $A_1^4:\quad F = G(x, u, u_x);$

5) $A_1^5:\quad F = h^{-1}h'' - 2h'h^{-1}u_x \ln|u| + (h'h^{-1})^2 u \ln^2|u| + uG(x, \omega_1, \omega_2),$
$\omega_1 = ue^{-ht}, \quad \omega_2 = u^{-1}u_x - h^{-1}h' \ln|u|;$

6) $A_1^6:\quad F = h^{-1}h'' - 2h'h^{-1}u_x \ln|u| + (h'h^{-1})^2 u \ln^2|u| + uG(t, x, \omega),$
$\omega = u^{-1}u_x - h'h^{-1} \ln|u|;$

7) $A_1^7:\quad F = r^{-1}(r_{tt} - r_{xx})u + G(t, x, \omega), \quad \omega = r_x u - ru_x.$

There are two real one- and two-dimensional Lie algebras:

1) $A_{2.1} = A_1 \oplus A_1 = \langle e_1, e_2 \rangle, \quad [e_1, e_2] = 0;$

2) $A_{2.2} = \langle e_1, e_2 \rangle, \qquad [e_1, e_2] = e_2.$

Studying their realizations in the class of the operators (2), on the base of the results of the Theorem 2 we can put one of the basis operators of one- and two-dimensional Lie algebras equal to one of operators, which are given in Theorem 2.

After the next steps, we have 19 realizations of the algebra $A_{2.1}$ and 15 realizations $A_{2.2}$, which are the algebras of invariance of equation (1). We give the realizations of the algebra $A_{2.1}$ and the corresponding values of the functions F in the equation (1).

$$A_{2.1}^1 = \langle ku\partial_u,\ t\partial_t + x\partial_x \rangle, \quad k \neq 0,$$
$$F = u^{-1}u_x^2 G(\omega, v), \quad \omega = tx^{-1}, \quad v = xu^{-1}u_x;$$

$$A_{2.1}^2 = \langle r(\zeta)\partial_u,\ t\partial_t + x\partial_x \rangle, \quad \zeta = tx^{-1},$$
$$\text{if}\quad r = \text{const}, \quad F = u_x^2 G(\omega, v), \quad \omega = \zeta, \quad v = xu_x,$$
$$\text{if}\quad r_\zeta \neq 0, \quad F = x^{-1}u_x(\zeta^2 - 1)r_{\zeta\zeta}(\zeta r_\zeta)^{-1} + x^{-2}G(\zeta, \omega), \quad \omega = \zeta r_\zeta u + rxu_x;$$

$$A_{2.1}^3 = \langle \partial_t + \beta\partial_x,\ ku\partial_u \rangle, \quad \beta > 0, \quad k \neq 0,$$
$$F = uG(v, \omega), \quad v = x - \beta t, \quad \omega = u^{-1}u_x;$$

$$A_{2.1}^4 = \langle \partial_t + \beta\partial_x,\ r(\zeta)\partial_u \rangle, \quad \zeta = x - \beta t, \quad \beta > 0,$$
$$F = r^{-1}r_{\zeta\zeta}(1 - \beta^2)u + G(\zeta, \omega), \quad \omega = r_\zeta u + ru_x;$$

$$A_{2.1}^5 = \langle \partial_t + \beta\partial_x,\ \partial_t + ku\partial_u \rangle, \quad k \neq 0,$$
$$F = uG(v, \omega), \quad v = \beta u + k\zeta, \quad \omega = u^{-1}u_x, \quad \zeta = x - \beta t;$$

$$A_{2.1}^6 = \langle \partial_t,\ \partial_x \rangle,$$
$$F = G(u, u_x);$$

$$A^7_{2.1} = \langle \partial_x,\ ku\partial_u \rangle\,, \quad k \neq 0,$$
$$F = uG(t, u^{-1}u_x);$$

$$A^8_{2.1} = \langle \partial_x,\ r(t)\partial_u \rangle\,, \quad r \neq 0,$$
$$F = r^{-1}\ddot{r}u + G(t, u_x), \quad \ddot{r} = \frac{d^2r}{dt^2};$$

$$A^9_{2.1} = \langle \partial_x,\ \partial_t + \lambda u\partial_u \rangle\,, \quad \lambda \neq 0,$$
$$F = uG(e^{-\lambda t}u, e^{-\lambda t}u_x);$$

$$A^{10}_{2.1} = \langle h(x)u\partial_u,\ \partial_t \rangle\,,$$
$$F = -2\frac{h'u_x}{h}\ln|u| + \frac{(h')^2}{h^2}u\ln^2|u| + \frac{h''}{h} + uG(x,\omega), \quad \omega = \frac{u_x}{u} - \frac{h'}{h}\ln|u|,$$

$$A^{11}_{2.1} = \langle \partial_u,\ \partial_t \rangle\,,$$
$$F = g(x, u_x);$$

$$A^{12}_{2.1} = \langle f(x)u\partial_u,\ \partial_t + ku\partial_u \rangle\,, \quad k \neq 0,$$
$$F = f^{-1}f''u\ln|u| - u^{-1}u_x^2 + uG(x,v), \quad v = kf^{-1}f't + u^{-1}u_x - f^{-1}f'\ln|u|;$$

$$A^{13}_{2.1} = \left\langle e^{\lambda t}\partial_u,\ \partial_t + \lambda\partial_u \right\rangle, \quad \lambda \neq 0,$$
$$F = \lambda^2 u + u_x G(x,\omega), \quad \omega = u_x e^{-\lambda t};$$

$$A^{14}_{2.1} = \langle f(x)u\partial_u,\ \partial_t + h(x)u\partial_u \rangle\,, \quad h' \neq 0,$$
$$F = -\omega^2 + G(x,V), \quad V = AT + x, \quad A = hf'f^{-1} - h', \quad A \neq 0,$$
$$\omega = u^{-1}u_x - f'f^{-1}\ln|u|, \quad h'' = hf^{-1}f'', \quad \frac{f'}{f} \neq \frac{h'}{h}, \quad \frac{h''}{h} = \frac{f''}{f};$$

$$A^{15}_{2.1} = \left\langle e^{h(x)t}\partial_u,\ \partial_t + f(x)u\partial_u \right\rangle, \quad f' \neq 0,$$
$$F = f^2 + 2f'f^{-1} - 2(f'f^{-1})^2 + 2f'f^{-1}\omega\ln|u| + (f'f^{-1})^2\ln^2|v|$$
$$+ [f^{-1}f'' - 2f'f^{-1} + 2(f'f^{-1})^2]\ln|v| + G(x,w), \quad w = \omega v + f^{-1}f'v\ln|v|,$$
$$v = ue^{-f(x)t}, \quad \omega = u^{-1}u_x + f^{-1}f'\ln|u|;$$

$$A^{16}_{2.1} = \langle \lambda xu\partial_u,\ ku\partial_u \rangle\,, \quad \lambda, k \neq 0,$$
$$F = -u^{-1}u_x^2 + uG(t,x);$$

$$A^{17}_{2.1} = \langle f(x)u\partial_u,\ h(x)u\partial_u \rangle\,, \quad h' \neq 0, \quad f' \neq 0,$$
$$F = -u^{-1}u_x^2 + uG(t,x), \quad f''f^{-1} = h''h^{-1};$$

$$A^{18}_{2.1} = \langle \frac{\varphi(t)}{\dot\varphi(t)}\partial_u,\ \frac{1}{\sqrt{\dot\varphi(t)}}\partial_u \rangle, \quad \dot\varphi \neq 0,$$
$$F = \frac{1}{4}\dot\varphi^{-2}[3(\ddot\varphi)^2 - 2\dot\varphi\dddot\varphi]u + G(t, x, \dot\varphi^{-1/2}u_x);$$

$$A^{19}_{2.1} = \langle \lambda(t\partial_t + x\partial_x,\ r(\zeta)\partial_u \rangle, \quad \zeta = tx^{-1},$$

$$1)\ r = k = \text{const}, \quad r \neq 0, \quad k \neq 0, \quad F = G(\zeta, xu_x),$$

$$2)\ r_\zeta \neq 0, \quad F = \zeta^{-1} r_\zeta^{-1} v^{-1}[(1-\zeta^2)r_{\zeta\zeta} - 2\zeta r_\zeta] + \tilde{G}(\zeta, \omega), \quad v = xu_x,$$

$$\omega = \zeta r_\zeta u + rv.$$

On base of the results of the Theorem 2 we proved the following theorem.

Theorem 3. *In the class of operators (2) there are no realizations of the algebras* $so(3)$ *and* $sl(2, R)$*.*

From this theorem we have the following results:

- in class of operators (2) there are no realizations of real semi-simple Lie algebras;
- there are not such equations (1) which have algebras of invariance, which isomorphic by real semi-simple algebras, or contain those algebras as subalgebras.

Thus, we must study of existence of realizations of only real solvable Lie algebras in the class of operators (2) for the complete group classification equation (1).

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