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The Integrability of Some Underdetermined Systems

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The problem of integrability of special nonholonomic systems with single-functional arbitrariness of solutions is studied. The algorithm and exact formulas are obtained. As an example the problem of "Integrating Wheel" motion is considered, and symmetry algebra for flat control system of n-th order are calculated.

1 Introduction

Mathematical models of various problems of science may be described by the systems of ordinary differential equations

$$F_j(t, x, \dot{x}, \ldots, x^{(n)}) = 0, \quad j = \overline{1, r}, \qquad x \subset X, \qquad \dim X = m, \quad m > r, \tag{1}$$

which contain more unknown functions (m) than equations (r). Similar systems are considered, for example, in geometry problems [1], problems of mathematical physics [2], nonholonomic mechanics [3], control theory [4]. Following [5], we will define such systems as "underdetermined systems".

In the present paper we will consider only underdetermined systems of the form

$$\begin{aligned} &\frac{d\xi^i}{du} - f^i(u)\frac{d\tau}{du} = 0, \qquad (\text{or} \quad \omega^i = d\xi^i - f^i(u)d\tau = 0), \\ &\tau = \tau(u), \qquad \xi^i = \xi^i(u), \qquad i = \overline{1, n} \end{aligned} \tag{2}$$

containing n equations and $(n + 1)$ unknown functions.

The aim of this research is to get exact formulas for the general solution of system (2). It is well known that sets of solutions of ordinary differential equations of n-th order are defined by n arbitrary constants. On the contrary the general solution of underdetermined systems may depends on arbitrary functions (not only constants). Let us consider a well-known example [3, 6, 7]. The motion of mechanical system with coordinates (x, y, z) is described by equation

$$\frac{dy}{dt} - z\frac{dx}{dt} = 0, \tag{3}$$

or in terms of differential forms

$$\omega = dy - zdx = 0, \qquad (\partial_z, \ \partial_x + z\partial_y). \tag{4}$$

The integrability conditions for this system are not fulfilled:

$$d\omega \wedge \omega = -dz \wedge dx \wedge dy \neq 0. \tag{5}$$

Therefore there does not exist any two-dimensional solutions of the form $\Phi(x, y, z) = C$. After H. Hertz such systems are known as "nonholonomic systems" [8]. But there exist one-dimensional solutions admitted by equation (3). For example, in [7] we can find a solution $\{x = t^2,\ y = t^4,\ z = 2t^2\}$. It is easy to construct the solution $\{x = \cos t,\ y = t\cos t - \sin t,\ z = t\}$. The question is: Can we construct a formula, which includes all one-dimensional solutions? We may find the positive answer in Pars' book [6]. His solution for (4) is

$$y = f(x), \qquad z = f'(x). \tag{6}$$

But this solution is only guess and we do not know what can we do in a more difficult situation. The algorithm for the general case is given by M. Gromov in his book [1]. Let us illustrate his algorithm for solving of the system

$$\begin{cases} \dfrac{d\xi^1}{du} = -u^2\dfrac{d\tau}{du}, \\ \dfrac{d\xi^2}{du} = u\dfrac{d\tau}{du}. \end{cases} \tag{7}$$

Following Gromov, rewrite system (7) in the form

$$\mathbf{B}\mathbf{x}' = 0, \tag{8}$$

where

$$\mathbf{B} = \begin{pmatrix} -u^2 & -1 & 0 \\ u & 0 & -1 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} \tau \\ \xi^1 \\ \xi^2 \end{pmatrix} \tag{9}$$

and take the solution in the form:

$$\mathbf{B}\mathbf{x} = \sigma, \qquad \sigma = \begin{pmatrix} \sigma^1 \\ \sigma^2 \end{pmatrix}, \tag{10}$$

where $\sigma^i(u)$ are arbitrary functions. Let us differentiate (10) taking into account (8). We obtain

$$\mathbf{B}'\mathbf{x} = \sigma'. \tag{11}$$

System (10), (11) is algebraic with respect to (τ, ξ^1, ξ^2)

$$\begin{cases} -u^2\tau - \xi^1 = \sigma^1, \\ u\tau - \xi^2 = \sigma^2, \\ -2u\tau = \dfrac{d\sigma^1}{du}, \\ \tau = \dfrac{d\sigma^2}{du}, \end{cases} \tag{12}$$

and has nontrivial solutions iff the condition

$$\frac{d\sigma^1}{du} + 2u\frac{d\sigma^2}{du} = 0. \tag{13}$$

takes place. Equation (13) is also underdetermined but it contains only 2 unknown functions. Proceeding in the similar way and making in (13) the substitution

$$\sigma^1 + 2u\sigma^2 = h, \tag{14}$$

where $h = h(u)$ is an arbitrary function we have

$$\sigma^2 = \frac{h'(u)}{2}, \tag{15}$$

and accordingly, from (14)

$$\sigma^1 = h - uh'(u). \tag{16}$$

At the last step we substitute (σ^1, σ^2) in (12) and finally obtain the solution in the form

$$\begin{cases} \tau = \dfrac{h''}{2}, \\ \xi^1 = -u^2 \dfrac{h''}{2} + uh' - h, \\ \xi^2 = u\dfrac{h''}{2} - \dfrac{h'}{2}. \end{cases} \tag{17}$$

Thus the Gromov anzats reduces underdetermined system also to underdetermined system with dimension is less than of the initial system. Therefore for solving system (2) we have to input consecutively $(n-1)$ times $n, (n-1), \ldots, 1$ new functions. But the role of these new functions is intermediate while the solution of initial problem may be defined by only one arbitrary function. Our goal is to exclude these intermediate calculations. With respect to this at first we have to calculate the number of arbitrary functions defining the general solution and then to get exact formulas.

2 General solution

In the general case we use the definition of "width of solution" which was introduced by E. Cartan in [9]. We will consider only nonholonomic systems (2), so $(n+1)$-dimensional integral manifolds are absent.

$$\Phi(\tau, \xi^i) = C. \tag{18}$$

At the following step we have to obtain for system (2) Cartan's characteristics (s_i). Direct calculations give

$$s = n, \qquad s_1 = 1. \tag{19}$$

Therefore, the general solution of system (2) depends on one arbitrary function $\sigma(u)$.

The next step is based on the following. The general solutions of simple cases show that the final formulas are linearly dependent on $\sigma(u)$ and its derivatives up to n-th order $(\sigma', \sigma'', \ldots, \sigma^{(n)})$. Hence we can try to find the general solution of (2) in the form

$$\begin{aligned} \tau &= \sum_{k=0}^{n} A_k \sigma^{(k)}, \qquad \left(\sigma^{(k)} = U^k \sigma\right), \\ \xi^i &= \sum_{k=0}^{n} B_k^i \sigma^{(k)}, \qquad U = \frac{d}{du}, \end{aligned} \tag{20}$$

with undefined coefficients (A^i, B_k^i). The substitution (20) into (2) leads us (after decomposition by $\sigma^{(k)}$) to the system

$$B_n^i = f^i A_n, \tag{21}$$

$$UB_k^i + B_{k-1}^i = f^i U A_k + f^i A_{k-1}, \qquad k = \overline{1,n}, \tag{22}$$

$$UB_0^i = f^i U A_0. \tag{23}$$

The substitution (21) into (22) gives us

$$\begin{aligned} B_{n-1}^i &= f^i A_{n-1} - A_n U f^i, \\ B_{n-2}^i &= f^i A_{n-2} - A_{n-1} U f^i + A_n U^2 f^i + U A_n U f^i, \\ B_{n-3}^i &= f^i A_{n-3} - A_{n-2} U f^i + U \left(A_{n-1} U f^i\right) - U^2 \left(A_n U f^i\right), \end{aligned} \tag{24}$$

and we may assume that

$$B_{n-k}^i = f^i A_{n-k} + \sum_{m=0}^{k-1} (-1)^{m+1} U^m \left(A_{n-k+m+1} U f^i\right), \tag{25}$$

or, redefining the subscript $(n-k \longrightarrow k)$,

$$B_k^i = f^i A_k + \sum_{m=0}^{n-k-1} (-1)^{m+1} U^m \left(A_{m+k+1} U f^i\right), \qquad k = \overline{1, n-1}. \tag{26}$$

In fact, we get identity via substitution (B_k^i) into (22). From (26) with respect to $k = 0$ we have

$$B_0^i = f^i A_0 + \sum_{m=0}^{n-1} (-1)^{m+1} U^m \left(A_{m+1} U f^i\right). \tag{27}$$

The substitution (27) into (23) gives us A_i:

$$\sum_{m=0}^{n} (-1)^m U^m \left(A_m U f^i\right) = 0. \tag{28}$$

Let us make the following transformations at (28):

1) rewrite according to Leibnitz formula (see, for example, [10]) the expression

$$U^m(A_m U f^i) = \sum_{s=0}^{m} \binom{m}{s} (U^{m-s} A_m) U^{m+1} f^i; \tag{29}$$

2) define

$$D_s = \sum_{m=s}^{n} (-1)^m \binom{m}{s} U^{m-s} A_m. \tag{30}$$

Then (28) takes the form

$$\sum_{s=0}^{n} D_s U^{s+1} f^i = 0, \qquad i = \overline{1,n}. \tag{31}$$

This system is linear algebraic one with respect to $(n+1)$ unknown variables D_s. The existence of solutions of the latter system is connected with the rank of functional $n \times (n+1)$ matrix

$$W(u) = a_j^i, \qquad a_j^i = \frac{d^{j+1} f^i}{du^{j+1}}. \tag{32}$$

The determinant of square matrix $\hat{W}(u)$ (which is $W(u)$ without last column) is a Wronskian for functions $\frac{df^i}{du}$ (see, for example, [11]). If rank of the matrix $W(u)$ is equal to n, then system (31) has a solution. We may get $D_n(u)$ as arbitrary function.

$$D_n(u) = h(u). \tag{33}$$

The remaining coefficients are defined from the system

$$\sum_{s=0}^{n-1} D_s U^{s+1} f^i = (-1) U^{n+1} f^i h(u). \tag{34}$$

In the particular case $U^{n+1} f^i = 0$ ($\forall\ i$) we have $D_s = 0$, $s = \overline{0, n-1}$. It is easy to show that function $h(u)$ is not essential, because of for any $h(u)$ the substitution $\hat{\sigma} = h\sigma$ leave only one arbitrary function in general solution. Therefore we may assume without loss of generality that

$$D_n(u) = h(u) = (-1)^n. \tag{35}$$

In this case we have from (28) $A_n = 1$, and from (21) $B_n^i = f^i$. By inverting formula (34) we can calculate the coefficients A_i

$$A_i = \sum_{m=i}^{n-1} (-1)^m \binom{m}{i} U^{m-i} D_m, \tag{36}$$

and according to (26) we can obtain B_i^k. As a result we may formulate the following theorem:

Theorem 1. *If Wronskian of functions $\varphi^i = \frac{df^i}{du}$ in system (2) is not equal to zero ($W(\varphi^i) \neq 0$), then the general solution of system (2) is given by the formulas*

$$\begin{cases} \tau = \sigma^{(n)} + \displaystyle\sum_{k=0}^{n-1} A_k \sigma^{(k)}, \\ \xi^i = f^i \sigma^{(n)} + \displaystyle\sum_{k=0}^{n-1} B_k^i \sigma^{(k)}, \end{cases} \tag{37}$$

where $\sigma = \sigma(u)$ is an arbitrary function and for calculating coefficients (A^i, B_k^i) one needs to follow the following algorithm:

1) solve the linear system

$$\sum_{s=0}^{n-1} (U^{s+1} f^i) D_s = (-1)^{n+1} U^{n+1} f^i \tag{38}$$

with respect to D_s;

2) calculate A_i from (36);

3) calculate B_k^i from recursion relations (21), (22) or from formulas (26).

In an important particular case $U^{n+1} f^i = 0$ ($\forall\ i$) (system (38) is homogeneous) the latter formulas simplify to

$$A_n = 1, \qquad A_i = 0, \qquad B_n = f^i, \qquad B_k^i = (-U)^{n-k} f^i, \qquad k = \overline{0, n-1}. \tag{39}$$

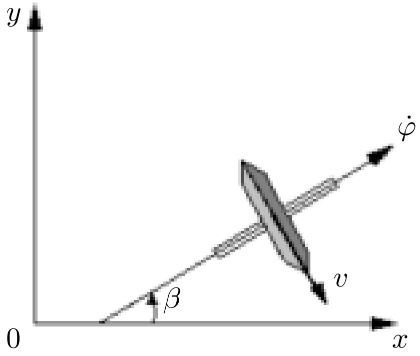

Figure 1. "Integrating Wheel"

3 Examples

Example 1 [3, p.28]. Let us consider the motion of a nonholonomic system ("Integrating Wheel") on the plane OXY (see Fig. 1).

During the rotation of wheel around its axis the coordinates x and y are bounded by following equations

$$\dot{x} = R\dot{\varphi}\sin\beta, \qquad \dot{y} = R\dot{\varphi}\cos\beta, \tag{40}$$

where $\dot{\varphi}$ is angular velocity, R is radius, β is angle of orientation of the wheel on the plane. Denoting by

$$\xi^1 = \frac{x}{R}, \qquad \xi^2 = \frac{y}{R}, \qquad \tau = \varphi, \qquad \beta = u, \tag{41}$$

we get the system

$$\begin{cases} \dfrac{d\xi^1}{du} = \sin u \dfrac{d\tau}{du}, \\ \dfrac{d\xi^2}{du} = \cos u \dfrac{d\tau}{du}. \end{cases} \tag{42}$$

According to Theorem 1 the result will be following. System (38) has the form

$$\begin{pmatrix} \cos u & \sin u \\ -\sin u & -\cos u \end{pmatrix} \begin{pmatrix} D_0 \\ D_1 \end{pmatrix} = \begin{pmatrix} \cos u \\ -\sin u \end{pmatrix}, \tag{43}$$

and its solution is

$$D_0 = 1, \qquad D_1 = 0. \tag{44}$$

By formulas (36) one can obtain

$$A_1 = -D_1 = 0, \qquad A_0 = D_0 - UD_1 = 1. \tag{45}$$

Finally, by using (26) we have

$$B_1^1 = -\cos u, \qquad B_0^1 = 0, \qquad B_1^2 = \sin u, \qquad B_0^2 = 0, \tag{46}$$

and the general solution takes the form

$$\begin{cases} \tau = \sigma_{uu} + \sigma, \\ \xi^1 = \sin u \sigma_{uu} - \cos u \sigma_u, \\ \xi^2 = \cos u \sigma_{uu} + \sin u \sigma_u, \end{cases} \tag{47}$$

where $\sigma = \sigma(u)$ is arbitrary function from $\mathbf{C}^3$.

Example 2. Let us calculate the symmetry algebra of a control system

$$x^{(n)} = u, \qquad x^{(n)} = \frac{d^n x}{dt^n}. \tag{48}$$

Rewrite system (48) in Cauchy form

$$\dot{x}^1 = x^2, \quad \dot{x}^2 = x^3, \quad \ldots, \quad \dot{x}^n = u, \tag{49}$$

where $x^1 = x$, $x^2 = \dot{x}$, …, $x^n = x^{(n-1)}$. Now with system (49) we can associate the differential operator

$$X_0 = \hat{X}_0 + u\partial_{x^n}, \qquad \hat{X}_0 = \partial_t + x^{i+1}\partial_{x^i}, \qquad i = \overline{1, n-1}. \tag{50}$$

The symmetry operator is

$$X = \tau(t, x, u)\partial_t + \xi^j(t, x, u)\partial_{x^j} + \varphi(t, x, u)\partial_u, \qquad j = \overline{1, n}. \tag{51}$$

Symmetry conditions give us the following determining equations

$$X f^j + f^j X_0 \tau - X_0 \xi^j = 0, \tag{52}$$

$$f^j U\tau - U\xi^j = 0, \qquad j = \overline{1, n}. \tag{53}$$

The last equation is the same as (2). From (53) we have

$$\xi^i_u - x^{i+1}\tau_u = 0, \tag{54}$$

$$\xi^n_u - u\tau_u = 0. \tag{55}$$

According to Theorem 1 the solution of (55) has the form

$$\tau = \sigma_u, \qquad \xi^n = u\sigma_u - \sigma, \tag{56}$$

where $\sigma = \sigma(t, x, u)$ is arbitrary function. Now for (54) we have

$$\xi^i = x^{i+1}\sigma_u + g^i, \tag{57}$$

where $g^i = g^i(t, x)$. Omitting the intermediate calculations, we can formulate the general result as following.

Theorem 2. *The maximal invariance algebra for control flat system (49) is infinite-dimensional and its infinitesimal operator is*

$$\begin{aligned} X = {} & -\frac{\partial}{\partial x^n}\left(\hat{X}_0^{n-2} g\right)\partial_t + \left(-x^{i+1}\frac{\partial}{\partial x^n}\left(\hat{X}_0^{n-2} g\right) + \hat{X}_0^{i-1} g\right)\partial_{x^i} \\ & + \left(\hat{X}_0^{n-1} g\right)\partial_{x^n} + \left(X_0^2 \hat{X}_0^{n-2}\right) g\partial_u, \qquad i = \overline{1, n-1}. \end{aligned} \tag{58}$$

where $g = g(t, x^1, \ldots, x^{n-1})$.

Important details and other examples the reader may find in [12].

4 Conclusion

The calculation of the general solution of system (2) according to Theorem 1 consists of only regular actions (solving of linear system and differentiation), so these procedures are easy realized in the analytical system **REDUCE**. Besides, exact formulas are very useful in supervising of control systems (see the details in [12]).

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