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Realizations of the Euclidean Algebra within the Class of Complex Lie Vector Fields

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We obtained a complete description of inequivalent realizations of the Euclidean algebra in the class of Lie vector fields with three independent and n dependent variables. In particular, principally new nonlinear realizations of the above algebra are constructed. We also construct functional bases of differential invariants for one realization of the Euclidean algebra and one realization of the extended Euclidean algebra.

As is well known, the problem of classifying linear and nonlinear partial differential equations (PDEs) admitting some Lie transformation group G is closely connected to that of describing inequivalent realizations of its Lie algebra AG in the class of differential operators of the first order or Lie vector fields (LVFs) [1–3]. Having realizations of the Lie algebra AG, we can to construct all PDEs admitting the group G by means of the Lie infinitesimal method [1, 2]. Fushchych and Yehorchenko found the complette set of first- and second-order differential invariants for the standard realizations of the Poincaré group $P(1,n)$, Euclidean group $E(n)$ [4] and for nonlinear realization of $P(1,n)$ [10, 11].

Rideau, Winternitz [5] and Zhdanov, Fushchych [6] have done complete description of inequivalent realizations of the Galilean group and its natural extensions in the class of LVFs with two independent and two dependent variables. Results are used in constructing of the general evolution equation of the second order

$$\Psi_t + F(t, x, \Psi, \Psi^*, \Psi_x, \Psi^*_x, \Psi_{xx}, \Psi^*_{xx}) = 0,$$

invariant under the Galilean, Galilean-similitude, and Schrödinger groups. All second-order PDEs, invariant under the Poincaré group, extended Poincaré group and conformal group in a two-dimensional space are constructed in [7, 8].

In this paper we study realizations of the Lie algebra $AE(3)$ of the Euclidean group $E(3)$ on the space $X \otimes U$ of complex variables $x = (x_1, x_2, x_3)$ and $u = (u_1, \dots, u_n)$.

1. Consider a problem of constructing realizations of the Lie algebra $AE(3)$ in the class of Lie vector fields (LVF realizations) of the form

$$Q_a = \xi^{ab}(x,u)\partial_{x_b} + \eta^{aj}(x,u)\partial_{u_j}, \tag{1}$$

Here ξ^{ab}, η^{aj} are some sufficiently smooth complex functions on the space $X \otimes U$. We use the notation $\partial_{x_b} = \frac{\partial}{\partial x_b}$, $\partial_{u_j} = \frac{\partial}{\partial u_j}$ and we sum over repeated indices ($a, b = 1, 2, 3$, $j = 1, 2, \dots, n$).

Definition 1. *We say that operators P_a, J_b $(a, b, c = 1, 2, 3)$ of the form (1) compose a realization of the Euclidean algebra $AE(3)$ in the class of Lie vector fields if*

- *they are linearly independent,*
- *they satisfy the following commutation relations:*

$$[P_a, P_b] = 0, \qquad [P_a, J_b] = i\varepsilon_{abc}P_c, \tag{2}$$

$$[J_a, J_b] = i\varepsilon_{abc}J_c. \tag{3}$$

In the above formulae $[Q_1, Q_2] \equiv Q_1Q_2 - Q_2Q_1$ is the commutator; $a, b, c = 1, 2, 3$; ε_{abc} is third order antisymmetic tensor with $\varepsilon_{123} = 1$; i is imaginary unit: $i^2 = -1$.

Let us note that linearly independent differential operators J_b satisfying commutation relations (3), compose a realization of the Lie algebra $AO(3)$ of the rotations group.

Algebra $AE(3)$ is a semi-direct sum of the Lie algebra of the rotation group $O(3)$ and the commutative ideal $I = \langle P_1, P_2, P_3\rangle$.

Here we study LVF realizations of the Euclidean algebra $AE(3)$, where translation generators P_a are of the form

$$P_a = i\partial_{x_a}, \qquad a = 1, 2, 3. \tag{4}$$

Precisely these LVF realizations of the Euclidean algebra $AE(3)$ are important in different problems of theoretical and mathematical physics (see, e.g., [9]).

Therefore, the problem of studying all LVF realizations of the Euclidean algebra $AE(3)$ is reduced to solving of relations (2) and (3) within the class of linear first-order differential operators, where P_a are of the form (4) and J_b are of the form (1). It is known [2] that commutation relations do not change after an arbitrary nondegenerate change of variables x, u

$$\tilde{x}_\alpha = f_\alpha(x, u), \qquad \alpha = 1, 2, 3, \tag{5}$$

$$\tilde{u}_\beta = g_\beta(x, u), \qquad \beta = 1, \ldots, n, \tag{6}$$

where f_α, g_β are sufficiently smooth complex functions defined on the space $X \otimes U$. Invertible transformations (5), (6) form a transformation group (a group of diffeomorphisms) and determine a natural equivalence relations of LVF realizations of the algebra $AE(3)$. Two realizations of the Euclidean algebra are called equivalent if the corresponding basis operators can be transformed one into another by a change of variables (5) and (6).

Let P_a, J_b $(a, b = 1, 2, 3)$ be differential operator of the form (4) and (1) respectively. From the commutation relations (2) we find that

$$J_a = -i\varepsilon_{abc}x_b\partial_{x_c} + \zeta_{ab}(u)\partial_{x_b} + A_a, \qquad a = 1, 2, 3, \tag{7}$$

where A_a are operators of the form

$$A_a = \tilde{\eta}_{aj}(u)\partial_{u_j}, \tag{8}$$

which are satisfing the commutation relations

$$[A_a, A_b] = i\varepsilon_{abc}A_c, \qquad a, b, c = 1, 2, 3. \tag{9}$$

In (7), (8) ζ_{ab} and $\tilde{\eta}_{aj}$ are some smooth functions.

Therefore, we begin the classification of LVF realizations of Euclidean algebra from construction of inequivalent realizations of the Lie algebra of the rotation group in the class of operators (8).

Theorem 1. *Let differential operators A_a $(a = 1, 2, 3)$ of the form (8) satisfy commutation relations (9). Then there exist changes of variables (6), reducing these operators to one of the following triplets of operators:*

$$A_a = 0, \qquad a = 1, 2, 3; \tag{10}$$

$$A_1 = \sin u_1 \partial_{u_1}, \qquad A_2 = \cos u_1 \partial_{u_1}, \qquad A_3 = i\partial_{u_1}; \tag{11}$$

$$\begin{aligned} A_1 &= -\sin u_1 \coth u_2 \partial_{u_1} + \cos u_1 \partial_{u_2} + \varepsilon \frac{\sin u_1}{\sinh u_2} \partial_{u_3}, \\ A_2 &= -\cos u_1 \coth u_2 \partial_{u_1} - \sin u_1 \partial_{u_2} + \varepsilon \frac{\cos u_1}{\sinh u_2} \partial_{u_3}, \\ A_3 &= i\partial_{u_1}, \qquad \varepsilon = 0, 1; \end{aligned} \tag{12}$$

$$\begin{aligned} A_1 &= \sin u_1 \partial_{u_1} + \cos u_1 \partial_{u_2}, \\ A_2 &= \cos u_1 \partial_{u_1} - \sin u_1 \partial_{u_2}, \\ A_3 &= i\partial_{u_1}; \end{aligned} \tag{13}$$

$$\begin{aligned} A_1 &= \sin u_1 \partial_{u_1} + u_2 \cos u_1 \partial_{u_2} + u_2 \sin u_1 \partial_{u_3}, \\ A_2 &= \cos u_1 \partial_{u_1} - u_2 \sin u_1 \partial_{u_2} + u_2 \cos u_1 \partial_{u_3}, \\ A_3 &= i\partial_{u_1}. \end{aligned} \tag{14}$$

It follows from the theorem 1 and definition of LVF realization of Euclidean algebra that the following statement is valid

Corollary. *The algebra $AO(3)$ possesses five nonequivalent LVF realizations presented by formulae (11)–(14).*

2. Now, using the results of Theorem 1, we shall construct inequivalent LVF realizations of the Euclidean algebra where P_a $(a = 1, 2, 3)$ are of the form (4) and J_b $(b = 1, 2, 3)$ are of the form (7). We will call these realizations of the Lie algebra $AE(3)$ *covariant* realizations.

Theorem 2. *Any covariant LVF realization of the Euclidean algebra $AE(3)$ is equivalent to one of the following realizations:*

1. $$P_a = i\partial_{x_a}, \qquad J_a = -i\varepsilon_{abc} x_b \partial_{x_c}, \qquad a, b, c, = 1, 2, 3; \tag{15}$$

2. $$\begin{aligned} P_a &= i\partial_{x_a}, \qquad a = 1, 2, 3, \\ J_1 &= i(x_3 \partial_{x_2} - x_2 \partial_{x_3}) + \sin u_1 \partial_{u_1}, \\ J_2 &= i(x_1 \partial_{x_3} - x_3 \partial_{x_1}) + \cos u_1 \partial_{u_1}, \\ J_3 &= i(x_2 \partial_{x_1} - x_1 \partial_{x_2}) + i\partial_{u_1}; \end{aligned} \tag{16}$$

3. $$\begin{aligned} P_a &= i\partial_{x_a}, \qquad a = 1, 2, 3, \\ J_1 &= i(x_3 \partial_{x_2} - x_2 \partial_{x_3}) + f\partial_{x_1} - i \sin u_1 \frac{\partial f}{\partial u_2} \partial_{x_3} - \sin u_1 \coth u_2 \partial_{u_1} + \cos u_1 \partial_{u_2}, \\ J_2 &= i(x_1 \partial_{x_3} - x_3 \partial_{x_1}) + f\partial_{x_2} - i \cos u_1 \frac{\partial f}{\partial u_2} \partial_{x_3} - \cos u_1 \coth u_2 \partial_{u_1} - \sin u_1 \partial_{u_2}, \\ J_3 &= i(x_2 \partial_{x_1} - x_1 \partial_{x_2}) + i\partial_{u_1}; \end{aligned} \tag{17}$$

4. $$\begin{aligned} P_a &= i\partial_{x_a}, \qquad a = 1, 2, 3, \\ J_1 &= i(x_3 \partial_{x_2} - x_2 \partial_{x_3}) + g\partial_{x_1} - i\left(\sin u_1 \frac{\partial g}{\partial u_2} - \frac{\cos u_1}{\sinh u_2} \frac{\partial g}{\partial u_3}\right) \partial_{x_3} \\ &\quad - \sin u_1 \coth u_2 \partial_{u_1} + \cos u_1 \partial_{u_2} + \frac{\sin u_1}{\sinh u_2} \partial_{u_3}, \end{aligned} \tag{18}$$

$$J_2 = i(x_1\partial_{x_3} - x_3\partial_{x_1}) + g\partial_{x_2} - i\left(\cos u_1 \frac{\partial g}{\partial u_2} + \frac{\sin u_1}{\sinh u_2}\frac{\partial g}{\partial u_3}\right)\partial_{x_3}$$
$$- \cos u_1 \coth u_2 \partial_{u_1} - \sin u_1 \partial_{u_2} + \frac{\cos u_1}{\sinh u_2}\partial_{u_3},$$
$$J_3 = i(x_2\partial_{x_1} - x_1\partial_{x_2}) + i\partial_{u_1};$$

5. $P_a = i\partial_{x_a}, \qquad a = 1, 2, 3,$

$$J_1 = i(x_3\partial_{x_2} - x_2\partial_{x_3}) + h\partial_{x_1} - i\sin u_1 \frac{\partial h}{\partial u_2}\partial_{x_3} + \sin u_1\partial_{u_1} + \cos u_1 \partial_{u_2},$$
$$J_2 = i(x_1\partial_{x_3} - x_3\partial_{x_1}) + h\partial_{x_2} - i\cos u_1 \frac{\partial h}{\partial u_2}\partial_{x_3} + \cos u_1\partial_{u_1} - \sin u_1 \partial_{u_2}, \qquad (19)$$
$$J_3 = i(x_2\partial_{x_1} - x_1\partial_{x_2}) + i\partial_{u_1};$$

6. $P_a = i\partial_{x_a}, \qquad a = 1, 2, 3,$

$$J_1 = i(x_3\partial_{x_2} - x_2\partial_{x_3}) + r\partial_{x_1} - iu_2\left(\cos u_1 \frac{\partial r}{\partial u_2} + \sin u_1 \frac{\partial r}{\partial u_3}\right)\partial_{x_3}$$
$$+ \sin u_1 \partial_{u_1} + u_2 \cos u_1 \partial_{u_2} + u_2 \sin u_1 \partial_{u_3},$$
$$J_2 = i(x_1\partial_{x_3} - x_3\partial_{x_1}) + r\partial_{x_2} - iu_2\left(\sin u_1 \frac{\partial r}{\partial u_2} - \cos u_1 \frac{\partial r}{\partial u_3}\right)\partial_{x_3} \qquad (20)$$
$$+ \cos u_1 \partial_{u_1} - u_2 \sin u_1 \partial_{u_2} + u_2 \cos u_1 \partial_{u_3},$$
$$J_3 = i(x_2\partial_{x_1} - x_1\partial_{x_2}) + i\partial_{u_1}.$$

Here $f = f(u_2, \ldots, u_n)$ and $h = h(u_2, \ldots, u_n)$ are given by the formulae

$$f = f_1 \cosh u_2 + f_2 \left(\cosh u_2 \ln\left|\tanh \frac{u_2}{2}\right| - 1\right),$$

and

$$h = h_1 e^{-u_2} + h_2 e^{2u_2}$$

respectively, where f_1, f_2, h_1, h_2 are arbitrary function of $u_3, \ldots, u_n$; $g = g(u_2, \ldots, u_n)$ is a solution of differential equation

$$\sinh^{-2} u_2 g_{u_3u_3} + g_{u_2u_2} + \coth u_2 g_{u_2} - 2g = 0$$

and $r = r(u_2, \ldots, u_n)$ is a solution of the equation

$$u_2^2\left(r_{u_3u_3} + r_{u_2u_2}\right) - 2r = 0.$$

3. Now, we use the obtained realizations to construct PDEs, invariant under the Euclidean group.

Let X_a ($a = 1, 2, \ldots, 6$) be basis operators of Lie algebra $AE(3)$ of the Euclidean group in the space of $X \otimes U$. A differential equation

$$F(x, u, \underset{1}{u}) = 0,$$

where $\underset{1}{u}$ is a set of the first derivatives of u, is invariant under group $E(3)$ if the function F satisfies the following relations [1, 2]

$$\mathrm{pr}^{(1)} X_a F\Big|_{F=0} = 0, \qquad a = 1, 2, \ldots, 6. \qquad (21)$$

Here $\mathrm{pr}^{(1)} X_a$ are first prolongations of the operators X_a.

Solving the system (21) we obtain the complete set of elementary differential invariants

$$I_r(x, u, \underset{1}{u}), \qquad r = 1, 2, \ldots, 4n-3,$$

and the invariant equation has the form

$$\Phi(I_1, I_2, \ldots, I_{4n-3}) = 0. \tag{22}$$

Hence to describe the general form of PDEs admitting Euclidean group, we must find a complete set of elementary differential invariants.

Let the basis operators of Lie algebra of Euclidean group be of the form (16). The prolongations of translation operators are equal to the P_a of (4) and prolongations of rotation operators read

$$\begin{aligned}\mathrm{pr}^{(1)} J_1 = {} & i(x_3\partial_{x_2} - x_2\partial_{x_3}) + \sin u^1 \partial_{u^1} + u_1^1 \cos u^1 \partial_{u_1^1} + (u_2^1 \cos u^1 + i u_3^1)\partial_{u_2^1} \\ & + (u_3^1 \cos u^1 - i u_2^1)\partial_{u_3^1} + i(u_3^k \partial_{u_2^k} - u_2^k \partial_{u_3^k}),\end{aligned} \tag{23}$$

$$\begin{aligned}\mathrm{pr}^{(1)} J_2 = {} & i(x_1\partial_{x_3} - x_3\partial_{x_1}) + \cos u^1 \partial_{u^1} - (u_1^1 \sin u^1 + i u_3^1)\partial_{u_1^1} \\ & - u_2^1 \sin u^1 \partial_{u_2^1} + (i u_1^1 - u_3^1 \sin u^1)\partial_{u_3^1} + i(u_1^k \partial_{u_3^k} - u_3^k \partial_{u_1^k}),\end{aligned} \tag{24}$$

$$\mathrm{pr}^{(1)} J_3 = i\{(x_2\partial_{x_1} - x_1\partial_{x_2}) + \partial_{u^1} + u_2^1 \partial_{u_1^1} - u_1^1 \partial_{u_2^1} + u_2^k \partial_{u_1^k} - u_1^k \partial_{u_2^k}\}, \quad k = 2, 3, \ldots, n. \tag{25}$$

Here and below we use the notation

$$u^a = u_a, \qquad u_b^a = \frac{\partial u^a}{\partial x_b}, \qquad \partial_{u_b^a} = \frac{\partial}{\partial u_b^a}.$$

Functions I_r are invariant under operators $P_a = i\partial_{x_a}$, $(a = 1, 2, 3)$, consequently they do not depend on x explictly.

Next, solving the system (21) for operator (23)–(25) we obtain the following elementary invariants:

$$I_1 = \frac{i\left(u_1^1 \sin u^1 + u_2^1 \cos u^1\right) - u_3^1}{\left(u_1^k\right)^2 + \left(u_2^k\right)^2 + \left(u_3^k\right)^2}, \qquad I_2^k = \frac{i\left(u_1^k \sin u^1 + u_2^k \cos u^1\right) - u_3^k}{\sqrt{\left(u_1^k\right)^2 + \left(u_2^k\right)^2 + \left(u_3^k\right)^2}},$$

$$I_3^k = \frac{u_3^k\left(u_2^1 \sin u^1 - u_1^1 \cos u^1\right) + u_3^1\left(u_1^k \cos u^1 - u_2^k \sin u^1\right) + i\left(u_1^1 u_2^k - u_1^k u_2^1\right)}{\left(u_1^k\right)^2 + \left(u_2^k\right)^2 + \left(u_3^k\right)^2},$$

$$I_4^k = u^k, \qquad I_5^k = \left(u_1^k\right)^2 + \left(u_2^k\right)^2 + \left(u_3^k\right)^2, \qquad k = 2, 3, \ldots, n.$$

The general form of the first order differential equation admitting the Euclidean group is given by (22).

4. The results of Theorem 2 can also be used for construction of inequivalent realization of the Lie algebra of the extended Euclidean group $\tilde{E}(3)$.

Definition 2. *We say that operators P_a, J_b, D $(a, b = 1, 2, 3)$ of the form (1) compose a realization of the extended Euclidean algebra $A\tilde{E}(3)$ in the class of Lie vector fields if*

- *they are linearly independent,*
- *they satisfy the commutation relations (2), (3) and the following ones*
  $$[P_a, D] = P_a, \qquad [J_b, D] = 0.$$

It is not difficult to make sure that operators (16) and the dilatation operator

$$D = i(x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3}) + \varepsilon\partial_v, \tag{26}$$

where $\varepsilon = 0$ or $\varepsilon = 1$, compose realization of algebra $A\tilde{E}(3)$.

Let us consider the problem of construction of the general form of PDEs, admitting this realization of the extended Euclidean group.

The prolongation of operators P_a and J_b $(a, b = 1, 2, 3)$ are of the form (4) and (23)–(25) respectively. Prolongation of operators D reads

$$D = i(x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3}) + \varepsilon\partial_v - i\left(u_1^j\partial_{u_1^j} + u_2^j\partial_{u_2^j} + u_3^j\partial_{u_3^j}\right), \qquad j = 1, \ldots, n.$$

Obviously, differential invariants of algebra $A\tilde{E}(3)$ are invariant with respect to the corresponding algebra $AE(3)$, hence, they are functions of I_1, I_2^k, I_3^k, I_4^k, I_5^k.

Taking into account all the above, instead of the operator D we can consider an operator

$$D' = \varepsilon\frac{\partial}{\partial I_4^k} + iI_1\frac{\partial}{\partial I_1} - 2iI_5\frac{\partial}{\partial I_5}.$$

Solving system (21), corresponding to above operator we construct the following differential invariant of extended Euclidean group

$$I_4^k + i\varepsilon\ln I_1, \quad I_2^k, \quad I_3^k, \quad (I_1)^2 I_5^k, \qquad k = 2, 3, \ldots, n.$$

The general form of invariant equations reads

$$\Phi\left(I_4^k + i\varepsilon\ln I_1, I_2^k, I_3^k, (I_1)^2 I_5^k\right) = 0.$$

Also we use results of Theorem 1 to construct inequivalent LVF realizations of the Lie algebra of the Poincaré group $P(1, 3)$.

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