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Transformations of Ordinary Differential Equations: Local and Nonlocal Symmetries

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The brief review of new methods of factorization, autonomization and exact linearization of the ordinary differential equations is represented. These methods along with the method of the group analysis based on using both point and nonpoint, local and nonlocal transformations are effective tools for study of nonlinear autonomous and nonautonomous dynamical systems. Thus a scope of exactly solvable problems of the Nonlinear analysis is extended.

Introduction

This paper is devoted to the analytical aspect of the problem of integrability of ordinary differential equations (ODE). There are two approaches to the problem one of which is related to the changes of variables and another to implied algebraic concepts. However, an application of substitutions as a rule had the heuristic nature. Such powerful methods as factorization were hardly extendable to differential equations, even linear ones; besides, they were inefficient. Plenty of expectations was connected to an application of Lie group and Lie algebra theory to differential equations (the group analysis), and it was not in vain. Conceptual and uniformizing role of this theory is now universally recognized. Algebraic approach became especially fruitful in solving mechanical and physical fundamental equations since invariance principles are background for the construction of these equations. However, its capability does not allow "to close" the integrability problem.

Being concerned with the integrability problem for ODE, the author has concluded that the key to its comprehension is contained in the ideas of **factorization and transformation** and in realizing the necessity of their combined application since the summarized results exceed the effect of a single idea. The uniform theory of a factorization and transformations of ODE allows to investigate structurally nonlinear and non-stationary problems of technology and natural sciences, what is especially important in connection with a continuous delinearization of Science in general and Physics in particular.

For a first time the author presented the factorization method for differential operators in connection with a transformation theory in 1967 [1]. Further logical development of this method has led to the extension of the factorization to the nonlinear equations and the creation of effective algorithms for searching of transformations. The author [2] incorporated the fundamentals of theories of factorization and transformations of n-th order ODE to uniform theory which structurally permitted to solve the problems of equivalence of various classes, i.e. the problems of their reduction to given (including canonical) prescribed form (see also [3–5]).

In the present paper the special attention is paid to autonomizable and linearizable equation classes.

The paper is organized as follows. In Section 1 we present the new method of exact linearization of nonlinear ODEs. We consider in detail the linearization of autonomous equations with the help of nonlocal transformation of variables. In Section 2 the example of the exponential nonlocal symmetry is given. In Section 3 we consider the class of linearizable equations of the third order and present some examples.

1 A new method for exact linearization of ODE

Theorem 1.1 [4]. *The equation*

$$y^{(n)} - f\left(x, y, y', \ldots, y^{(n-1)}\right) = 0 \tag{1.1}$$

is reducible to the linear autonomous form

$$M_n z \equiv z^{(n)}(t) + \sum_{k=1}^{n} \binom{n}{k} b_k z^{(n-k)}(t) = 0, \qquad b_k = \text{const.} \tag{1.2}$$

by means of the reversible transformation

$$y = v(x,y)z, \qquad dt = u_1(x,y)dx + u_2(x,y)dy, \tag{1.3}$$

where v, u_1, and u_2 are sufficiently smooth functions and $v(u_1 + u_2 y') \neq 0$ in a domain $\Gamma(x,y)$ iff (1.1) admits the noncommutative factorization

$$\prod_{k=n}^{1} \left[D - \frac{v_x + v_y y'}{v} - (k-1)\frac{D(u_1 + u_2 y')}{u_1 + u_2 y'} - r_k(u_1 + u_2 y')\right] y = 0, \tag{1.4}$$

or the commutative one

$$\prod_{k=1}^{n} \left[\frac{1}{u_1 + u_2 y'} D - \frac{v_x + v_y y'}{v(u_1 + u_2 y')} - r_k\right] y = 0, \qquad D = d/dx, \qquad v_x = \partial v/\partial x, \qquad v_y = \partial v/\partial y, \tag{1.5}$$

where $D = d/dx$, and r_k are the roots of the characteristic equation

$$M_n(r) \equiv \sum_{k=0}^{n} \binom{n}{k} b_k r^{n-k} = 0, \qquad b_0 = 1. \tag{1.6}$$

Necessity. Factorizing (1.2) we obtain

$$\prod_{k=n}^{1} (D_t - r_k) z = 0, \qquad D_t = d/dt. \tag{1.7}$$

We apply the transformation, inverse to (1.3), to (1.7):

$$z = v^{-1} y, \qquad dx = 1/(u_1 + u_2 y')dt:$$

$$\prod_{k=n}^{1} \left[\frac{1}{u_1 + u_2 y'} D - r_k\right] \frac{y}{v} = \prod_{k=n}^{2} \left[\frac{1}{u_1 + u_2 y'} D - r_k\right] \left[\frac{1}{u_1 + u_2 y'} D - r_k\right] \frac{y}{v}$$

$$= \prod_{k=n}^{2} \left[\frac{1}{u_1 + u_2 y'} D - r_k\right] \frac{1}{v} \left[\frac{1}{u_1 + u_2 y'} D - \frac{Dv}{v(u_1 + u_2 y')} - r_1\right] y = 0, \quad Dv = v_x + v_y y'.$$

Using the operator identity

$$\left(\frac{1}{u_1+u_2y'}D-r_k\right)\frac{1}{v}=\frac{1}{v}\left[\frac{1}{u_1+u_2y'}D-\frac{Dv}{v(u_1+u_2y')}-r_k\right],$$

we obtain the expression

$$\frac{1}{v}\prod_{k=1}^{n}\left[\frac{1}{u_1+u_2y'}D-\frac{v_x+v_yy'}{v(u_1+u_2y')}-r_k\right]y=0,$$

that corresponds to the factorization (1.4). The factorization (1.5) can be obtained from (1.4) as follows. If we apply an easily verifiable identity

$$\left[\frac{1}{u_1+u_2y'}D-\frac{v_x+v_yy'}{v(u_1+u_2y')}-r_s\right]\frac{1}{(u_1+u_2y')^{s-1}}$$
$$=\frac{1}{(u_1+u_2y')^{s}}\left[D-\frac{v_x+v_yy'}{v}-(s-1)\frac{D(u_1+u_2y')}{u_1+u_2y'}-r_s(u_1+u_2y')\right],$$

$s=\overline{1,n}$, we get a noncommutative factorization

$$\frac{1}{v(u_1+u_2y')^n}\prod_{k=n}^{1}\left[D-\frac{Dv}{v}-(k-1)\frac{D(u_1+u_2y')}{u_1+u_2y'}-r_k(u_1+u_2y')\right]y,$$

that corresponds to (1.5).

Sufficiency. Let take place the factorization (1.5) takes place. We apply transformation (1.3), sequentially changing the dependent and independent variables: a) $y=vz$; b) $dt=(u_1+u_2y')dx$, $(D=(u_1+u_2y')D_t)$. Let $U=u_1+u_2y'$.

$$\prod_{k=n}^{1}\left[D-\frac{Dv}{v}-(k-1)\frac{DU}{U}-r_kU\right]vz=\prod_{k=n}^{2}\left[D-\frac{Dv}{v}-(k-1)\frac{DU}{U}-r_kU\right]v(D-r_1U)z.$$

By virtue of the identity

$$\left[D-\frac{Dv}{v}-(s-1)\frac{DU}{U}-r_sU\right]v=v\left[D-(s-1)\frac{DU}{U}-r_sU\right]v,\ \ s=\overline{1,n}$$

we obtain

$$\prod_{k=n}^{1}\left[D-\frac{Dv}{v}-(k-1)\frac{DU}{U}-r_kU\right]vz=v\prod_{k=n}^{1}\left[D-(k-1)\frac{DU}{U}-r_kU\right]z.$$

Further, changing the independent variable we get:

$$v\prod_{k=n}^{1}\left[D-(k-1)\frac{DU}{U}-r_kU\right]z=v\prod_{k=n}^{1}\left[UD_t-(k-1)\frac{DU}{U}-r_kU\right]z$$
$$=v\prod_{k=n}^{2}\left[UD_t-(k-1)\frac{DU}{U}-r_kU\right]U(D_t-r_1)z.$$

Applying the operator identity:

$$\left[D-(s-1)\frac{DU}{U}-r_sU\right]U^{s-1}=U^s(D_t-r_s),\qquad s=\overline{1,n}$$

we have as a result the factorization

$$\prod_{k=n}^{1}\left[D-\frac{Dv}{v}-(k-1)\frac{DU}{U}-r_kU\right]y=vU^n\prod_{k=n}^{1}(D_t-r_k)z=0,$$

that corresponds to the equation (1.2).

The transformation (1.3) encloses the following important ones: Kummer–Liouville transformation (KLT)

$$y(x)=v(x)z,\quad dt=u(x)dx,\quad vu\neq 0,\quad \forall\, x\in \mathbf{i_0}\subset \mathbf{i},\quad v,u\in \mathbf{C}^n(\mathbf{i_0}), \tag{1.8}$$

exact nonlocal linearization of nonlinear autonomous equations

$$y=v(y)z,\quad dt=u(y)dx,\quad u(y(x))v(y(x))\neq 0,\quad \forall\, x\in \mathbf{I}=\{x\mid a\leq x\leq b\}; \tag{1.9}$$

the general point linearization

$$t=f(x,y),\qquad z=\varphi(x,y),\qquad \det\left(\frac{t,z}{x,y}\right)=t_xz_y-t_yz_x\neq 0, \tag{1.10}$$

corresponding to (1.3) for $u_{1y}=u_{2x}$; the point linearization

$$t=f(x),\qquad z=\varphi(x,y), \tag{1.11}$$

preserving fibering; the linearization

$$y=v(x,y)z,\qquad dt=u(x,y)dx, \tag{1.12}$$

connected with arbitrary point Lie symmetry; and finally, the general nonlocal linearization (1.3).

Theorem 1.2. *After the sequential application the composition of the transformations (1.8) and (1.9), i.e. of the transformations*

$$y=v_1(x)v_2(y/v_1(x))z,\qquad dt=u_1(x)u_2(y/v_1(x))dx \tag{1.13}$$

the equation (1.1) is reducible to (1.2) iff the commutative factorization

$$\prod_{k=1}^{n}\left[\frac{1}{u_1u_2}D-\frac{v_1'v_2+v_1v_2^*Y'}{v_1v_2u_1u_2}-r_k\right]y=0,\qquad Y=\frac{y}{v_1},\quad (')=\frac{d}{dx},\quad (*)=\frac{d}{dY}; \tag{1.14}$$

or the noncommutative one

$$\prod_{k=n}^{1}\left[D-\frac{v_1'}{v_1}-(k-1)\frac{u_1'}{u_1}-\frac{v_2^*}{v_2}Y'-(k-1)\frac{u_2^*}{u_2}Y'-r_ku_1u_2\right]y=0, \tag{1.15}$$

takes place; and the diagram

$$\begin{array}{ccc}\mathbf{A} & \xrightarrow{f} & \mathbf{B}\\ \downarrow\varphi & & \downarrow g\\ \mathbf{C} & \xrightarrow{\psi} & \mathbf{D}\end{array}$$

is commutative, i.e. $g\circ f=\psi\circ\varphi$.

The formulas (1.14) and (1.15) easily follow from (1.4) and (1.5) by virtue of (1.13). a commutativity of the diagram or realization of the condition $f\circ g=\varphi\circ\psi$, is checked immediately.

The transformations f, g, φ and ψ have the following form

$$f: \ y = v_1(x)Y, \quad ds = u_1(x)dx; \qquad g: \ Y = v_2(Y)z, \quad dt = u_2(Y)ds;$$

$$\varphi: \ y = v_2(Y)P, \quad dq = u_2(Y)dx; \qquad \psi: \ P = V_1(q)z, \quad dt = U_1(q)dq.$$

Here **A** denotes the set of equations (1.1), (1.15), **B** denotes the set of nonlinear autonomous equations having the factorized form

$$\prod_{k=n}^{1}\left[D_s - \frac{v_2^*}{v_2}\frac{dY}{ds} - (k-1)\frac{u_2^*}{u_2}\frac{dY}{ds} - r_k u_2(Y)\right]Y = 0, \qquad D_s = \frac{d}{ds},$$

C is a set of the linear nonautonomous reducible equations

$$\prod_{k=n}^{1}\left[D_q - \frac{1}{V_1}\frac{dV_1}{dq} - (k-1)\frac{1}{U_1}\frac{dU_1}{dq} - r_k U_1(q)\right]P = 0,$$

$$V_1(q(x)) = v_1(x), \qquad U_1(q(x)) = u_1(x)$$

and **D** denotes the set of the linear equations (1.3), (1.7).

Remark 1.1. Theorems 1.1 and 1.2 were announced in [6]. a linearization through the transformation of unknown function was applied in [7], and through the transformation of independent variable was used in [8, 9]. The examples can be found in [10]. In cited works [7–10], as a rule, the considered equations had the second order. Linearization of equations of order $n > 2$ is considered in [11]. Group analysis of ODE of the order $n > 2$ is considered in [12].

It should be mentioned, that the fact of existence of the indicated factorizations for the differential equations allows to discover required transformations.

Theorem 1.3. *The equation*

$$y^{(n)} = F\left(y, y', \ldots, y^{(n-1)}\right), \qquad n > 2 \tag{1.16}$$

is reducible to the linear autonomous form

$$M_n z \equiv z^{(n)}(t) + \sum_{k=1}^{n}\binom{n}{k} b_k z^{(n-k)} + c = 0, \qquad b_k, c = \text{const}, \tag{1.2$'$}$$

by means of the transformation (1.9) iff (1.16) admits the noncommutative factorization

$$\begin{aligned}\prod_{k=n}^{1}\left[D - \left(\frac{1}{y} - \left(\log\int\varphi^{\frac{n^2-n+2}{2n}}\exp\left(\int f dy\right)dy\right)^* + (k-1)\frac{u^*}{u}\right)y' - r_k u\right]y \\ + \frac{c}{\beta}\varphi^{\frac{n^2+n-2}{2n}}\exp\left(-\int f dy\right) = 0.\end{aligned} \tag{1.17}$$

In expanded form it is written as

$$\begin{aligned}&y^{(n)} + nf(y)y'y^{(n-1)} + \cdots + nb_1\varphi(y)y^{(n-1)} + \cdots \\ &+\sum_{m=1}^{n-1}\binom{n}{m}b_m\varphi^m \sum_{s_1+2s_2+\cdots+(n-m)s_{n-m}=n-m}\psi^{12\ldots n-m}_{s_1 s_2\ldots s_{n-m}}y^{(1)s_1}y^{(2)s_2}\cdots y^{(n-m)s_{n-m}} \\ &+\varphi^{\frac{n^2+n-2}{2n}}\exp\left(-\int f dy\right)\left(b_n\int\varphi^{\frac{n^2-n+2}{2n}}\exp\left(\int f dy\right)dy + \frac{c}{\beta}\right) = 0,\end{aligned} \tag{1.18}$$

where the coefficients ψ are the differential expressions, depending from f and φ, $\psi^{12\ldots n-m}_{00\ldots 1}$=1.

In addition we have the linearized transformation

$$z = \beta \int \varphi^{\frac{n^2-n+2}{2n}} \exp\left(\int f(y)dy\right) dy, \qquad dt = \varphi(y)dx, \tag{1.19}$$

and also (for $c = 0$) the one-parameter set of solutions

$$\int \frac{\varphi^{\frac{n^2-3n+2}{2n}} \exp\left(\int f dy\right) dy}{\int \varphi^{\frac{n^2-n+2}{2n}} \exp\left(\int f dy\right) dy} = r_k x + C, \tag{1.20}$$

where r_k are distinct roots of the characteristic equation (1.6).

The equation (1.16) admits a factorization:

$$\prod_{k=n}^{1} \left[D - \left(\frac{v^*}{v} + (k-1)\frac{u^*}{u} \right) y' - r_k u \right] y + cu^n v = 0. \tag{1.21}$$

At first, writing down the product in (1.21), we obtain the expression

$$\left(1 - \frac{v^*}{v}y\right) y^{(n)} - \left[n\frac{v^{**}}{v}y + \left(1 - \frac{v^*}{v}y\right)\left(2n\frac{v^*}{v} + \frac{n^2-n+2}{2}\frac{u^*}{u}\right)\right] y'y^{(n-1)} + \cdots, \tag{1.22}$$

what is proved by induction for $n \geq 3$. Really, let the formula (1.22) hold for $n = m$. Then for $n = m+1$ we have:

$$\left[D - \left(\frac{v^*}{v} + m\frac{u^*}{u}\right) y' \right] \left\{ \left(1 - \frac{v^*}{v}y\right) y^{(m)} \right.$$
$$\left. - \left[m\frac{v^{**}}{v}y + \left(1 - \frac{v^*}{v}y\right)\left(2m\frac{v^*}{v} + \frac{m^2-m+2}{2}\frac{u^*}{u}\right)\right] y'y^{(m-1)} + \cdots \right\}.$$

Collecting the terms at $y^{(m+1)}$ and $y'y^{(m)}$, we obtain the first terms of the new expression:

$$\left(1 - \frac{v^*}{v}y\right) y^{(m+1)} - \left[(m+1)\frac{v^{**}}{v}y \right.$$
$$\left. + \left(1 - \frac{v^*}{v}y\right)\left(2(m+1)\frac{v^*}{v} + \frac{m^2+m+2}{2}\frac{u^*}{u}\right)\right] y'y^{(m)} + \cdots.$$

This prove (1.22). Let us introduce the notation

$$n\frac{v^{**}}{v}y + \left(1 - \frac{v^*}{v}y\right)\left(2n\frac{v^*}{v} + \frac{n^2-n+2}{2}\frac{u^*}{u}\right) = -nf(y)\left(1 - \frac{v^*}{v}y\right).$$

We have the second order nonlinear nonautonomous equation for $v(y)$

$$v^{**} - \frac{2}{v}v^{*2} + \left(\frac{2}{y} - \frac{n^2-n+2}{2n}\frac{u^*}{u} - f\right) v^* + \left(\frac{n^2-n+2}{2n}\frac{u^*}{u} + f\right)\frac{1}{y}v = 0.$$

After the substitution $v = V^{-1}$ this equation is reduced to the linear nonautonomous equation

$$V^{**} + \left(\frac{2}{y} - \frac{n^2-n+2}{2n}\frac{u^*}{u} - f\right) V^* - \frac{1}{y}\frac{n^2-n+2}{2n}\frac{u^*}{u}V = 0,$$

admitting the factorization

$$\left(D_y + \frac{1}{y} - \frac{n^2-n+2}{2n}\frac{u^*}{u} - f\right)\left(D_y + \frac{1}{y}\right) V = 0, \qquad D_y = d/dy$$

and having the solution

$$V = \frac{1}{y}\beta \int u^{\frac{n^2-n+2}{2n}} \exp\left(\int f dy\right) dy.$$

Then we get

$$v(y) = y\left(\beta \int u^{\frac{n^2-n+2}{2n}} \exp\left(\int f dy\right) dy\right)^{-1}. \tag{1.23}$$

(In particular, for $u = \exp\left(-\frac{2n}{n^2-n+2}\int f dy\right)$ we get $v = y(\beta y+\gamma)^{-1}$, $\gamma = \text{const} \neq 0$.) Substituting (1.23) in (1.21), we obtain (1.17). Putting $u = \varphi(y)$, in accordance with (1.23) we get (1.19). Writing down explicitly the product in (1.17), we have (1.18). The equation (1.21) is consisted with the first order equation

$$\left(1 - \frac{v^*}{v}y\right)y' - r_k u y = 0 \tag{1.24}$$

for $c = 0$. Put $u = \varphi(y)$ and substitute (1.23) in (1.24), we obtain (1.20).

Remark 1.2. Rather wide class of n-th order nonlinear autonomous equations can be tested by the method of the exact linearization. The tests can be specializations of the theorem 1.3 for concrete values of n.

2 The example of nonlocal symmetry

Nonlocal symmetries are considered in [13–17] and other works.

Example 2.1 [17]. The equation

$$y'' = y^{-1}y'^2 + pg(x)y^p y' + g'(x)y^{p+1}, \tag{2.1}$$

where p is a nonzero constant and $g(x)$ a nonzero arbitrary function, does not possess a Lie point symmetries except special cases. However, it has the first integral $I = y'/y - g(x)y^p$. The equation (2.1) admits the factorization $D\left(y'/y - g(x)y^p\right) = 0$ and has the exponential nonlocal symmetry

$$G = y\exp\left(\int g(x)y^p dx\right)\frac{\partial}{\partial y}.$$

The author is not going to develop this theme in detail in this paper because he hopes to develop it in other papers.

3 Linearization of the autonomous equations of the third order

Proposition 3.1. *a third order autonomous equation in the form*

$$y''' + f_5(y)y'y'' + f_4(y)y'' + f_3(y)y'^3 + f_2(y)y'^2 + f_1(y)y' + f_0(y) = 0 \tag{3.1}$$

is linearizable by the transformation (1.9)

$$\dddot{z} + 3b_1\ddot{z} + 3b_2\dot{z} + b_3 z + c = 0, \qquad b_1, b_2, b_3, c = \text{const}, \tag{3.2}$$

iff it can be represented in the form

$$y''' + 3f(y)y'y'' + \left(\frac{1}{3}\frac{\varphi^{**}}{\varphi} - \frac{5}{9}\frac{\varphi^{*2}}{\varphi^2} - \frac{1}{3}f\frac{\varphi^*}{\varphi} + f^2 + f^*\right)y'^3$$
$$+3b_1\varphi\left[y'' + \left(f + \frac{1}{3}\frac{\varphi^*}{\varphi}\right)y'^2\right] + 3b_2\varphi^2 y' \tag{3.3}$$
$$+\varphi^{5/3}\left(b_3 \exp\left(-\int f dy\right)\int \varphi^{4/3}\exp\left(\int f dy\right)dy + \frac{c}{\beta}\right) = 0,$$

which is reduced to (3.2) by the substitution

$$z = \beta\int \varphi^{4/3}\exp\left(\int f dy\right)dy, \qquad dt = \varphi(y)dx \tag{3.4}$$

and we have one-parameter families of solutions as $c = 0$

$$\int \frac{\varphi^{1/3}\exp\left(\int f dy\right)dy}{\int \varphi^{4/3}\exp\left(\int f dy\right)dy} = r_k x + C, \tag{3.5}$$

where r_k *are the distinct roots of the characteristic equation*

$$r^3 + 3b_1 r^2 + 3b_2 r + b_3 = 0. \tag{3.6}$$

Remark 3.1. Equations of the type

$$y''' + \varphi(y)y'y'' + \psi(y)y'' + \sum_{k=0}^{3} f_k(y)y'^k = 0 \tag{3.7}$$

can be tested by the method of the exact linearization.

Example 3.1. It is known that the *sin-Gordon* equation

$$u_{xt} = \sin u \tag{3.8}$$

has a generalized nonlocal symmetry of the third order $\left(u_{xxx} + \frac{1}{2}u_x^3\right)\partial_u$, which is connected (see, for example, [18], p. 117–119) with ODE

$$y''' + 1/2y'^3 = 0. \tag{3.9}$$

About this equation it is said: "Unfortunately, determination of general solutions for higher order ODE is very complicated problem". But the equation (3.9) can be integrated. It is a special case of (3.3), admits the factorization

$$2iy\left(y''' + \frac{1}{2}y'^3\right) \equiv \left[D - \left(i + \frac{1}{y}\right)y'\right]\left[D + \left(\frac{1}{2}i - \frac{1}{y}\right)y'\right]\left[D + \left(2i - \frac{1}{y}\right)y'\right]y = 0$$

and is linearized to $\dddot{z} = 0$ by the substitution $z = \exp(2iy)$, $dt = \exp\left(\frac{3}{2}iy\right)dx$. The general solution of (3.9) in the parametric form is $x = \int \left(c_1 + c_2 t + c_3 t^2\right)^{-3/4} dt$, $y = -1/2i\ln(c_1 + c_2 t$ $+ c_3 t^2)$.

Consider third order equation

$$y''' = F(x, y, y', y''), \tag{3.10}$$

which by the transformation of the form

$$y = v_1(x)v_2(y/v_1(x))z, \qquad dt = u_1(x)u_2(y/v_1(x))dx \tag{3.11}$$

can be reduced to the linear autonomous form (3.2).

About an integration of the generalized Emden–Fowler equation (EFE)

For example, let us consider one of possible generalizations of the EFE, i.e.

$$y''' + bx^s y^n = 0, \qquad n \neq 0, \quad n \neq 1. \tag{3.12}$$

We use a test of the autonomization. Equation (3.12) by means of the transformation $y = x^{\frac{s+3}{1-n}} z$, $dt = x^{-1}dx$ is reduced to an autonomous form

$$\dddot{z} + (3k-3)\ddot{z} + [k(k-1) + k(k-2) + (k-1)(k-2)]\dot{z} + k(k-1)(k-2)z + bz^n = 0$$

and has the exact solutions

$$y = \rho x^k, \qquad k(k-1)(k-2)\rho + b\rho^n = 0, \qquad k = \frac{s+3}{1-n}.$$

So, the equation (3.12) is reduced to the autonomous form

$$Y'''(\tau) + bY^n = 0 \tag{3.13}$$

by the transformations $y = x^2 Y$, $d\tau = x^{-2}dx$. For thus obtained equation we apply the test of the linearization, i.e. we use the proposition 3.1. Equation (3.13) can be related to the class of (3.3) iff it can be represented in the form

$$Y''' + \varphi^{5/3}\left(b_3 \int \varphi^{4/3} dY + \frac{c}{\beta}\right) = 0, \tag{3.14}$$

where $b_1 = b_2 = 0$ in (3.3) and φ satisfies the equation

$$\frac{1}{3}\frac{\varphi^{**}}{\varphi} - \frac{5}{9}\frac{\varphi^{*2}}{\varphi^2} = 0. \tag{3.15}$$

The solution of equation (3.15) is a function $\varphi = Y^{-3/2}$. Then equation (3.16) is in the form

$$Y''' - b_3 Y^{-7/2} + \frac{c}{\beta} Y^{-5/2} = 0, \qquad \beta = -1. \tag{3.16}$$

Two cases are possible: $b_3 = 0$, $c \neq 0$ and $b_3 \neq 0$, $c = 0$. Let us consider the first one:

$$Y''' - cY^{-5/2} = 0, \qquad (b_3 = 0). \tag{3.17}$$

At $n = -5/2$ we have $-4 = s - 5$, then $s = 1$. The input equation is:

$$y''' + bxy^{-5/2} = 0, \qquad (c = -b). \tag{3.18}$$

Let $c = 0$. Then equation (3.16) takes the form

$$Y''' + bY^{-7/2} = 0, \qquad (b_3 = -b). \tag{3.19}$$

For $n = -7/2$ we obtain $s = -4 - 2(-7/2) = 3$. Then the equation (3.12) gets the form:

$$y''' + bx^3 y^{-7/2} = 0. \tag{3.20}$$

Let us apply to equations (3.18) and (3.20) the following substitutions

$$y = x^2 Y, \qquad d\tau = x^{-2}dx, \qquad Y = Y^2 z, \qquad dt = Y^{-3/2} d\tau$$

or the resulting substitutions in the transformed form

$$y = x^2 Y^2 z, \quad dt = x^{-2} Y^{-3/2} dx, \quad \left(Y = yx^{-2}\right), \quad \text{i.e.} \quad y = x^{-2} y^2 z, \quad dt = xy^{-3/2} dx,$$

we obtain respectively $\dddot{z} - b = 0$, $\dddot{z} - bz = 0$. The factorizations of equations (3.18) and (3.20) have respectively the forms:

$$-\left(D + \frac{y'}{y}\right)\left(D + \frac{1}{x} - \frac{1}{2}\frac{y'}{y}\right)\left(D + \frac{2}{x} - \frac{2y'}{y}\right)y + bxy^{-5/2} = 0,$$

$$\left(D + \frac{y'}{y} - r_3xy^{-3/2}\right)\left(D + \frac{1}{x} - \frac{1}{2}\frac{y'}{y} - r_2xy^{-3/2}\right)\left(D + \frac{2}{x} - \frac{2y'}{y} - r_1xy^{-3/2}\right)y = 0,$$

where $r_k, k = \overline{1,3}$, satisfies to a characteristic equation $r^3 - b = 0$.

Now let us consider the linear equation $y''' + bx^s y = 0$, $s \neq 0$, i.e. equation (3.12) for $n = 1$. Then we get two values for s: $s = -3$, $s = -6$. At $s = -3$ we have Euler's equation $y''' + bx^{-3}y = 0$, and we have the Halphen's equation $y''' + bx^{-6}y = 0$ for $s = -6$.

Let us note that the asymptotic solutions of the equation (3.14) were considered in ([19], p. 261–265).

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