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Towards a Classification of Realizations of the Euclid Algebra $e(3)$

V. LAHNO † *and R.Z. ZHDANOV* ‡

† *Pedagogical Institute, 2 Ostrogradskogo Street, 314000 Poltava, Ukraine*
E-mail: laggo@poltava.bank.gov.ua

‡ *Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivska Street, 01004 Kyiv, Ukraine*
E-mail: renat@imath.kiev.ua

We classify realizations of the Lie algebras of the rotation $O(3)$ and Euclid $E(3)$ groups within the class of first-order differential operators in arbitrary finite dimensions. It is established that there are only two distinct realizations of the Lie algebra of the group $O(3)$ which are inequivalent within the action of a diffeomorphism group. Using this result we describe a special subclass of realizations of the Euclid algebra which are called covariant.

1. In the present paper we study realizations of the Lie algebra of the Euclid group $E(3)$ (which will be called in the sequel the Euclid algebra $e(3)$) within the class of Lie vector fields on the space $V = X \otimes U$ of independent and dependent variables. In the case under study X is the three-dimensional Euclid space having the coordinates $x = (x_1, x_2, x_3)$; U is the space of real-valued scalar functions $u(x) = (u_1(x), u_2(x), \ldots, u_n(x))$, and Lie vector fields are first-order differential operators of the form

$$Q = \xi_a(x,u)\partial_{x_a} + \eta_i(x,u)\partial_{u_i}, \tag{1}$$

where ξ_a, η_i ($a = 1,2,3$; $i = 1,\ldots,n$) are some sufficiently smooth real-valued functions defined on the space V, $\partial_{x_a} = \frac{\partial}{\partial x_a}$, $\partial_{u_i} = \frac{\partial}{\partial u_i}$. Hereafter, we use the summation convention for the repeated indices.

We say that the operators P_a, J_b ($a,b = 1,2,3$) belonging to class (1) form a basis of the realization of the Euclid algebra $e(3)$ if (a) they are linearly independent, and (b) they satisfy the following commutation relations:

$$[P_a, P_b] = 0, \tag{2}$$

$$[J_a, P_b] = \varepsilon_{abc} P_c, \tag{3}$$

$$[J_a, J_b] = \varepsilon_{abc} J_c, \tag{4}$$

where

$$\varepsilon_{abc} = \begin{cases} 1, & (abc) = \text{cycle}\,(123), \\ -1, & (abc) = \text{cycle}\,(213), \\ 0, & \text{in the remaining cases.} \end{cases}$$

The realization of the Euclid algebra $e(3)$ within the class of Lie vector fields (1) is called covariant if coefficients of the basis elements

$$P_a = \xi^{(1)}_{ab}(x,u)\partial_{x_b} + \eta^{(1)}_{ai}(x,u)\partial_{u_i} \qquad (a,b = 1,2,3;\ i = 1,\ldots,n) \tag{5}$$

satisfy the following condition:

$$\operatorname{rank}\left\|\begin{matrix} \xi_{11}^{(1)} & \xi_{12}^{(1)} & \xi_{13}^{(1)} & \eta_{11}^{(1)} & \dots & \eta_{1n}^{(1)} \\ \xi_{21}^{(1)} & \xi_{22}^{(1)} & \xi_{23}^{(1)} & \eta_{21}^{(1)} & \dots & \eta_{2n}^{(1)} \\ \xi_{31}^{(1)} & \xi_{32}^{(1)} & \xi_{33}^{(1)} & \eta_{31}^{(1)} & \dots & \eta_{3n}^{(1)} \end{matrix}\right\| = 3. \tag{6}$$

It is easy to check that the relations (2)–(4) are invariant with respect to an arbitrary invertible transformation of variables x, u

$$y_a = f_a(x,u), \quad a = 1,2,3; \qquad v_i = g_i(x,u) \quad i = 1,\dots,n, \tag{7}$$

where f_a, g_i are sufficiently smooth functions defined on the space V. That is why we can introduce on the set of realizations of the Euclid algebra $e(3)$ the following relation: two realizations of the algebra $e(3)$ are called equivalent if they are transformed one into another by means of an invertible transformation (7). As invertible transformations of the form (7) form a group (called diffeomorphism group), this relation is the equivalence relation. It divides the set of all realizations of the Euclid algebra into equivalence classes $A_1,\dots,A_r$. Consequently, to describe all possible realizations of $e(3)$ it suffices to construct one representative of each equivalence class A_j, $j = 1,\dots,r$.

2. As it follows from commutation relations (2)–(4) of the algebra $e(3)$, the latter is the semi-direct sum of the commutative ideal $t^3 = \langle P_1, P_2, P_3\rangle$ and of the simple algebra $so(3) = \langle J_1, J_2, J_3\rangle$. That is why we start investigation of covariant realizations of the algebra $e(3)$ by studying realizations of the translation generators P_a $(a = 1,2,3)$ within the class of operators (1). To this end we will make use of the following lemma.

Lemma 1. *Let the operators P_a $(a = 1,2,3)$ of the form (5) satisfy relation (6). Then there exists a transformation of the form (7) reducing the operators P_a to become $P'_a = \partial_{y_a}$, $a = 1,2,3$.*

Proof. In view of (6) $P_a \neq 0$ for all $a = 1,2,3$. It is well-known [1] that a non-zero operator

$$P_1 = \xi_{1b}^{(1)}(x,u)\partial_{x_b} + \eta_{1i}^{(1)}(x,u)\partial_{u_i}$$

can be always reduced to the form $P'_1 = \partial_{y_1}$ by transformation (7). If we denote by P'_2, P'_3 the operators P_2, P_3 written in the new variables y, v, then owing to commutation relations (2) they commute with the operator $P'_1 = \partial_{y_1}$. Hence, we conclude that their coefficients are independent of y_1.

Furthermore, due to the condition (6) at least one of the coefficients $\xi_{22}^{\prime(1)}$, $\xi_{23}^{\prime(1)}$, $\eta_{21}^{\prime(1)}$, …, $\eta_{2n}^{\prime(1)}$ of the operator P'_2 is not equal to zero.

Summing up, we conclude that the operator P'_2 is of the form

$$P'_2 = \xi_{2b}^{\prime(1)}(y_2,y_3,v)\partial_{y_b} + \eta_{2i}^{\prime(1)}(y_2,y_3,v)\partial_{v_i},$$

not all the functions $\xi_{22}^{\prime(1)}$, $\xi_{23}^{\prime(1)}$, $\eta_{21}^{\prime(1)},\dots,\eta_{2n}^{\prime(1)}$ being identically equal to zero.

Making a transformation

$$\begin{aligned} &z_1 = y_1 + F(y_2,y_3,v), \qquad z_2 = G(y_2,y_3,v), \\ &z_3 = \omega_0(y_2,y_3,v), \qquad \omega_i = \omega_i(y_2,y_3,v), \quad i = 1,\dots,n, \end{aligned} \tag{8}$$

where the functions F, G are particular solutions of differential equations

$$\xi_{22}^{\prime(1)}(y_2,y_3,v)F_{y_2} + \xi_{23}^{\prime(1)}(y_2,y_3,v)F_{y_2} + \eta_{2i}^{\prime(1)}(y_2,y_3,v)F_{u_i} + \xi_{21}^{\prime(1)}(y_2,y_3,v) = 0,$$

$$\xi_{22}^{\prime(1)}(y_2,y_3,v)G_{y_2} + \xi_{23}^{\prime(1)}(y_2,y_3,v)G_{y_3} + \eta_{2i}^{\prime(1)}(y_2,y_3,v)G_{u_i} = 1$$

and $\omega_0, \omega_1, \ldots, \omega_n$ are functionally-independent first integrals of the Euler–Lagrange system

$$\frac{dy_2}{\xi_{22}'^{(1)}} = \frac{dy_3}{\xi_{23}'^{(1)}} = \frac{dv_1}{\eta_{21}'^{(1)}} = \cdots = \frac{dv_n}{\eta_{2n}'^{(1)}},$$

which has exactly $n+1$ functionally-independent integrals, we reduce the operator P_2' to the form $P_2'' = \partial_{z_2}$. It is easy to check that transformation (8) does not alter the form of the operator P_1'. Being rewritten in the new variables z, ω it reads as $P_1'' = \partial_{z_1}$.

As the right-hand sides of (8) are functionally-independent by construction, transformation (8) is invertible. Consequently, operators P_a are equivalent to operators P_a'', where $P_1'' = \partial_{z_1}$, $P_2'' = \partial_{z_2}$ and

$$P_3'' = \xi_{3b}''^{(1)}(z_3, \omega)\partial_{z_b} + \eta_{3i}''^{(1)}(z, \omega)\partial_{\omega_i} \neq 0$$

(coefficients of the above operator are independent of z_1, z_2 because of the fact that it commutes with the operators P_1'', P_2''). And what is more, due to (6) at least one of the coefficients $\xi_{33}''^{(1)}, \eta_{31}''^{(1)}, \ldots, \eta_{3n}''^{(1)}$ of the operator P_3'' is not identically equal to zero.

It is not difficult to verify that there exists the invertible transformation

$$Z_1 = z_1 + F(z_3, \omega), \qquad Z_2 = z_2 + G(z_3, \omega),$$
$$Z_3 = H(z_3, \omega), \qquad W_i = \Omega_i(z_3, \omega), \quad i = 1, \ldots, n,$$

which reduces the operators P_a'', $a = 1, 2, 3$ to the form $P_a''' = \partial_{z_a}$, $a = 1, 2, 3$.

Lemma is proved.

Due to Lemma 1 the operators P_a can be reduced to the form $P_a = \partial_{x_a}$ by means of a properly chosen transformation (7). Inserting the operators

$$P_a = \partial_{x_a}, \qquad J_a = \xi_{ab}(x, u)\partial_{x_b} + \eta_{ai}(x, u)\partial_{u_i}, \qquad a, b = 1, 2, 3; \quad i = 1, \ldots, n,$$

into commutation relations (3) and equating the coefficients of the linearly-independent operators ∂_{x_a}, ∂_{u_i} ($a = 1, 2, 3$; $i = 1, \ldots, n$) we arrive at the system of partial differential equations for the functions $\xi_{ab}(x, u)$, $\eta_{ai}(x, u)$

$$\xi_{acx_b} = -\varepsilon_{abc}, \qquad \eta_{aix_b} = 0, \qquad a, b, c = 1, 2, 3, \quad i = 1, \ldots, n.$$

Integrating the above system we conclude that the operators J_a have the form

$$J_a = -\varepsilon_{abc} x_b \partial_{x_c} + j_{ab}(u)\partial_{x_b} + \tilde{\eta}_{ai}(u)\partial_{u_i}, \qquad a, b = 1, 2, 3, \quad i = 1, \ldots, n, \tag{9}$$

where j_{ab}, $\tilde{\eta}_{ab}$ are arbitrary smooth functions.

Inserting (9) into the commutation relations (4) and equating the coefficients of ∂_{u_i} ($i = 1, \ldots, n$) show that the operators $\mathcal{J}_a = \tilde{\eta}_{ai}\partial_{u_i}$, ($a = 1, 2, 3$) have to fulfill (4) with $J_a \to \mathcal{J}_a$.

Lemma 2. *Let first-order differential operators*

$$\mathcal{J}_a = \eta_{ai}(u)\partial_{u_i}, \qquad a = 1, 2, 3, \quad i = 1, \ldots, n, \tag{10}$$

satisfy commutation relations (4) of the Lie algebra so(3). Then either all of them are equal to zero, i.e.

$$\mathcal{J}_a = 0, \qquad a = 1, 2, 3, \tag{11}$$

or there exists a transformation

$$v_i = F_i(u), \qquad i = 1, \ldots, n,$$

reducing these operators to one of the following forms:

$$\begin{aligned} 1.\quad & \mathcal{J}_1 = -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2}, \\ & \mathcal{J}_2 = -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2}, \\ & \mathcal{J}_3 = \partial_{u_1}; \end{aligned} \tag{12}$$

$$\begin{aligned} 2.\quad & \mathcal{J}_1 = -\sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2} + \sin u_1 \sec u_2 \partial_{u_3}, \\ & \mathcal{J}_2 = -\cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2} + \cos u_1 \sec u_2 \partial_{u_3}, \\ & \mathcal{J}_3 = \partial_{u_1}. \end{aligned} \tag{13}$$

The proof of Lemma 2 requires long cumbersome calculations which are omitted here.

Notice that the set of inequivalent realizations of the Lie algebra $so(3)$ within the class of first-order differential operators (10) is exhausted by the realizations given in (12), (13).

Hence, taking into account Lemma 2 we conclude that any covariant realization of the algebra $e(3)$ is equivalent to the following one:

$$P_a = \partial_{x_a}, \qquad J_a = -\varepsilon_{abc} x_b \partial_{x_c} + j_{ab}(u)\partial_{x_b} + \mathcal{J}_a, \qquad a,b,c = 1,2,3, \tag{14}$$

operators $\mathcal{J}_a$ being given by one of formulae (11)–(13).

Making a transformation

$$y_a = x_a + F_a(u), \qquad v_i = u_i, \qquad a = 1,2,3, \quad i = 1,\ldots,n,$$

we reduce operators J_a from (14) to become

$$\begin{aligned} & J_1 = -y_2\partial_{y_3} + y_3\partial_{y_2} + A\partial_{y_1} + B\partial_{y_2} + C\partial_{y_3} + \mathcal{J}_1, \\ & J_2 = -y_3\partial_{y_1} + y_1\partial_{y_3} + F\partial_{y_2} + G\partial_{y_3} + \mathcal{J}_2, \\ & J_3 = -y_1\partial_{y_2} + y_2\partial_{y_1} + H\partial_{y_3} + \mathcal{J}_3, \end{aligned} \tag{15}$$

where A, B, C, F, G, H are arbitrary smooth functions of $v_1,\ldots,v_n$.

Substituting the operators (15) into (4) and equating the coefficients of linearly-independent operators ∂_{y_1}, ∂_{y_2}, ∂_{y_3}, ∂_{v_i} $(i = 1,\ldots,n)$ result in the following system of partial differential equations:

$$\begin{aligned} & \mathcal{J}_2 A = -C, \qquad \mathcal{J}_3 C - \mathcal{J}_1 H = G, \qquad \mathcal{J}_3 F = -B, \\ & \mathcal{J}_1 G - \mathcal{J}_2 C = H - A - F, \qquad \mathcal{J}_3 A = B, \qquad \mathcal{J}_3 B = F - A - H, \\ & \mathcal{J}_1 F - \mathcal{J}_2 B = G, \qquad A - F - H = 0, \qquad \mathcal{J}_2 H - \mathcal{J}_3 G = C. \end{aligned} \tag{16}$$

Analyzing system (16) we arrive at the following assertion.

Theorem 1. *Any covariant realizations of the algebra $e(3)$ within the class of first-order differential operators is equivalent to one of the following realizations:*

$$\begin{aligned} 1.\quad & P_a = \partial_{x_a}, \qquad J_a = -\varepsilon_{abc} x_b \partial_{x_c}, \qquad a,b,c = 1,2,3; \\ 2.\quad & P_a = \partial_{x_a}, \qquad a = 1,2,3, \\ & J_1 = -x_2\partial_{x_3} + x_3\partial_{x_2} + f\partial_{x_1} - f_{u_2}\sin u_1 \partial_{x_2} - \sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2}, \\ & J_2 = -x_3\partial_{x_1} + x_1\partial_{x_3} + f\partial_{x_2} - f_{u_2}\cos u_2 \partial_{x_3} - \cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2}, \\ & J_3 = -x_1\partial_{x_2} + x_2\partial_{x_1} + \partial_{u_1}; \end{aligned}$$

3. $$\begin{aligned} & P_a = \partial_{x_a}, \qquad a = 1,2,3, \\ & J_1 = -x_2\partial_{x_3} + x_3\partial_{x_2} + g\partial_{x_1} - (\sin u_1 g_{u_2} + \cos u_1 \sec u_2 g_{u_3})\partial_{x_3} \\ & \qquad - \sin u_1 \tan u_2 \partial_{u_1} - \cos u_1 \partial_{u_2} + \sin u_1 \sec u_2 \partial_{u_3}, \\ & J_2 = -x_3\partial_{x_1} + x_1\partial_{x_3} + g\partial_{x_2} - (\cos u_1 g_{u_2} - \sin u_1 \sec u_2 g_{u_3})\partial_{x_3} \\ & \qquad - \cos u_1 \tan u_2 \partial_{u_1} + \sin u_1 \partial_{u_2} + \cos u_1 \sec u_2 \partial_{u_3}, \\ & J_3 = -x_1\partial_{x_2} + x_2\partial_{x_1} + \partial_{u_1}. \end{aligned}$$

Here $f = f(u_2, \ldots, u_n)$ is given by the formula

$$f = \alpha \sin u_2 + \beta \left(\sin u_2 \ln \frac{\sin u_2 + 1}{\cos u_2} - 1 \right),$$

α, β are arbitrary smooth functions of $u_3, \ldots, u_n$ and $g = g(u_2, \ldots, u_n)$ is a solution of the following linear partial differential equation:

$$\cos^2 u_2 g_{u_2u_2} + g_{u_3u_3} - \sin u_2 \cos u_2 g_{u_2} + 2\cos^2 u_2 g = 0.$$

3. Summarizing the results obtained in the previous section yields the following structure of realizations of the Lie algebra $so(3)$ by Lie vector fields in n variables.

1. If $n = 1$, then there are no non-zero realizations.
2. As there is no realization of $so(3)$ by real non-zero 2×2 matrices, the only non-zero realizations is given by (12).
3. In the case $n = 3$ there are two more inequivalent realizations (12) and (13).
4. Provided $n > 3$, there is no new realizations of $so(3)$ and, furthermore. any realization can be reduced to a linear one.

Notice that a complete description of covariant realizations of the conformal algebra $c(n, m)$ in the space of $n+m$ independent and one dependent variables was obtained in [2, 3]. Some new realizations of the Galilei algebra $g(1, 3)$ were suggested in [4]. Yehorchenko [5], and Fushchych, Tsyfra and Boyko [6] have constructed new (nonlinear) realizations of the Poincaré algebras $p(1, 2)$ and $p(1, 3)$ correspondingly. Complete description of realizations of the Galilei algebra $g_2(1, 1)$ in the space of two dependent and two independent variables was obtained in [7, 8].

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