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The Ovsjannikov's Theorem on Group Classification of a Linear Hyperbolic Type Partial Differential Equation Revisited

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The group classification of linear hyperbolic partial differential equation is carried out with the use of the new approach to solving group classification problems suggested recently by Zhdanov and Lahno (**J. Phys. A: Math. Gen., V.32, 7405 (1999)**).

1. Consider a partial differential equation of the hyperbolic type

$$u_{tx} + A(t,x)u_t + B(t,x)u_x + C(t,x)u = 0, \tag{1}$$

where $u = u(t,x)$, $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$, $u_{tx} = \frac{\partial^2 u}{\partial t \partial x}$. Group classification of equations (1) admitting non-trivial (finite-parameter) symmetry group has been performed by L.V. Ovsjannikov [1, 2]. His classification scheme is based on using the Laplace invariants

$$h = A_t + AB - c, \qquad k = B_x + AB - C.$$

The results obtained can be formulated as follows.

Theorem 1 (Ovsjannikov [1, 2]). *Equation (1) admits a Lie algebra of the dimension higher than 1 if and only if the functions*

$$p = \frac{k}{h}, \qquad q = \frac{\partial_x \partial_y (\ln h)}{h}$$

are constant. If p and q are constant, then equation (1) is equivalent either to the Euler–Poisson equation $(q \neq 0)$

$$u_{tx} - \frac{2u_t}{q(t+x)} - \frac{2pu_x}{q(t+x)} + \frac{4pu}{q^2(t+x)^2} = 0 \tag{2}$$

or to equation $(q = 0)$

$$u_{tx} + tu_t + pxu_x + ptxu = 0 \tag{3}$$

and its symmetry algebra is a three-dimensional Lie algebra L^3.

What is more, Ovsjannikov has proved that the basis of the Lie algebra L^3 is formed by the operators

$$\partial_t - \partial_x, \qquad t\partial_t + x\partial_x, \qquad t^2\partial_t - x^2\partial_x + \frac{2}{q}(pt - x)u\partial_u$$

for equation (2), and by the operators

$$t\partial_t - x\partial_x, \qquad \partial_t - xu\partial_u, \qquad \partial_x - ptu\partial_u$$

for equation (3).

In this paper we perform group classification of equation (1) by using an alternative approach suggested in [3].

2. Using the infinitesimal Lie method we obtain that equation (1) is invariant under infinite-dimensional transformation group, which is generated by the operator

$$X_\infty = \omega(t,x)\partial_u, \qquad \omega_{tx} + A\omega_t + B\omega_x + C\omega = 0,$$

and under the one-parameter transformation group, whose infinitesimal operators reads as

$$X = f(t)\partial_t + g(x)\partial_x + h(t,x)u\partial_u, \tag{4}$$

where

$$\begin{aligned} &h_t + B\dot{f} + fB_t + gB_x = 0,\\ &h_x + Ag' + gA_x + fA_t = 0,\\ &h_{tx} + C\dot{f} + fC_t + Cg' + gC_x + Ah_t + Bh_x = 0. \end{aligned} \tag{5}$$

In (5) the following notations are used, $\dot{f} = \frac{df}{dt}$, $g' = \frac{dg}{dx}$.

Furthermore, as the direct calculations show, the equivalence group of the equation (1) is a superposition of the following transformations:

$$\begin{aligned} &(a) \quad \tau = \alpha(t), \quad \xi = \beta(x), \quad v = \theta(t,x)u + \rho(t,x),\\ &(b) \quad \tau = \alpha(x), \quad \xi = \beta(t), \quad v = \theta(t,x)u + \rho(t,x), \end{aligned} \tag{6}$$

where α and β are arbitrary smooth functions, $\theta \neq 0$ and θ, ρ satisfy the condition

$$\theta_t\rho_x + \rho_t\theta_x - \theta\rho_{tx} + \rho\theta_{tx} - 2\rho\theta^{-1}\theta_t\theta_x - C\theta\rho = 0.$$

In order to perform group classification of equation (1), we start with studying realizations of real Lie algebras within the class of operators (4) up to the equivalence relation determined by transformations (6). As a next step, we select those realizations, that form bases of invariance algebras of equations (1).

Remark 1. We use the known classification of non-isomorphic real Lie algebras (see, for example, [4, 5]).

Remark 2. Equation

$$u_{tx} = 0 \tag{7}$$

is invariant under infinite-dimensional transformation group, which is generated by the operator

$$X_\infty = f(t)\partial_t + g(x)\partial_x + \lambda u\partial_u,$$

where f and g are arbitrary smooth functions and $\lambda = \mathrm{const}$. What is more, its general solution reads as

$$u = \varphi(t) + \psi(x)$$

with arbitrary smooth functions φ, ψ. Furthermore, the equation

$$u_{tx} + B(x)u_x = 0, \qquad B \neq 0, \tag{8}$$

has the following general solution:

$$u = \int \varphi(x)e^{-tB(x)}dx + \psi(t),$$

where φ, ψ are arbitrary smooth functions.

Therefore, we consider only equations of the form (1), which are inequivalent to (7) and (8).

It is well-known, that a linear partial differential equation of the form (1) is invariant under the operator $u\partial_u$ and this operator satisfies the following commutation relation:

$$[X, u\partial_u] = 0,$$

where X has the form (4).

Consequently, the function $h(t, x)$ in operator (3) is determined up to a constant summand.

The list of non-isomorphic two-dimensional real Lie algebras is exhausted by the following two algebras:

$$A_{2.1} = \langle e_1, e_2\rangle, \qquad [e_2, e_2] = 0;$$
$$A_{2.2} = \langle e_1, e_2\rangle, \qquad [e_2, e_2] = e_2.$$

If these algebras are maximal invariance algebras of equation (1), then one of their basis operators must coincide with the operator $u\partial_u$. Consequently, we have to consider realizations of the algebra $A_{2.1}$ only.

Proposition 1. *Let the algebra $A_{2.1}$ be invariance algebra of equation (1). The set of inequivalent realizations of this algebra is exhausted by the following two realizations:*

$$A_{2.1} = \langle u\partial_u, \partial_t\rangle;$$
$$A_{2.2} = \langle u\partial_u, \partial_t + \partial_x\rangle.$$

The corresponding invariant equations can be taken in the following form:

$$A^1_{2.1}: \ u_{tx} + B(x)u_x + u = 0; \tag{9}$$

$$A^2_{2.2}: \ u_{tx} + B(z)u_x + C(z)u = 0, \qquad z = t - x, \quad C \neq 0. \tag{10}$$

Proof. First of all we note that the operator $u\partial_u$ is invariant under action of the changes of variables (6). Choose $e_1 = u\partial_u$ as the first basis operator of the Let in the algebra $A_{2.1}$ and let the second basis operator e_2 have the general form (4).

If $f \cdot g \neq 0$ in the operator e_2, then making the change of variables (6), where α, β, θ, ρ are solutions of the system of differental equations

$$\dot\alpha f = 1, \qquad \beta' g = 1, \qquad f\theta_t + g\theta_x + h\theta = 0, \qquad \theta \neq 0, \qquad f\rho_t + g\rho_x = 0,$$

reduces this operator to the operator

$$e_2' = \partial_\tau + \partial_\xi.$$

If $f \neq 0$, $g = 0$ in the operator e_2, then performing the change of variables (6), where $\beta = \beta(x)$, $\rho = \rho(x)$ and functions α, θ are solutions of system of differential equations

$$\dot\alpha f = 1, \qquad f\theta_t + h\theta = 0, \qquad \theta \neq 0,$$

reduces this operator to become

$$e_2' = \partial_\tau.$$

If $f = 0$, $g \neq 0$ in the operator e_2, then making another change of variables (6) ($t \to x$, $x \to t$) reduces this case to the previous one.

If, finally, $f = g = 0$ in the operator e_2, then $h \neq 0$ and

$$e_2 = h(t, x)u\partial_u, \qquad h \neq \text{const}.$$

Thus, we obtain three inequivalent realizations of the algebra $A_{2.1}$ within the class of operators (3):

$$A^1_{2.1} = \langle u\partial_u, \partial_t\rangle,$$
$$A^2_{2.1} = \langle u\partial_u, \partial_t + \partial_x\rangle,$$
$$A^3_{2.1} = \langle u\partial_u, h(t, x)u\partial_u\rangle, \qquad h \neq \text{const}.$$

The direct verification of conditions (5) for the obtained realizations yields the following results:

- Invariant equations for the first and second realizations have the form:

$$A^1_{2.1}: \ u_{tx} + A(x)u_t + B(x)u_x + C(x)u = 0, \qquad C \neq 0; \tag{11}$$

$$A^2_{2.1}: \ u_{tx} + A(z)u_t + B(z)u_x + C(z)u = 0, \qquad z = t - x. \tag{12}$$

- If the realization $A^3_{2.1}$ is invariance algebra of an equation of the form (1), then $h = \text{const}$.

Furthermore, it is not difficult to verify that the realization $A^1_{2.1}$ is invariant with respect to the change of variables

$$\tau = t + \lambda, \qquad \lambda = \text{const}, \qquad \xi = \beta(x), \qquad v = \theta(x)u + \rho(x), \qquad \theta \neq 0. \tag{13}$$

If in (13) β, θ and ρ are solutions of the system of differential equations

$$\theta_x = \theta A, \qquad \beta' = C - \theta^{-1}\partial_x B, \qquad B\theta^{-1}\theta_x\rho - B\rho_x - C\rho = 0,$$

then this change of variables reduces equation (11) to equation of the form (9).

Using analogous reasonings, it is not difficult to show that equation (12) is equivalent to (10).

Proposition 1 is proved.

Thus obtained classification of equations (1), which are invariant under two-dimensional Lie algebras, permits realizing further group classification of equation (1) by the method suggested in [3].

The system of determining equations (5) for equation (9) reads as

$$h_t + B\dot{f} + gB_x = 0, \qquad h_x = 0, \qquad \dot{f} + g' = 0. \tag{14}$$

The second and third equations from (14) imply that $h = h(t)$, $f = \lambda_1 t + \lambda_2$, $g = -\lambda_1 x + \lambda_3$, where $\lambda_1, \lambda_2, \lambda_3 = \text{const}$.

Consequently, extension of the symmetry of equation (9) is only possible, if the function B in first equation (14) within the equivalence relation has the form

$$B = mx, \qquad m = \text{const}, \qquad m \neq 0,$$

which means that equation (9) reads as

$$u_{tx} + mxu_x + u = 0, \qquad m = \text{const}. \tag{15}$$

Its invariance algebra is the four-dimensional Lie algebra

$$\langle u\partial_u, \partial_t, t\partial_t - x\partial_x, \partial_x - mtu\partial_u\rangle.$$

Analogously, we verify that extension of symmetry of equation (10) is only possible, if it has the form

$$u_{tx} + \frac{m}{z}u_x + \frac{k}{z^2}u = 0, \qquad m, k = \mathrm{const}, \quad k \neq 0, \quad z = t - x. \tag{16}$$

The invariance algebra of equation (16) is the four-dimensional Lie algebra

$$\langle u\partial_u, \partial_t + \partial_x, t\partial_t + x\partial_x + \frac{1}{2}mu\partial_u, t^2\partial_t + x^2\partial_x + mtu\partial_u\rangle.$$

Cosequently, the following assertion holds true:

Proposition 2. *Equation (1) admits a Lie algebra of infinitesimal operators (4), whose dimension is higher than two, if it is either equivalent to equation (15) or to (16), its invariance algebra being necessarily four-dimensional.*

It is straightforward to verify that the results obtained in Proposition 2 are equivalent to results obtained by Ovsjannikov.

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