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Construction of Invariants for a System of Differential Equations in the $(n + 2m)$-Dimensional Space

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In this paper an algorithm of construction of infinitesimal operator and invariants for $(n + 2m)$-dimensional space is represented.

Let us consider the system of differential equations of the following form:

$$\begin{aligned}
&\frac{dx}{dt} = -\lambda y + X(x, y, z_1, \ldots, z_n),\\
&\frac{dy}{dt} = \lambda x + Y(x, y, z_1, \ldots, z_n),\\
&\frac{dz_j}{dt} = \sum_{i=1}^{n} r_{ji} z_i + Z_j(x, y, z_1, \ldots, z_n), \qquad j = 1, \ldots, n,
\end{aligned} \tag{1}$$

where $x, y \in \mathbb{R}^m$. Hence, the system of differential equations (1) is in $(n + 2m)$-dimensional space.

V.I. Zubov [1] suggested the following theorem:

Theorem. *A necessary and sufficient condition for the family (1) to have a family of limited solutions in a neighborhood of point $(0, \ldots, 0, z_1^o, \ldots, z_n^o)$, is the existence of m holomorphic integrals of (1) the form:*

$$c_s^2 = x_s^2 + y_s^2 + \Psi_s(x, y, z_1, \ldots, z_n).$$

By substitution $y_s = \rho_s \cos\varphi_s$, $x_s = \rho_s \sin\varphi_s$, $s = 1, \ldots, m$ we transform (1) to the form:

$$\begin{aligned}
&\frac{d\rho_s}{dt} = R_s,\\
&\frac{d\varphi_s}{dt} = \lambda_s + \Theta_s, \qquad s = 1, \ldots, m,\\
&\frac{dz_j}{dt} = \sum_{i=1}^{n} r_{ji} z_i + P_j(\rho_1 \ldots, \rho_m, \varphi_1, \ldots, \varphi_m, z_1, \ldots, z_n), \qquad j = 1, \ldots, n,
\end{aligned} \tag{2}$$

where

$$R_s = \cos\varphi_s X_s + \sin\varphi_s Y_s, \qquad \Theta_s = \frac{\cos\varphi_s Y_s - \sin\varphi_s X_s}{\rho_s},$$

$$\begin{aligned}
&P_j(\rho_1 \ldots, \rho_m, \varphi_1, \ldots, \varphi_m, z_1, \ldots, z_n)\\
&\qquad = Z_j(\rho_1 \cos\varphi_1, \ldots, \rho_m \cos\varphi_m, \rho_1 \sin\varphi_1, \ldots, \rho_m \sin\varphi_m, z_1, \ldots, z_n).
\end{aligned}$$

We will seek a solution in the following form of rows:

$$\rho_s = c_s + \sum_{k=2}^{\infty} r_s^{(k)}(\varphi_1, \dots, \varphi_m, c_1, \dots, c_m),$$

$$z_j = \sum_{k=2}^{\infty} z_s^{(k)}(\varphi_1, \dots, \varphi_m, c_1, \dots, c_m).$$

By making an appropriate substitution into (2) and stating the coefficients at equal degrees as equal, we receive functions $r_s^{(k)}$, $z_j^{(k)}$. If all such functions are periodical with respect to $\varphi_1, \dots, \varphi_m$ and at sufficiently small $\|c_s\|$, we obtain the following family of solutions:

$$\rho_s = c_s + F_s(z_1, \dots, z_n, \varphi_1, \dots, \varphi_m, c_1, \dots, c_m), \qquad s = 1, \dots, m.$$

However, in order to find ρ_s, an infinite number of differential equation needs to be solved.

To solve this problem, we will use invariants with respect to transformations of $SO(2)$. If we obtain the whole system of invariants, their number will define the number of equations, which have to be solved in order to find a solution to the original problem. In a particular case, when (1) is in the form:

$$\begin{aligned}
\frac{dx_s}{dt} &= -\lambda_s y_s + X_s(x_s, y_s, z_1, \dots, z_n), \\
\frac{dy_s}{dt} &= \lambda_s x_s + Y_s(x_s, y_s, z_1, \dots, z_n), \qquad s = 1, \dots, m, \\
\frac{dz_j}{dt} &= \sum_{i=1}^{n} r_{ji} z_i + Z_j(x_1, \dots, x_m, y_1, \dots, y_m, z_1, \dots, z_n), \qquad j = 1, \dots, n
\end{aligned}$$

the quantity of invariants for each pair of imaginary numbers is obtained in [2, 3].

Let us build an infinitesimal operator for finding invariants of (1).

To do this, we will consider one pair of imaginary numbers and corresponding equations:

$$\begin{aligned}
\frac{dx_s}{dt} &= -\lambda_s y_s + X_s(x_1, \dots, x_m, y_1, \dots, y_m, z_1, \dots, z_n), \\
\frac{dy_s}{dt} &= \lambda_s x_s + Y_s(x_1, \dots, x_m, y_1, \dots, y_m, z_1, \dots, z_n), \qquad s = 1, \dots, m.
\end{aligned} \tag{3}$$

If X_s and Y_s are viewed as polynomials of the variables $z_1, \dots, z_n$, let us consider X_s and Y_s at random monomial $z_1, \dots, z_n$. Suppose that

$$X_s = \sum_{i_1+i_2+\dots+i_{2m}=\{l\}} c_{i_1\dots i_{2m}} x_1^{i_1} \dots x_m^{i_m} y_1^{i_1} \dots y_m^{i_m},$$

$$Y_s = \sum_{i_1+i_2+\dots+i_{2m}=\{l\}} b_{i_1\dots i_{2m}} x_1^{i_1} \dots x_m^{i_m} y_1^{i_1} \dots y_m^{i_m}.$$

For building an infinitesimal operator, we use the same method, with is used in case $m = 1$ [3]. The right part of the system of equations is written in a matrix form G_{sl}. Then, after a transformation at the $SO(2)$ (a rotation by $\delta = (\delta_1, \dots, \delta_m)$), the variables will change

$$\begin{pmatrix} x_s \\ y_s \end{pmatrix} = \Delta_s \begin{pmatrix} \bar{x}_s \\ \bar{y}_s \end{pmatrix},$$

where

$$\Delta_s = \begin{pmatrix} \cos\delta_s & -\sin\delta_s \\ \sin\delta_s & \cos\delta_s \end{pmatrix}.$$

Accordingly, the matrix G_{sl} will also change. The new matrix will be in the form:

$$\bar{G}_{sl}(\delta) = \Delta_s^{-1} G_{sl} D_s(\delta).$$

Respectively, each element of $\bar{G}_{sl}$ is a linear transformation of the elements of G_{sl}

$$\bar{g}_{sl}^{(ij)} = B(\delta) g_{sl}.$$

In $\delta = 0$, we seek the differential $\bar{g}_{sl}^{(ij)}$

$$d\bar{g}_{sl}^{(ij)} = \frac{\partial B(\delta)}{\partial \delta} G_{sl} \,|_{\delta=0}\, d\delta = k_{sl}^{(ij)} d\delta.$$

Therefore, the infitesimal operator of the $SO(2)$ group for the right part of (3) with homogeneous polynomials of degree l, may be written in the following way:

$$U_s = \frac{\partial}{\partial \delta} + \sum_{i,j} k_{sl}^{(ij)} \frac{\partial}{\partial g_{sl}^{(ij)}},$$

where $\sum\limits_{i,j} k_{sl}^{(ij)} \frac{\partial}{\partial g_{sl}^{(ij)}}$ is a linear transformation of the elements $k_{sl}^{(ij)} \frac{\partial}{\partial g_{sl}^{(ij)}}$.

If the right hand part represents a sum of homogeneous spaces $L_s = L_{\otimes p_1} + L_{\otimes p_2} + \dots + L_{\otimes p_s}$, then the infinitesimal operator we seek will be expressed as a sum of infinitesimal operators (as shown in [3]) from respective spaces $L_{\otimes l}$

$$U_s = \frac{\partial}{\partial \delta} + \sum_{l=p_1}^{p_s} \sum_{i,j} k_{sl}^{(ij)} \frac{\partial}{\partial g_{sl}^{(ij)}}, \qquad s = 1, \dots m. \tag{4}$$

Now we assume that all previous conditions, valid for the pair of imaginary solutions, are preserved. Let us consider a general infinitesimal operator for the whole system. Then, for each pair of imaginary solutions $i\lambda$, $s = 1, \dots, m$, the infinitesimal operator will be obtained using the same method and will have the form (4). Thus, the general infinitesimal operator will look:

$$U = \sum_{s=1}^{m} U_s. \tag{5}$$

Theorem. *Let the right hand side of (1) be fixed with respect to the variables $z_1, \dots, z_n$. We will consider a part of the system, which is a system of degree $2m$. Then the invariants of the $SO(2)$ group in the space of coefficients are solutions for the differential equation $Uf = 0$, where U is expressed in (5).*

Example. Consider the system

$$\frac{dx_1}{dt} = -y_1 + a_{11}x_1 + a_{12}y_1 + a_{13}x_2 + a_{14}y_2,$$
$$\frac{dy_1}{dt} = x_1 + a_{21}x_1 + a_{22}y_1 + a_{23}x_2 + a_{24}y_2,$$
$$\frac{dx_2}{dt} = -y_2 + a_{31}x_1 + a_{32}y_1 + a_{33}x_2 + a_{34}y_2,$$
$$\frac{dy_2}{dt} = x_2 + a_{41}x_1 + a_{42}y_1 + a_{43}x_2 + a_{44}y_2.$$

By substitution

$$x_i = \frac{1}{2}(\bar{w}_i + w_i), \qquad y_i = \frac{i}{2}(\bar{w} - w)$$

we transform the system to the form

$$\begin{aligned}
\frac{dw_1}{dt} &= \frac{1}{2}\bar{w}_1(a_{11} + ia_{12} + ia_{21} - a_{22}) + \frac{1}{2}w_1(a_{11} - i_{12} + ia_{21} + a_{22}) \\
&\quad + \frac{1}{2}\bar{w}_2(a_{13} + ia_{14} + ia_{23} - a_{24}) + \frac{1}{2}w_2(a_{13} - ia_{14} + ia_{23} + a_{24}), \\
\frac{dw_2}{dt} &= \frac{1}{2}\bar{w}_1(a_{31} + ia_{32} + ia_{41} - a_{42}) + \frac{1}{2}w_1(a_{31} - i_{32} + ia_{41} + a_{42}) \\
&\quad + \frac{1}{2}\bar{w}_2(a_{33} + ia_{34} + ia_{43} - a_{44}) + \frac{1}{2}w_2(a_{33} - ia_{34} + ia_{43} + a_{44}).
\end{aligned}$$

After substitution

$$w_j = w_j' \exp i\varphi_j, \qquad j = 1, 2$$

we receive the following form:

$$\begin{aligned}
\frac{dw_1}{dt} &= \frac{1}{2}\bar{w}_1 e^{-2i\varphi_1} z_{11} + \frac{1}{2}w_1 z_{12} + \frac{1}{2}\bar{w}_2 e^{-i(\varphi_1+\varphi_2)} z_{13} + \frac{1}{2}w_2 e^{i(\varphi_2-\varphi_1)} z_{14}, \\
\frac{dw_2}{dt} &= \frac{1}{2}\bar{w}_1 e^{-i(\varphi_1+\varphi_2)} z_{21} + \frac{1}{2}w_1 e^{i(\varphi_1-\varphi_2)} z_{22} + \frac{1}{2}\bar{w}_2 e^{-2i\varphi_2} z_{23} + \frac{1}{2}w_2 z_{24}.
\end{aligned}$$

Find differentials

$$\begin{aligned}
&dz_{11} = -2ie^{-2i\varphi_1} z_{11} d\varphi_1, && dz_{13} = -iz_{13} e^{-i(\varphi_1+\varphi_2)}(d\varphi_1 + d\varphi_2), \\
&dz_{14} = ie^{i(\varphi_2-\varphi_1)} z_{14}(d\varphi_2 - d\varphi_1), && dz_{21} = -ie^{-i(\varphi_1+\varphi_2)} z_{21}(d\varphi_1 + d\varphi_2), \\
&dz_{22} = ie^{i(\varphi_1-\varphi_2)} z_{22}(d\varphi_2 - d\varphi_2), && dz_{23} = -2ie^{-2i\varphi_2} z_{23} d\varphi_2.
\end{aligned}$$

Build infitesimal operator

$$\begin{aligned}
U = \frac{\partial}{\partial\varphi} &- 2ie^{-2i\varphi_1} z_{11}\frac{\partial}{\partial z_{11}} - ie^{-i(\varphi_1+\varphi_2)} z_{13}\frac{\partial}{\partial z_{13}} + ie^{i(\varphi_2-\varphi_1)} z_{14}\frac{\partial}{\partial z_{14}} \\
&- ie^{-i(\varphi_1+\varphi_2)} z_{21}\frac{\partial}{\partial z_{21}} + ie^{i(\varphi_1-\varphi_2)} z_{22}\frac{\partial}{\partial z_{22}} + dz_{23} = -2ie^{-2i\varphi_2} z_{23}\frac{\partial}{\partial z_{23}}.
\end{aligned}$$

The invariants

$$z_{12},\ z_{24},\ z_{14}z_{22},\ z_{11}\bar{z}_{11},\ z_{23}\bar{z}_{23},\ z_{13}\bar{z}_{21}$$

are solutions of equations $Uz = 0$.

Hence, for the existence of family of limited solutions in a neighborhood of point $(0, 0, 0, 0)$, we have only 6 conditions.

References

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