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About Symmetries of Exterior Differential Equations, Appropriated to a System of Quasilinear Differential Equations of the First Order

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The symmetries for a quasilinear system of first order partial differential equations are determined. The transformation to a system of exterior differential equations is used. Is shown that the use of this method allows to simplify a problem of defining equations determination.

1 Introduction

Most calculations of symmetries of differential equations are done with the classical L.V. Ovsyannikov method [1]. In 1965 K.P. Surovikhin published a paper [2], in which the differential forms were applied for searching symmetries. In the K.P. Surovikhin paper the system of hyperbolic type equations was considered, and the canonization method for finding symmetries of a exterior differential equations system was applied. In 1971 F.B. Estabrook and B.K. Harrison published a paper [3] in which the Lie derivatives were used for finding symmetries of the exterior differential equations system. This method is easier and universal compared to a canonization method. This method has also certain advantages compared to the L.V. Ovsyannikov method.

However, as it was noted by B.K. Harrison [4], this method was not used widely in the literature. The author also developed a method for finding the symmetries of exterior differential equations with use of the Lie derivatives [5] (the author did not know about the F.B. Estabrook and B.K. Harrison method). In the present paper this method is applied to finding symmetries of quasilinear partial differential equations of the first order. The advantages of this method on a comparison with the L.V. Ovsyannikov method are considered.

2 System of exterior differential equations

A quasilinear system of the first order partial differential equations is considered as a submainfold (surface) Σ in 1-jets space $J^1(\pi)$ of a bundle $\pi : E \longrightarrow M$ local cuts [6]. This submainfold is determined by the system of equations

$$F^k\left(x^i, u^j, p_i^j\right) = 0, \tag{1}$$

where $x^i \in M \subset R^n$, $u^j \in U \subset R^m$, $p_i^j \in J^1(\pi)$, $E = M \times U$. Thus in the space $J^1(\pi)$ there is a Cartan distribution C, defined by Cartan 1-forms

$$\Omega^j = du^j - \sum_{i=1}^{n} p_i^j dx^i. \tag{2}$$

The surface Σ is integral variety Cartan's distribution. Therefore together with (1) should be fulfilled

$$\Omega^j = 0. \tag{3}$$

Thus, a cut $u : M \longrightarrow E$ is a solution of the system (1) if the relations (1), (3) are fulfilled. We shall designate system of relations (1), (3) as $C\Sigma$.

For the quasilinear system of equations (1), we have

$$F^k = c^k_{ji}(x,u)p^j_i + c^k_0(x,u), \tag{4}$$

where $c^k_{ji}(x,u)$, $c^k_0(x,u)$ are continuous functions.

We can obtained now a system of exterior differential equations. For this purpose we shall multiply each equation of the system (1) and the base volume M

$$\omega^k_F = F^k dx^1 \wedge \ldots \wedge dx^n. \tag{5}$$

From Cartan 1-forms we can obtain the n-forms

$$\Omega^j_i = \Omega^j \wedge (dx^1 \wedge \ldots \wedge dx^n)_{\bar{i}} = du^j \wedge (dx^1 \wedge \ldots \wedge dx^n)_{\bar{i}} + p^j_i(-1)^i dx^1 \wedge \ldots \wedge dx^n, \tag{6}$$

where $(dx^1 \wedge \ldots \wedge dx^n)_{\bar{i}} = dx^1 \wedge \ldots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \ldots \wedge dx^n$.

The system of exterior differential equations $\Lambda(\Sigma)$ is obtained by the following method

$$\omega^k = \omega^k_F - (-1)^i c^k_{ji}\Omega^j_i = 0. \tag{7}$$

After substitution (5), (6) we have

$$\omega^k = -(-1)^i c^k_{ji}(x,u) du^j \wedge (dx^1 \wedge \ldots \wedge dx^n)_{\bar{i}} + c^k_0(x,u) dx^1 \wedge \ldots \wedge dx^n. \tag{8}$$

Let us note, that the system $\Lambda(\Sigma)$ on the space E is determined.

Thus initial system of equations $C\Sigma$, defined as the surface Σ with Cartan distribution C, appropriated by the system of the exterior differential equations $\Lambda(C\Sigma)$

$$\Omega^j = 0, \qquad \omega^k = 0. \tag{9}$$

From $\Omega^j = 0$ and $\omega^k = 0$ it follows that $F^k = 0$. Hence the systems $C\Sigma$ and $\Lambda(C\Sigma)$ are equivalent and any integrated variety $C\Sigma$ is an integrated variety for $\Lambda(C\Sigma)$ and vice versa.

3 About symmetries for $C\Sigma$ and $\Lambda(C\Sigma)$

Let us consider now a problem of symmetries for $C\Sigma$ and $\Lambda(C\Sigma)$.

According to [1] and [6], the classical infinitesimal symmetry of the equations $C\Sigma$ is Lie vector field $\overline{X}$, such that

$$\overline{X}(\Omega^k) = \lambda^j \Omega^j, \tag{10}$$

$$\overline{X}(F^k) = \alpha^j F^j. \tag{11}$$

Here λ^j, α^j are some functions, and $\overline{X}(\Omega^k)$ is determined by the Lie derivative

$$\overline{X}(\Omega^k) = d(\overline{X}\rfloor\Omega^k) + \overline{X}\rfloor d(\Omega^k),$$

where $\rfloor$ is an interior product. The Lie vector field $\overline{X}$ belongs to a space, tangent to $J^1(\pi)$, and $\overline{X}$ is a lift of a vector field X, tangent to a space of a bundle E

$$\overline{X} = X + X^{(1)}, \tag{12}$$

where

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \phi^j(x,u)\frac{\partial}{\partial u^j}, \qquad X^{(1)} = \zeta_i^j(x,u,p)\frac{\partial}{\partial p_i^j}, \tag{13}$$

ξ^i, ϕ^j, ζ_i^j are some functions. We shall designate the Lie algebra of vector fields X as $\mathrm{sym}(\Sigma)$.

Theorem. *For point infinitesimal symmetries of systems $C\Sigma$ and $\Lambda(C\Sigma)$ the relation is fulfilled*

$$\mathrm{sym}(\Sigma) = \mathrm{cosym}(\Sigma).$$

Proof. The vector field X of a point symmetry is uniquely determined by the Lie vector field $\overline{X}$. Therefore it is enough to show that any symmetry of a system $C\Sigma$ is a symmetry of a system $\Lambda(C\Sigma)$ and vice versa.

Let at first $\overline{X}$ is infinitesimal symmetry $C\Sigma$, i.e.,

$$\overline{X}(F^k)\mid_{C\Sigma} = (X + X^{(1)})(F^k)\mid_{C\Sigma} = 0. \tag{14}$$

Let us show, that

$$X(\omega^k)\mid_{\Lambda(C\Sigma)} = X(\omega^k)\mid_{\Lambda(\Sigma)} = 0. \tag{15}$$

Taking into account (6) and (10), we have

$$\overline{X}(\Omega_i^k)\mid_{\Lambda(C\Sigma)} = [\lambda^j\Omega^j \wedge (dx^1 \wedge \ldots \wedge dx^n)_{\bar{i}} + \gamma^l\Omega^k \wedge (dx^1 \wedge \ldots \wedge dx^n)_{\bar{l}}]\mid_{\Lambda(C\Sigma)} = 0, \tag{16}$$

where γ is some function. Taking into account (5) and (11), we have

$$\begin{aligned}\overline{X}(\omega_F^k) &= \overline{X}(F^k)(dx^1 \wedge \ldots \wedge dx^n) + F^k\overline{X}(dx^1 \wedge \ldots \wedge dx^n)\\ &= \alpha^j F^j(dx^1 \wedge \ldots \wedge dx^n) + \beta F^k(dx^1 \wedge \ldots \wedge dx^n) = (\alpha^j\omega_F^j + \beta\omega_F^k),\end{aligned}$$

and, thus

$$\overline{X}(\omega_F^k) = \mu^j\omega_F^j,$$

where μ^j, β are some functions. Hence, taking into account (7) and (17)

$$\overline{X}(\omega^k)\mid_{\Lambda(C\Sigma)} = [\overline{X}(\omega_F^k) - (-1)^i\overline{X}(c_{ji}^k\Omega_i^j)]\mid_{\Lambda(C\Sigma)} = \mu^j\omega_F^j\mid_{\Lambda(C\Sigma)} = 0.$$

As the forms ω^k are defined in coordinates of space E, then $X^{(1)}(\omega^k) = 0$. Therefore

$$X(\omega^k)\mid_{\Lambda(C\Sigma)} = 0.$$

We can write the latter equality as

$$X(\omega^k) = \rho_j^k\omega^j + \sigma_j^k\Omega^j,$$

where ρ_j^k, σ_j^k are some functions. As the forms ω^j and vector field X are defined on a space of a bundle E, then, $\sigma_j^k \equiv 0$ and, therefore, (16) is fulfilled.

Let now X be an infinitesimal symmetry of $\Lambda(\Sigma)$, i.e., (16) is fulfilled. Let us define the vector field $\overline{X} = X + X^{(1)}$ so, that it is a Lie field (saves the Cartan distribution). Let us show that (15) also is fulfilled.

Taking into account (17) we have

$$X(\omega^k)\mid_{\Lambda(\Sigma)} = X(\omega^k)\mid_{\Lambda(C\Sigma)} = X(\omega^k)\mid_{C\Sigma} =$$
$$= \overline{X}(\omega^k)\mid_{C\Sigma} = [\overline{X}(\omega^k_F) - (-1)^i \overline{X}(c^k_{ji}\Omega^j_i)]\mid_{C\Sigma} = [\overline{X}(\omega^k_F)]\mid_{C\Sigma} = 0$$

and we can write

$$\overline{X}(\omega^k_F) = \mu^j \omega^j_F.$$

From here follows, that

$$\overline{X}(F^k) = \alpha^j F^j$$

and consequently (15) is fulfilled. The theorem is proved.

So the problem of searching infinitesimal symmetries for the given class of equations system $C\Sigma$ is equivalent to a problem of searching infinitesimal symmetries for the system $\Lambda(C\Sigma)$, to be exact systems $\Lambda(\Sigma)$. Thus infinitesimal symmetries are vector fields, tangents to base of bundle E, and not to a space of a bundle $J^1(\pi)$. As it is shown below, such lowering of a vector field space dimensionality reduces to some decreasing of difficulty in construction of the defining equations system.

4 Example

As example, let us consider a system of two equations

$$\frac{\partial u^k}{\partial x^1} + c^k_j\left(x^1, x^2, u^1, u^2\right)\frac{\partial u^j}{\partial x^2} + c^k_0\left(x^1, x^2, u^1, u^2\right) = 0, \tag{17}$$

where $i, j, k = 1, 2$.

The system of the exterior differential equations $\Lambda(\Sigma)$ will look like

$$\omega^k = du^k \wedge dx^2 - c^k_j du^j \wedge dx^1 + c^k_0 dx^1 \wedge dx^2 = 0. \tag{18}$$

Infinitesimal symmetry of a system $\Lambda(\Sigma)$ will be a vector field

$$X = \xi^1\left(x^1, x^2, u^1, u^2\right)\frac{\partial}{\partial x^1} + \xi^2\left(x^1, x^2, u^1, u^2\right)\frac{\partial}{\partial x^2}$$
$$+ \phi^1\left(x^1, x^2, u^1, u^2\right)\frac{\partial}{\partial u^1} + \phi^2\left(x^1, x^2, u^1, u^2\right)\frac{\partial}{\partial u^2}.$$

The defining equations for $\operatorname{cosym}(\Sigma)$ are obtained from a condition (16). We have

$$\begin{aligned} X(\omega^k) = {} & d\phi^k \wedge dx^2 + du^k \wedge d\xi^2 - c^k_j\left(d\phi^j \wedge dx^1 + du^j \wedge d\xi^1\right) \\ & - X\rfloor d(c^k_j) du^j \wedge dx^1 + c^k_0\left(d\xi^1 \wedge dx^2 + dx^1 \wedge d\xi^2\right) + X\rfloor d(c^k_0) dx^1 \wedge dx^2. \end{aligned} \tag{19}$$

The system of defining equations for functions ξ^i, ϕ^j is obtained by a substitution (20) to (16). The decomposition is carried on under the forms: $du^1 \wedge du^2$, $du^1 \wedge dx^1$, $du^2 \wedge dx^1$, $dx^1 \wedge dx^2$.

We have after decomposition from the first equation of the system (19)

$$\frac{\partial \xi^2}{\partial u^2} - c^1_1\frac{\partial \xi^1}{\partial u^2} + c^1_2\frac{\partial \xi^1}{\partial u^1} = 0,$$

$$c_2^1\frac{\partial\phi^2}{\partial u^1}-c_1^2\frac{\partial\phi^1}{\partial u^2}-\frac{\partial\xi^2}{\partial x^1}+c_1^1\left(\frac{\partial\xi^1}{\partial x^1}-\frac{\partial\xi^2}{\partial x^2}\right)+\xi^i\frac{\partial c_1^1}{\partial x^i}+\phi^j\frac{\partial c_1^1}{\partial u^j}$$
$$+\left[\left(c_1^1\right)^2+c_2^1c_1^2\right]\frac{\partial\xi^1}{\partial x^2}+c_0^1\left(c_1^1\frac{\partial\xi^1}{\partial u^1}+c_1^2\frac{\partial\xi^1}{\partial u^2}-\frac{\partial\xi^2}{\partial u^1}\right)=0,$$

$$c_2^1\left(\frac{\partial\phi^2}{\partial u^2}-\frac{\partial\phi^1}{\partial u^1}\right)-\left(c_2^2-c_1^1\right)\frac{\partial\phi^1}{\partial u^2}+c_2^1\left(\frac{\partial\xi^1}{\partial x^1}-\frac{\partial\xi^2}{\partial x^2}\right)+c_2^1\left(c_1^1+c_2^2\right)\frac{\partial\xi^1}{\partial x^2}$$
$$+\xi^i\frac{\partial c_2^1}{\partial x^i}+\phi^j\frac{\partial c_2^1}{\partial u^j}++c_0^1\left(c_2^1\frac{\partial\xi^1}{\partial u^1}+c_2^2\frac{\partial\xi^1}{\partial u^2}-\frac{\partial\xi^2}{\partial u^2}\right)=0,$$

$$\frac{\partial\phi^1}{\partial x^1}+c_j^1\frac{\partial\phi^j}{\partial x^2}+\xi^i\frac{\partial c_0^1}{\partial x^i}+\phi^j\frac{\partial c_0^1}{\partial u^j}+c_0^1\left(\frac{\partial\xi^1}{\partial x^1}+\frac{\partial\xi^2}{\partial x^2}-c_0^1\frac{\partial\xi^1}{\partial u^1}-c_0^2\frac{\partial\xi^1}{\partial u^2}\right)=0.$$

From the second equation we have

$$\frac{\partial\xi^2}{\partial u^1}-c_1^2\frac{\partial\xi^1}{\partial u^2}+c_2^2\frac{\partial\xi^1}{\partial u^1}=0,$$

$$c_1^2\left(\frac{\partial\phi^2}{\partial u^2}-\frac{\partial\phi^1}{\partial u^1}\right)-\left(c_1^1-c_2^2\right)\frac{\partial\phi^2}{\partial u^1}+c_1^2\left(\frac{\partial\xi^1}{\partial x^1}-\frac{\partial\xi^2}{\partial x^2}\right)+c_1^2\left(c_1^1+c_2^2\right)\frac{\partial\xi^1}{\partial x^2}$$
$$+\xi^i\frac{\partial c_1^2}{\partial x^i}+\phi^j\frac{\partial c_1^2}{\partial u^j}+c_0^2\left(c_1^1\frac{\partial\xi^1}{\partial u^1}+c_1^2\frac{\partial\xi^1}{\partial u^2}-\frac{\partial\xi^2}{\partial u^1}\right)=0,$$

$$-c_2^1\frac{\partial\phi^2}{\partial u^1}+c_1^2\frac{\partial\phi^1}{\partial u^2}-\frac{\partial\xi^2}{\partial x^1}+c_2^2\left(\frac{\partial\xi^1}{\partial x^1}-\frac{\partial\xi^2}{\partial x^2}\right)+\xi^i\frac{\partial c_2^2}{\partial x^i}+\phi^j\frac{\partial c_2^2}{\partial u^j}$$
$$+\left[\left(c_2^2\right)^2+c_2^1c_1^2\right]\frac{\partial\xi^1}{\partial x^2}+c_0^2\left(c_2^1\frac{\partial\xi^1}{\partial u^1}+c_2^2\frac{\partial\xi^1}{\partial u^2}-\frac{\partial\xi^2}{\partial u^2}\right)=0,$$

$$\frac{\partial\phi^2}{\partial x^1}+c_j^2\frac{\partial\phi^j}{\partial x^2}+\xi^i\frac{\partial c_0^2}{\partial x^i}+\phi^j\frac{\partial c_0^2}{\partial u^j}+c_0^2\left(\frac{\partial\xi^1}{\partial x^1}+\frac{\partial\xi^2}{\partial x^2}-c_0^1\frac{\partial\xi^1}{\partial u^1}-c_0^2\frac{\partial\xi^1}{\partial u^2}\right)=0.$$

Thus we have obtained a system of defining equations for determination of symmetries of the system (19). The system of defining equations is over-determined. The number N_d of the defining system equations is determined by expression $N_d=mN_c-N_l$. Here $m=2$ is the number of the initial system equations, N_c is number of decomposition conditions, N_l is number of linearly dependent equations for the defining equations system. For the considered system $N_c=4$ and $N_l=0$ (all equations of a defining system are linearly independent). Therefore we have $N_d=8$.

If the system of defining equations obtained by L.V. Ovsjannikov's technique [2], then the number of decomposition conditions $N_c=6$ (decomposition under p_1^1, p_1^2, $\left(p_1^1\right)^2$, $\left(p_1^2\right)^2$, $p_1^1p_1^2$ and under absolute terms). Therefore $2N_c=12$. Thus the general number of the equations will be also eight, as $N_l=4$ (four equations will linearly depend on other equations).

5 About number of decomposition conditions

In more common case the number of decomposition conditions N_c for the exterior differential equations system corresponding quasilinear first order system is determined by expression

$$N_c=C_{m+n}^m-m,$$

where m and n are numbers of dependent and independent variables, C^m_{m+n} is the number of combinations from $n+m$ elements under n,

$$C^n_{m+n} = \frac{(n+m)!}{n!m!}.$$

Thus, the transformation to a system of the exterior differential equations (for a considered class of the equations) allows to lower the number of decomposition conditions for searching of symmetries and to eliminate from consideration linearly dependent equations of a defining system. In some cases it reduces complexity at deriving of the defining equations system.

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