Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 127–130.

Oscillation of Solutions of Ordinary Differential Equations Systems Generated by Finite-Dimensional Group Algebra

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The class of nonlinear second order systems having oscillation solutions has been described. Let us take note that periodic solution is particular case of oscillation solution. The algorithm of construction of a reducible transformation transforming initial system to system generated by finite-dimensional group algebra has been developed. It is important that initial systems can be essentially nonlinear. The class of relaxation oscillations has been reduced to the considered case.

Among the variety of second order systems is of interest to select systems in which periodic or "similar" to it change of system state almost periodic, recurrent or oscillatory can take place.

Consider a nonlinear autooscillatory system of differential equations with one degree of freedom

$$\begin{cases} \dot{x} = f_1(x, y), \\ \dot{y} = f_2(x, y), \end{cases} \tag{1}$$

where $(x, y) \in \mathrm{R}^2$, the overdot in (1) means derivative "d/dt" with respect to $t \in [0, +\infty)$, functions $f_i(x, y)$, $i = 1, 2$ are arbitrary analytical functions in some open domain D of plane (x, y) that satisfy Lipschitz condition in any bounded closed region that is subset of D.

The mathematical model of autooscillatory system is essentially nonlinear. Restriction of amplitude of autooscillations takes place in autooscillatory system due to its nonlinearity. The form of them can be diverse including nonusual. Among similar class of nonlinear systems, the relaxation systems, are of special class

$$\begin{cases} \varepsilon\dot{x} = f_1(x, y), \\ \dot{y} = f_2(x, y), \end{cases} \qquad 0 < \varepsilon \ll 1,$$

where oscillations are very far from harmonic. The construction of approximative analytical expressions for them cannot be obtained within the limits of classical methods of perturbation theory and requires special methods. This problem was solved for indicated class of systems.

Suppose that system (1) has integral in the form of curve

$$F(x, y) = C, \qquad 0 \le C \le C^*, \tag{2}$$

that satisfies the following conditions:

1. Curve (2) is sectionally smooth;
2. The curve surrounding a system's state of equilibrium (x_0, y_0) is closed;
3. Curve (2) restricts some simply connected domain D^* that is subset of determination region of system (1) $D^* \subseteq D$;

4. Curve (2) does not have points of self-intersection that means that for given implicit function $F(x,y)=C$ the following condition takes place:

$$\Delta = \left(\frac{\partial^2 F(x,y)}{\partial x^2}\right)_{(0,0)} \cdot \left(\frac{\partial^2 F(x,y)}{\partial y^2}\right)_{(0,0)} - \left(\frac{\partial^2 F(x,y)}{\partial x \partial y}\right)^2_{(0,0)} \geq 0. \tag{3}$$

Theorem 1. *If an autooscillatory nonlinear system with one power of freedom (1) has an integral (2) in the form of sectionally smooth closed curve for which the conditions 1–4 are satisfied then system (1) has a restricted and oscillatory solution*

$$u = \sum_{i=1}^{m} A_i \sum_{\substack{j=2p \\ p=0,1,\ldots,[\frac{i}{2}]}} C_i^j \cdot (-1)^{\frac{j}{2}} \cdot \cos^{i-j}\varphi \cdot \sin^j \varphi,$$

$$v = \sum_{i=1}^{m} A_i \sum_{\substack{j=2p+1 \\ p=0,1,\ldots,[\frac{i-1}{2}]}} C_i^j \cdot (-1)^{\frac{j-1}{2}} \cdot \cos^{i-j}\varphi \cdot \sin^j \varphi.$$

And, vice versa, a restricted and oscillatory solution of system (1) corresponds to a phase trajectory in the form of sectionally smooth closed curve $F(x,y)=C$ that does not have the points of self-intersection.

Represent system (1) in complex plane by means of the change of variables

$$\begin{cases} x = \dfrac{1}{2}(w+\overline{w}), \\ y = -\dfrac{i}{2}(w-\overline{w}), \end{cases} \tag{4}$$

where $w=u+iv$, $\overline{w}=u-iv$.

In view of special properties of the change the system (1) may be rewriten as

$$\begin{cases} \dot{u} = f_1(u,v), \\ \dot{v} = f_2(u,v), \end{cases} \tag{5}$$

where $u+iv=w$ and integral (2) in complex variables shall respectively have the form of

$$F(u,v)=C, \qquad u+iv=w, \tag{6}$$

where sectionally smooth closed curve (6) will restrict respective one-connected domain D_w of complex plane w.

Theorem 2 (approximative transformation of nonlinear system). *Suppose that autooscillatory nonlinear system (5) where functions $f_i(u,v)$, $i=1,2$ are analytical in some domain D of complex plane w has an integral (6) in the form of sectionally smooth closed curve that satisfies conditions 1–4.*

By the method of trigonometric interpolation we construct the power function mapping unit circle $|W|=1$ on curve the (6) to some fixed value of parameter C

$$w = \sum_{n=1}^{m} A_n W^n, \qquad w=u+iv, \quad W=U+iV, \quad m\to\infty. \tag{7}$$

The inverse function

$$W = U + iV = G(w) = \sqrt{c}\sum_{n=1}^{m} B_n w^n \tag{8}$$

transforms integral (6) to the canonical form

$$W\overline{W} = C, \qquad U^2 + V^2 = C. \tag{9}$$

Then transformation (8) reduces system (5) to a system generated by finite-dimensional group algebra $so(2)$

$$\begin{cases} \dot{U} = -\alpha(U, V)V, \\ \dot{V} = \alpha(U, V)U. \end{cases} \tag{10}$$

Remark 1. The system (10) has an oscillation solution

$$\begin{cases} U = \rho\cos\varphi(t), \\ V = \rho\sin\varphi(t), \end{cases} \tag{11}$$

where the functions $\varphi(t)$ and $\alpha(U, V)$ satisfy differential equation for phase $\varphi(t)$ and amplitude ρ of oscillations

$$\frac{d\varphi}{dt} = \alpha(\rho, \varphi). \tag{12}$$

Thus we determine a transformation of coordinates in form of power series for receiving phase trajectory of system (1) in the form of a family of concentric circles with centre in the origin. But a representative point moves along one of phase trajectory with variable angular velocity $\frac{d\varphi}{dt} = \text{var}$. Remark that in particular case with $\alpha = 1$ the point moves on the circle uniformly. In this case we have a harmonic solution

$$\begin{cases} U = A\cos(t + \phi), \\ V = A\sin(t + \phi), \end{cases}$$

where A is the amplitude of oscillations and ϕ is the phase of oscillations.

Remark 2. The solution (11) is periodic if function $\varphi(t)$ is periodic or $\varphi(t) = t$.

The Riemannian theorem, the theorem of conformity of domain boundaries at one-to-one conformal mapping of domains and Christoffel–Schwarz integral [5] realizing mapping of unit circle $|W| \leq 1$ on internal region of polygon are theoretical base for transformation (7). The constants of integral will be unit circle points which correspond to vertices of a polygon when mapping.

For numerical solution we use of stated problem Filchakov method of trigonometric interpolation of conformal mapping of domains. This method allows to obtain with help of simple formulas any given accuracy of construction of function mapping unit circle on internal region of any previously given simply connected and one-sheet domain D_w restricted by curve (6).

It is of great importance that the method of trigonometric interpolation does not give any restrictions on the manner of setting of contour what means that curve (6) can be given analytically, graphically or tabular, only by a discrete series of points.

Remark 3. Taking into consideration that in power series (8) the coefficients are imaginary

$$B_n = B_n^{(1)} + iB_n^{(2)}, \qquad w = u + iv,$$

and using the Newton binomial formula it is possible to determine real and imaginary parts for the transformation $W = G(w)$:

$$U = \sqrt{c}\sum_{n=1}^{m} B_n^{(1)} \sum_{\substack{k=2l \\ l=0,1,\ldots,\left[\frac{n}{2}\right]}} C_n^k(-1)^{\frac{k}{2}} u^{n-k}v^k - \sqrt{c}\sum_{n=1}^{m} B_n^{(2)} \sum_{\substack{k=2l+1 \\ l=0,1,\ldots,\left[\frac{n-1}{2}\right]}} C_n^k(-1)^{\frac{k-1}{2}} u^{n-k}v^k.$$

$$V = \sqrt{c}\sum_{n=1}^{m} B_n^{(1)} \sum_{\substack{k=2l+1 \\ l=0,1,\ldots,\left[\frac{n-1}{2}\right]}} C_n^k(-1)^{\frac{k-1}{2}} u^{n-k}v^k + \sqrt{c}\sum_{n=1}^{m} B_n^{(2)} \sum_{\substack{k=2l \\ l=0,1,\ldots,\left[\frac{n}{2}\right]}} C_n^k(-1)^{\frac{k}{2}} u^{n-k}v^k.$$

Similar problem for analysis of autonomous second order systems that are closed to nonlinear conservative is solved in [4]. The main result of this paper is a considerable extension of the class of studied systems was without essential restrictions for the functions $f_1(x, y)$, $f_2(x, y)$. Moreover there is a possibility of generalization of theory in case $n > 2$.

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