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# Nonlocal Symmetries of Nonlinear Integrable Models

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In this paper, nonlocal symmetries are considered for some integrable equations including the first equation of the AKNS hierarchy, the so-called breaking soliton equation, the Boussinesq equation and the Toda equation. Besides, using invariant transformations of corresponding spectral problems, more nonlocal symmetries can be produced from one seed symmetry.

## 1 Introduction

Symmetries and conservation laws for differential equations are the central themes of perpetual interest in mathematical physics. During past thirty years, the study of symmetries has been connected with the development of soliton theory and, in fact, it constitutes an indispensable and important part of soliton theory.

Let us begin with the celebrated Korteweg de Vries equation

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1}$$

where the subscripts represent derivatives. A symmetry of the KdV equation (1) is defined as a solution of its linearized equation

$$\sigma_t + 6(u\sigma)_x + \sigma_{xxx} = 0. \tag{2}$$

It is well known that $x$-translation and $t$-translation invariance of (1) leads to the following symmetries: $u_x$, $u_t$ of the KdV equation (1). In order to find more generalized symmetries, the concepts of recursion operators or strong symmetries, and hereditary symmetries were introduced by Olver and Fuchssteiner and used to find these symmetries [1, 2]. Furthermore, Galilean invariance of the KdV equation (1) leads to symmetry $tu_x - \frac{1}{6}$, which may be viewed as the origin of active research on the time-dependent symmetries and the corresponding Lie algebraic structures for nonlinear equations; and these time-dependent symmetries are connected with nonisospectral problems (see, e.g. [3–6]). Apart from the symmetries mentioned above, there exist so-called nonlocal symmetries expressed by spectral functions, e.g., $\sigma = (\phi^2)_x$ is a symmetry of the KdV equation (1), where $\phi$ is a spectral function of Lax pair

$$\phi_{xx} + u\phi = \lambda\phi, \tag{3}$$

$$\phi_t = u_x\phi - (2u + 4\lambda)\phi_x. \tag{4}$$

To search for nonlocal symmetries is an interesting topic. On one hand, these nonlocal symmetries enlarge class of symmetries. Besides, nonlocal symmetries are connected with integrable models. Such an example is the nonlocal symmetry $\sigma = (\phi^2)_x$ generates well-known sinh-Gordon equation and Liouville equation [7]. Further more examples can be found in [8–10]. A natural problem arises now: how to find nonlocal symmetries? An effective method to find nonlocal symmetries seems to find inverse of the corresponding recursion operators (see [11]). However, to find inverse of recursion operators is a difficult problem by itself. Recently one of authors (Lou) re-obtained the nonlocal symmetry $\sigma = (\phi^2)_x$ from the conformal invariance of the Schwartz form of the KdV equation (1) [12]. It is an interesting result. As explained below, this nonlocal symmetry is basic one of the KdV equation, from which all the known nonlocal symmetries can be obtained. In fact, we know from [12] that $\frac{d^n}{d\lambda^n}(\phi^2)_x$ is also a symmetry and inverse recursion operator of the KdV equation (1) appears naturally when $\frac{d^n}{d\lambda^n}(\phi^2)_x$ is rewritten as a single multiplication form. Secondly, the other two nonlocal seed symmetries $\partial_x\phi^2\partial_x^{-1}\phi^{-2}$ and $\partial_x\phi^2\partial_x^{-1}\phi^{-2}\partial_x^{-1}\phi^{-2}$ are easily obtained from $(\phi^2)_x$ by considering the fact that Lax pair (3), (4) of the KdV equation is invariant under transformation $\phi \longrightarrow \phi\partial_x^{-1}\phi^{-2}$ and (3), (4) are linear differential equations with respect to $\phi$. That means all the known nonlocal symmetries of the KdV equation in literature can be obtained from one seed symmetry $(\phi^2)_x$.

In this paper, we intend to search for nonlocal seed symmetries of some integrable models. It is noticed that recently there have been active research on nonlinearization of spectral problems and generation of finite dimensional integrable systems (see, e.g. [13]). In literature, there are two cases to be considered: Bargmann and Neumann constraints. For the KdV equation, it is obvious that Bargmann constraint is equivalent to symmetry constraint $u_x = (\phi^2)_x$. With this observation in mind, we are going to derive nonlocal symmetries along this line. Besides, using invariance of spectral problem, more nonlocal symmetries can be produced from one seed symmetry.

## 2 The AKNS case

The AKNS hierarchy is

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = L^n K_0 = L^n \begin{pmatrix} -iq \\ ir \end{pmatrix}, \tag{5}$$

where

$$L = \begin{pmatrix} -D + 2qD^{-1}r & 2qD^{-1}q \\ -2rD^{-1}r & D - 2rD^{-1}q \end{pmatrix}$$

with $D = \frac{\partial}{\partial x}$, $D^{-1} = \int_{-\infty}^{x} dx$. In what follows, we only consider $n = 2$ case for the sake of convenience in calculation. In this case, (5) becomes

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = i \begin{pmatrix} -q_{xx} + 2q^2 r \\ r_{xx} - 2r^2 q \end{pmatrix}, \qquad i = \sqrt{-1}. \tag{6}$$

Its Lax pair is [14]

$$\begin{pmatrix} \phi_{1x} \\ \phi_{2x} \end{pmatrix} = \begin{pmatrix} -i\lambda & q \\ r & i\lambda \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{7}$$

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}_t = \begin{pmatrix} 2i\lambda^2 + iqr & -2q\lambda - iq_x \\ -2r\lambda + ir_x & -2i\lambda^2 - iqr \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{8}$$

It is known that the Bargmann constraint is [15]

$$\begin{pmatrix} q \\ r \end{pmatrix} = c_0 \begin{pmatrix} \phi_1^2 \\ -\phi_2^2 \end{pmatrix}$$

and $K_0 = \begin{pmatrix} -iq \\ ir \end{pmatrix}$ is a symmetry of (6). Thus $\sigma = \begin{pmatrix} \phi_1^2 \\ \phi_2^2 \end{pmatrix}$ is possible to become a symmetry of (6). Indeed, a direct calculation shows that $\sigma = \begin{pmatrix} \phi_1^2 \\ \phi_2^2 \end{pmatrix}$ is a symmetry (see also [16]). In order to obtain more seed symmetries, we now consider the invariance property of (7) and (8). To this end, we have

**Proposition 1**. *Lax pair (7) and (8) is invariant under transformation*

$$\phi_1 \longrightarrow F(t)\phi_1 + (\alpha-1)\frac{1}{\phi_2} + \phi_1 \left( \alpha \int_{x_0}^{x} \frac{q}{\phi_1^2}\, dx + (\alpha-1) \int_{x_0}^{x} \frac{r}{\phi_2^2}\, dx \right),$$

$$\phi_2 \longrightarrow F(t)\phi_2 + \alpha\frac{1}{\phi_1} + \phi_2 \left( \alpha \int_{x_0}^{x} \frac{q}{\phi_1^2}\, dx + (\alpha-1) \int_{x_0}^{x} \frac{r}{\phi_2^2}\, dx \right),$$

*where $\alpha$ is an arbitrary constant and $F(t)$ is a function of $t$ defined by*

$$F(t) = -\int^{t} \left[ \alpha \frac{iq_x + 2\lambda q}{\phi_1^2} + (1-\alpha) \frac{ir_x - 2\lambda r}{\phi_2^2} \right]_{x=x_0} dt.$$

**Proof**: direct calculation.

**Proposition 2**. *Suppose that* $\begin{pmatrix} \phi_1^{(i)} \\ \phi_2^{(i)} \end{pmatrix}$ $(i = 1, 2)$ *is a solution of (7) and (8). Then* $\begin{pmatrix} \phi_1^{(1)}\phi_1^{(2)} \\ \phi_2^{(1)}\phi_2^{(2)} \end{pmatrix}$ *is a symmetry of (6).*

Using Proposition 1 and 2, we know that

$$\begin{pmatrix} F(t)\phi_1^2 + (\alpha-1)\dfrac{\phi_1}{\phi_2} + \phi_1^2 \left( \alpha \displaystyle\int_{x_0}^{x} \frac{q}{\phi_1^2}\, dx + (\alpha-1) \int_{x_0}^{x} \frac{r}{\phi_2^2}\, dx \right) \\ F(t)\phi_2^2 + \alpha\dfrac{\phi_2}{\phi_1} + \phi_2^2 \left( \alpha \displaystyle\int_{x_0}^{x} \frac{q}{\phi_1^2}\, dx + (\alpha-1) \int_{x_0}^{x} \frac{r}{\phi_2^2}\, dx \right) \end{pmatrix}$$

and

$$\begin{pmatrix} \left[ F(t)\phi_1 + (\alpha-1)\dfrac{1}{\phi_2} + \phi_1 \left( \alpha \displaystyle\int_{x_0}^{x} \frac{q}{\phi_1^2}\, dx + (\alpha-1) \int_{x_0}^{x} \frac{r}{\phi_2^2}\, dx \right) \right]^2 \\ \left[ F(t)\phi_2 + \alpha\dfrac{1}{\phi_1} + \phi_2 \left( \alpha \displaystyle\int_{x_0}^{x} \frac{q}{\phi_1^2}\, dx + (\alpha-1) \int_{x_0}^{x} \frac{r}{\phi_2^2}\, dx \right) \right]^2 \end{pmatrix}$$

are symmetries of (6). Furthermore, in [17], the inverse of recursion operator $L$ was obtained. Thus more symmetries can be obtained from seed symmetries and inverse recursion operator $L^{-1}$.

# 3 The breaking soliton equation

The breaking soliton equation under consideration is

$$u_{xt} = 4u_x u_{xy} + 2u_y u_{xx} - u_{xxxy} \tag{9}$$

which was first introduced by Calogero and Degasperis [18]. Set $v = u_x$, then (9) can be written as

$$v_t = 4vv_y + 2(\partial_x^{-1} v_y)v_x - v_{xxy}. \tag{10}$$

Its bi-Hamiltonian structure and the Lax pair equations with non-isospectral problem have been discussed in [19]. In [10], one of authors (Lou) found a nonlocal symmetry of (9)

$$\sigma = 2\phi_x \phi(1 + \partial_x^{-1}\phi^{-3}\phi_y) + \phi^{-1}\phi_y$$

with

$$-\phi_{xx} + v\phi = 0, \tag{11}$$

$$\phi_t = -v_y \phi + 2\phi_x \partial_x^{-1} v_y. \tag{12}$$

It is easily verified that Lax pair (11), (12) is invariant under the transformation $\phi \longrightarrow \phi\partial_x^{-1}\frac{1}{\phi^2}$. Besides, we have

**Proposition 3**. *Suppose $\phi_1$ and $\phi_2$ are two solutions of (11), (12). Then*

$$\begin{aligned}\sigma(\epsilon,\delta) = {} & 2(\epsilon\phi_1 + \delta\phi_2)_x(\epsilon\phi_1 + \delta\phi_2)\left(1 + \partial_x^{-1}(\epsilon\phi_1 + \delta\phi_2)^{-3}(\epsilon\phi_1 + \delta\phi_2)_y\right) \\ & + (\epsilon\phi_1 + \delta\phi_2)^{-1}(\epsilon\phi_1 + \delta\phi_2)_y\end{aligned}$$

*and $\frac{\partial^{m+n}}{\partial\epsilon^m \partial\delta^n}\sigma(\epsilon,\delta)$ are symmetries of (10) (here $\epsilon, \delta$ are arbitrary constants).*

Using these results, many nonlocal symmetries can be obtained.

# 4 The Boussinesq equation

The Boussinesq equation under consideration is [20]

$$u_{tt} + \left(u^2\right)_{xx} + \frac{1}{3}u_{xxxx} = 0. \tag{13}$$

The corresponding Lax pair is

$$\phi_{xxx} + \frac{3}{2}u\phi_x + \left(\frac{3}{4}u_x - \frac{3}{4}\partial_x^{-1}u_t\right)\phi = 0, \tag{14}$$

$$\phi_t = -\phi_{xx} - u\phi \tag{15}$$

and its adjoint version is

$$\phi^*_{xxx} + \frac{3}{2}u\phi^*_x + \left(\frac{3}{4}u_x + \frac{3}{4}\partial_x^{-1}u_t\right)\phi^* = 0, \tag{16}$$

$$\phi^*_t = \phi^*_{xx} + u\phi^*. \tag{17}$$

Just as the KP case [12, 21], it is easily verified that $(\phi\phi^*)_x$ is a symmetry of (13). In the following, we want to give more symmetries of (13) by considering invariance property of (14), (15) and (16), (17). To this end, we obtain

**Proposition 4**. *Suppose $\phi_1$ and $\phi_2$ are two linearly independent spectral functions of (14), (15) (or (16), (17)) corresponding to $u$. Then*

$$\Phi = \psi_1(t)\phi_1 - \psi_2(t)\phi_2 + \phi_1 \int_{x_0}^{x} \frac{\phi_2}{W^2(\phi_1,\phi_2)}\, dx - \phi_2 \int_{x_0}^{x} \frac{\phi_1}{W^2(\phi_1,\phi_2)}\, dx$$

*is also a spectral function of (14), (15) (or (16), (17)) corresponding to $u$, where*

$$W(\phi_1,\phi_2) \equiv \phi_{1x}\phi_2 - \phi_1\phi_{2x}, \tag{18}$$

$$\psi_1(t) = \int^t \left[\frac{\phi_{2x}}{W^2(\phi_1,\phi_2)}\right]_{x=x_0} dt, \tag{19}$$

$$\psi_2(t) = \int^t \left[\frac{\phi_{1x}}{W^2(\phi_1,\phi_2)}\right]_{x=x_0} dt. \tag{20}$$

**Proof**: direct calculation.

From Proposition 4, we know

$$\begin{aligned}\sigma = \Bigg[\Bigg(& c_0\phi_1 + c_1\phi_2 + c_2\psi_1(t)\phi_1 - c_2\psi_2(t)\phi_2 + c_2\phi_1 \int_{x_0}^{x} \frac{\phi_2}{W^2(\phi_1,\phi_2)}\, dx \\ & -c_2\phi_2 \int_{x_0}^{x} \frac{\phi_1}{W^2(\phi_1,\phi_2)}\, dx\Bigg)(c_3\phi_1^* + c_4\phi_2^* + c_5\psi_1^*(t)\phi_1^* - c_5\psi_2^*(t)\phi_2^* \\ & +c_5\phi_1^* \int_{x_0^*}^{x} \frac{\phi_2^*}{W^2(\phi_1^*,\phi_2^*)}\, dx - c_5\phi_2^* \int_{x_0^*}^{x} \frac{\phi_1^*}{W^2(\phi_1^*,\phi_2^*)}\, dx\Bigg)\Bigg]_x\end{aligned} \tag{21}$$

is also a symmetry of (13), where $\phi_i$ and $\phi_i^*$ $(i = 1, 2)$ are spectral functions of (14), (15) and (16), (17) respectively, $\psi_i(t)$ $(i = 1, 2)$ is defined by (19), (20) and

$$\psi_1^*(t) = \int^t \left[\frac{\phi_{2x}^*}{W^2(\phi_1^*,\phi_2^*)}\right]_{x=x_0^*} dt, \tag{22}$$

$$\psi_2^*(t) = \int^t \left[\frac{\phi_{1x}^*}{W^2(\phi_1^*,\phi_2^*)}\right]_{x=x_0^*} dt. \tag{23}$$

# 5 The Toda equation

The Toda equation under consideration is [22]

$$\frac{d^2}{dt^2} \ln v(n) = v(n-1) - 2v(n) + v(n+1) \tag{24}$$

or equivalently

$$\frac{dv(n)}{dt} = v(n)\partial_t^{-1}[v(n-1) - 2v(n) + v(n+1)] \tag{25}$$

which may be rewritten in a coupled form

$$\frac{dp(n)}{dt} = v(n) - v(n+1), \tag{26}$$

$$\frac{dv(n)}{dt} = v(n)(p(n-1) - p(n)). \tag{27}$$

It is known that (25) or (26), (27) has a Lax pair

$$y_{n+1} + p(n)y_n + v(n)y_{n-1} = \lambda y_n, \tag{28}$$

$$y_{nt} = v(n)y_{n-1} - \frac{1}{2}\lambda y_n \tag{29}$$

and its adjoint version is

$$y^*_{n-1} + p(n)y^*_n + v(n+1)y^*_{n+1} = \lambda y^*_n, \tag{30}$$

$$-y^*_{nt} = v(n+1)y^*_{n+1} - \frac{1}{2}\lambda y^*_n. \tag{31}$$

Here the adjoint operator of a difference operator is defined by

$$\left(a(n)e^{k\partial_n}\right)^* = e^{-k\partial_n}a(n).$$

A symmetry of the Toda equation (25) is defined as a solution of its linearized equation

$$\frac{d\sigma(n)}{dt} = \sigma(n)\partial_t^{-1}[v(n-1) - 2v(n) + v(n+1)] + v(n)\partial_t^{-1}[\sigma(n-1) - 2\sigma(n) + \sigma(n+1)]. \tag{32}$$

Just as the two-dimensional Toda equation [23], it is easily verified that $\sigma(n) = (y_n y^*_{n-1})_t$ is a symmetry of the Toda equation (25). To obtain more seed symmetries,we now consider invariance property of (28), (29) and (30), (31). We obtain

**Proposition 5.** *Suppose $y_n$ is a spectral function of (28), (29) and $\lim_{n\to-\infty} p(n) = 0$. Then*

$$\bar{y}_n = y_n \sum_{k=-\infty}^{n} \frac{\prod_{i=-\infty}^{k-1} v(i)}{y_k y_{k-1}}$$

*is also a spectral function of (28), (29).*

**Proof**: direct calculation.

Similarly, we have

**Proposition 6**. *Suppose $y^*_n$ is a spectral function of (30), (31) and $\lim_{n\to\infty} p(n) = 0$. Then*

$$\bar{y}^*_n = y^*_n \sum_{k=n}^{\infty} \frac{\prod_{i=k+2}^{\infty} v(i)}{y^*_k y^*_{k+1}}$$

*is also a spectral function of (30), (31).*

From Proposition 5 and 6, we know

$$\left[\left(c_1 y_n + c_2 y_n \sum_{k=-\infty}^{n} \frac{\prod_{i=-\infty}^{k-1} v(i)}{y_k y_{k-1}}\right)\left(c_3 y^*_{n-1} + c_4 y^*_{n-1} \sum_{k=n-1}^{\infty} \frac{\prod_{i=k+2}^{\infty} v(i)}{y^*_k y^*_{k+1}}\right)\right]_t$$

is also a symmetry of (25), where $y_n$ and $y^*_n$ are spectral functions of (28), (29) and (30), (31) respectively and $c_i$ is an arbitrary constant ($i = 1, 2, 3, 4$).

## 6 Summary

In this paper, nonlocal symmetries are considered for four integrable equations as examples which include the first equation of the AKNS hierarchy, the so-called breaking soliton equation, the Boussinesq equation and the Toda equation. Besides, using invariance properties of corresponding spectral problems under suitable transformations, more nonlocal symmetries can be produced from one seed symmetry.

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