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Nonlinear Integrable Models with Higher d'Alembertian Operator in Any Dimension

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We consider a nonlinear $\mathbf{C}\mathrm{P}^1$-model on Minkowski space of any dimension. To solve its equations of motion is in general not easy, so we study not the full equations but subequations. It is well known that they have rich solutions and an infinite number of conserved currents.

We extend the subequations and show that extended ones also have rich solutions and an infinite number of conserved currents.

1 Introduction

Nonlinear sigma models play an important role in field theory. They are very interesting objects (toy models) to study not only at the classical but also the quantum level, [1, 2].

The $\mathbf{C}\mathrm{P}^1$-model in (1+1)-dimensions is particularly well understood (Belavin–Polyakov et al). The one in $(1+2)$-dimensions is also relatively well studied. But the $\mathbf{C}\mathrm{P}^1$-model in any higher dimensions has not been studied sufficiently because of difficulties arising from higher dimensionality.

We consider the $\mathbf{C}\mathrm{P}^1$-model in $(1+n)$-dimensions. But it is not easy to solve its equations of motion directly, so we change our strategy. We decompose the full equations into subequations (those determine a submodel in the terminology of [3]). By the way these equations have a long history since [4, 5]. Smirnov and Sobolev have constructed (maybe) general solutions for them. At the same time we can construct an infinite number of conserved currents for them. In this sense the submodel is integrable. The construction by Smirnov and Sobolev (S–S construction in our terminology) is clear and suggestive. Getting a hint from S–S construction we extend the submodel stated above, [9]. For our extended submodel we can construct

(A) (maybe) general solutions,

(B) an infinite number of conserved currents

similarly to the case of submodel. That is, our extended system is also integrable.

In this talk I will discuss (A) and (B) in some detail.

2 $\mathbf{CP}^1$-Models in Any Dimension and Submodels

Let M^{1+n} be a $(1+n)$-dimensional Minkowski space and $\eta = (\eta_{\mu\nu}) = \mathrm{diag}(1, -1, \ldots, -1)$ its metric. For a function

$$u : M^{1+n} \to \mathbf{C} \tag{2.1}$$

an action $\mathcal{A}(u)$ is defined as

$$\mathcal{A}(u) \equiv \int d^{1+n}x \, \frac{\partial^\mu u \, \partial_\mu \bar{u}}{(1+|u|^2)^2}. \tag{2.2}$$

This action is invariant under the transformation

$$u \to \frac{1}{u}. \tag{2.3}$$

Therefore u can be lifted from $\mathbf{C}$ to $\mathbf{C}\mathrm{P}^1$, [6]. The equations of motion of (2.2) read

$$\left(1+|u|^2\right)\partial^\mu\partial_\mu u - 2\bar{u}\partial^\mu u\partial_\mu u = 0. \tag{2.4}$$

We want to solve (2.4) completely, but it is not easy. When $n = 1$ many solutions were constructed by Belavin and Polyakov, [1]. For $n \geq 2$ much is not known about the construction of solutions as far as we know.

Here changing the way of thinking, we try to solve not the full equations (2.4) but

$$\partial^\mu\partial_\mu u = 0 \quad \text{and} \quad \partial^\mu u\partial_\mu u = 0. \tag{2.5}$$

Of course if u is a solution of (2.5), then u satisfies (2.4). We call (2.5) subequations of (2.4) (or submodel of $\mathbf{C}\mathrm{P}^1$-model in the terminology of [3]). (2.5) is much milder than (2.4).

Now our aim in the following is

(A) to write down all solutions of (2.5),

(B) to write down all conserved currents of (2.5).

Here a conserved current is a vector $(V_\mu)_{\mu=0,\ldots,n}$ satisfying

$$\partial^\mu V_\mu(u,\bar{u}) = 0. \tag{2.6}$$

3 Construction of Solutions and Conserved Currents

(A) has a long history [4, 5]. Now we make a short review of Smirnov–Sobolev's construction (S–S construction as abbreviated). For u let $a_0(u), a_1(u), \ldots, a_n(u)$ and $b(u)$ be functions given and we set

$$\delta \equiv \sum_{\mu=0}^{n} a_\mu(u)x_\mu - b(u) \tag{3.1}$$

and

$$a^\mu(u)a_\mu(u) \equiv a_0^2(u) - \sum_{j=1}^{n} a_j^2(u) = 0. \tag{3.2}$$

Putting $\delta = 0$, we solve as

$$\delta = 0 \quad \Rightarrow \quad u = u(x_0, x_1, \ldots, x_n) \tag{3.3}$$

by the inverse function theorem. Then

Proposition 3.1 ([4, 5]). *u is a solution of (2.5).*

Next we turn to (B). In [3, 6, 7] an infinite number of conserved currents was constructed by the representation theory of Lie algebras ($su(2)$ or $su(1,1)$). But their results are extended further.

Let f be a function of C^2-class on $\mathbf{C}$. For $\tilde{f}(x_\mu) \equiv f(u(x_\mu), \bar{u}(x_\mu))$ we set

$$V_\mu(\tilde{f}) \equiv \partial_\mu u\frac{\partial f}{\partial u} - \partial_\mu \bar{u}\frac{\partial f}{\partial \bar{u}}. \tag{3.4}$$

Then

Proposition 3.2 ([9]). *$V_\mu(\tilde{f})$ is a conserved current of (2.5).*

From this proposition we find that (2.5) has uncountably many conserved currents (all C^2-class functions on $\mathbf{C}$).

Remark 3.1. (A) and (B) seem at first sight unrelated. But the existence of an infinite number of conserved currents implies the infinite number of symmetries, therefore they give a strict restriction on ansatz of construction of solutions. As a consequence we have only Smirnov–Sobolev's construction. This is our story (conjecture). We want to prove this at any cost.

4 New Models with Higher Order Derivatives

Let us extend the results in Section 2 and Section 3. For that we look over the S–S construction again.

$$\delta \equiv \sum_{\mu=0}^{n} a_\mu(u) x_\mu - b(u), \tag{4.1}$$

$$a_0^2(u) - \sum_{j=1}^{n} a_j^2(u) = 0. \tag{4.2}$$

Here we try to change the power in (4.2) from 2 to an arbitrary integer p ($p \geq 2$)

$$a_0^p(u) - \sum_{j=1}^{n} a_j^p(u) = 0. \tag{4.3}$$

Under this condition we solve (4.1) as

$$\delta = 0 \quad \Rightarrow \quad u = u(x_0, x_1, \ldots, x_n). \tag{4.4}$$

We call this an extended S–S construction.

Problem. What are differential equations which u in (4.4) satisfies?

We are considering the converse of Section 2 and Section 3. That is, first of all a "solution" is given and next we look for a system of equations which u satisfies. But it is not easy to extend subequations (2.5) in this fashin. Trying to transform (2.5) in an equivalent manner we reach

Lemma 4.1. *(2.5) is equivalent to*

$$\Box_2 u \equiv \partial^\mu \partial_\mu u = 0 \quad \textit{and} \quad \Box_2(u^2) = 0. \tag{4.5}$$

This form is very clear and suggestive. We can extend (4.5) to obtain

Definition 4.1. $$\Box_p(u^k) \equiv \left(\frac{\partial^p}{\partial x_0^p} - \sum_{j=1}^{n} \frac{\partial^p}{\partial x_j^p} \right)(u^k) = 0 \quad \text{for} \quad 1 \leq k \leq p. \tag{4.6}$$

Next let F_n be a Bell polynomial (see [9, 11] for details) and we set $F_{n,\mu}$ as

$$F_{n,\mu} \equiv: F_n\left(\partial_\mu u \frac{\partial}{\partial u}, \partial_\mu^2 u \frac{\partial}{\partial u}, \cdots, \partial_\mu^n u \frac{\partial}{\partial u}\right) : \tag{4.7}$$

Here : : is the normal ordering (moving differentials to right end). Let $\bar{F}_{n,\mu}$ be the complex conjugate of $F_{n,\mu}$ ($u \to \bar{u}$). For f any C^p-class function on $\mathbf{C}$ we set

$$V_{p,\mu}(\tilde{f}) \equiv \sum_{k=0}^{p-1} (-1)^k : F_{p-1-k,\mu} \bar{F}_{k,\mu} : (f). \tag{4.8}$$

For examples when $p = 2$ and 3

$$V_{2,\mu}(\tilde{f}) \equiv F_{1,\mu}(f) - \bar{F}_{1,\mu}(f) = \partial_\mu u \frac{\partial f}{\partial u} - \partial_\mu \bar{u} \frac{\partial f}{\partial \bar{u}}, \tag{4.9}$$

$$\begin{aligned} V_{3,\mu}(\tilde{f}) &\equiv F_{2,\mu}(f) - : F_{1,\mu} \bar{F}_{1,\mu} : (f) + \bar{F}_{2,\mu}(f) \\ &= \partial_\mu^2 u \frac{\partial f}{\partial u} + (\partial_\mu u)^2 \frac{\partial^2 f}{\partial u^2} - \partial_\mu u \partial_\mu \bar{u} \frac{\partial^2 f}{\partial u \partial \bar{u}} + \partial_\mu^2 \bar{u} \frac{\partial f}{\partial \bar{u}} + (\partial_\mu \bar{u})^2 \frac{\partial^2 f}{\partial \bar{u}^2}. \end{aligned} \tag{4.10}$$

Under the preparations mentioned above

Theorem 4.1 ([9, 10]). *We have the following:*

(A) u in (4.4) is a solution of (4.6),

(B) $V_{p,\mu}(\tilde{f})$ in (4.8) is a conserved current of (4.6).

We could extend the results corresponding to $p = 2$ in Section 3 to ones corresponding to any p in a complete manner.

Remark 4.1. Our extended S–S construction may give general solutions. And moreover the statement in the comment in Section 3 may hold even in this case.

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