Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 109–115.

On Symmetry Reduction and Some Exact Solutions of the Multidimensional Born–Infeld Equation

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Using the subgroup structure of the extended generalized Poincaré group $\tilde{P}(1,4)$, symmetry reduction of the multidimensional Born–Infeld equation to differential equations with fewer independent variables is made. Some classes of exact solutions of the equation under investigation are constructed.

The Born–Infeld equation in spaces of various dimensions has many applications (see, for example, [1–8]).

The symmetry properties of the multidimensional Born–Infeld equation were studied in [9–11]. In these works, multiparameter families of exact solutions were constructed using special ansatzes.

Let us consider the equation

$$\Box u \left(1 - u_\nu u^\nu\right) + u^\mu u^\nu u_{\mu\nu} = 0, \tag{1}$$

where $u = u(x)$, $x = (x_0, x_1, x_2, x_3) \in M(1,3)$, $u_\mu \equiv \frac{\partial u}{\partial x^\mu}$, $u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x^\mu \partial x^\nu}$, $u^\mu = g^{\mu\nu} u_\nu$, $g_{\mu\nu} = (1,-1,-1,-1)\delta_{\mu\nu}$, $\mu,\nu = 0,1,2,3$, $\Box$ is the d'Alembert operator.

The symmetry group of equation (1) is the group $\tilde{P}(1,4)$ [9–11].

On the basis of the subgroup structure of the group $\tilde{P}(1,4)$ and the invariants of its subgroups [12], the symmetry reduction of the investigated equation to differential equations with fewer independent variables was done. In many cases the reduced equations are linear ODEs. Taking into account solutions of the reduced equations, we found multiparameter families of exact solutions of the equation under consideration. Below we only present ansatzes which reduce equation (1) to ordinary differential equations, and we list the ODEs obtained as well as some exact solutions of the Born–Infeld equation

1. $u^2 = -x_2^2 \varphi^2(\omega) + x_0^2, \quad \omega = \dfrac{x_3}{x_2},$

   $\left(\varphi^2 - \omega^2 - 1\right)\varphi\varphi'' - \left(\varphi^2 - \omega^2 - 1\right)\varphi'^2 - 2\varphi'^2 - 2\omega\varphi\varphi' = 1;$

2. $u^2 = -\left(x_1^2 + x_2^2\right)\varphi^2(\omega) + x_0^2, \quad \omega = \dfrac{x_3}{\left(x_1^2 + x_2^2\right)^{1/2}};$

   $\left(\varphi^2 - \omega^2 - 1\right)\varphi\varphi'' + \left(\left(3\varphi^2 + 1\right)\left(\omega^2 + 1\right) - \varphi\right)\varphi'^2 - 3\omega\left(\varphi^2 + 1\right)\varphi\varphi' + \varphi^4 = 1;$

3. $u = -x_2^{\frac{\alpha+1}{\alpha}}\left(\varphi(\omega)\right)^{\frac{\alpha+1}{\alpha}} + x_0, \quad \omega = \dfrac{x_3}{x_2}, \ \alpha \neq 0;$

   $\varphi'' = 0; \qquad u = x_0 - \left(c_1 x_2 + c_2 x_3\right)^{\frac{\alpha+1}{\alpha}};$

4. $u = -\varphi(\omega) - 2\ln x_2 + x_0, \quad \omega = \dfrac{x_3}{x_2};$

   $2\varphi'' + \varphi'^2 = 0; \qquad u = 2c_1 e^{-\frac{x_3}{2x_2}} - 2\ln x_2 + x_0 - c_2;$

5. $u=-\varphi(\omega)+x_0-\ln\left(x_1^2+x_2^2\right)+2\alpha\operatorname{arctg}\dfrac{x_2}{x_1},\quad \omega=\dfrac{x_3}{\left(x_1^2+x_2^2\right)^{1/2}},\ \alpha\geq 0;$

$4\left(\alpha^2\omega^2+\alpha^2+1\right)\varphi''-\omega\left(\omega^2+1\right)\varphi'^3+2\varphi'^2=0;$

6. $u=-\varphi(\omega)+x_0+2\operatorname{arctg}\dfrac{x_2}{x_1},\quad \omega=\dfrac{x_3}{\left(x_1^2+x_2^2\right)^{1/2}};$

$4\left(\omega^2+1\right)\varphi''-\omega\left(\omega^2+1\right)\varphi'^3+4\omega\varphi'=0;$

7. $u^2=x_0^2\varphi^2(\omega)-x_3^2,\quad \omega=\dfrac{\left(x_1^2+x_2^2\right)^{1/2}}{x_0};$

$\omega\left(\varphi^2+\omega^2-1\right)\varphi\varphi''+\left(\omega^2-1\right)\varphi\varphi'^3-\omega\left(2\varphi^2+\omega^2-1\right)\varphi^2+$
$+\left(\varphi^2+\omega\left(2\omega-1\right)\right)\varphi\varphi'-\omega\left(\varphi^2-1\right)=0;$

8. $u^2=-\left(x_1^2+x_2^2\right)\varphi^2(\omega)+x_0^2-x_3^2,\quad \omega=\ln\dfrac{\left(x_0-u\right)^2}{x_1^2+x_2^2}-2c\operatorname{arctg}\dfrac{x_2}{x_1},\quad c>0;$

$4\left(c^2\varphi^4-\left(c^2-1\right)\varphi^2-1\right)\varphi\varphi''-8\left(c^2+1\right)\varphi^3\varphi'^3+\left(\varphi^4+8c^2\varphi^2-4\right)\varphi'^4-$
$-\left(6\varphi^4+4\varphi^2-10\right)\varphi\varphi'+\varphi^2\left(\varphi^4+\varphi^2-2\right)=0;$

9. $u^2=-\left(x_1^2+x_2^2\right)\varphi^2(\omega)+x_0^2-x_3^2,$

$\omega=(1+\alpha)\ln\left(x_0^2-x_3^2-u^2\right)-2\alpha\ln\left(x_0-u\right)-2\beta\operatorname{arctg}\dfrac{x_2}{x_1},\quad \alpha\neq 0,\quad \beta\geq 0;$

$4\left(\alpha^2+\left(\beta^2-\alpha^2+1\right)\varphi^2-\beta^2\varphi^4\right)\varphi^2\varphi''+8\left(\alpha^2+1\right)\left(\beta^2\varphi^2+\alpha^2+\alpha-2\right)\varphi'^3-$
$-4\left(\left(2\beta^2-\alpha^2+1\right)\varphi^2+3\alpha^2-2\alpha-6\right)\varphi\varphi'^2+$
$+2\left(\varphi^2-\alpha-6\right)\varphi^2\varphi'-\varphi^3\left(\varphi^4+\varphi^2-2\right)=0;$

10. $u^2=-\left(x_1^2+x_2^2\right)\varphi^2(\omega)+x_0^2-x_3^2,\quad \omega=\ln\left(x_0^2-x_3^2-u^2\right)-2\alpha\operatorname{arctg}\dfrac{x_2}{x_1},\quad \alpha>0;$

$4\left(\alpha^2+1-\alpha^2\varphi^2\right)\varphi^4\varphi''+8\left(\alpha^2\varphi^2-2\right)\varphi'^3-4\left(\left(2\alpha^2+1\right)\varphi^2-6\right)\varphi\varphi'^2+$
$+2\left(\varphi^2-6\right)\varphi^2\varphi'-\varphi^3\left(\varphi^4+\varphi^2-2\right)=0;$

11. $u^2=-\left(x_1^2+x_2^2\right)\varphi^2(\omega)+x_0^2-x_3^2,\quad \omega=2\operatorname{arctg}\dfrac{x_2}{x_1}-\ln\left(x_1^2+x_2^2\right);$

$4\varphi\left(\alpha^2\left(\varphi^2-1\right)-1\right)\varphi''-8\left(\alpha^2+1\right)\varphi\varphi'^3+4\left(3\varphi^2+2\left(\alpha^2+1\right)\right)\varphi'^2-$
$-2\left(3\varphi^2+2\right)\varphi\varphi'+\varphi^2\left(\varphi^2+1\right)=2.$

The ansatzes (1)–(11) can be written in the following form: $h(u)=f(x)\varphi(\omega)+g(x)$, where $h(u)$, $f(x)$, $g(x)$ are given functions, $\varphi(\omega)$ is an unknown function. $\omega=\omega(x,u)$ are one-dimensional invariants of subgroups of the group $\tilde{P}(1,4)$.

12. $x_2\omega=x_3\varphi(\omega),\quad \omega=\dfrac{x_3}{(x_0-u)},$

$\varphi''=0,\qquad u=x_0+c_1x_2+c_2x_3;$

13. $x_3\omega=\left(x_1^2+x_2^2\right)^{1/2}\varphi(\omega),\quad \omega=\dfrac{\left(x_1^2+x_2^2\right)^{1/2}}{u};$

$\omega\left(\varphi^2+\omega^2+1\right)\varphi''+\left(\omega^2+1\right)\varphi'^3-2\omega\varphi\varphi'^2+\left(\varphi^2+1\right)\varphi'=0;$

14. $x_3\omega = \left(x_1^2 + x_2^2\right)^{1/2} \varphi(\omega), \quad \omega = \dfrac{\left(x_1^2 + x_2^2\right)^{1/2}}{x_0 - u};$

$$\omega\varphi'' + \varphi'^3 + \varphi' = 0,$$

$$\frac{x_3}{x_0 - u} = \ln\left(\frac{2\sqrt{x_1^2 + x_2^2}}{x_0 - u} + 2\sqrt{\frac{x_1^2 + x_2^2}{(x_0 - u)^2} - 1}\right) + c;$$

15. $\dfrac{x_3}{\left(x_3\left(2x_0\omega - x_3\right) - x_1^2\omega^2\right)^{1/2}} = \varphi(\omega), \quad \omega = \dfrac{x_3}{x_0 - u},$

$$\left(\varphi^2\varphi^2 - 1\right)\varphi\varphi'' + \left(4 - 3\omega^2\varphi^2\right)\varphi'^2 + 2\omega\varphi\varphi^3\varphi' + \varphi^4 = 0;$$

16. $\omega\left(\dfrac{x_3}{\omega}\left(2x_0 - \dfrac{x_3}{\omega}\right) - x_1^2 - x_2^2\right)^{1/2} = x_3\varphi(\omega), \quad \omega = \dfrac{x_3}{x_0 - u};$

$$\left(\omega^2 - \varphi^2\right)\varphi^2\varphi'' + \left(\omega^2 - 3\varphi^2\right)\varphi'^2 + 4\omega\varphi\varphi' - 2\varphi^2 = 0;$$

17. $\left(x_0^2 - x_1^2 - x_2^2\right)^{1/2}\omega = x_3\varphi(\omega), \quad \omega = \dfrac{x_3}{u},$

$$\left(\varphi^2 - \omega^2 - 1\right)\varphi''\varphi + 2\left(\omega^2 + 1\right)\varphi'^2 - 4\omega\varphi\varphi' + 2\varphi^2 - 2 = 0;$$

18. $\ln\dfrac{x_3}{\omega} + \dfrac{x_1\omega}{x_3} = \varphi(\omega), \quad \omega = \dfrac{x_3}{x_0 - u};$

$$\varphi'' = 0; \qquad \frac{c_1x_3 - x_1}{x_0 - u} - \ln\left(x_0 - u\right) + c_2 = 0;$$

19. $\ln x_3^2\omega - \dfrac{x_3^2}{\omega} = \varphi(\omega) - 2x_0, \quad \omega = \dfrac{x_3^2}{x_0 - u};$

$$4\omega\left(\omega + 1\right)\varphi'' - 2\omega\varphi'^2 + 2\left(\omega + 1\right)\varphi' = -1;$$

20. $\dfrac{x_3}{\omega x_2 - x_1} = \varphi(\omega), \quad \omega = x_0 - u;$

$$\varphi = 0; \qquad u = x_0 - \frac{x_1 + x_3}{x_2};$$

21. $\ln\dfrac{x_3^2}{\omega} - \dfrac{x_1^2\omega}{x_3^2} - \dfrac{x_3^2}{\omega} = \varphi(\omega) - 2x_0, \quad \omega = \dfrac{x_3^2}{x_0 - u};$

$$\omega\left(\omega + 4\right)\varphi'' - 4\omega\varphi'^2 + 2\left(2\omega + 1\right)\varphi' = -3;$$

22. $\dfrac{x_3^2}{\left(\omega + 1\right)\left(\omega\left(2x_0 - \omega\right) - x_1^2\right) - \omega x_2^2} = \varphi(\omega), \quad \omega = x_0 - u;$

$$\left(\omega\left(\omega + 1\right)\right)^2\left(\varphi\varphi'' - 2\varphi'^2\right) + 4\omega\left(\omega + 1\right)\left(2\omega + 1\right)\varphi\varphi' + 2\omega\left(\omega + 1\right)\varphi' -$$
$$-2\left(7\omega^2 + 7\omega + 2\right)\varphi^2 + 6\left(2\omega + 1\right)\varphi = 0;$$

23. $\ln\dfrac{x_3^2}{\omega} - \dfrac{\left(x_1^2 + x_2^2\right)\omega}{x_3^2} - \dfrac{x_3^2}{\omega} = \varphi(\omega) - 2x_0, \quad \omega = \dfrac{x_3^2}{x_0 - u};$

$$\omega\left(\omega + 4\right)\varphi'' - 6\omega\varphi'^2 + 2\left(3\omega + 1\right)\varphi' = -5;$$

24. $x_3\omega = \left(x_1^2 + x_2^2\right)^{1/2} \varphi(\omega) - \left(x_1^2 + x_2^2\right)^{1/2} \operatorname{arctg} \dfrac{x_2}{x_1}, \quad \omega = \dfrac{\left(x_1^2 + x_2^2\right)^{1/2}}{x_0 - u};$

$\omega\left(\omega^2 + 1\right)\varphi'' + \omega^2\varphi'^3 + \left(\omega^2 + 2\right)\varphi' = 0;$

$$\frac{x_3}{x_0 - u} + \operatorname{arctg}\frac{x_2}{x_1} = i\left(\ln\frac{\sqrt{x_1^2 + x_2^2}}{\sqrt{x_1^2 + x_2^2 + (x_0 - u)^2} + x_0 - u} + \frac{\sqrt{x_1^2 + x_2^2 + (x_0 - u)^2}}{x_0 - u} + c\right);$$

25. $\left(2x_0\omega\left(x_1^2 + x_2^2\right)^{1/2} - x_3^2\omega^2 - x_1^2 - x_2^2\right)^{1/2} = \left(x_1^2 + x_2^2\right)^{1/2}\varphi(\omega), \quad \omega = \dfrac{\left(x_1^2 + x_2^2\right)^{1/2}}{x_0 - u};$

$\omega\left(\varphi^3 - 2\varphi^2 + \omega^2\right)\varphi'' - \left(\varphi\varphi'\right)^3 + \omega^3\varphi'^2 - \left(\varphi^2 - 2\omega^2\right)\varphi\varphi' - 5\omega\varphi^2 = 0;$

26. $x_3\omega + \sqrt{x_1^2 + x_2^2}\ln\dfrac{\sqrt{x_1^2 + x_2^2}}{\omega} = \left(x_1^2 + x_2^2\right)^{1/2}\varphi(\omega) - \alpha\left(x_1^2 + x_2^2\right)^{1/2}\operatorname{arctg}\dfrac{x_2}{x_1},$

$\omega = \dfrac{\left(x_1^2 + x_2^2\right)^{1/2}}{x_0 - u}, \quad \alpha \geq 0;$

$\omega\left(\omega^2 + \alpha^2\right)\varphi'' + \omega^2\varphi'^3 + \left(\omega^2 + 2\alpha^2\right)\varphi' = 0;$

$$\frac{x_3}{x_0 - u} + \ln(x_0 - u) + \alpha\operatorname{arctg}\frac{x_2}{x_1} =$$
$$= i\left(\alpha^2\ln\frac{\sqrt{x_1^2 + x_2^2}}{\sqrt{x_1^2 + x_2^2 + \alpha^2(x_0 - u)^2} + \alpha^2(x_0 - u)} + \frac{\sqrt{x_1^2 + x_2^2 + \alpha^2(x_0 - u)^2}}{x_0 - u} + c\right);$$

27. $3\ln\left(\omega\left(x_1^2 + x_2^2\right)^{1/2}\right) - 2\ln\left(12x_0 - \left(\omega\left(x_1^2 + x_2^2\right)^{1/2} + 4x_3\right)^{1/2} \times\right.$

$\left.\times\left(6 + 2x_3 - \omega\left(x_1^2 + x_2^2\right)^{1/2}\right)\right) = \varphi(\omega), \quad \omega = \dfrac{(x_0 - u)^2 - 4x_3}{\left(x_1^2 + x_2^2\right)^{1/2}};$

$144\left(\left(\omega^2 + 16\right)e^\varphi - \omega^2\right)\omega\varphi'' + \omega^4\left(\omega^2 + 16\right)\varphi'^3 -$

$-24\omega\left(3\left(\omega^2 + 16\right)e^\varphi + 2\omega^2\right)\varphi'^2 + 144\left(\left(40 + \omega^2\right)e^\varphi - \omega^2\right)\varphi' = 0;$

28. $\ln\left(\omega\left(x_1^2 + x_2^2\right)\right) - \omega\left(x_1^2 + x_2^2\right) - \dfrac{x_3^2}{\omega\left(x_1^2 + x_2^2\right)} = \varphi(\omega) - 2\alpha\operatorname{arctg}\dfrac{x_2}{x_1} - 2x_0,$

$\omega = \dfrac{x_0 - u}{x_1^2 + x_2^2}, \quad \alpha \geq 0;$

$\omega^2\left(4\omega\left(1 - \alpha^2\omega\right) + 1\right)\varphi'' + 2\omega^5\varphi'^3 - 2\omega^3\varphi'^2 - 2\left(\alpha^2\omega^2 - 2\omega + 1\right)\varphi'\omega - 2\alpha^2\omega + 3 = 0;$

29. $\omega\left(x_1^2 + x_2^2\right) + \dfrac{x_3^2}{\omega\left(x_1^2 + x_2^2\right)} = \varphi(\omega) + 2\operatorname{arctg}\dfrac{x_2}{x_1} + 2x_0, \quad \omega = \dfrac{x_0 - u}{x_1^2 + x_2^2},$

$\omega^2\left(1 - 4\omega^2\right)\varphi'' + 2\omega^5\varphi'^3 - 2\omega^3\varphi'^2 - 2\omega\left(\omega^2 + 1\right)\varphi' - 2\omega + 1 = 0;$

30. $\ln\left((\omega + \alpha)\left(\omega(2x_0 - \omega) - x_1^2\right) - x_2^2\omega\right) - 2\ln\left(\dfrac{\beta x_2}{\omega + \alpha} + \dfrac{x_1}{\omega} - x_3\right) = \varphi(\omega),$

$\omega = x_0 - u, \quad \alpha > 0, \quad \beta \geq 0;$

$$\omega^4(\omega+\alpha)^4\varphi'' + \omega^4(\omega+\alpha)^4\varphi'^2 - 4\omega^3(\omega+\alpha)^3(2\omega+\alpha)\varphi' + 2\omega(\omega+\alpha)\beta^2\omega^2 + (\omega+\alpha)^2+$$
$$+\left(\omega(\omega+\alpha)^2\right)e^{\varphi}\varphi' - 2e^{\varphi}\left(\beta^2\omega^2(7\omega+3\alpha) + 3(2\omega+\alpha)\left(\omega+\alpha^2\right)^2+\right.$$
$$\left.+(4\omega+\alpha)(\omega+\alpha)^2\omega^2\right) = 0;$$

31. $\ln\left(\omega\left(2x_0-\omega\right) - x_1^2 - x_2^2\right) - 2\ln\left(x_3\omega - x_1\right) = \varphi(\omega), \quad \omega = x_0 - u;$
$$\omega^2\varphi'' + \omega^2\varphi'^2 + 2\omega\left(e^{\varphi}\left(\omega^2+1\right)-1\right)\varphi' - 2\left(\omega^2+2\right)e^{\varphi} = 4;$$

32. $\ln\left(\omega\left(2x_0-\omega\right) - x_1^2 - \dfrac{\omega}{\omega+1}x_2^2\right) - 2\ln\left(\dfrac{\alpha x_2}{\omega+1} + \dfrac{x_1}{\omega} - x_3\right) = \varphi(\omega),$
$\omega = x_0 - u, \quad \alpha > 0;$
$$\omega^4\left(\omega+1\right)^3\left(\varphi''+\varphi'^2\right) + 2\left(\omega+1\right)\omega\left(e^{\varphi}\left(\omega^2+\left(\omega^2+1\right)^3\right) - 2\omega^2\left(\omega+1\right)^2 - \omega^3\right)\varphi'+$$
$$+2e^{\varphi}\left(\left(\omega+1\right)^2\left(\omega^4-3\omega^3-2\omega^2-4\omega-4\right) - 2\alpha^2\left(2\omega+1\right)\omega^2\right)+$$
$$+2\left(3\omega+2\right)\left(\omega+1\right)^2\omega^2 = 0.$$

The ansatzes (12)–(32) can be written in the following form: $h(x,\omega) = f(x)\varphi(\omega) + g(x)$, where $h(x,\omega)$, $f(x)$, $g(x)$ are given functions, $\varphi(\omega)$ is an unknown function. $\omega = \omega(x,u)$ are one-dimensional invariants of subgroups of the group $\tilde{P}(1,4)$.

33. $2\ln\left(x_0-u\right) = \varphi(\omega) + \ln\left(x_1^2+x_2^2\right) + 2c\,\mathrm{arctg}\,\dfrac{x_2}{x_1}, \quad \omega = \dfrac{x_3}{\left(x_1^2+x_2^2\right)^{1/2}};$
$$4\left(c^2\left(\omega^2+1\right)+1\right)\varphi'' - \omega\left(\omega^2+1\right)\varphi'^3 + 2\left(3\omega^2+2\right)\varphi'^2 - 12\omega\varphi' + 8\left(c^2+1\right) = 0;$$

34. $\dfrac{x_0}{\left(x_1^2+x_2^2+u^2\right)^{1/2}} = \varphi(\omega), \quad \omega = \dfrac{x_3}{x_0};$
$$\left(\varphi^2-\omega^2-1\right)\varphi^2\varphi'' + 2\omega\varphi'^3 - 4\varphi\varphi'^2 - 2\omega\varphi^4\varphi' + 2\varphi^7 = 0;$$

35. $\dfrac{\left(u-x_0\right)^{\alpha-1}}{\left(u+x_0\right)^{\alpha+1}} = \varphi(\omega), \quad \omega = \dfrac{x_3}{\left(x_0^2-u^2\right)^{1/2}}, \quad \alpha > 0;$
$$\left(\alpha^2-1-\omega^2\right)\varphi^2\varphi'' + \omega\left(1-\omega^2\right)\varphi'^3 + \left(4\alpha^2-8+\left(4\alpha^2+6\right)\omega^2\right)\varphi\varphi'^2-$$
$$-2\left(\alpha^2-2\alpha+7\right)\omega\varphi'\varphi^2 - 8\left(\alpha^2-1\right)\varphi^3 = 0;$$

36. $2\alpha\ln\left(x_0-u\right) = \varphi(\omega) + (1+\alpha)\ln\left(x_1^2+x_2^2\right) - 2\beta\,\mathrm{arctg}\,\dfrac{x_2}{x_1},$
$\omega = \dfrac{x_3}{\sqrt{x_1^2+x_2^2}}, \quad \beta \geq 0, \quad \alpha^2+\beta^2 \neq 0;$
$$4\alpha\left(\left(\alpha+1\right)^2 + \beta^2\left(\omega^2+1\right)\right)\varphi'' - \alpha\omega\left(\omega^2+1\right)\varphi'^3+$$
$$+2\left(3\alpha(\alpha+1)\omega^2 + \alpha^2 - \beta^2 - 1\right)\varphi'^2 + 4\left(2\beta\left(\alpha+1-\beta^2\right) - 3\alpha(\alpha+1)^2\right)\omega\varphi'+$$
$$+4\alpha(\alpha+1)\left(2(\alpha+1)^2 + 2\beta^2 + 1\right) + 2\left(\beta^2-\alpha^2+1\right) = 0;$$

37. $\dfrac{x_3}{\sqrt{x_0^2-x_1^2-u^2}} = \varphi(\omega), \quad \omega = 2\gamma\ln\left(x_0-u\right) - (1+\gamma)\ln\left(x_0^2-x_1^2-u^2\right), \quad \gamma \neq 0;$
$$2\left(\gamma^2\varphi^2 + (\gamma+1)(3\gamma+1)\right)\varphi'' - 4(\gamma+1)\left(\gamma^2+\gamma-2\right)\varphi'^3+$$
$$+2\left(\gamma^2+\gamma-2\right)\varphi\varphi'^2 + \left((\gamma+6)\varphi^2 - 3\gamma - 1\right)\varphi' - \varphi^3 + \varphi = 0;$$

38. $\dfrac{x_3}{\sqrt{x_0^2-x_1^2-x_2^2-u^2}}=\varphi(\omega)$,

$\omega=2\alpha\ln(x_0-u)-(1+\alpha)\ln\left(x_0^2-x_1^2-x_2^2-u^2\right),\ \alpha\neq 0$;

$4\left(\alpha^2\varphi^2+(\alpha+1)(3\alpha+1)\right)\varphi''-8(\alpha+1)\left(\alpha^2+\alpha-3\right)\varphi'^3+$
$+4\left(2\alpha^2-6\alpha-9\right)\varphi\varphi'^2+2\left((2\alpha+9)\varphi^2-2\alpha-4\right)\varphi'-3\left(\varphi^3-\varphi\right)=0$;

39. $\dfrac{x_3}{(x_0-u)^2-4x_1}=\varphi(\omega)$,

$\omega=3\ln\Big((x_0-u)^2-4x_1\Big)-2\ln\Big(6\,(x_0+u)-6x_1\,(x_0-u)+(x_0-u)^3\Big)$;

$144\left(\left(16\varphi^2+1\right)e^{\omega}-1\right)\varphi''-2592e^{\omega}\varphi'^3+432e^{\omega}\varphi\varphi'^2+9\left(\left(16\varphi^2+1\right)e^{\omega}+2\right)\varphi'-8\varphi=0$;

40. $(x_0-u)^2=x_2\varphi(\omega)+4x_1,\quad \omega=\dfrac{x_3}{x_2};\quad \left(\varphi^2+16\omega^2+16\right)\varphi''=0$;

$u=x_0-(4x_1+c_1x_2+c_2x_3)^{1/2},\quad u=x_0-2\sqrt{x_1+i\left(x_2^2+x_3^2\right)^{1/2}}$;

41. $\left(\dfrac{x_1^2+x_2^2}{x_3^2+u}\right)^{1/2}=\varphi(\omega),\quad \omega=c\,\mathrm{arctg}\,\dfrac{x_2}{x_1}-\mathrm{arctg}\,\dfrac{u}{x_3},\quad 0<c\leq 1$;

$\left(\varphi^2+1\right)\left(\varphi^2+c^2\right)\varphi\varphi''-\left(\varphi^2+c^2\left(\varphi+2\right)\right)\varphi'^2+\varphi^2\left(\varphi^4-1\right)=0$;

42. $\left(\dfrac{x_1^2+x_2^2}{x_3^2+u}\right)^{1/2}=\varphi(\omega),\quad \omega=2\alpha\,\mathrm{arctg}\,\dfrac{x_2}{x_1}+2\beta\,\mathrm{arctg}\,\dfrac{u}{x_3}-\ln\left(x_1^2+x_2^2\right),\quad \alpha>0,\quad \beta\geq 0$;

$4\varphi\left(\varphi^4+\left(\alpha^2+\beta^2+1\right)\varphi^2+\alpha^2\right)\varphi''-4\left(\varphi^2\left(\alpha^2+\beta^2+1\right)+2\alpha+1\right)\varphi'^2-$
$-4\varphi\varphi'+\varphi^2\left(\varphi^4-1\right)=0$;

43. $\dfrac{\left(x_0^2-x_1^2-x_2^2-x_3^2-u^2\right)^{1/2}}{x_0-u}=\varphi(\omega),\quad \omega=\dfrac{x_1}{x_0-u}+\ln(x_0-u)$;

$\varphi\left(\varphi^2-1\right)\varphi''+\left(3\varphi^2-1\right)\varphi'^2+5\varphi\varphi'+3\varphi^2=0$;

44. $(x_0-u)^2=\left(x_1^2+x_2^2\right)^{1/2}\varphi(\omega)+4x_3,\quad \omega=4c\,\mathrm{arctg}\,\dfrac{x_2}{x_1}+\ln\left(x_1^2+x_2^2\right),\quad c>0$;

$16\left(c^2\varphi^2+4\left(4c^2+1\right)\right)\varphi''+8\left(4c^2+1\right)\varphi'^3+12\varphi\varphi'^2+2\left(32+3\varphi^2\right)\varphi'+\varphi^3+16\varphi=0$;

45. $\dfrac{x_1+x_2+x_3^2}{x_0-u}=\varphi(\omega),\quad \omega=x_0+u+\ln(x_0-u)$;

$\varphi\,(4+\varphi)\,\varphi''-\varphi'^3-2\,(4+\varphi)\,\varphi'^2-4\varphi\varphi'+6\varphi=0.$

The ansatzes (33)–(45) can be written in the following form: $h(x,u)=f(x)\varphi(\omega)+g(x)$, where $h(x,u)$, $f(x)$, $g(x)$ are given functions, $\varphi(\omega)$ is an unknown function, $\omega=\omega(x,u)$ are one-dimensional invariants of subgroups of the group $\tilde{P}(1,4)$.

Let us note that the equation (1) was also studied with the help of the subgroup structure of the group $P(1,4)$ as well as invariants of its nonconjugate subgroups. Some of the results we obtained were published in [13, 14].

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