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On Symmetry Reduction of the Five-Dimensional Dirac Equation

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Using the subgroup structure of the generalized Poincaré group $P(1,4)$, the symmetry reduction of the five-dimensional Dirac equation to systems of differential equations with fewer independent variables is done.

The Dirac equation in spaces of various dimensions has many applications (see, for example, [1–9]).

Let us consider the equation

$$\left(\gamma_k P^k - m\right)\psi(x) = 0, \tag{1}$$

where $x = (x_0, x_1, x_2, x_3, x_4)$, $P_k = i\frac{\partial}{\partial x_k}$, $k = 0,1,2,3,4$; γ_k are 4×4 Dirac matrices. Equation (1) is invariant under the generalized Poincaré group $P(1,4)$. Continuous subgroups of the group $P(1,4)$ were in [10–14]. For all continuous subgroups of the group $P(1,4)$ invariants in five-dimensional Minkowski space are constructed. The majority of these invariants has been presented in [15, 16].

Following [17–23] and using the subgroup structure of the group $P(1,4)$ ansatzes which reduce equation (1) to systems of differential equations with fewer independent variables were constructed and the corresponding symmetry reduction has been obtained. Some of these results were presented in [24].

In the present paper we give some new results which are obtained on the basis of subgroup structure of the group $P(1,4)$ and invariants of its nonconjugate subgroups.

First we present some ansatzes which reduce the equation (1) to systems of ordinary differential equations and we give the corresponding systems of reduced equations.

1. $$\psi(x) = \exp\left[-\frac{1}{2}\gamma_4\gamma_0\ln(x_0+x_4) + \frac{1}{2}\gamma_2\gamma_1\arcsin\frac{x_1}{\sqrt{x_1^2+x_2^2}}\right]\varphi(\omega), \quad \omega = \left(x_1^2+x_2^2\right)^{1/2};$$

$$i\left[\gamma_2\varphi' + \frac{1}{2}\left(\gamma_0+\gamma_4+\frac{1}{\omega}\gamma_2\right)\varphi\right] - m\varphi = 0.$$

2. $$\psi(x) = \exp\left[\frac{1}{2}(\gamma_2\gamma_1 + d\gamma_4\gamma_0)\arcsin\frac{x_1}{\sqrt{x_1^2+x_2^2}}\right]\varphi(\omega), \quad \omega = \left(x_1^2+x_2^2\right)^{1/2}, \quad d>0;$$

$$i\left[\gamma_2\varphi' + \frac{1}{2\omega}(\gamma_2 + d\gamma_0\gamma_1\gamma_4)\varphi\right] - m\varphi = 0.$$

3. $$\psi(x) = \exp\left\{\frac{1}{2}\left[\gamma_2\gamma_1 + \varepsilon(\gamma_0+\gamma_4)\gamma_3\right]\arcsin\frac{x_1}{\sqrt{x_1^2+x_2^2}}\right\}\varphi(\omega), \quad \omega = \left(x_1^2+x_2^2\right)^{1/2}, \ \varepsilon=\pm 1;$$

$$i\left[\gamma_2\varphi' + \frac{1}{2\omega}\left(\gamma_2 + \varepsilon\gamma_1(\gamma_0+\gamma_4)\gamma_3\right)\varphi\right] - m\varphi = 0.$$

4. $\psi(x) = \exp\left[\frac{1}{2}\gamma_4\gamma_0\left(-\frac{1}{a}x_3 - \frac{d}{a}\arcsin\frac{x_1}{\sqrt{x_1^2+x_2^2}}\right) + \frac{1}{2}\gamma_2\gamma_1\arcsin\frac{x_1}{\sqrt{x_1^2+x_2^2}}\right]\varphi(\omega),$

$\omega = \left(x_1^2+x_2^2\right)^{1/2}, \quad a \neq 0, \quad d \neq 0;$

$i\left\{\gamma_2\varphi' - \frac{1}{2}\left[\frac{1}{\omega}\left(\frac{d}{a}\gamma_0\gamma_1\gamma_4 - \gamma_2\right) + \frac{1}{a}\gamma_0\gamma_3\gamma_4\right]\varphi\right\} - m\varphi = 0.$

5. $\psi(x) = \exp\left[\frac{1}{2}\gamma_2\gamma_1\arcsin\frac{x_1}{\sqrt{x_1^2+x_2^2}} + \frac{1}{2}(\gamma_0+\gamma_4)\gamma_3(x_0+x_4)\right]\varphi(\omega), \quad \omega = \left(x_1^2+x_2^2\right)^{1/2};$

$i\left(\gamma_2\varphi' + \frac{1}{2\omega}\gamma_2\varphi\right) - m\varphi = 0.$

6. $\psi(x) = \exp\left[\frac{1}{2}\gamma_2\gamma_1\arcsin\frac{x_1}{\sqrt{x_1^2+x_2^2}} - \frac{1}{2}(\gamma_0+\gamma_4)\gamma_3(x_0+x_4)\right]\varphi(\omega), \quad \omega = \left(x_1^2+x_2^2\right)^{1/2};$

$i\left(\gamma_2\varphi' + \frac{1}{2\omega}\gamma_2\varphi\right) - m\varphi = 0.$

7. $\psi(x) = \exp\left[-\frac{1}{2a}\gamma_4\gamma_0x_3 + \frac{1}{2}\gamma_2\gamma_1\arcsin\frac{x_1}{\sqrt{x_1^2+x_2^2}}\right]\varphi(\omega), \quad \omega = \left(x_1^2+x_2^2\right)^{1/2}, \quad a \neq 0;$

$i\left\{\gamma_2\varphi' + \frac{1}{2}\left[\frac{1}{\omega}\gamma_2 - \frac{1}{a}\gamma_0\gamma_3\gamma_4\right]\varphi\right\} - m\varphi = 0.$

8. $\psi(x) = \exp\left[-\frac{1}{2e}(\gamma_2\gamma_1 + e\gamma_4\gamma_0)\ln(x_0+x_4)\right]\varphi(\omega), \quad \omega = \left(x_0^2-x_4^2\right)^{1/2}, \quad e > 0;$

$\frac{i}{2}\left\{\left[\omega(\gamma_0+\gamma_4) + \frac{1}{\omega}(\gamma_0-\gamma_4)\right]\varphi' + (\gamma_0+\gamma_4)(1-\frac{1}{e}\gamma_2\gamma_1)\varphi\right\} - m\varphi = 0.$

9. $\psi(x) = \exp\left\{-\frac{1}{2}\gamma_4\gamma_0\ln(x_0+x_4) + \frac{1}{2\mu}\gamma_2\gamma_1[\alpha\ln(x_0+x_4) - x_3]\right\}\varphi(\omega),$

$\omega = \left(x_0^2-x_4^2\right)^{1/2}, \quad \alpha \neq 0, \quad \mu \neq 0;$

$\frac{i}{2}\left\{\left[\omega(\gamma_0+\gamma_4) + \frac{1}{\omega}(\gamma_0-\gamma_4)\right]\varphi' + \left[(\gamma_0+\gamma_4)\left(1+\frac{\alpha}{\mu}\gamma_2\gamma_1\right) + \frac{1}{\mu}\gamma_1\gamma_2\gamma_3\right]\varphi\right\} - m\varphi = 0.$

10. $\psi(x) = \exp\left[-\frac{1}{2}\gamma_4\gamma_0\ln(x_0+x_4) - \frac{1}{2\alpha}\gamma_2\gamma_1x_3\right]\varphi(\omega), \quad \omega = \left(x_0^2-x_4^2\right)^{1/2}, \quad \alpha \neq 0;$

$\frac{i}{2}\left\{\left[\omega(\gamma_0+\gamma_4) + \frac{1}{\omega}(\gamma_0-\gamma_4)\right]\varphi' + \left[\gamma_0+\gamma_4+\frac{1}{\alpha}\gamma_1\gamma_2\gamma_3\right]\varphi\right\} - m\varphi = 0.$

11. $\psi(x) = \exp\left(\frac{1}{2}\gamma_4\gamma_3\arcsin\frac{x_3}{\sqrt{x_3^2+x_4^2}} + \frac{1}{2d}\gamma_2\gamma_1x_0\right)\varphi(\omega), \quad \omega = \left(x_3^2+x_4^2\right)^{1/2}, \quad d \neq 0;$

$i\left[\gamma_4\varphi' + \frac{1}{2}\left(\frac{1}{d}\gamma_0\gamma_2\gamma_1 + \frac{1}{\omega}\gamma_4\right)\varphi\right] - m\varphi = 0.$

12. $\psi(x) = \exp\left[\frac{1}{2}\left(\gamma_4\gamma_3 + \frac{e}{2}\gamma_2\gamma_1\right)\arcsin\frac{x_3}{\sqrt{x_3^2+x_4^2}}\right]\varphi(\omega), \quad \omega = \left(x_3^2+x_4^2\right)^{1/2}, \quad e \neq 0;$

$$i\left[\gamma_4\varphi' + \frac{1}{2\omega}\left(\gamma_4 + \frac{e}{2}\gamma_3\gamma_2\gamma_1\right)\varphi\right] - m\varphi = 0.$$

13. $\psi(x) = \exp\left(-\frac{1}{2\alpha}\gamma_2\gamma_1 x_3\right)\varphi(\omega), \quad \omega = x_0, \quad \alpha \neq 0;$

$$i\left(\gamma_0\varphi' + \frac{1}{2\alpha}\gamma_1\gamma_2\gamma_3\varphi\right) - m\varphi = 0.$$

14. $\psi(x) = \exp\left[-\frac{1}{2}(\gamma_0+\gamma_4)\gamma_3 x_1\right]\varphi(\omega), \quad \omega = x_2;$

$$i\left[\gamma_2\varphi' - \frac{1}{2}\gamma_1(\gamma_0+\gamma_4)\gamma_3\varphi\right] - m\varphi = 0.$$

15. $\psi(x) = \exp\left[\frac{1}{2}(\gamma_0+\gamma_4)\gamma_3(x_0+x_4)\right]\varphi(\omega), \quad \omega = x_2;$

$$i\gamma_2\varphi' - m\varphi = 0.$$

16. $\psi(x) = \exp\left[-\frac{1}{2}\gamma_4\gamma_0\ln(x_0+x_4)\right]\varphi(\omega), \quad \omega = x_3;$

$$i\left[\gamma_3\varphi' + \frac{1}{2}(\gamma_0+\gamma_4)\varphi\right] - m\varphi = 0.$$

17. $\psi(x) = \exp\left(-\frac{1}{2\tilde{a}_2}\gamma_4\gamma_0 x_2\right)\varphi(\omega), \quad \omega = x_3, \quad \tilde{a}_2 > 0;$

$$i\left(\gamma_3\varphi' - \frac{1}{2\tilde{a}_2}\gamma_0\gamma_2\gamma_4\varphi\right) - m\varphi = 0.$$

18. $\psi(x) = \exp\left(\frac{1}{d}\gamma_2\gamma_1 x_0\right)\varphi(\omega), \quad \omega = x_3, \quad d > 0;$

$$i\left(\gamma_3\varphi' - \frac{1}{d}\gamma_0\gamma_1\gamma_2\varphi\right) - m\varphi = 0.$$

19. $\psi(x) = \exp\left[-\frac{1}{2e}(\gamma_2\gamma_1 + e\gamma_4\gamma_0)\ln(x_0+x_4)\right]\varphi(\omega), \quad \omega = x_3, \quad e > 0;$

$$i\left[\gamma_3\varphi' + \frac{1}{2}(\gamma_0+\gamma_4)\left(1 - \frac{1}{e}\gamma_2\gamma_1\right)\varphi\right] - m\varphi = 0.$$

20. $\psi(x) = \exp\left(-\frac{1}{2d_3}\gamma_2\gamma_1 x_3\right)\varphi(\omega), \quad \omega = x_4, \quad d_3 \neq 0;$

$$i\left(\gamma_4\varphi' + \frac{1}{2d_3}\gamma_1\gamma_2\gamma_3\varphi\right) - m\varphi = 0.$$

21. $\psi(x) = \exp\left[-\frac{1}{2}\gamma_4\gamma_0 \ln(x_0+x_4)\right]\varphi(\omega), \quad \omega = \ln(x_0+x_4) - \frac{x_3}{a_3}, \quad a_3 > 0;$

$$i\left[\left(\gamma_0+\gamma_4-\frac{1}{a_3}\gamma_3\right)\varphi' + \frac{1}{2}(\gamma_0+\gamma_4)\varphi\right] - m\varphi = 0.$$

22. $\psi(x) = \exp\left[\frac{1}{2}\gamma_4\gamma_0 \ln(x_0-x_4)\right]\varphi(\omega), \quad \omega = \ln(x_0-x_4) + \frac{x_3}{a_3}, \quad a_3 > 0;$

$$i\left[\left(\gamma_0-\gamma_4+\frac{1}{a_3}\gamma_3\right)\varphi' + \frac{1}{2}(\gamma_0-\gamma_4)\varphi\right] - m\varphi = 0.$$

23. $\psi(x) = \exp\left[\frac{1}{2}(\gamma_0+\gamma_4)\gamma_3(x_0+x_4)\right]\varphi(\omega), \quad \omega = x_1 - \frac{1}{b}x_3 + \frac{1}{2b}(x_0+x_4)^2, \quad b \neq 0;$

$$i\gamma_1\varphi' - m\varphi = 0.$$

24. $\psi(x) = \exp\left[\frac{1}{2}(\gamma_0+\gamma_4)\frac{\gamma_1 x_1 + (x_2-x_3)\gamma_2}{x_0+x_4}\right]\varphi(\omega), \quad \omega = x_0+x_4;$

$$i\left\{(\gamma_0+\gamma_4)\varphi' + \frac{1}{\omega}(\gamma_0+\gamma_4)\left[1-\frac{1}{2}\gamma_2\gamma_3\right]\varphi\right\} - m\varphi = 0.$$

25. $\psi(x) = \exp\left[\frac{1}{2}(\gamma_0+\gamma_4)\gamma_3\frac{x_3 - bx_1}{x_0+x_4}\right]\varphi(\omega), \quad \omega = x_0+x_4, \quad b \neq 0;$

$$i\left[(\gamma_0+\gamma_4)\varphi' + \frac{1}{2\omega}(\gamma_0+\gamma_4)(1+b\gamma_1\gamma_3)\varphi\right] - m\varphi = 0.$$

26. $\psi(x) = \exp\left[\frac{1}{2\varepsilon}\left(\gamma_2\gamma_1 + \varepsilon(\gamma_0+\gamma_4)\gamma_3\right)\frac{x_3}{x_0+x_4}\right]\varphi(\omega), \quad \omega = x_0+x_4, \quad \varepsilon = \pm 1;$

$$i\left[(\gamma_0+\gamma_4)\varphi' + \frac{1}{2\varepsilon\omega}\left(\varepsilon(\gamma_0+\gamma_4) - \gamma_1\gamma_2\gamma_3\right)\varphi\right] - m\varphi = 0.$$

27. $\psi(x) = \exp\left[-\frac{1}{2}(\gamma_0+\gamma_4)\gamma_3 x_2\right]\varphi(\omega), \quad \omega = x_0+x_4;$

$$i\left[(\gamma_0+\gamma_4)\varphi' + \frac{1}{2}(\gamma_0+\gamma_4)\gamma_2\gamma_3\varphi\right] - m\varphi = 0.$$

28. $\psi(x) = \exp\left[\frac{1}{2}\gamma_2\gamma_1\left(x_0 - x_4 - \frac{x_3^2}{x_0+x_4}\right) + \frac{1}{2}(\gamma_0+\gamma_4)\gamma_3\frac{x_3}{x_0+x_4}\right]\varphi(\omega), \quad \omega = x_0+x_4;$

$$i\left[(\gamma_0+\gamma_4)\varphi' + \frac{1}{2}\left(\frac{1}{\omega}(\gamma_0+\gamma_4) + \gamma_0 - \gamma_4\right)\varphi\right] - m\varphi = 0.$$

29. $\psi(x) = \exp\left[\frac{1}{2}\gamma_2\gamma_1\left(x_4 - x_0 + \frac{x_3^2}{x_0+x_4}\right) + \frac{1}{2}(\gamma_0+\gamma_4)\gamma_3\frac{x_3}{x_0+x_4}\right]\varphi(\omega), \quad \omega = x_0+x_4;$

$$i\left[(\gamma_0+\gamma_4)\varphi' + \frac{1}{2}\left(\frac{1}{\omega}(\gamma_0+\gamma_4) - \gamma_0 + \gamma_4\right)\varphi\right] - m\varphi = 0.$$

30. $\psi(x) = \exp\left[\frac{1}{2}\gamma_2\gamma_1(x_4 - x_0)\right]\varphi(\omega), \quad \omega = x_0 + x_4;$

$$i\left[(\gamma_0+\gamma_4)\varphi' + \frac{1}{2}\gamma_2\gamma_1(\gamma_4-\gamma_0)\varphi\right] - m\varphi = 0.$$

31. $\psi(x) = \exp\left[\frac{1}{2}\gamma_2\gamma_1(x_0 - x_4)\right]\varphi(\omega), \quad \omega = x_0 + x_4;$

$$i\left[(\gamma_0+\gamma_4)\varphi' + \frac{1}{2}\gamma_2\gamma_1(\gamma_0-\gamma_4)\varphi\right] - m\varphi = 0.$$

32. $\psi(x) = \exp\left[\frac{1}{2\delta}(\gamma_0+\gamma_4)\left((\delta\gamma_2-\gamma_1)\frac{x_2}{x_0+x_4} - \frac{x_3}{2}\gamma_1\right)\right]\varphi(\omega), \quad \omega = x_0 + x_4, \quad \delta > 0;$

$$i(\gamma_0+\gamma_4)\left[\varphi' - \frac{1}{2\delta\omega}\left(\gamma_1(\gamma_2+\omega\gamma_3)-\delta\right)\varphi\right] - m\varphi = 0.$$

33. $\psi(x) = \exp\left[\frac{1}{2}(\gamma_0+\gamma_4)\left(\frac{x_2}{x_0+x_4}\gamma_2 - x_3\gamma_1\right)\right]\varphi(\omega), \quad \omega = x_0 + x_4;$

$$i(\gamma_0+\gamma_4)\left[\varphi' + \frac{1}{2}\left(\frac{1}{\omega} - \gamma_1\gamma_3\right)\varphi\right] - m\varphi = 0.$$

34. $\psi(x) = \exp\left[\frac{1}{2}(\gamma_0+\gamma_4)\left(\frac{\mu x_1 - x_3}{1+\mu(x_0+x_4)}\gamma_1 + \frac{x_2\gamma_2}{x_0+x_4}\right)\right]\varphi(\omega), \quad \omega = x_0 + x_4, \quad \mu > 0;$

$$i(\gamma_0+\gamma_4)\left\{\varphi' + \frac{1}{2}\left[\frac{1}{\mu\omega+1}\left(\mu - \gamma_1\gamma_3\right) + \frac{1}{\omega}\right]\varphi\right\} - m\varphi = 0.$$

35. $\psi(x) = \exp\left[\frac{1}{2}\gamma_2\gamma_1\left(\frac{2}{3\alpha^2}(x_0+x_4)^3 - \frac{2}{\alpha}x_3(x_0+x_4) + x_0 - x_4\right) + \right.$

$\left. + \frac{1}{2\alpha}(\gamma_0+\gamma_4)\gamma_3(x_0+x_4)\right]\varphi(\omega), \quad \omega = \alpha x_3 - \frac{1}{2}(x_0+x_4)^2, \quad \alpha \neq 0;$

$$i\left\{\alpha\gamma_3\varphi' + \left[\left(\frac{\omega}{\alpha^2} + \frac{1}{2}\right)\gamma_0 + \left(\frac{\omega}{\alpha^2} - \frac{1}{2}\right)\gamma_4\right]\gamma_2\gamma_1\varphi\right\} - m\varphi = 0.$$

36. $ds\psi(x) = \exp\left[\frac{1}{2}\gamma_2\gamma_1\left(-\frac{2}{3\alpha^2}(x_0+x_4)^3 + \frac{2}{\alpha}x_3(x_0+x_4) - x_0 + x_4\right) + \right.$

$\left. + \frac{1}{2\alpha}(\gamma_0+\gamma_4)\gamma_3(x_0+x_4)\right]\varphi(\omega), \quad \omega = \alpha x_3 - \frac{1}{2}(x_0+x_4)^2, \quad \alpha \neq 0;$

$$i\left\{\alpha\gamma_3\varphi' + \left[\left(\frac{\omega}{\alpha^2} - \frac{1}{2}\right)\gamma_0 + \left(\frac{\omega}{\alpha^2} + \frac{1}{2}\right)\gamma_4\right]\gamma_2\gamma_1\varphi\right\} - m\varphi = 0.$$

Let us note that the ansatzes (1)–(36) are obtained with the help of four-dimensional non-Abelian subalgebras of the Lie algebra of the group $P(1,4)$. The basis elements of these subalgebras commute if they belong to the Lie algebra of the group $SO(1,4)$.

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