Proceedings of Institute of Mathematics of NAS of Ukraine 2000, Vol. 30, Part 1, 17–24.

Matrix Methods of Searching for Lax Pairs and a Paper by Estevez

B. Kent HARRISON

Department of Physics and Astronomy, Brigham Young University, Provo, Utah 84602 U.S.A.
E-mail: harrison@physics.byu.edu

Lax pairs are useful in studying nonlinear partial differential equations, although finding them is often difficult. A standard approach for finding them was developed by Wahlquist and Estabrook [1]. It was designed to apply to for equations with two independent variables and generally produces incomplete Lie algebras (called "prolongation structures"), which can be written as relations among certain matrices and their commutators. Extending the method to three variable problems is more difficult. One still gets matrix equations, but now with a more complicated structure. Exploration of a Lax pair in a paper by Estevez [3] suggested a variation of the method. This paper will discuss how that can be used to obtain her Lax pair.

P.G. Estevez [3] published a recent paper dealing with a particular nonlinear partial differential equation (NLPDE):

$$0 = m_y^2(n_{yt} - m_{xxxy}) + m_{xy}\left(n_y^2 - m_{xy}^2\right) + 2m_y(m_{xy}m_{xxy} - n_y n_{xy}) - 4m_y^3 m_{xx} \tag{1}$$

with $m_t = n_x$. In this paper, subscripts mean derivatives.

This equation was a reformulation of a set of equations

$$\begin{aligned} 0 &= v_y - (uw)_x, \\ 0 &= \lambda u_t + u_{xx} - 2uv, \\ 0 &= \lambda w_t - w_{xx} + 2vw \end{aligned} \tag{2}$$

which were obtained earlier by other authors. These equations have the Painleve property, which was used by Estevez in an investigation of Eq. (1) by the singular manifold method in which she, among other things, found a Lax pair. Her treatment is fairly complicated. It is not obvious from the original equations that a Lax pair exists.

Study of that Lax pair led this author to try a matrix approach to try to find the same result. This is basically a version of the Wahlquist–Estabrook method [1] that this author spoke about at the first Kyiv Conference "Symmetry in Nonlinear Mathematical Physics" four years ago [4]. The matrix equations are quite complicated but can be simplified, with guidance from already known results. Here some earlier results are reviewed, with particular attention to the use of matrices.

Lax pairs have been known since 1968, when Lax discussed them in terms of operators in his paper of that year on the KdV equation (the 12 included here did not occur in Lax's version) [6]

$$u_t + 12uu_x + u_{xxx} = 0. \tag{3}$$

In the later treatment by Wahlquist and Estabrook [1, 2] (WE), the Lax pair may be expressed in terms of linear matrix equations for two auxiliary variables, with coefficients involving the variable u, whose integrability condition gives the KdV equation.

Wahlquist and Estabrook used differential forms in analyzing partial differential equations. We show here the definition of new variables z and p, introduced to reduce equations to first derivatives, with the KdV equation using the new variables:

$$z = u_x, \qquad p = z_x, \qquad u_t + p_x + 12uz = 0. \tag{4}$$

Then we write these three equations in terms of three differential forms in the five variables x, t, u, p, z (the set of these is called the ideal I of forms, $I = \{\alpha, \beta, \gamma\}$:

$$\begin{aligned} \alpha &= du\,dt - z\,dx\,dt, \\ \beta &= dz\,dt - p\,dx\,dt, \\ \gamma &= -du\,dx + dp\,dt + 12uz\,dx\,dt, \end{aligned} \tag{5}$$

where the hook product $\wedge$ between basis forms such as du and dt is understood. (If one now assumes that the field variables u, z, p are functions of x and t and requires these differential forms to vanish, one recovers the original equations.)

Next, WE assume the existence of a variable y and an auxiliary 1-form, called a prolongation form,

$$\omega = -dy + f(y, u, p, z)\,dx + g(y, u, p, z)\,dt. \tag{6}$$

The exterior derivative of this form is to lie in the "augmented" ideal of forms $I' = \{I, \omega\}$:

$$d\omega \subset \{I, \omega\} \tag{7}$$

so that when I and ω vanish, this amounts to an integrability condition.

Now take y to be a column vector and assume f and g to be linear in the components of y; then we can rewrite (6) as a matrix equation:

$$\omega = -dy + \alpha y, \tag{8}$$

where

$$\alpha = F\,dx + G\,dt \tag{9}$$

is a matrix 1-form and where F and G are matrices and are functions of u, z and p. The integrability condition (7) is then expressed by

$$\begin{aligned} d\omega &= d\alpha\,y - \alpha \wedge dy, \\ &= d\alpha\,y - \alpha \wedge (-\omega + \alpha y), \\ &= (d\alpha - \alpha \wedge \alpha)y \bmod \omega \end{aligned} \tag{10}$$

so that

$$d\alpha - \alpha \wedge \alpha \subset I \tag{11}$$

or

$$G_x - F_t - [F, G] = 0 \tag{12}$$

which is to be satisfied if the original field equations hold.

The use of differential forms gives some insight into how the problem might be formulated and presents an elegant structure. However, the problem can be formulated without forms. Write

$$y_x = F\,y, \qquad y_t = G\,y \tag{13}$$

then

$$\begin{aligned} y_{xt} &= F_t\, y + F y_t = F_t\, y + FGy \\ &= G_x\, y + G y_x = G_x\, y + GFy, \end{aligned} \tag{14}$$

giving Eq. (12) as before. For the KdV case this equation becomes simply:

$$\begin{aligned} &F_p = F_z = 0, \qquad G_p + F_u = 0, \\ &z\, G_u + p\, G_z + 12uz\, F_u = [F, G]. \end{aligned} \tag{15}$$

Solution of these equations leads eventually to the relations

$$\begin{aligned} &F = Au^2 + Bu + C, \\ &G = -p(2uA + B) + z^2 A + 6zD + K(u), \end{aligned} \tag{16}$$

where

$$[B, C] = 6D, \qquad [A, B] = [A, C] = 0 \tag{17}$$

and

$$K(u) = 2u^3([A, D] - 4A) + 3u^2([B, D] - 2B) + 6u[C, D] + E \tag{18}$$

and where there are six more equations among A, B, C, D and E, which are constant matrices. These equations involve commutators of commutators and will not be given here.

The set of equations for A, B, C, D and E constitutes an incomplete Lie algebra (called a "prolongation structure" by WE). It is of interest in its own right; however, we wish to find a representation in order to find the Lax pair. Closure of the algebra can be achieved by Ansatz, as WE show. A two-dimensional representation for these five matrices is, where λ is constant:

$$A = 0, \quad B = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \lambda \\ 0 & 0 \end{bmatrix}, \quad D = 1/3 \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E = -4\lambda C, \tag{19}$$

giving

$$F = \begin{bmatrix} 0 & \lambda - 2u \\ 1 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} -2z & 4(u+\lambda)(2u-\lambda) + 2p \\ -4(u+\lambda) & 2z \end{bmatrix}, \tag{20}$$

with a Lax pair written as differentials of the components of y:

$$\begin{aligned} &dy_1 = (\lambda - 2u) y_2\, dx + \{[4(u+\lambda)(2u-\lambda) + 2p] y_2 - 2zy_1\}\, dt, \\ &dy_2 = y_1\, dx + [2zy_2 - 4(u+\lambda) y_1]\, dt. \end{aligned} \tag{21}$$

It should be noted here that WE use the auxiliary variables in the Lax pair equations (which they call "pseudopotentials") to help derive Bäcklund transformations.

One can see that this method is most suited for differential equations in two independent variables, since the prolongation form is simply a 1-form. The three independent variable case is much harder. We can see why, from a differential form standpoint, by noting that the ideal of forms representing equations with n independent variables generally requires n-forms (although there are exceptions). As an example, we give the KP equation

$$3u_{tt} + 6(uu_x)_x + u_{xxx} + 3u_{xy} = 0, \tag{22}$$

and with new variables p, r, z and w defined by

$$p = u_x, \qquad r = p_x, \qquad z = w_x = -(3/4)u_t \tag{23}$$

this becomes

$$w_t = (3/2)up + (1/4)r_x + (3/4)u_y. \tag{24}$$

An ideal of 3-forms representing the KP equation is (where the $\wedge$ is suppressed):

$$\begin{aligned}
&(du\,dt - p\,dx\,dt)\,dy,\\
&(dp\,dt - r\,dx\,dt)\,dy,\\
&(dw\,dt - z\,dx\,dt)\,dy,\\
&(dp\,dx - (4/3)\,dz\,dt)\,dy,\\
&(dw\,dx + (3/2)up\,dx\,dt + (1/4)\,dr\,dt)\,dy + (3/4)\,du\,dx\,dt.
\end{aligned} \tag{25}$$

The difficulty in such cases arises in trying to construct a prolongation form. If we simply write it as a 1-form in three variables, then its exterior derivative is a 2-form, and its vanishing cannot be achieved with the 3-forms in the ideal.

H. Morris' approach [5], motivated by the WE method and here called the MWE method, does not use forms. He proceeded by assuming the equations

$$\begin{aligned}
&\zeta_x = -F\zeta - A\zeta_y,\\
&\zeta_t = -G\zeta - B\zeta_y,
\end{aligned} \tag{26}$$

where A and B are constant matrices, with F and G being matrix functions of u, p, r, z and w, and by assuming integrability. After writing out the integrability condition and substituting from Eq. (26) where possible, he set the coefficients of ζ, ζ_y and ζ_{yy} to zero, yielding the following equations (to be taken modulo the field equations, in other words to be satisfied if the field equations are satisfied):

$$\begin{aligned}
&[A, B] = 0,\\
&[G, A] + [B, F] = 0,\\
&F_t - G_x + [G, F] + BF_y - AG_y = 0.
\end{aligned} \tag{27}$$

Note that the equations are now more complicated than just relations among commutators.

Morris' approach suggested to the author an approach to the three-variable problem using differential forms [4]. While this has some interest, it appears rather artificial. It is not needed here.

A solution of Morris' equations, given by himself and corrected in [4], has this set of matrices, where k is a constant:

$$\begin{aligned}
&A = (3/4)\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0 \end{bmatrix}, \qquad B = -(3/4)\begin{bmatrix} 0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 1 & 0 \end{bmatrix},\\
&F = \begin{bmatrix} 0 & -1 & 0\\ 3u/4 & 0 & -1\\ w-k & 3u/4 & 0 \end{bmatrix}, \qquad G = \begin{bmatrix} u/4 & 0 & 1\\ -w+k+p/4 & -u/2 & 0\\ r/4 + 9u^2/16 & -w+k-p/4 & u/4 \end{bmatrix}.
\end{aligned} \tag{28}$$

Equation (26), with these matrices, now constitutes a Lax pair for the KP equation.

We now go back to Estevez' paper and equation. We write her Lax pair in matrix form, defining new variables:

$$C_t = M(C_{xx} + 2qC), \qquad C_{xy} = QC_y - pC, \tag{29}$$

with

$$q = m_x, \quad r = m_{xy}, \quad p = m_y, \quad z = n_y = \int m_{ty}\, dx, \quad s = r_x \tag{30}$$

and

$$Q = (r1 + zM)/(2p), \qquad M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{31}$$

where 1 in Q is the 2×2 unit matrix and C is a 2-component column matrix. We note that her original equation, in the new variables, is

$$0 = p^2(z_t - s_x) + r\left(z^2 - r^2\right) + 2p(rs - zz_x) - 4p^3 q_x, \tag{32}$$

where $z_x = p_t$.

Let us now attempt to use the MWE method to find this Lax pair. We assume equations exactly like Eq. (26),

$$\begin{aligned} \zeta_x &= -A\zeta_y - F\zeta, \\ \zeta_t &= -B\zeta_y - G\zeta, \end{aligned} \tag{33}$$

where F and G are matrix functions of m, n, p, q, r, s and z. The equations easily show that F is independent of n, q, r and s and is linear in z. Continuing the process eventually leads to a trivial solution. Interchanging independent variables in Eq. (33) does not lead to a solution either. Thus, the MWE method does not work, indicating that there is not a Lax pair of the form (33).

So we attempt to generalize the MWE method (denote this by GMWE). We try the following, noting that it uses a particular assumed structure, motivated by knowing the answer already!

Assume a pair of differential equations for the column vector C from above:

$$\begin{aligned} C_t &= FC_{xx} + GC_x + HC, \\ C_{xy} &= KC_x + LC_y + NC, \end{aligned} \tag{34}$$

where F, G, H, K, L and N are matrix functions of m, n, p, q, r, s and z, and assume integrability: $(C_t)_{xy} = (C_{xy})_t$, substituting for C_t and C_{xy} from these equations, wherever possible. One gets a complicated matrix equation with terms linear in C_{xxx}, C_{xx}, C_x, C_y and C. We equate the coefficients to zero. After some simplification, such as substitution of F_y from the first of these into other equations, we get

$$\begin{aligned} F_y &= [K, F], \\ G_y &= [K, G] + [LK + N, F] - FK_x - K_x F, \\ L_t &= [H + GL + FL^2, L] + F_x(L_x + L^2) + F(L_{xx} + 2L_x L) \\ &\quad + [F, L]L_x + G_x L + GL_x + H_x, \\ N_t &= (2FL_x + F_x L)N + [H, N] + [F, L](N_x + LN) + FN_{xx} \\ &\quad + F_x N_x + G_x N + GN_x + H_{xy} - LH_y - KH_x + [G, L]N, \\ K_t &= [G, N + LK] + [F, LN + L^2 K] + [H, K] + K_x G + (N + LK - K_x)F_x \\ &\quad + (L_x K + 2LK_x - K_{xx} + N_x)F + F(N_x + L_x K) + H_y. \end{aligned} \tag{35}$$

Obviously further simplification is needed. We take some hints from the known answer; choose $K = G = 0$. Then $F_y = 0$, yielding

$$0 = F_m p + F_n z + F_p p_y + F_q r + F_r r_y + F_s s_y + F_z z_y. \tag{36}$$

The coefficients of p, z, etc., must vanish, giving all derivatives of F zero, so that F is constant. The remaining equations become

$$\begin{aligned}
&0 = [F, N],\\
&0 = H_y + N_x F + F N_x + [F, LN],\\
&L_t = F(L_{xx} + 2L_x L + L L_x) - L F L_x + H_x + [F L^2 + H, L],\\
&N_t = F(N_{xx} + L N_x + 2L_x N) - L(F N_x + H_y) + H_{xy} + [F, L]LN + [H, N].
\end{aligned} \tag{37}$$

The second of these suggests that N and H be taken as functions of separate variables whose x- and y-derivatives are equal. p and q appear to be the obvious variables. We write $H = H(q)$ and $N = N(p)$, substitute, and cancel $q_y(= p_x)$. Then the second equation yields

$$H' = -N'F - FN', \qquad [F, LN] = 0. \tag{38}$$

Since F is constant, the fact that it commutes with N also means that it commutes with N'. We assume F to have an inverse. Then separation of variables in the first of Eq. (38) and dropping matrix integration constants gives

$$H = aq, \qquad N = cp, \tag{39}$$

where a and c are constant matrices and $c = -(1/2)aF^{-1}$. We note that F, a and c now all mutually commute. We assume that c has an inverse; then F commutes with L and with L_x as well.

Substitution of these expressions into the last of Eq. (37) gives, after simplification,

$$2FL_x cp = -(Fc + a)p_{xx} + Lam_{xy} + cz_x. \tag{40}$$

Multiplying by the inverses of F and c gives the equation

$$2pL_x = 1p_{xx} - 2Lp_x + F^{-1}z_x \tag{41}$$

which can be integrated and solved for L, giving

$$L = (1r + F^{-1}z + U)/(2p), \tag{42}$$

where U is a matrix integration constant satisfying $[F, U] = 0$.

The third of Eq. (37) now becomes, after using $[F, L] = 0$,

$$F(L_{xx} + 2L_x L) + H_x + [H, L] = L_t. \tag{43}$$

We substitute for L from Eq. (42) and find, after substituting for various derivatives, for z_t from Eq. (32), multiplying by $2p^3$, and canceling some terms,

$$\begin{aligned}
&F\left[-3prs + 2r^3 + \left(2r^2 - ps\right)U\right] + 1\left(-2prz_x - psz + 2zr^2\right)\\
&\quad + \left[F\left(ps - r^2\right) + 1(pz_x - rz) - rFU\right]\left(1r + F^{-1}z + U\right) + 2p^3 q_x a + p^2 q[a, U]\\
&\quad = -1prz_x - pz_x U + F^{-1}\left(4p^3 q_x + pzz_x - 2prs + r^3 - rz^2\right).
\end{aligned} \tag{44}$$

By comparing terms we see immediately that $F = F^{-1}$ and $a = 2F$, so that $c = -1$. The z_x term shows that $U = 0$; then the remaining terms cancel identically. If one now takes

$$F = M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{45}$$

one gets Estevez' Lax pair Eq. (29).

What could be done to try simplifying Eq. (35) in some other way? We can assume that all matrices commute. Then F is constant for the same reason as before. However, one gets this equation for G and K:

$$G_y = -FK_x - K_xF. \tag{46}$$

By the same argument used for H and N above, we may write

$$K = Ap, \qquad G = -2FAq, \tag{47}$$

where A is constant. We assume that F and A have inverses. The remaining equations are

$$\begin{aligned} K_t &= K_xG + 2KFL_x + 2LFK_x + F(2N_x - K_{xx}) + H_y, \\ L_t &= F(L_{xx} + 2LL_x) + LG_x + GL_x + H_x, \\ N_t &= F(N_{xx} + 2NL_x) + NG_x + GN_x + H_{xy} - KH_x - LH_y. \end{aligned} \tag{48}$$

Motivated by the first of these equations we take $H = H(q)$ since y-derivatives of other variables cannot be expressed in terms of the variables we are using. We expand N_x in terms of derivatives with respect to z, p, q and r and set coefficients of z_x and q_x to zero, giving expressions for N_z and N_q. Integration of those equations yields

$$N = (1/2)F^{-1}Az - pAL + W(p, r). \tag{49}$$

Substitution into the remaining part of the equation yields an equation linear in s. Setting the coefficients equal to zero gives finally

$$\begin{aligned} N &= (1/2)F^{-1}Az - pAL + rA - pC + E, \\ H &= FA^2q^2 + 2FCq + D, \end{aligned} \tag{50}$$

where C, D and E are constant.

Substitution of these results into the equation for N_t with elimination of L_t and z_t (from Eq. (32)) gives an equation which could be integrated on x to give L, were it not for a term $-2F^{-1}Apq_x$. This fact seems to show that A must be zero after all, giving a contradiction. Thus, at the least, A does not have an inverse and perhaps should be taken to be zero, leading to the previous case.

We can approach this from a slightly different point of view. Let us ask what NLPDE is consistent with the Lax pair (34) we have assumed. To simplify we will assume that all matrices commute, as before. We get a constant F as before. We assume some basic field m with $q = m_x$, $p = m_y$. We get $G = -2qFA + B$ and $K = qA + C$, where B and C are constant, similar to previous results. For reasons similar to the previous ones, we take $A = 0$. Thus G and K are constant. Furthermore, it seems appropriate and useful to take $K = 0$. Calculation for H, N and L proceeds much as before. We finally get an equation where, in order to make all terms proportional to the same matrix, we merely need to assume F^{-1} is proportional to F and c is proportional to 1, giving $F^2 = \lambda 1$ and $c = \mu 1$, where λ and μ are constants. μ can be absorbed by change of variables. So we have a slightly more general equation than Estevez:

$$\begin{aligned} 0 = &-m_y^2 m_{xxxy} + m_{xy}\left(2m_y m_{xxy} - m_{xy}^2\right) - 4m_y^3 m_{xx} \\ &+\lambda^{-1}\left(m_{xy}n_y^2 - 2n_y m_y m_{yt} + m_y^2 n_{yt}\right). \end{aligned} \tag{51}$$

We can also generalize by taking $H = H(q, m)$, $N = N(p)$ when solving for those two quantities. We get

$$N = cp, \qquad H = -(Fc + cF)q + Q(m). \tag{52}$$

We assume that F and c have inverses, and this enables explicit solution for several quantities. This all reduces eventually to the same equation as before.

One can ask what the most general equation is that is consistent with a generalized Lax pair of the type, say,

$$\begin{aligned} &C_t = FC_{xx} + GC_x + HC + AC_{yy} + BC_y, \\ &C_{xy} = KC_x + LC_y + NC. \end{aligned} \tag{53}$$

The equations resulting from this are very complicated and nothing has been done with them.

In summary, one can see that trial of a Lax pair of the generalized form (34) or something like it could perhaps work for some equations, as a generalization of (33). Chances are that any particular guess will not work for a new NLPDE that one might have; but this at least gives some suggestions for how one might look for a Lax pair using matrices. Defining new variables might motivate the linear structure that one might try. A general approach is not available.

References

[1] Wahlquist H.D. and Estabrook F.B., Prolongation structures of nonlinear evolution equations, *J. Math. Phys.*, 1975, V.16, N 1, 1–7.

[2] Wahlquist H.D. and Estabrook F.B., Bäcklund transformation for solutions of the Korteweg-de Vries equation, *Phys. Rev. Lett.*, 1973, V.31, N 23, 1386–1390.

[3] Estévez P.G., Darboux transformation and solutions for an equation in $2 + 1$ dimensions, *J. Math. Phys.*, 1999, V.40, N 3, 1406–1419.

[4] Harrison B.K., On methods of finding Bäcklund transformations in systems with more than two independent variables, *J. Nonlin. Math. Phys.*, 1995, V.2, N 2–3, 201–215.

[5] Morris H.C., Prolongation structures and nonlinear evolution equations in two spatial dimensions, *J. Math. Phys.*, 1976, V.17, N 10, 1870–1872.

[6] Lax P.D., Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure and Appl. Math.*, 1968, V.21, N 5, 467–490.