Ukrainian Mathematical Congress

Dedicated to 200th Anniversary of Mykhailo Ostrohrads'kyi

Proceedings

of the Fourth International Conference

SYMMETRY

in Nonlinear Mathematical Physics

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Цей том “Праць Інституту математики НАН України” є збірником статей учасників Четвертої міжнародної конференції “Симетрія у нелінійній математичній фізиці”. Збірник складається з двох частин, кожна з яких видана окремою книгою.

Дане видання є першою частиною і включає праці, присвячені подальшому розвитку та застосуванню теоретико-групових методів у математичній фізиці та теорії диференціальних рівнянь. Додатково, як окрема секція представлені статі, присвячені методу оберненої задачі розсіювання.

Розраховано на наукових працівників, аспірантів, які цікавляться новими тенденціями симетрійного аналізу і побудови точних розв'язків нелінійних рівнянь.

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This volume of the Proceedings of Institute of Mathematics of NAS of Ukraine includes papers of participants of the Fourth International Conference “Symmetry in Nonlinear Mathematical Physics”. The collection consists of two parts which are published as separate issues.

This issue is the first part which contains papers devoted to further development and applications of group-theoretical methods in mathematical physics and differential equations. In addition, the inverse scattering problem approach is represented in a separate section.

The book may be useful for researchers and post graduate students who are interested in modern trends in symmetry analysis and construction of exact solutions of nonlinear equations.

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Preface

The Fourth International Conference "Symmetry in Nonlinear Mathematical Physics" was traditionally organized by the Institute of Mathematics of the National Academy of Sciences of Ukraine and M. Dragomanov National Pedagogical University. It continues the series of the the scientific meetings started in 1995 due to efforts of Professor Wilhelm Fushchych. The specific feature of the conference held in Kyiv, July 9–15, 2001 was that it was a part of the Ukrainian Mathematical Congress devoted to 200 Anniversary of the great Ukrainian mathematician Mykhailo Ostrohrads'kyi.

The conference generated significant interest of mathematicians and physicists. More than 120 participants from 26 countries presented their talks the majority of which is included in these Proceedings. The Proceedings contain also papers whose authors were unable to come to conference but submitted their papers.

The Proceedings are published in two parts and contain 109 papers. The titles of conference talks not presented for publication at the Proceedings are given in the end of part 2.

A number of papers included into Proceedings is devoted to traditional subjects of Lie theory, i.e., analysis of symmetries of nonlinear partial differential equations, symmetry reduction and construction of exact solutions. In these papers which are collected in the first part both classical Lie methods and modern trends in symmetry analysis are represented. The first part contains also papers related to the inverse scattering approach. The first paper includes the biographical essay of M. Ostrohrads'kyi.

The trend of our conference is continuous increase of contributions devoted to problems of algebra, group theory and symmetries in physics and other natural sciences. These contributions are collected in the second part which includes papers devoted to representations and applications of classical and deformed Lie algebras, supersymmetry and its various generalizations.

We believe that all papers present a valuable contribution to the symmetry analysis of equations of mathematical physics and other applications of symmetry.

Anatoly NIKITIN

March, 2002

Fourth International Conference

SYMMETRY IN NONLINEAR MATHEMATICAL PHYSICS

July 9–15, 2001, Kyiv, Ukraine

Organized by

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July 9–15, 2001, Kyiv, Ukraine

TOPICS

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* Supersymmetry and Parasupersymmetry

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We invite everybody to participate in the next Conference planned for July, 2003.

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Mykhailo Vasyl'ovych Ostrohrads'kyi

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The paper describes life and research of the great Ukrainian mathematician Mykhailo Ostrohrads'kyi, whose 200th anniversary was marked in 2001. His development as a mathematician and his joint work with other most prominent scientists of his time are presented. Mykhailo Ostrohrads'kyi published more than 50 research papers, and laid foundation to many areas in calculus, differential equations and mathematical physics. The paper gives a review of some important results and formulae by Ostrohrads'kyi, shows their importance to further development of science.

On September 12, 2001 (it is September 24 according to the new calendar) there is 200th anniversary of Mykhailo Vasyl'ovych Ostrohrads'kyi, an outstanding Ukrainian mathematician, whose papers rightfully belong to the treasury of science and profoundly influenced development of mathematical analysis, the theory of differential equations, mathematical physics, and mechanics and rightfully belong to the treasury of science. His attention was always concentrated on extremely important problems of his time of both theoretical and practical nature. Similarly to Lagrange, he sought general approaches to the investigation of problems of different nature, discovering, as a result, original ways of reaching his goal.

Investigations of Ostrohrads'kyi embraced the entire spectrum of problems studied by prominent European mathematicians of that time, such as N. Abel, W. Hamilton, C. Gauss, A. Cauchy, J. Lagrange, P. Laplace, J. Liouville, S. Poisson, J. Fourier, C. Jacobi, etc. For this reason, his results had certain intersections with results of these scientists, but they were never inferior to them in the generality of problems considered the rigorousness and originality of exposition, and efficiency of applications. He was a star of the first magnitude in the constellation of these outstanding personalities.

For better understanding of the significance of the scientific heritage of Ostrohrads'kyi, one must characterize, at least in general terms, the epoch in which his views were formed and his scientific activity developed. It fell mainly to the first half of the 19th century, one of the most remarkable and productive periods in the history of exact natural sciences. At the beginning of the 19th century, many fundamental works were published, such as the five-volume *Treatise on Celestial Mechanics* (1799–1825) by Laplace and the two-volume *Analytical Mechanics* (1811–1815) by Lagrange. In these works, deep in content and masterful in exposition, the results of predecessors in mechanics and astronomy were outlined and systematized, and foundations for future investigations in these branches of science were laid. It is quite possible that the general methods presented in the indicated work of Lagrange most strongly affected the formation of scientific views of Ostrohrads'kyi. He substantially developed and generalized them in numerous papers and lectures.

As V.A. Steklov noted about the first half of the 19th century in his speech at the celebration of the centenary of Ostrohrads'kyi, "this was an exciting period of time, when almost every day brought new ideas and new discoveries in various areas of mathematical physics and, with it, in mathematical analysis. Without no overstatement one can say that little has been added to the scientific ideas of that era, and today's efforts are mainly at streamlining and developing the theories of the great thinkers of that time, of extending their applications, and of perfecting the proofs". Mykhailo Ostrohrads'kyi participated in almost all mentioned areas of mathematical physics. His works in the theories of heat, elasticity, and attraction, as well as hydrodynamics, are not inferior in their significance to the works of the luminaries of science of that time.

Mykhailo Vasyl'ovych Ostrohrads'kyi, a son of a landowner Vasyl Ivanovych and of Iryna Andriyivna (born Ustymovych) was born in his father's estate in the village Pashenna of Kobelyaky povit (district), Poltava region[1].

At 9 Myshko entered the pension at the Poltava Gymnasium named *House for Education of Poor Gentry*, where one of the tutors was another famous person from Poltava, a poet I.P. Kotlyarevs'kyi. From here Mykhailo Ostrohrads'kyi was transferred to Romny Postal Office where he soon was awarded with a civil title of Collegial Registrar. In 1815 14-year old civil servant was dismissed from his position and entered the same pension once more. Ostrohrads'kyi was not distinguished by a particular diligence, but was noticed as a lively, capable and smart boy.

The following data with respect to first years of Ostrohrads'kyi's studies at the Poltava Gymnasium were discovered in its archive. During the first month of his studies at the Gymnasium he was marked as "average" by capability, "diligent" by diligence and "fair" by manners, and at the end of the first year of studies – "smart" by capability and "courteous" by manners.

During 1813 Ostrohrads'kyi, with the 9-grade system, had the following grades: 6 for psychology, 7 for moral philosophy, 2 for history and geography, 0 for Latin, French and German. We can see that Ostrohrads'kyi did not like languages, especially Latin.

In 1816 father took him to St. Petersburg to enlist him into one of the guard 's regiment, but, at the advice of P.A. Ustymovych, M.V. Ostrohrads'kyi's uncle, changed his decision and decided to enlist him to Kharkiv University.

Ostrohrads'kyi attended the university at first as a free listener, but later, in 1817, entered as a student of the department of physics and mathematics. During the first year of the university course and the first half of the second year he studied badly and continued to dream about the military service, and every moment he was ready to exchange the university for any regiment.

At the age of 17 on October 3, 1818 Ostrohrads'kyi completed his studies at the university, and received his student's diploma noting that he studied algebra, trigonometry, curve-line geometry, civil architecture, practical geometry, history, statistics of Russia and world history with very good success, and military studies, function theory, integral and variation calculus and Russian language and literature with excellent success.

In 1820 he had exams together with other students, and at the general meeting of the university his name was distinguished. At the time, seeing Ostrohrads'kyi's success, the professor of mathematics T.F. Osipovs'kyi wanted to award Ostrohrads'kyi with the candidate's degree. To get this degree Ostrohrads'kyi had to take the exam on philosophy, but the philosophy professor refused to take an exam for the reason he did not attend lectures on philosophy.

The Ostrohrads'kyi went to the university management, produced his diploma and gave it to professors who had a meeting, with the request to "remove his name from the student list".

Mykhailo Vasyl'ovych went to the village to his father, stating his firm intent to go abroad and to study with famous French mathematicians. Father listened to his son and quite favored his intent.

In May 1822 Ostrohrads'kyi started his journey, but he was robbed in Chernihiv. Father gave him money once more, and in August of the same year he was in Paris already. Having reached his goal with great difficulties, Ostrohrads'kyi attended lectures in Sorbonna and Collége de

[1]Biographical data were taken from the sketch by P.I. Trypols'kyi [1].

France, and his bright talent attracted attention of famous French mathematicians: Laplace, Fourier, Ampere, Poisson, Cauchy etc. He was very friendly with the two latter researchers, and later exchanged letters with them, and he was accepted in Laplace's home as a family member.

The scientific talent of Ostrohrads'kyi was powerful, versatile, and, at the same time, original. In only six years of his stay in Paris, which was the center of mathematical research at that time, he got well informed about diverse new ideas and theories, concentrated on the most important problems that were the object of the work of the constellation of French genii (Laplace, Poisson, Cauchy, Fourier, etc.), and succeeded in their solution, getting ahead of these scientists in many issues. In the memoir *On Definite Integrals Taken between Imaginary Limits* submitted to the Académie Française in 1825, Cauchy expressed this by the following words: "Monsieur Ostrohrads'kyi, a young Russian man gifted by an extraordinary insight and very skillful in the analysis of infinitesimals, also applied these integrals and, transforming them to ordinary ones, gave a new proof of the formulae mentioned above, and communicated other formulae, which I now present ..." In his works, as we already noted, Cauchy repeatedly referred to Ostrohrads'kyi. He reviewed his scientific works and was one of the numerous French mathematicians who enthusiastically supported the candidacy of Ostrohrads'kyi for the election as an *Immortal* of the Académie Française. In 1856, he was elected the corresponding member of this academy.

The real mathematical debut of Ostrohrads'kyi took place in 1826, when he submitted his *Mémoire sur la Propagation des Ondes dans un Bassin Cylindrique* to the Académie Française. Under various additional physical assumptions, the problem of wave propagation on the surface of water was studied by Newton, Laplace, Lagrange, Cauchy, and Poisson. The main input of M.V.Ostrohrads'kyi to this issue was that he was the first who considered this problem in a closed cylinder of finite depth. Poisson and Cauchy, who were present at the talk given by Ostrohrads'kyi, highly evaluated the results presented, after which it was decided to publish them in *Mémoires Présentées par Divers Savants*. This was a great honor for Ostrohrads'kyi, who was only 25 years of age. This success strengthened the reputation of the scientist. Warm relationships between him and French mathematicians, such as Cauchy, Poisson, J. Sturm, G. Lamé, etc., were established and lasted for many years.

In 1828, Ostrohrads'kyi moved to St. Petersburg. Only at that moment, after coming back to Russia, Ostrohrads'kyi was appreciated by his compatriots, and a circle of people who loved mathematics was established around him at once, who wanted to find out about new views and methods in calculus. In the same year, in 1928 (on December 17) the Imperial Academy of Science elected him as Adjunct of Applied Mathematics, in 1830 he received the title of Extraordinary Academician, and in a year – the title of Ordinary Academician. In July 1830 he was sent to Paris with a research purpose and at that time presented to the Paris institute his course of celestial mechanics, where he showed great independence, mainly in simplification of explanation of general methods. Arago and Poisson, having considered this work at the request of the Paris Academy, awarded Ostrohrads'kyi with a praising reference that was finished by the following words: "We believe that the paper by Ostrohrads'kyi deserves the Academy's praise and approval"; in this Arago puts Ostrohrads'kyi's name along with that of immortal Laplace.

Inspired by the first successes, Ostrohrads'kyi set the grand problem of presentation of various sections of mathematical physics by means of mathematical methods. In one of his reports submitted to the St. Petersburg Academy in 1830, he wrote: "The followers of Newton developed the great law of universal gravitation in detail and applied mathematical analysis to numerous important problems in general physics and physics of weightless substances. The collection of their works about the system of universe forms the immortal folios of *Celestial Mechanics*, from which astronomers will take the elements for their tables for a long time. However, physical and mathematical theories are still not unified; they are distributed over numerous collections of academic memoirs and are investigated by different methods, often very doubtful and imperfect;

moreover, there are theories developed but never presented. I set it as my aim to combine these theories, present them by using a uniform method, and indicate their most important applications. I already collected the necessary materials on the motion and equilibrium of elastic bodies, propagation of waves on the surface of incompressible liquids and propagation of heat inside solid bodies and, in particular, inside the globe. However, these theories will constitute only the necessary part of the entire work, which will also embrace the distribution of electricity and magnetism in bodies capable of being electrified or magnetized through electrodynamical influence, motion of electric fluids, motion and equilibrium of liquids, action of capillarity, distribution of heat in liquids, and probability theory; in this last part, I will dwell upon several issues in which the famous author of Celestial Mechanics was apparently wrong."

In this respect it is interesting to recall that D. Hilbert, who, as a true mathematician, was concerned with the absence of order in the triumphal progress of physics at the beginning of the 20th century and decided to give a mathematical presentation of physics by using the axiomatic approach (the sixth Hilbert problem). However, despite his deep faith in the omnipotence of the axiomatic method and its capability to bring an order into chaos, Hilbert realized that mathematics alone is insufficient for the solution of all physical problems. Although Hilbert spent a lot of effort and time to be well informed about new physical investigations, he failed to implement his plan concerning physics.

In the most general statement, the problem indicated was formulated by Ostrohrads'kyi in his report made at a session of the St. Petersburg Academy of Sciences on November 5, 1828, and published as an academic edition in 1831 in French under the title *Note sur la Theorie de la Chaleur*. Steklov wrote the following words with respect to this paper: "After Fourier constructed the differential equation of heat propagation in solids, the need for formulation of techniques to determine the temperature of a body that is sought for according to conditions of the problem.

Fourier himself and also Poisson considered the cases of cooling of a solid ball, cylinder, cube and rectangular parallelepiped.

In all these cases Fourier employed the same technique knows now as the Fourier method, but he was unlikely to see the property of its generality in its total. At least we cannot see that from Fourier's research papers, and I am hardly wrong to say that the Fourier method in all its generality was first formulated by Ostrohrads'kyi, and then (in 1829) by Lamé and Duhamel."

In this research Ostrohrads'kyi in part went ahead of Cauchy who in 15 years in his memoir *Recherches sur les intégrales des équations linéares aux differences partielles* obtained the same results once more, and in the note to this memoir Cauchy said: "I would like to compare the theorems I found with those obtained by Ostrohrads'kyi in one of his memoirs, but having bad memory and even not knowing whether this memoir by Ostrohrads'kyi was published anywhere, I am unable to do that". Evidently that the memoir by Ostrohrads'kyi being considered contains just the same conclusions Cauchy was interested in, or at least, part of them.

Ideas of Ostrohrads'kyi's report of 1828 were continued in his two *Notes on the Theory of Heat*, submitted to the St. Petersburg Academy of Sciences on September 5, 1828, and July 8, 1829. Maybe, this title does not adequately reflect the content of these notes, but it indicates that, in the 1820s, the analytical theory of heat was the leading topic in mathematical physics (for the most part, this is true for Paris, where Ostrohrads'kyi worked in 1822—1828).

The results of the *Notes* are important not only from the viewpoint of their significance for physics. It is difficult to overestimate their general mathematical significance because, on the one hand, they laid the foundation for important theories, which has been successfully developed up to now, and, on the other hand, the statements obtained therein constitute a part of the foundations of contemporary mathematical analysis.

In this context, the first note is the most important. It consists of two parts. In the first part, the general scheme of the solution of boundary-value problems in mathematical physics is described. The formula for the transformation of the volume integral of the divergence type into a surface integral was derived that now is an integral part of any calculus textbook, and is called the Ostrohrads'kyi–Gauss formula.

The appearance of this formula was stimulated by the needs of potential theory, theory of heat, and variational calculus. The first steps related to volume integrals were made by Lagrange, who found a method for their calculation and gave a formula for a change of variables that generalizes the corresponding Euler formula for double integrals. As for the surface integrals, the *Analytical Mechanics* of Lagrange (1813) contains only certain notes related to specific cases. However, the development of electrostatics and the theory of magnetism expanded the circle of problems of potential theory. Furthermore, the investigation of the distribution of static electricity over the surface of a body led to the necessity of introducing the notion of surface integral. It first appeared in the paper by Gauss published in 1813 and related to potential theory, and some theorems from this work can be regarded as partial cases of Ostrohrads'kyi formula. The formula itself was not shown in the indicated work by Gauss.

Hence, the first great merit of Ostrohrads'kyi lies in the fact that he was the first who realized the mentioned formula (Ostrohrads'kyi–Gauss formula) is of independent interest and indicated its general mathematical importance. In his prominent work of 1834 on variational calculus, he extended this formula to the case of arbitrarily many variables. Its vector interpretation was given in the *Treatise on Electricity and Magnetism* by J. Maxwell, who stressed the priority of Ostrohrads'kyi in the discovery of this formula.

The second substantial result, which is also contained in the first part of the mentioned note, is the introduction of an adjoint operator L^* for a linear differential operator L of arbitrary order with constant coefficients and the derivation of the integral relation for them. For many years, numerous mathematicians worked on the generalization of this formula, and today it is one of the cornerstones of the entire theory of boundary-value problems for differential and difference equations.

The second part of the note was devoted to the application of the general scheme presented in the first part to problems of heat propagation in solid bodies of arbitrary form, namely, to the solution of the mixed heat propagation problem in a bounded domain G with smooth boundary ∂G.

The key point of the work considered was the hypothesis that the spectrum of problem under consideration is discrete and that spectral decomposition of an arbitrary function $f(x)$ inside the domain G. M.V. Ostrohrads'kyi understood that decomposition gives mapping of the function only inside the domain G.

On this occasion, Ostrohrads'kyi wrote: "I think that the series of the decomposition obtained always converges, but it is very difficult to prove this wonderful property in the general case". These words indicate that Ostrohrads'kyi was aware of the complexity of the problem of convergence of such series. Indeed, at those times, numerous fields of mathematical analysis did not have necessary tools not only for solving this problem, but even for getting started with it. The validity of the hypothesis advanced by Ostrohrads'kyi was completely confirmed in the 1960s.

Ostrohrads'kyi's decomposition formula has a universal character because it is also applicable to non-self adjoint boundary value problems and for the case when the domain G is not bounded.

"Finally, note that the the eigenvalues of problem under consideration are always real, which is a consequence of the law of propagation of heat, but even this general fact must be established by mathematical analysis", Ostrohrads'kyi wrote in the same note. This means that, unlike many known scientists (Poisson, Laplace, Fourier, Poincaré, etc.) who worked in the field of mathematical physics and mechanics and thought that the rigorousness requirements can be weakened in these fields science, Ostrohrads'kyi had an opposite opinion consonant with the convictions of Gauss, Cauchy, and Abel.

In the second of his *Notes on the Theory of Heat*, Ostrohrads'kyi, for the first time, solved a mixed heat propagation problem with the difference that a function $T(t,x)$ instead of zero enters the right-hand side of the boundary condition, i.e., in the case where this condition is inhomogeneous. This problem was considered earlier by Laplace and Poisson in the case where $T(t,x)$ does not depend on t. Ostrohrads'kyi reduced the problem with an inhomogeneous boundary condition to a problem with a homogeneous boundary condition, but for an inhomogeneous equation whose solution was sought in the form of an infinite series. The Ostrohrads'kyi method of reduction of an inhomogeneous boundary-value problem to a homogeneous one is presented in modern textbooks on mathematical physics as the Duhamel principle. Indeed, J. Duhamel solved this problem simultaneously with Ostrohrads'kyi, but he published his result in 1833, whereas Ostrohrads'kyi published his note in the *Mémoires de l'Academie des Sciénces de St.-Pétersbourg* in 1831.

The systematic investigation of the problem of expansions of the mentioned type in the eigenfunctions of the operator was continued by Ostrohrads'kyi's followers, in particular, by M.G. Krein, O.Ya. Povzner, I.M. Glazman, Yu.M. Berezans'kyi, V.O. Marchenko and by their students.

Among other works of Ostrohrads'kyi that significantly influenced the subsequent development of the theory of partial differential equations and variational calculus, a special place belongs to his fundamental work *Mémoire sur le Calcul des Variations des Integrales Multiples* submitted to the St. Petersburg Academy of Sciences on January 24, 1834. This memoir immediately drew the attention of mathematicians. In 1836, it was reprinted by the known Crelle's *Journal für die reine und angewandte Mathematik*, and its complete English version appeared in 1861 in *A History of the Calculus of Variations during the 19th Century* by I. Todhunter. It was the paper where fundamental results on the integral calculus of functions of many variables were presented. These results are regarded as classical for a long time already, and, up to now, they serve as the main tool in the theory of partial differential equations. First of all, this concerns Gauss–Ostrohrads'kyi formula in the case of arbitrary multiplicity n, the rule of location of the integration limits with respect to each variable when passing from an n-fold integral to a repeated integral, and the method for finding the derivative with respect to a parameter of a multidimensional volume integral with a variable limit of integration that, together with the integrand, depends on this parameter. In the same work, for the first time, Ostrohrads'kyi introduced (simultaneously with Jacobi) the notion of functional determinants (Jacobians). The developed foundations of integral calculus enabled Ostrohrads'kyi to completely solve the problem of calculation of the variation of an n-fold integral with variable limits of integration. Note that, under certain restrictions on the domain of integration, a formula for the first variation was obtained earlier by Euler for $n = 2$ and by Lagrange for $n = 3$. Without additional restrictions, in the case $n = 2$, the corresponding formula was established by Poisson simultaneously with the general case considered by Ostrohrads'kyi.

In the same memoir, Ostrohrads'kyi actually showed that the problem of variational calculus on the extremum of a multiple integral is equivalent to the problem of finding a certain solution of a partial differential equation. Later, this fact, which Riemann called the Dirichlet principle, drew the attention of Gauss, Thomson, and Dirichlet. It was established that this principle plays a key role in numerous variational methods for the solution of boundary-value problems for differential equations. A considerable contribution to the development of these methods for various classes of equations was made by mathematicians from Ukraine such as M.M. Bogolyubov, M.M. Krylov, M.P. Kravchuk, N.I. Pol's'kyi, Yu.D. Sokolov, and their followers.

In connection with the investigations carried out by Ukrainian mathematicians, in particular, at the Institute of Mathematics of the Ukrainian Academy of Sciences, it is reasonable to recall Ostrohrads'kyi's work *Note sur la Méthode des Approximations Successives* (1835) devoted to the integration of the nonlinear Duffing equation using the expansion in the small parameter a.

Later, it became clear that this equation plays an important role in the investigation of the process of pitching and rolling of a ship. Much later, the method of a small parameter received wide recognition due to the works of Poincaré, O.M. Lyapunov, M.M. Krylov, M.M. Bogolyubov, Yu.A. Mitropol'skii, and their students. Thus, the Ostrohrads'kyi method was a predecessor of the theory of nonlinear oscillations.

Besides the aforementioned programmatic works, in which Ostrohrads'kyi laid the foundations of the theory of partial differential equations, he also wrote many papers related to the integration of specific equations of mathematical physics and mechanics. Among them, one should mention his large (100 p.) work *Memoir on Differential Equations Related to the Isoperimetric Problem*. Among other important results presented in this memoir, it was shown that all differential equations of variational (Euler–Lagrange) problems with one independent variable can be reduced to canonical systems. Most textbooks on the theory of ordinary differential equations contain the Ostrohrads'kyi–Liouville formula published by M.V. Ostrohrads'kyi in his note *On Linear Differential Equation of the n-th Order* (1838) (in the case $n = 2$, it was obtained by Abel in 1827). In the works of Liouville, there is no this very formula.

M.V. Ostrohrads'kyi wrote 54 research memoirs, all in French, 50 of these were read at the meetings of the Russian Academy of Sciences and published in its editions, and others were published in the editions of the Paris Academy of Sciences.

As to the appearance of the manuscripts of the great mathematician we can say the following. He was very unwilling to do any rewriting.

He was a brilliant lecturer, and merits of his lectures were dependent a lot on his mood. Sometimes he gave the whole lecture on mechanics or higher mathematics not using a blackboard, if even complicated formulae were to be introduced.

He lectured with a great passion; wrote huge letters and for this reason made the blackboard full very fast, and then rushed to a large table covered by black impregnated fabric, continued to write at it and then lifted it to show to the listeners what was written. With his passionate lecturing he got tired very soon and sat to rest for a few minutes, drinking a lot of water.

He had a very good memory, remembered many historical and literature works that he read when he was young: knew many poems by heart, his favorite poet was T.H. Shevchenko, almost all poems of which he knew also by heart. His handwriting was so bad that even his close relatives could not read it.

He did not interfere into household issues at all, his wife dealt with that; he preferred walks at hand with his servant Shchak and philosophizing on different issues.

He rarely got ill, and with no problem sustained severe Petersburg climate after the south. He could be often seen at the Neva embankment under a strong rain without an umbrella and galoshes; note that he hated polished boots. In 1830 he had to be treated maybe for the first time. The matter was that during his trip to Paris he injured his eye because of careless using a phosphor match, and he had to go to a doctor. But prompt departure to Russia did not allow him to complete his treatment in Paris, and he got a cold at his eye while going back by sea and after his return to Paris lost his eye at all because of unsuccessful treatment.

In 1831 Ostrohrads'kyi got married, secretly to his father, to Maria Vasylivna Kupfer from Livland that brilliantly wrote verses in German, played and sang, and he encouraged her in all ways to perfect herself in these arts. At the end of his life Ostrohrads'kyi became very religious, and he had an icon-lamp burning even during not so important holidays. His mother's shadow reportedly told him: "Mykhailo, believe and pray!" From that time he became religious.

Many foreign scientific institutions elected Ostrohrads'kyi as their member: he was awarded by one of the most honorary titles for a scientists – a title of a corresponding member of the Paris Academy of Sciences, and titles of a member of Turin, Rome, American Academies, and a title of Honorary Doctor of the Alexander University. Among all that he was especially proud by the title of the member of the American Academy.

Ostrohrads'kyi made history not only as a first-rank scientist. He was a great teacher whose activity had a decisive influence on the increase of the level and role of science, first of all, mathematics, mechanics, and engineering, in the Russian Empire. Any other scientist and pedagogue of the first half of the 19th century can hardly be compared with him in this respect. The time of Euler with his fundamental achievements in the Russian mathematics and mechanics was followed by a certain fall. No systematic investigations were performed in these directions. This fall lasted till the appearance of Ostrohrads'kyi. After he moved to St. Petersburg, this city became the center of the mathematical life of Russia. His scientific works, inimitable lectures, and gifted disciples indicated the rise of the Russian science. Ostrohrads'kyi was an active promoter of new physical and mathematical achievements and the author of many textbooks on mathematics and mechanics, which were used by several generations of scientists and engineers. It is difficult to find a scientific institution of St. Petersburg where he did not give lectures. He devoted much time to pedagogic activity, and, thus, less time left for his scientific work. "A man, without doubt, of brilliant mind", P.L. Chebyshev wrote about Ostrohrads'kyi, "he did not accomplish even a half of what he could have done if he were not "bogged down" with tiresome permanent teaching work". However, this teaching "bog" made its great input into the progress of mathematics and physics in Russia. Under the influence of two Ukrainians, namely, Ostrohrads'kyi and V.Ya. Buniakowski, the first scientific schools were created in these directions, whose branches gave the world such renowned scientists as P.L. Chebyshev, M.E. Zhukovs'kyi, A.M. Lyapunov, V.A. Steklov, G.F. Voronoi, S.A. Chaplygin, etc.

Mykhailo Vasyl'ovych died unexpectedly. In summer of 1861 he came to his own estate in Ukraine and caught cold. Instead of treatment he decided to go to St. Petersburg, but had to stay in Poltava because of his illness. He died on December 20, 1861 (1 January, 1862) at midnight.

The last will of Ostrohrads'kyi was to be buried, as Shevchenko, in Ukraine. In accordance with his will, he was buried in his home village of Pashenna (now Pashenivka), Poltava province. Two hundred years ago, this land gave the world Ostrohrads'kyi, who, in the period of rapid development of science at the beginning of the 19th century, was the only Slavonian who, together with the glorious team of West-European scientists, created the foundations of modern mathematics, physics, and mechanics.

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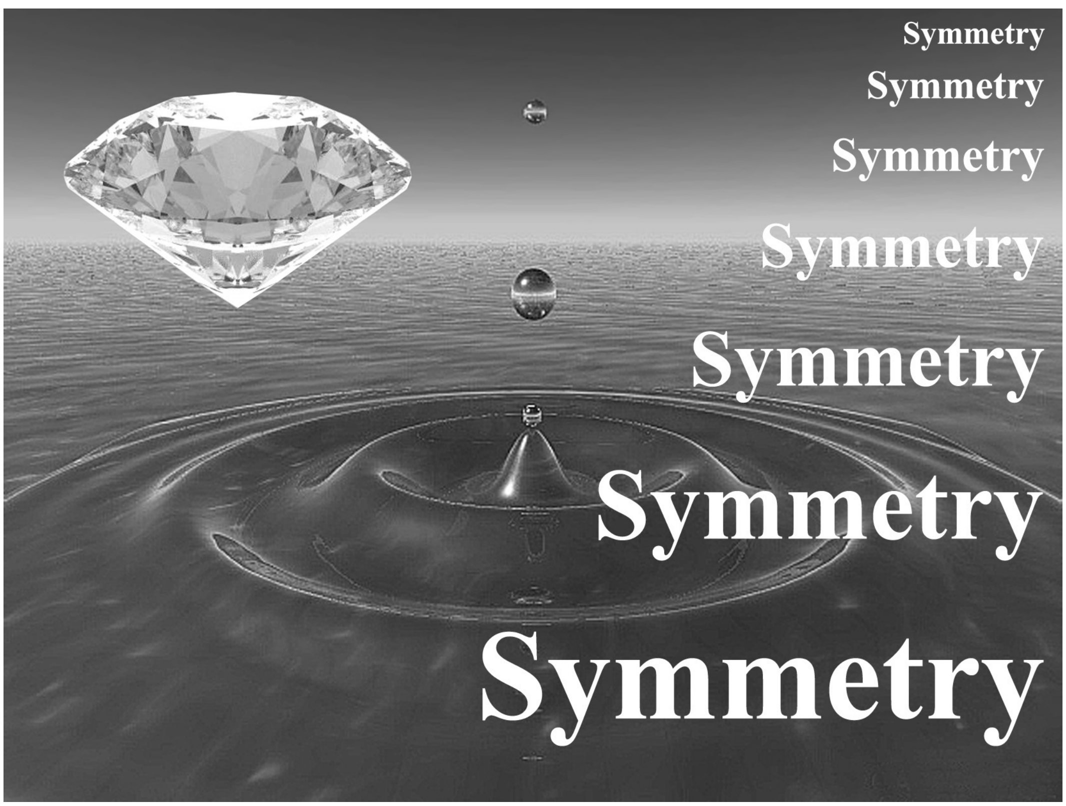
Symmetry of Differential Equations



Symmetry in Nonlinear Mathematical Physics

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An Old Problem Newly Treated with Differential Forms: When and How Can the Equation $y'' = f(x, y, y')$ Be Linearized?

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Sophus Lie, more than a century ago, investigated the problem of linearization of the equation $y'' = f(x, y, y')$, where $()'$ means d/dx [1]. Originally, he investigated the necessary conditions for linearization by a point transformation and showed that f must be a cubic in y' and that other conditions must be satisfied. Later, he and others such as Tresse [2] worked out actual construction of the linearizing transformations, often using group theory. The present author will show a method of construction using differential forms, suitable when certain intermediate equations can be integrated explicitly.

1 Introduction

The possible linearization of the equation

$$y'' = f(x, y, y'), \tag{1}$$

where the prime indicates differentiation with respect to x, might be considered a simple problem, but it is actually rather complex. It is a very old problem, having been investigated by Sophus Lie [1] and by other subsequent authors (for example Tresse [2], Ibragimov [3, 4, 5], Berkovich [6], Grissom et al [7], Kamran et al [8, 9], Bocharov et al [10], Schwarz [11], Steeb [12], and N. Euler [13]). These methods use group theory or approach the problem as a Cartan equivalence problem. There are also treatments that consider equivalence of nonlinear and linear partial differential equations, such as those by Kumei and Bluman [14, 15].

In 1998 the author spent a month at the University of Witwatersrand in South Africa as the guest of Fazal Mahomed. During that month this was one of the problems that we looked at, and it became intriguing as the possibility of using differential forms in its treatment emerged. This talk is a detailed report on that research. The author has reported on it before in a summary fashion [16].

The previous papers that treat this as a Cartan equivalence problem use the Cartan theory. The differential forms used here are not part of that theory, but are used to make the treatment simpler and more obvious. One can carry out the same calculations without forms. The virtue of this approach is that the linearization can be achieved, in principle, by solving some intermediate linear differential equations. We will see how this can be done.

2 Basic theory and conditions for linearizability

We begin by adopting Lie's approach: assume a point transformation given by new variables

$$X = F(x, y), \qquad Y = G(x, y), \tag{2}$$

and require that

$$d^2Y/dX^2 = 0. \tag{3}$$

We note that this is a special case of a linear equation. Lie does not consider other cases, with terms in dY/dX and Y. We will comment on this later.

Now consider the conditions imposed on $F(x,y)$ and $G(x,y)$ by this requirement. We first construct, using equation (2),

$$dY/dX = (G_x dx + G_y dy)/(F_x dx + F_y dy) = \left(G_x + G_y y'\right) / \left(F_x + F_y y'\right),$$

where subscripts x and y denote differentiation. Now the second derivative equation may be written simply in terms of a differential $d(dY/dX) = 0$, or

$$\left(F_x + F_y y'\right) d\left(G_x + G_y y'\right) - \left(G_x + G_y y'\right) d\left(F_x + F_y y'\right) = 0,$$

which now may be treated as a differential form equation. We expand the differentials and obtain

$$\left(F_x + F_y y'\right)\left(dG_x + y' dG_y + G_y dy'\right) - \left(G_x + G_y y'\right)\left(dF_x + y' dF_y + F_y dy'\right) = 0$$

or

$$T dy' + \rho {y'}^2 + (\lambda + \delta) y' + \sigma = 0, \tag{4}$$

where

$$T = F_x G_y - F_y G_x$$

and we have the $1-$forms

$$\begin{aligned} \rho &= F_y dG_y - G_y dF_y, \qquad & \lambda &= F_y dG_x - G_y dF_x, \\ \sigma &= F_x dG_x - G_x dF_x, \qquad & \delta &= F_x dG_y - G_x dF_y. \end{aligned} \tag{5}$$

We note that

$$dT = \delta - \lambda. \tag{6}$$

Rewrite equation (4) as

$$dy' = \alpha + \beta y' + \gamma {y'}^2, \tag{7}$$

where

$$\alpha = -\sigma/T, \qquad \beta = -(\lambda + \delta)/T, \qquad \gamma = -\rho/T. \tag{8}$$

This sort of equation has occurred in other contexts, such as in searching for Bäcklund transformations, where y' may be viewed as a fiber coordinate on a base space parameterized by x and y.

We remember from differential form calculus that $dd\omega = 0$, where ω is any form, and that $1-$forms anticommute under the hook product $\wedge$. For integrability of equation (7), we ask $ddy' = 0$, or

$$0 = d\alpha + dy' \wedge \beta + y' d\beta + 2y' dy' \wedge \gamma + {y'}^2 d\gamma,$$

and with substitution from equation (7) we have

$$0 = d\alpha + \left(\alpha + \beta y' + \gamma {y'}^2\right) \wedge \beta + y' d\beta + 2y' \left(\alpha + \beta y' + \gamma {y'}^2\right) \wedge \gamma + {y'}^2 d\gamma.$$

The y'^3 term vanishes because $\gamma \wedge \gamma = 0$; we equate the coefficients of the other powers of y' to zero and get

$$d\alpha = \beta \wedge \alpha, \qquad d\beta = 2\gamma \wedge \alpha, \qquad d\gamma = \gamma \wedge \beta. \tag{9}$$

Now we go back to equations (5) and expand the differentials:

$$\begin{aligned}
\rho &= F_y(G_{xy}dx + G_{yy}dy) - G_y(F_{xy}dx + F_{yy}dy),\\
\lambda &= F_y(G_{xx}dx + G_{xy}dy) - G_y(F_{xx}dx + F_{xy}dy),\\
\sigma &= F_x(G_{xx}dx + G_{xy}dy) - G_x(F_{xx}dx + F_{xy}dy),\\
\delta &= F_x(G_{xy}dx + G_{yy}dy) - G_x(F_{xy}dx + F_{yy}dy),
\end{aligned}$$

or

$$\rho = Adx + Bdy, \qquad \lambda = Cdx + Ady, \qquad \sigma = Ddx + Edy, \qquad \delta = Edx + Hdy, \tag{10}$$

where

$$\begin{aligned}
A &= F_yG_{xy} - G_yF_{xy}, \qquad B = F_yG_{yy} - G_yF_{yy}, \qquad C = F_yG_{xx} - G_yF_{xx},\\
D &= F_xG_{xx} - G_xF_{xx}, \qquad E = F_xG_{xy} - G_xF_{xy}, \qquad H = F_xG_{yy} - G_xF_{yy}.
\end{aligned}$$

Thus, from equations (8) and (10),

$$\begin{aligned}
&\alpha = -(Ddx + Edy)/T, \qquad \beta = -(Cdx + Edx + Ady + Hdy)/T,\\
&\gamma = -(Adx + Bdy)/T.
\end{aligned} \tag{11}$$

We now substitute α, β, and γ into equation (7) for dy', divide by dx to convert the differential forms to functions, and rewrite it as:

$$y'' + f_0 + f_1y' + f_2y'^2 + f_3y'^3 = 0, \tag{12}$$

where the f_k are given by

$$f_0 = D/T, \qquad f_1 = (C + 2E)/T, \qquad f_2 = (H + 2A)/T, \qquad f_3 = B/T.$$

We define K and L as

$$K = E/T, \qquad L = A/T, \tag{13}$$

and replace D, C, H, and B in the $1-$forms in equation (11) in favor of the f_k, K, and L, obtaining

$$\alpha = -f_0dx - Kdy, \qquad \beta = (K - f_1)dx + (L - f_2)dy, \qquad \gamma = -Ldx - f_3dy. \tag{14}$$

We also note from equation (6) for dT that now

$$dT/T = (3K - f_1)dx + (f_2 - 3L)dy. \tag{15}$$

We see from the above that it is necessary, for the original assumption of linearizability to hold, that the expression $f(x, y, y')$ in equation (1) be a cubic in y'. Thus the original form of the equation which we have is to be that in equation (12) above, with the f_k known functions of x and y. We see that the $1-$forms α, β, γ, and dT/T are now expressed in terms of these four known functions f_k and two other functions K and L. The first three of these $1-$forms can now be substituted into equations (9) to find conditions on the various functions.

If we do that, the first equation, for $d\alpha$, gives the equation

$$f_{0y} - K_x = -K(K - f_1) + f_0(L - f_2),$$

which is nonlinear in K. The other equations give similar results. However, we can simplify the situation by defining new variables:

$$T = 1/W^3, \qquad E = U/W^4, \qquad A = V/W^4,$$

so that from equation (13)

$$K = U/W, \qquad L = V/W. \tag{16}$$

Equation (15) now becomes

$$3dW/W = (f_1 - 3K)dx + (3L - f_2)dy. \tag{17}$$

We now have this situation. The dW equation (17) gives expressions for W_x and W_y. The $d\alpha$ equation in equation (9) gives, after substitution for W_x, an expression for U_x which is linear in U, V, and W. The $d\gamma$ equation gives an expression for V_y, which is also linear. The $d\beta$ equation gives a linear expression for $V_x - U_y$. The integrability condition on W, $ddW = 0$, gives a linear expression for $V_x + U_y$. The latter two equations can be solved for V_x and U_y. Thus we have expressions for all derivatives of U, V, and W, all of which are linear and homogeneous (no constant terms) in the same variables.

We summarize all these relations in a nice matrix equation

$$dr = Mr, \tag{18}$$

where

$$r = \begin{bmatrix} U \\ V \\ W \end{bmatrix} \qquad \text{and} \qquad M = Pdx + Qdy, \tag{19}$$

where

$$P = (1/3)\begin{bmatrix} -2f_1 & -3f_0 & 3f_{0y} + 3f_0 f_2 \\ 0 & f_1 & 2f_{2x} - f_{1y} - 3f_0 f_3 \\ -3 & 0 & f_1 \end{bmatrix}$$

and

$$Q = (1/3)\begin{bmatrix} -f_2 & 0 & 2f_{1y} - f_{2x} + 3f_0 f_3 \\ 3f_3 & 2f_2 & 3f_{3x} - 3f_1 f_3 \\ 0 & 3 & -f_2 \end{bmatrix}.$$

For integrability, $ddr = 0$, or $0 = dMr - M \wedge dr = dMr - M \wedge Mr$, giving

$$dM = M \wedge M$$

which is not zero since M is a matrix. Substitution for M in terms of P and Q gives the condition

$$Q_x - P_y = [P, Q],$$

the necessary condition on the f_k for linearization to be possible. This matrix condition reduces to two equations:

$$f_{0yy} + f_0(f_{2y} - 2f_{3x}) + f_2 f_{0y} - f_3 f_{0x} + (1/3)(f_{2xx} - 2f_{1xy} + f_1 f_{2x} - 2f_1 f_{1y}) = 0 \tag{20}$$

and

$$f_{3xx} + f_3(2f_{0y} - f_{1x}) + f_0 f_{3y} - f_1 f_{3x} + (1/3)(f_{1yy} - 2f_{2xy} + 2f_2 f_{2x} - f_2 f_{1y}) = 0. \tag{21}$$

To summarize, we note that linearizability requires the original differential equation to be a cubic in y', with the coefficients satisfying equations (20) and (21). These conditions are written out in Lie [1] and in Ibragimov [3], for example.

3 Construction of the linearizing point transformations

In the following, we will need U, V, and W, so we will need to solve equations (18). It is important to note that the most general solution is apparently not necessary; special solutions will suffice. Thus one can make simplifying assumptions in the solution. Once the equations are solved, then we construct K and L from equations (16).

In order to find the $F(x, y)$ and $G(x, y)$ for which we are seeking, we revert to equations (5) and solve for dF_x, dF_y, dG_x, and dG_y. Solution for the first two gives

$$dF_x = (F_y\sigma - F_x\lambda)/T, \qquad dF_y = (F_y\delta - F_x\rho)/T.$$

Solution for the second two, dG_x and dG_y, shows that they satisfy the same equation, so we will write only equations for the derivatives of F. We note that $\delta + \lambda = -T\beta$ and that $\delta - \lambda = dT$, so we can solve these equations for δ and λ. We can also substitute for σ and ρ in terms of α and γ. We get finally

$$dF_x = -F_y\alpha + F_x(\beta + dT/T)/2, \qquad dF_y = F_x\gamma + F_y(-\beta + dT/T)/2.$$

We substitute for α, β, γ, and dW in terms of the expressions obtained above, with the f_k, K, and L. The dW terms disappear and we are left with two equations which we can express in matrix form as follows.

Write

$$R = \begin{bmatrix} F_x \\ F_y \end{bmatrix} \qquad \text{and} \qquad S = \begin{bmatrix} G_x \\ G_y \end{bmatrix}.$$

Now

$$dR = ZR, \qquad dS = ZS, \tag{22}$$

where

$$Z = \begin{bmatrix} (2K - f_1)dx - Ldy & f_0 dx + Kdy \\ -Ldx - f_3 dy & Kdx + (f_2 - 2L)dy \end{bmatrix}.$$

This linear equation set can be solved for R; there will be two independent solutions, which can be taken as R and S. See equation (22). (Integrability is guaranteed by the previous conditions, as can be seen by setting $ddR = 0$.) Finally, one can solve

$$dF = [dx\ dy]R, \qquad dG = [dx\ dy]S \tag{23}$$

for F and G.

We can summarize the procedure.

1. Make sure that the original differential equation is a cubic in y'.
2. Test the coefficients f_k to see whether they satisfy equations (20) and (21). If equations (1) and (2) are satisfied, then the equation is linearizable in principle.
3. Construct the 3×3 matrix M and solve equation (18) (linear!) for the three components of r – a special solution is usually sufficient – and construct K and L.
4. Construct the 2×2 matrix Z and solve equation (22) (linear!) for R or S.
5. Solve equation (23); the two independent solutions may be taken as F and G.

Steps (1) and (2) test for linearizability; steps (3)–(5) perform the construction (in principle).

4 Examples

4.1 The general linear equation

We first consider the equation

$$y'' + a(x)y' + b(x)y + c(x) = 0.$$

We see that $f_2 = f_3 = 0$, $f_1 = a(x)$, and $f_0 = b(x)y + c(x)$. Equations (20) and (21) are satisfied, so this equation can in principle be cast into the form (3). However, when one writes out equation (18), one sees quickly that the resulting linear equations give a second-order equation for U, say, which is as difficult to solve as the original equation. Thus this method is not a magic way to simplify the general linear second-order equation.

4.2 An equation considered by Ibragimov

Ibragimov [3] considered the equation

$$y'' = x^{-1}\left[ay'^3 + by'^2 + \left(1 + b^2/3a\right)y' + b/3a + b^3/27a^2\right],$$

which has $f_3 = -a/x$, $f_2 = -b/x$, $f_1 = -\left(1 + b^2/a\right)/x$, $f_0 = -\left(b/3a + b^3/27a^3\right)/x$, and which satifies the linearizability conditions. Inspection shows that one may take $a = 1$ without loss of generality, and that, by defining new variables U/x, V/x, and W/x^2, one can write an equation like the r equation (18) for which the matrix coefficients are constants, so that it can be solved directly. The details are rather messy, but one eventually gets the linearizing transformation

$$X = F = y + cx, \qquad Y = G = [y + c(x-1)]^2 + x^2,$$

where $c = b/3$. However, this does not save any labor, because the original equation is separable in y' and x and can be integrated quickly!

4.3 A trial equation

We consider the equation

$$y'' + (2/x)y' + \left(18x^2y^3 - 2x/y^2\right)y'^3 = 0, \tag{24}$$

which satisfies the linearizability conditions. We see that $f_0 = f_2 = 0$, $f_1 = 2/x$, and $f_3 = 18x^2y^3 - 2x/y^2$. Thus the matrices P and Q are

$$P = \begin{bmatrix} -4/3x & 0 & 0 \\ 0 & 2/3x & 0 \\ -1 & 0 & 2/3x \end{bmatrix} \qquad \text{and} \qquad Q = \begin{bmatrix} 0 & 0 & 0 \\ 18x^2y^3 - 2x/y^2 & 0 & 2/y^2 \\ 0 & 1 & 0 \end{bmatrix}.$$

From equations (18) and (19), we see that $dU = -(4U/3x)dx$; so we take $U = 0$. Then we have

$$dV = (2V/3x)dx + \left(2W/y^2\right)dy \qquad \text{and} \qquad dW = (2W/3x)dx + V dy,$$

so that $W_x = 2W/3x$. Integrating, we get $W = x^{2/3}a(y)$, for some function $a(y)$. We also see that $V = W_y = x^{2/3}a'(y)$, and further that $a'' = 2a/y^2$. We use the special solution $a(y) = y^2$, yielding finally

$$U = 0, \qquad V = 2x^{2/3}y, \qquad W = x^{2/3}y^2, \qquad \text{so that} \qquad K = 0, \qquad L = 2/y.$$

We can now construct the matrix Z. It is

$$Z = \begin{bmatrix} -2dx/x - 2dy/y & 0 \\ -2dx/y - \left(18x^2\,y^3 - 2x/y^2\right)dy & -4dy/y \end{bmatrix}.$$

Write $R = \begin{bmatrix} b \\ c \end{bmatrix}$. Then from equation (22) we have $db = -2(dx/x + dy/y)b$, which enables immediate integration: $b = k/\left(x^2y^2\right)$, where k is a constant. We also have $c_x = b_y$, which when integrated gives $c = 2k/\left(xy^3\right) + g(y)$.

Finally, we have $c_y = \left(18x^2\,y^3 - 2x/y^2\right)b - 4c/y$, or, after simplification, $g' + 4g/y = -18ky$. Solution gives $g(y) = -3ky^2 + m/y^4$, where m is another constant. Integration of equation (23), $dF = [dx\ dy]R$, now gives two solutions, one proportional to k and the other proportional to m. We take these two solutions as F and G:

$$X = F(x, y) = 1/\left(xy^2\right) + y^3, \qquad Y = G(x, y) = 1/y^3,$$

the linearizing transformation. Construction of d^2Y/dX^2 shows that it is zero provided the original differential equation (24) is satisfied.

Equation (24) was constructed by trial and error in order to provide a useful example of the use of the method. It turns out to have a eight-parameter symmetry group. One can naively try a reduction of order based on a scale transformation together with the usual tricks. Inspection of scale in the equation shows that y has the scale $x^{-1/5}$, so that $y = x^{-1/5}u$ produces the equation

$$x^2u'' + (8/5)xu' - (4/25)u + \left(18u^3 - 2/u^2\right)(xu' - u/5)^3 = 0.$$

We continue by defining $s = \ln x$, $v = du/ds$, and by converting the independent variable to u, with the dependent variable v. We find

$$vdv/du + 3u/5 - 4v/25 + \left(18u^3 - 2/u^2\right)(v - u/5)^3 = 0,$$

a rather nasty Abel equation.

Of course, this naive procedure applied to second order equations in general produces an Abel equation. Application of more sophisticated techniques such as used by Stephani [17] may produce a solution more easily when there are a number of symmetries (which has not been tried here). But the matter does raise the question, is it necessary for an equation to have a certain number of symmetries in order for this method to work well? Ibragimov [3] and Euler [13] note that the answer is yes; a necessary and sufficient condition for linearization by a point transformation is that the equation admit the $sl(3, \mathbb{R})$ Lie point symmetry algebra, or that it admit eight point symmetries. So this is another way to test for linearizability, although the calculation of the symmetries may be lengthy.

4.4 The general Kepler problem

The radial Newtonian central force equation, after substitution for the angular momentum and change of independent variable to θ, can be written

$$\left(\ell/r^2\right)(d/d\theta)\left[\left(\ell/mr^2\right)dr/d\theta\right] - \ell^2/\left(mr^3\right) - f(r) = 0,$$

where $f(r)$ is the force. If $f(r) = -(\ell^2 A/m)r^n$, where A is constant, and we let $r \to y$, $\theta \to x$, we have, where $k = n + 4$,

$$y'' - (2/y)y'^2 - y + Ay^k = 0.$$

Thus $f_3 = 0$, $f_2 = -2/y$, $f_1 = 0$, $f_0 = Ay^k - y$. The linearizability conditions require $k = 1$ or 2 so that $n = -3$ or -2. Why are not more values of n allowed? Because of the restriction of the original assumption of equations (2) and (3).

4.5 Geodesics on a sphere

This equation,

$$y'' = 2y'^2 \cot y + \sin y \cos y,$$

(which also has an eight-parameter symmetry algebra) is treated by Stephani [17, p. 78]. We have

$$f_1 = f_3 = 0, \qquad f_2 = -2\cot y, \qquad f_0 = -\sin y \cos y,$$

and it is easily seen that the linearization conditions are satisfied. A special solution for r gives $V = 0$, $U = \sin x(\sin y)^{2/3}$, $W = \cos x(\sin y)^{2/3}$, so that $K = \tan x$ and $L = 0$. The components of R may be found to be $(b - a\sin x \cot y)(\sec x)^2$ and $a\sec x(\csc y)^2$. We may take the coefficients of a and b to be two independent solutions; then integration for F and G gives $X = F = \tan x$ and $Y = G = \cot y \sec x$, and integration of equation (2) gives

$$\cot y = c\sin x + d\cos x,$$

where c and d are constants, the known solution.

4.6 Example from Stephani

The equation,

$$y'' = (x - y)y'^3,$$

is also treated in Stephani [17] and has an eight-parameter symmetry algebra. One sees easily that the linearization conditions are satisfied. The equation for U is $dU = 0$, so that we may take $U = 0$. Then $dV = -Wdy$ and $dW = Vdy$, which are satisfied by $V = \sin y$ and $W = -\cos y$, giving $K = 0$ and $L = -\tan y$. Solution for R and S, and then for F and G, gives $F = X = \tan y$ and $G = Y = (x - y)\sec y$. Now $Y = aX + b$, where a and b are constants, gives the solution of the equation essentially as suggested by Stephani:

$$x = y + a\sin y + b\cos y.$$

5 Third-order equation

Some authors have studied the third-order ordinary differential equation [18, 12]. The approach used in the present paper, however, does not readily yield a solution to the third-order problem.

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Symmetries and Integrability Properties of Generalized Fisher Type Nonlinear Diffusion Equation

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Nonlinear reaction-diffusion systems are known to exhibit very many novel spatiotemporal patterns. Fisher equation is a prototype of diffusive equations. In this contribution we investigate the integrability properties of the generalized Fisher type equation to obtain physically interesting solutions using Lie symmetry analysis. In particular, we report several travelling wave patterns, static patterns and localized structures depending upon the choice of the parameters involved.

1 Introduction

Nonlinear partial differential equations are frequently used to model a wide variety of phenomena in physics, chemistry, biology and other fields [1, 2, 3]. In such models, when large aggregates of microstructures consisting of particles, atoms, molecules, defects, dislocations, etc. are able to move and/or interact, the evolution of the concentration of the species can be shown to obey nonlinear diffusion equations of reactive type. These equations play an important role in dissipative dynamical systems. Many interesting physical phenomena, such as wall propagation in liquid crystals, nerve impulse propagation in nerve fibres, pattern formation in dissipative systems, nucleation kinetics and neutron action in the reactor, are closely connected with the study of nonlinear diffusion equations. The underlying systems give rise to very many simple/complex patterns which are essentially distinct structures on a suitable space-time scale and they arise as collective and cooperative phenomena due to the underlying large number of constituent subsystems. These structures tell us a lot about the dynamics as well as about the microscopic behaviour of the underlying systems to some extent. As the interactions among the constituents are nonlinear, novel structures which can mimic naturally occurring patterns arise. These structures can be stationary or changing with time.

Generally, in the study of dissipative systems, one of the challenging problems is the selection mechanism. That is, one would like to know the kinds of evolving velocity and emerging patterns that would be selected in a kinetic process when the system is suddenly quenched into an unstable state. Aronson and Weinberger's work on the Fisher type nonlinear diffusion equation [4] has shown the existence of distinct selection mechanism, that is the solution $u(x,t)$ of the Fisher equation in $(1+1)$ dimensions,

$$u_t = u_{xx} + u(1-u), \tag{1}$$

converges to a local travelling wave with a definite speed from a wide class of initial data. Further it is known that equation (1) has a travelling wave solution called a cline [3] which is nothing but a wave travelling in the x-direction with $c \geq c_{\min} = 2$. However, the first explicit analytic form for a cline solution was obtained by Ablowitz and Zeppetella [5], who showed that an exact propagating wavefront solution (see Fig. 1) is of the form

$$u(x,t) = 1 - \left[1 + \frac{k}{\sqrt{6}} \exp\left(\frac{x - \frac{5}{\sqrt{6}}t}{\sqrt{6}}\right)\right]^{-2}, \tag{2}$$

where k is an arbitrary constant. Here the authors made use of the Painlevé singularity structure analysis of equation (1) to find the exact solution in the year 1979.

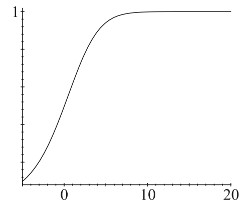

Figure 1. An exact wavefront solution [5] of the Fisher equation $\left(\xi = x - \frac{5}{\sqrt{6}}t\right)$.

There is continuing interest in recent literature [6] to investigate more general forms of the Fisher equation. For instance, there is an interesting generalization of the Fisher equation in the description of bacterial colony growth, chemical kinetics and many other natural phenomena and it is of the general form [10]

$$\frac{\partial u(\vec{r},t)}{\partial t} = D\triangle u(\vec{r},t) + \Lambda(u)[\nabla u(\vec{r},t)]^2 + \lambda u(\vec{r},t)G(u,\vec{r},t), \tag{3}$$

where D is the diffusion coefficient, Λ is the nonlocal growth rate, λ is the local growth rate, $G(u,\vec{r},t)$ is the local growth function, ∇ and $\triangle$ are gradient and Laplacian operators respectively. As a special case of equation (3), we obtain the generalized Fisher type equation

$$u_t - \triangle u - \frac{m}{1-u}(\nabla u)^2 - u(1-u) = 0, \tag{4}$$

where the subscript denotes partial differentiation with respect to time. In the study of population dynamics, $u(\vec{r},t)$ refers to the population density at point $\vec{r}$ at time t. In equation (4), the linear term modelling the birth rate gives rise to an exponential growth in time while the quadratic term that models competition between individuals for food, etc. leads to a stable, homogeneous value $u = 1$ at long times and the diffusion term models the spatial variation of the population. This introduces the possibility of spatial pattern formation between the homogeneous regions with $u = 1$ and $u = 0$ for appropriate initial conditions. Further, the classical Fisher equation ($m = 0$) occurs in models of population growth [3], neurophysiology [7], Brownian motion [8] and nuclear reactors [9]. Besides allowing for exact solutions, the $m = 2$ case finds its application in real systems such as the bacterial colony growth [10] where the square-gradient term corresponds to the nonlocal growth occuring at concentration gradients which is similar to the nonlinear terms in the Kuramoto–Sivashinsky equation for propagating flame and in the theory of growing interfaces. Moreover, models which admit exact solutions are of considerable importance for understanding general behaviour of nonlinear dissipative systems. In one dimensional space, such models have received considerable attention. But many realistic models are two or three dimensional in nature and in this direction, Brazhnik and Tyson [6] considered equation (4) in two spatial dimensions and explored five kinds of travelling wave patterns namely plane, V and Y-waves, a separatrix and space oscillating propagating structures. All these structures were found when the medium is unbounded and spatially homogeneous. Further they show that when the medium is bounded and no flux is allowed through the boundaries, only plane and oscillating waves survive because the frontline of the wave must approach the boundary orthogonally.

In general, obtaining solutions for reaction-diffusion systems is more complex than that for pure dispersive systems. For the latter there are several analytical methods like the inverse scattering transform method [11], the Hirota method [12], Bäcklund transformation method [13], Lie–Bäcklund symmetries method and so on. On the other hand, for nonlinear diffusive systems, no such formal techniques are available to solve them analytically. Very often perturbation analysis or numerical techniques are used to treat them. There is therefore an urgent need to isolate and identify integrable nonlinear reaction-diffusion systems which can act as model systems to deal with more complicated cases. In this connection, symmetry analysis can play a very crucial role.

Consider for example, the well known case of Burgers equation

$$u_t = \nu u_{xx} + u u_x,$$

where ν is the diffusion coefficient. It can be considered to be integrable in the sense that it is linearizable: Under the Cole–Hopf transformation $u = -\nu v_x/v$, it reduces to the linear heat equation. It possesses interesting Lie point symmetry structures and infinite number of Lie–Bäcklund symmetries. So, it will be quite interesting to know about other such integrable reaction-diffusion equations and the role of symmetries that allows the system to exhibit different spatiotemporal patterns and structures which usually possess some kind of symmetry. In this direction, the method of Lie groups is the most powerful method to analyse nonlinear partial differential equations (PDEs) and hence we make use of it and the singularity structure analysis to investigate the integrability properties and hence the dynamics/patterns of (4). We report in this paper that the $m = 2$ case of equation (4) possesses infinite dimensional Lie symmetry structure, which allows one to linearize it both in $(1+1)$ and $(2+1)$ dimensions and to obtain a large class of exact solutions. We also obtain several exact solutions for the $m \neq 2$ case.

The plan of the paper is as follows. In Section 2, we briefly recall some of the important reaction-diffusion equations exhibiting novel/complex patterns. Then in Section 3, by carrying out the singularity structure analysis, we point out that the PDE (4) is free from movable critical singular manifolds for the specific value $m = 2$. More interestingly, we point out that the Bäcklund transformation deduced from the Laurent expansion gives rise to the linearizing transformation for this case in a natural way. In Sections 4 and 5, we discuss different underlying patterns via symmetry analysis and similarity reductions for the generalized Fisher type equation in 1- and 2-spatial dimensions, respectively. Finally we summarize our results in Section 6.

2 Reaction-diffusion systems and various patterns

The general form of the nonlinear reaction-diffusion equation is given by

$$\frac{\partial \underline{C}}{\partial t} = \vec{\nabla}\cdot(D\vec{\nabla}\underline{C}) + \vec{F}(\underline{C}^T, \vec{r}, t), \qquad \underline{C} = (c_1, c_2, \ldots, c_n)^T,$$
$$D = \mathrm{diag}\,(D_1, D_2, \ldots, D_n), \qquad \vec{F} = (f_1, f_2, \ldots, f_n)^T.$$

Here $\underline{C}$ represents the population or concentration densities of the species and D and $\vec{F}$ are, in general, nonlinear functions of $\underline{C}$ representing the diffusivity and the reaction kinetics respectively. In such a case, the dynamics is dominated by the onset of patterns. Inspite of the absence of rigorous analytical tools as in the case of soliton systems, combined local analysis and numerical investigations on such systems have been found to exhibit a number of important spatiotemporal patterns.

Some of the dominant patterns exhibited by these systems are homogeneous or uniform steady states, travelling waves, spiral waves, Turing patterns (rolls, stripes, hexagons, rhombs, etc.), localized structures, spatiotemporal chaos and so on. A few of the well known models include the following:

2.1 The Oregonator model

This model explains the various features of the Belousov–Zhabotinsky reaction and was introduced by Fields, Körös and Noyes of University of Oregon, USA in 1972. In its simplest version it reads as [14]

$$\begin{aligned} u_{1t} &= D_1\nabla^2 u_1 + \eta^{-1}\left[u_1(1-u_1) - \frac{bu_2(u_1-a)}{(u_1+a)}\right], \\ u_{2t} &= D_2\nabla^2 u_2 + u_1 - u_2. \end{aligned} \tag{5}$$

Here u_1 is the concentration of the autocatalytic species $HBrO_2$, u_2 is the concentration of the transition ion catalyst in the oxidised state Ce^{3+} and Fe^{3+} and η, a and b are parameters. This model is the most popular among the pattern forming chemical reactions. In particular, (5) exhibits 'propagating pulse solutions' that can travel through the system without attenuation. Besides, it admits periodic wave trains, target patterns and in two dimensions they generate spiral waves.

2.2 Gierer–Meinhardt model

It describes possible interaction between an activator a and a rapidly diffusing inhibitor h and is of the form [15]

$$\begin{aligned} a_t &= D_a\nabla^2 a + \rho_a\frac{a^2}{(1+K_a a^2)} - \mu_a a + \sigma_a, \\ h_t &= D_h\nabla^2 h + \rho_h a^2 - \mu_h h + \sigma_h, \end{aligned}$$

where D_a and D_b are the two diffusion coeffcients, ρ_a and ρ_b are the removal rates and σ_a and σ_b are the basic production terms of the activator and inhibitor respectively. Further K_a corresponds to the saturation constant. This model is mainly used in the study of the development of an organism in biological pattern formation. They are also used to model cell differentiation, cell movement, shape changes of cells and tissues and so on.

2.3 Brusselator model

Among the various reaction-diffusion type model systems, this is one of the best studied models for the formation of chemical patterns theoretically [16]. It is based on the chemical reactions

$$A \longrightarrow X, \qquad B+X \longrightarrow Y, \qquad 2X+Y \longrightarrow 3X, \qquad X \longrightarrow E,$$

where the concentration of the species A, B and E are maintained constant. Thus they form the real constant parameters of the system. The evolution of the active species X and Y can be described by

$$\begin{aligned} X_t &= A - (B+1)X + X^2Y + D_X\nabla^2 X, \\ Y_t &= BX - X^2Y + D_Y\nabla^2 Y, \end{aligned} \tag{6}$$

after proper rescaling. Here D_X and D_Y are diffusion coefficients. This model exhibits heterogeneous patterns through Turing instability.

2.4 Lotka–Volterra predator-prey model

Taking into consideration the interaction of two species in which the population of the prey is dependent on the predator and vice-versa, the model equations [17] become

$$\begin{aligned} S_{1t} &= D_1 S_{1xx} + a_1 S_1 - b_1 S_1 S_2, \\ S_{2t} &= D_2 S_{2xx} - a_2 S_2 + b_2 S_1 S_2. \end{aligned}$$

Here S_1 and S_2 are the population densities of prey and predator. D_1 and D_2 are the diffusivities of the two populations, respectively. The parameters a_1, a_2 are the linear ratio of birth and death rates of the individual species while b_1, b_2 are the nonlinear decay and growth factors due to interaction.

2.5 FitzHugh–Nagumo nerve conduction model

The Hodgkin–Huxley model describes the propagation of the electrical impulses along the axonal membrane of a nerve fibre. FitzHugh–Nagumo nerve conduction equation [18] is the simplest version of the above model and is represented by the following set of equations:

$$\begin{aligned} V_t &= V_{xx} + V - \frac{V^3}{3} - R + I(x,t), \\ R_t &= c(V + a - bR). \end{aligned} \tag{7}$$

Here the membrane potential is $V(x,t)$, R corresponds to the lumped refractory variable and $I(x,t)$ is the external injected current. The parameters a and b are positive constants while c stands for the temperature factor. The above model has been widely used to study various phenomena in neurophysiology and cardiophysiology. This system exhibits travelling wave pulses [3]. In particular, the two-dimensional version of (7) admits ring wave patterns as well as spiral wave patterns for a variety of special initial conditions.

As mentioned in the introduction, symmetries can play a very important role in determining the underlying dynamics of nonlinear systems. Particularly they can help to identify integrable cases of the above type of reactive-diffusive systems, if they exist. As an important case study, we now investigate integrability and symmetry properties of the generalized Fisher type equation (4).

3 Singularity structure analysis

This analysis separates out the $m = 2$ case for both the $(1 + 1)$ and $(2 + 1)$ dimensions as the only system for which the Fisher equation (4) is free from movable critical singular manifolds satisfying the Painlevé property [13]. By locally expanding the solution in the neighbourhood of the non-characteristic singular manifold $\phi(x,t) = 0$, $\phi_x, \phi_t \neq 0$ in the form of the Laurent series [19]

$$u = \sum_{j=0}^{\infty} u_j \phi^{j+p},$$

the possible values of the power of the leading order term are found to be

$$\begin{aligned} &(i)\ p = -2, \\ &(ii)\ p = \frac{1}{1-m}, \qquad m \neq 1, \\ &(iii)\ p = 0. \end{aligned}$$

For all these leading orders, only for the value $m = 2$ the solution is free from movable critical singular manifolds since for $p = -1$ the leading order coefficient u_0 becomes arbitrary besides the arbitrary singular manifold ϕ. In all other cases only one arbitrary function exists for $m = 2$ thereby leading to special solutions.

More interestingly, from the Laurent series expansion if we cut off the series at "constant" level term, that is $j = -p$ for the leading order $p = 1/(1-m) = -1$, $m = 2$, one can deduce the Bäcklund transformation that gives rise to the linearizing transformation in a natural way. Thus, defining the relation

$$u = \frac{u_0}{\phi} + u_1, \tag{8}$$

we demand that if u_1 is a solution of equation (4) for the case $m = 2$, then u is also a solution, from which the Bäcklund transformation is deduced. Now starting from the trivial solution, $u_1 = 0$ of (4), we find that the equations for u_0 and ϕ in equation (8) are consistent for the choice $u_0 = \phi$, giving rise to the new solution $u = 1$. This is nothing but an exact solution of equation (4). Then with $u_1 = 1$ as the new seed solution, one can check from equations satisfied by u_0 and ϕ that

$$u_0 = -1, \qquad \phi_t - \phi_{xx} - \phi + 1 = 0. \tag{9}$$

Choosing $\phi = 1 + \chi$, equation (9) can be rewritten as the linear heat equation,

$$\chi_t - \chi_{xx} - \chi = 0. \tag{10}$$

Thus the transformation

$$u = 1 - \frac{1}{1+\chi}, \tag{11}$$

where χ satisfies the linear heat equation (10), is the linearizing transformation for equation (1) in $(1+1)$ dimensions for the choice $m = 2$ in an automatic way. We note that this is exactly the transformation given in ref. [20] in an adhoc way. Here we have given an interpretation for the transformation in terms of the Bäcklund transformation. The same transformation (11) linearizes equation (3) in $(2+1)$ dimensions (for $m = 2$) as well, where χ satisfies the two dimensional linear heat equation $\chi_t - \chi_{xx} - \chi_{yy} - \chi = 0$. Further equation (11) transforms equation (4) in $(3+1)$ dimensions to the 3-dimensional heat equation as well; however, we do not study the case further here.

4 Symmetries and integrability properties of $(1+1)$ dimensional generalized Fisher equation

The generalized Fisher equation (4) in its $(1+1)$ dimensional form reads as

$$u_t - u_{xx} - \frac{m}{1-u} - u + u^2 = 0. \tag{12}$$

An invariance analysis of equation (12) under the infinitesimal transformations

$$\begin{aligned} x &\longrightarrow X = x + \varepsilon\xi(t,x,u), \qquad && t \longrightarrow T = t + \varepsilon\tau(t,x,u), \\ u &\longrightarrow U = u + \varepsilon\phi(t,x,u), \qquad && \varepsilon \ll 1, \end{aligned}$$

separates out the $m = 2$ case in that it possesses a nontrivial infinite-dimensional Lie algebra of symmetries

$$\tau = a, \qquad \xi = b, \qquad \phi = c(t,x)(1-u)^2.$$

Here a, b are arbitrary constants and $c(t,x)$ is any solution of the linear heat equation $c_t - c_{xx} - c = 0$. For all other values of m in equation (12) one gets trivial translation symmetries

$$\tau = a, \qquad \xi = b, \qquad \phi = 0.$$

In order to obtain solutions of physical importance and corresponding patterns, we make use of the method of similarity reductions. This leads to the similarity reduced variables for the $m = 2$ case as

$$z = ax - bt, \qquad u = 1 - \frac{a}{a + v(z) + \int c(t,x)dt}. \tag{13}$$

Using (13), equation (12) can be reduced for the $m = 2$ case to the similarity reduced ordinary differential equation (ODE)

$$a^2 v'' + bv' + v = 0$$

whose general solution is

$$v = I_1 e^{m_1 z} + I_2 e^{m_2 z}, \qquad m_{1,2} = \frac{-b \pm \sqrt{b^2 - 4a^2}}{2a^2},$$

where I_1 and I_2 are integration constants thereby leading to

$$u = \begin{cases} 1 - \dfrac{a}{a + I_1 e^{m_1(ax-bt)} + I_2 e^{m_2(ax-bt)} + \int c(t,x)dt}, & b^2 - 4a^2 > 0; \\ 1 - \dfrac{a}{a + e^{p(ax-bt)}\left(I_1 + I_2(ax-bt)\right) + \int c(t,x)dt}, & b^2 - 4a^2 = 0; \\ 1 - \dfrac{a}{a + e^{p(ax-bt)}\left(I_1 \cos q(ax-bt) + I_2 \sin q(ax-bt)\right) + \int c(t,x)dt}, & b^2 - 4a^2 < 0 \end{cases}$$

with $p = -b/2a^2$, $q = \sqrt{4a^2 - b^2}/2a^2$, as the solution to the original PDE (12). Here the similarity reduced variable (13) is nothing but the linearizing transformation (11).

Proceeding in a similar fashion for all the other (nonintegrable) cases ($m \neq 2$), the similarity variables $z = ax - bt$ and $u = w(z)$ reduce equation (12) to the ODE

$$a^2 vv'' - ma^2 v'^2 + bvv' - (1-v)v^2 = 0, \qquad v = 1 - w, \tag{14}$$

which is in general nonintegrable except for $m = 0$ and $b/a = 5/\sqrt{6}$. This special choice leads to the cline solution (2) obtained by Ablowitz and Zeppetella [5]. In the static case ($b = 0$), one obtains elliptic function solutions. Besides, a particular solitary wave solution

$$u = 1 - \frac{(3-2m)}{(2-2m)}\left[\operatorname{sech}^2\left(I_2 - \frac{x}{2}\sqrt{\frac{1}{1-m}}\right)\right], \qquad m < 1$$

with I_2 as the second integration constant, which is a limiting case of a elliptic function solution, is also obtained (refer Fig. 2). In the general case, as equation (14) is of nonintegrable nature, we make use of numerical techniques to study the underlying dynamics. Here we obtain typical periodic wave trains for $b/a = 0$ which is in accordance with the fact that reaction-diffusion systems exhibiting limit cycle motion in the absence of diffusion exhibits travelling wave patterns (Fig. 3a,b). For $b/a = 1$, we get a propagating pulse (Fig. 3c) and the corresponding phase portrait ($v - v'$) shows a stable spiral equilibrium point (Fig. 3d). On increasing the value of b/a, that is, at $b/a \geq 2$ ($b = 2.041$) [5], the system supports a travelling wave front (Fig. 3e) and the trajectories in the phase plane ($v - v'$) correspond to a stable node (Fig. 3f).

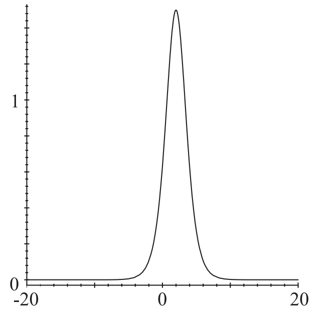

Figure 2. A static solitary wave pulse for $m = 1/2$ of the generalized Fisher equation (12).

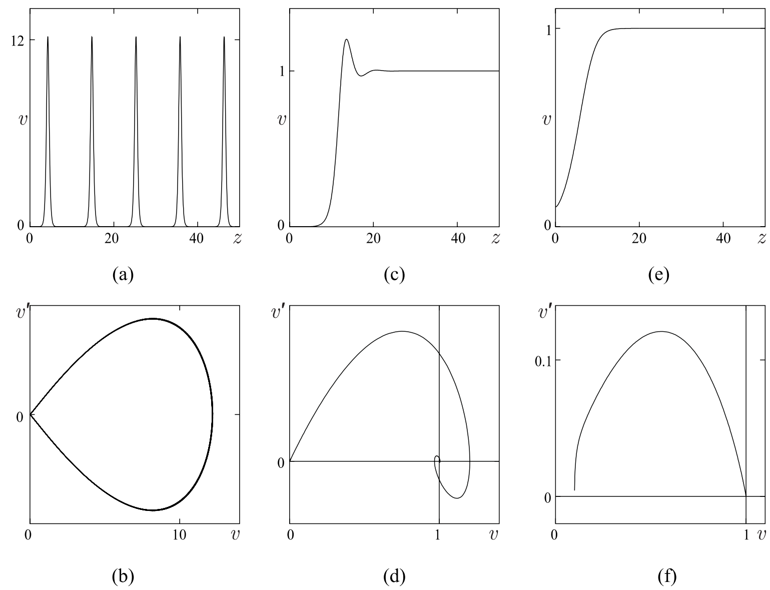

Figure 3. Propagating patterns and corresponding phase portraits in the $v - v'$ plane of equation (14): (a) periodic pulses; (b) limit cycle; (c) travelling pulse; (d) stable spiral; (e) travelling wavefront; (f) stable node.

5 The $(2+1)$ dimensional generalized Fisher equation

Extending a similar analysis to the $(2+1)$ dimensional case of the generalized Fisher equation

$$u_t - u_{xx} - u_{yy} - \frac{m}{1-u}\left(u_x^2 + u_y^2\right) - u + u^2 = 0, \tag{15}$$

one finds that the invariance analysis of equation (15) under the infinitesimal transformation singles out the special value $m = 2$ for which the Lie point symmetries are

$$\tau = a, \qquad \xi = b_3 y + b_4, \qquad \eta = -b_3 x + d_4, \qquad \phi = c(t,x,y)(1-u)^2,$$

where η is the infinitesimal symmetry associated with the variable y, $c(t,x,y)$ is the solution of the two dimensional linear heat equation $c_t - c_{xx} - c_{yy} - c = 0$ and b_3, b_4 and d_4 are arbitrary constants. But for all other choices of m $(\neq 2)$ we get

$$\tau = a, \qquad \xi = b_3 y + b_4, \qquad \eta = -b_3 x + d_4, \qquad \phi = 0.$$

In a similar fashion as that for the $(1+1)$ dimensional case, the similatiry variables for the $m = 2$ case

$$z_1 = \frac{b_3}{2}(x^2 + y^2) + b_4 y - d_4 x, \qquad z_2 = -t - \frac{a}{b_3}\sin^{-1}\left(\frac{d_4 - b_3 x}{\sqrt{d_4^2 + 2b_3 z_1 + b_4^2}}\right),$$

$$u = 1 - \frac{a}{w(z_1, z_2) + \int c(t, x, y) dt} \tag{16}$$

reduce the PDE (15) to

$$w_{z_2} + 2b_3 w_{z_1} + \left(2b_3 z_1 + b_4^2 + d_4^2\right) w_{z_1 z_1} + \frac{a^2 w_{z_2 z_2}}{2b_3 z_1 + b_4^2 + d_4^2} + w - a = 0. \tag{17}$$

Here too one can obtain the linear heat equation

$$\chi_t - \chi_{xx} - \chi_{yy} - \chi = 0,$$
$$\chi = \frac{1}{a}\left[w(z_1, z_2) + \int c(t, x, y) dt\right],$$

from the similarity form (16). Such a transformation can be interpreted as the linearizing transformation from a group theoretical point of view.

Carrying out a Lie symmetry analysis for equation (17) also, one can obtain the new similarity variables

$$\zeta = \bar{z}_1, \quad w = a + e^{\left(\frac{c_1 \bar{z}_2}{c_3}\right)}\left[f(\zeta) + \frac{1}{c_3}\int \hat{c}_2(\bar{z}_1, \bar{z}_2) e^{\left(-\frac{c_1}{c_3}\bar{z}_2\right)} d\bar{z}_2\right],$$
$$\bar{z_1} = 2b_3 z_1 + b_4^2 + d_4^2, \qquad \bar{z}_2 = z_2, \qquad b_3, d_4 \neq 0,$$

where f satisfies the linear second order ODE of the form

$$\zeta^2 f'' + \zeta f' + (A + B\zeta) f = 0,$$
$$A = (ac_1/2b_3c_3)^2, \qquad B = (1 + c_1/c_3)/4b_3^2, \tag{18}$$

with prime denoting differentiation w.r.t. ζ. Thus the solution to the original PDE reads as

$$u = 1 - a\Bigg[a + e^{\left(\frac{c_1}{c_3}\bar{z}_2\right)}\Bigg(I_1 Z_1\left(2\sqrt{B\bar{z}_1}\right) + I_2 Z_2\left(2\sqrt{B\bar{z}_1}\right)$$
$$- \int \frac{\hat{c}_2(\bar{z}_1, \bar{z}_2)}{c_3} e^{\left(\frac{c_1}{c_3}\bar{z}_2\right)} d\bar{z}_2\Bigg) + \int c(t, x, y) dt\Bigg]^{-1}. \tag{19}$$

In the limit $b_4 = d_4 = c_1 = 0$ the system is found to exhibit circularly symmetric structures given in Fig. 4.

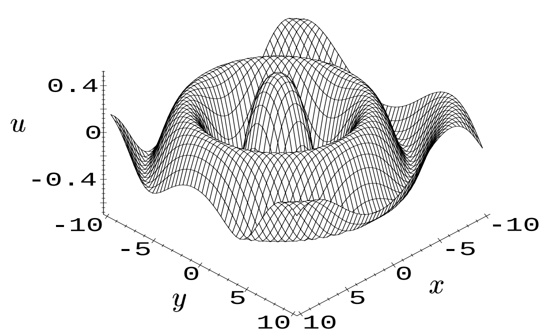

Figure 4. Circularly symmetric patterns of the $(2+1)$ dimensional generalized Fisher equation (15) for $m = 2$.

More interestingly, in the special case $b_3 = 0$, $d_4 = 0$ the system exhibits propagating wave structures and the corresponding forms are

$$u = \begin{cases} 1 - a\Bigg\{a + \exp\left[-k\left(\dfrac{a}{b_4}x - t\right)\right]\Bigg[I_1\cos(\sqrt{k_1}c_5b_4y) + I_2\sin\left(\sqrt{k_1}c_5b_4y\right) \\ \qquad + \displaystyle\int \dfrac{\hat{c}_3(z_1,z_2)}{c_5}e^{kz_2}dz_2\Bigg] + \displaystyle\int c(t,x,y)dt\Bigg\}^{-1}, \quad k_1 < 0, \\ 1 - a\Bigg\{a + \exp\left[-k\left(\dfrac{a}{b_4}x - t\right)\right]\Bigg[I_1e^{\sqrt{k_1}c_5b_4y} + I_2e^{-\sqrt{k_1}c_5b_4y} \\ \qquad + \displaystyle\int \dfrac{\hat{c}_3(z_1,z_2)}{c_5}e^{kz_2}dz_2\Bigg] + \displaystyle\int c(t,x,y)dt\Bigg\}^{-1}, \quad k_1 > 0, \\ 1 - a\Bigg\{a + \exp\left[-k\left(\dfrac{a}{b_4}x - t\right)\right]\Bigg[I_1c_5b_4y + I_2 \\ \qquad + \displaystyle\int \dfrac{\hat{c}_3(z_1,z_2)}{c_5}e^{kz_2}dz_2\Bigg] + \displaystyle\int c(t,x,y)dt\Bigg\}^{-1}, \quad k_1 = 0, \end{cases} \tag{20}$$

where the parameter $k_1 = \frac{1}{b_4^2c_5^2}\left[k - \left(\frac{ak}{b_4}\right)^2 - 1\right]$ with $k = -c_2/c_5$ and $z_1 = b_4y$, $z_2 = \frac{a}{b_4}x - t$. Here c_2, c_4, c_5 are arbitrary constants of integration. Equation (20), in particular exhibits the five classes of bounded travelling wave solutions reported by Brazhnik and Tyson [6] for certain choice of the parameters involved along with the specific assumptions of the functions $\hat{c}_3(z_1,z_2) = 0$ and $c(t,x,y) = 0$. The corresponding solutions are given below.

Among the classes of solutions, the simplest travelling wave solution (Fig. 5a)

$$u = 1 - \frac{1}{1 + A\exp\left[-k\left(\frac{a}{b_4}x - t\right) \pm \sqrt{k_1}c_5b_4y\right]}, \qquad k_1 > 0$$

can be constructed by assuming either $I_1 = 0$ or $I_2 = 0$. For $I_1 = I_2$ ($\neq 0$), we obtain a V-wave pattern (Fig. 5b)

$$u = 1 - \frac{1}{1 + A\exp\left[-k\left(\frac{a}{b_4}x - t\right)\right]\cosh\left(\sqrt{k_1}c_5b_4y\right)}, \qquad k_1 > 0.$$

Again the case $I_2 = 0$ and $k_1 < 0$ leads to a wave front oscillating in space (Fig. 5c) and is represented by

$$u = 1 - \frac{1}{1 + A\exp\left[-k\left(\frac{a}{b_4}x - t\right)\right]\left|\cos\left(\sqrt{k_1}c_5b_4y\right)\right|}.$$

But when $I_1 \neq 0$ and $I_2 = 0$ we get a separatrix (Fig. 5d)

$$u = 1 - \frac{1}{1 + A|y|\exp\left[-k\left(\frac{a}{b_4}x - t\right)\right]}.$$

Finally for positive k_1 and $I_1 = -I_2$ the Y-wave solution (Fig. 5e) becomes

$$u = 1 - \frac{1}{1 + A\exp\left[-k\left(\frac{a}{b_4}x - t\right)\right]\left|\sinh\left(\sqrt{k_1}c_5b_4y\right)\right|}.$$

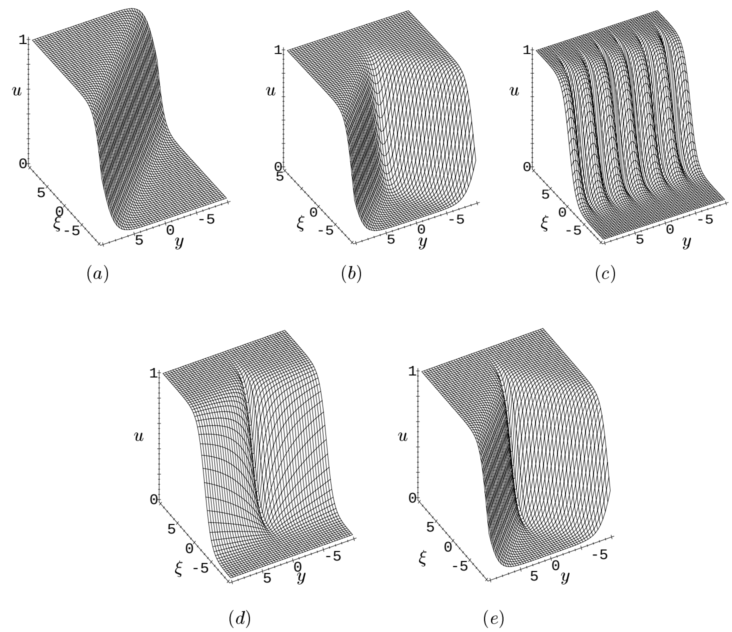

Figure 5. Five interesting classes of propagating wave patterns as obtained in ref. [6], which follow from equations (20): (a) travelling waves; (b) V-waves; (c) oscillating front; (d) separatrix solution; (e) Y-waves with $\xi = -k(\frac{a}{b_4}x - t)$.

In each of the above solutions A is a positive constant. It is a well known fact about Fisher equation is that it forms a basis for many nonlinear models of different nature. As a result, the above solutions are reminiscent of patterns from different fields. In particular, V-waves are characterized in the framework of geometrical crystal growth related models [21] and in excitable media [22] while space oscillating fronts are relevant to cellular flame structures and patterns in chemical reaction diffusion systems [23]. Further it has been shown in [24] with a geometrical model that excitable media can support space-oscillating fronts. Several static structures can also be obtained as limiting cases of the above solutions (19) and (20).

Finally, a similar analysis for the nonintegrable $(m \neq 2)$ case yields static patterns/structures in (x, y) variables. Here one has to look for certain special solutions due to its nonintegrable nature. That is, for $b_3 = 0$ and $d_4 = 0$ with the similarity variables $z_1 = b_4 y$, $z_2 = \frac{a}{b_4}x - t$, $u = w(z_1, z_2)$, the reduced ODE reads as

$$
\begin{aligned}
&Df'' + \frac{Dm}{1-f}f'^2 - c_1 f' + f(1-f) = 0,\\
&D = \left(\frac{a^2}{b_4^2}{c_1}^2 + b_4^2{c_2}^2\right), \qquad ()' = d/d\zeta,
\end{aligned}
$$

with $\zeta = -c_1\left(\frac{a}{b_4}x - t\right) + c_2 b_4 y$ and $w = f(\zeta)$, giving rise to plane wave structures. For $b_3 = 0$, the similarity variables $z_1 = d_4 x - b_4 y$, $z_2 = ax - b_4 t$ and $u = w(z_1, z_2)$ reduces the PDE to an ODE

$$
Af_1'' + Bf_1' - \frac{Am}{f_1}f_1'^2 - f_1 + f_1^2 = 0, \qquad ()' = d/d\zeta
$$

with $f_1 = 1 - f$, $A = a^2\left(c_1^2 d_4^2 + c_2^2 b_4^2\right)$, $B = -d_4 b_4(c_1 + c_2)$ and $\zeta = ac_2(d_4 x - b_4 y) - d_4(c_1 + c_2)(ax - b_4 t)$, $w = f(\zeta)$. Then the system is found to possess elliptic function solutions including the limiting case of the solitary pulse for certain choices of the constants involved.

6 Conclusion

Our studies on the integrability/symmetry properties of the the generalized Fisher type nonlinear reaction-diffusion equation show that the system under consideration possesses interesting Lie point symmetries that could form infinite dimensional Lie algebra for the particular choice of the system parameter $m = 2$, thereby exhibiting various interesting patterns and dynamics. Besides, the singularity structure analysis singles out the $m = 2$ case as the only system parameter for which the generalized Fisher type equation is free from movable critical singular manifolds. The generalized Fisher equation is found to possess a large number of interesting wave patterns. It will be of interest to consider other physically interesting reaction-diffusion systems from the Lie symmetry point of view and to study the underlying patterns.

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Solution of the Rayleigh Problem for a Power Law Non-Newtonian Conducting Fluid via Group Method

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The magnetic Rayleigh problem where a semi-infinite plate is given an impulsive motion and thereafter moves with constant velocity in a non-Newtonian power law fluid of infinite extent is studied. The solution of this highly non-linear problem is obtained by means of the transformation group theoretic approach. The one-parameter group transformation reduces the number of independent variables by one and the governing partial differential equation with the boundary conditions reduce to an ordinary differential equation with the appropriate boundary conditions. Effect of the parameters and time on the velocity has been studied and the results are plotted.

1 Introduction

An investigation is made of the magnetic Rayleigh problem where a semi-infinite plate is given an impulsive motion and thereafter moves with constant velocity in a non-Newtonian power law fluid of infinite extent. We will study the non-stationary flow of an electrically conducting non-Newtonian fluid of infinite extent in a transverse external magnetic field. The rheological model of this fluid is given by the well-known expression for a power law fluid [13]

$$\tau_{ij} = -p\delta_{ij} + k\left|\frac{1}{2}I_2\right|^{\frac{n-1}{2}} e_{ij},$$

where τ_{ij} is the shear stress, p is the pressure, δ_{ij} is the Kronecker symbol, k the coefficient of consistency, I_2 the second strain rate invariant, e_{ij} the strain rate tensor and n is a parameter characteristic of the non-Newtonian behavior of the fluid.

For $n = 1$, the behavior of the fluid is Newtonian, for $n > 1$, the behavior is dilatant and for $0 < n < 1$, the behavior is pseudo-plastic.

The equation of motion of the semi-infinite flat plate in the infinite power law non-Newtonian fluid after an impulsive end loading and maintaining constant velocity thereafter is

$$\frac{\partial u}{\partial t} - \gamma\frac{\partial}{\partial y}\left\{\left[\left(\frac{\partial u}{\partial y}\right)^2\right]^{\frac{n-1}{2}}\frac{\partial u}{\partial y}\right\} + MH^2u = 0,$$

where $u(y,t)$ is the velocity of the fluid flow in the horizontal direction, V is the steady state velocity of the plate, t is the time, y is the coordinate normal to the plate, n is constant, $\gamma\left(=\frac{k}{\rho}\right)$ is constant, k is the coefficient of consistency, ρ is the density of the fluid, $M\left(=\frac{\sigma\mu^2}{\rho}\right)$ is constant, σ is the magnetic conductivity, μ is the magnetic permeability and H is the magnetic field strength and is function of time $H = H(t)$.

The solution of this highly non-linear problem is obtained by means of the transformation group theoretic approach. The one-parameter group transformation reduces the number of independent variables by one and the governing partial differential equation with the boundary conditions reduce to an ordinary differential equation with the appropriate boundary conditions. Effect of the parameters M, w $(=\gamma V^{n-1})$, n and time t on the velocity $u(y,t)$ has been studied and the results are plotted.

Fluids that obey Newton's law of viscosity are called Newtonian fluids.

Newton's law of viscosity is $\tau = \mu \frac{du}{dy}$, where τ is the shear stress and μ is the viscosity. Not all fluids follow the Newtonian stress-strain relation. Some fluids, such as "Ketchup" are "shear-thinning"; that is the coefficient of resistance decreases with increasing strain rate. Fluids that do not follow the Newtonian relation are called non-Newtonian fluids. Viscosity of non-Newtonian fluids is a function of the strain rate [8].

In 1970, Sapunkov [11] studied non-Newtonian flow of an electrically conducting fluid. He obtained approximate solution to the problem solved in this paper but only in the special case of very strong or very weak magnetic fields. The solution was obtained only for a power law fluid for $n = 2$. In 1971, Vujanovic [12] obtained approximate solution by means of a new and effective variational method. In 1972, Vujanovic, Strauss and Djukic [13] used a new variational principle. This new principle allows one to obtain the solution in a straightforward manner. The mathematical technique used in the present analysis is the one-parameter group transformation. The group methods, as a class of methods, which lead to reduction of the number of independent variables, were first introduced by Birkhoff [4] in 1948, where he made use of one-parameter transformation groups. In 1952, Morgan [10] presented a theory, which has led to improvements over earlier similarity methods. The method has been applied intensively by Abd-el-Malek et al. [1, 2, 5, 7], Ames [3], Morgan and Gaggioli [9] and A.J.A. Morgan [10]. In this work, we present a general procedure for applying a one-parameter group transformation to the Raylrigh problem for a power law non-Newtonian conducting fluid.

Under the transformation, the partial differential equation with boundary conditions is reduced to an ordinary differential equation with the appropriate corresponding conditions. The equation is then solved numerically using non-linear finite difference method applied to the non linear second order boundary value problem [6] to calculate approximated value of the velocity of the fluid $u(y,t)$.

The fluid studied here is assumed to be incompressible and such that the electric and polarization effects can be neglected.

2 Formulation of the problem and the governing equation

Consider the equation of motion of the semi-infinite flat plate in the infinite power law non-Newtonian fluid (Rayleigh problem) of the form:

$$\frac{\partial u}{\partial t} - \gamma \frac{\partial}{\partial y} \left\{ \left[\left(\frac{\partial u}{\partial y} \right)^2 \right]^{\frac{n-1}{2}} \frac{\partial u}{\partial y} \right\} + MH^2 u = 0 \tag{1}$$

with the boundary conditions

$$(i) \quad u(0,t) = V, \qquad t > 0,$$
$$(ii) \quad u(\infty,t) = 0, \qquad t > 0$$

and initial condition

$$u(y,0) = 0, \qquad y > 0.$$

Equation (1) can be written as:

$$\frac{\partial u}{\partial t} - n\gamma \left(\frac{\partial u}{\partial y}\right)^{n-1} \left(\frac{\partial^2 u}{\partial y^2}\right) + MH^2 u = 0. \tag{2}$$

Assume

$$u(y,t) = VF(y,t), \tag{3}$$

where $F(y,t)$ is unknown function and its proper form will be determined later on.
Substitution from (3) into (2) yields

$$V\frac{\partial F}{\partial t} - n\gamma V^n \left(\frac{\partial F}{\partial y}\right)^{n-1} \left(\frac{\partial^2 F}{\partial y^2}\right) + MH^2 VF = 0,$$

which can be written as:

$$\frac{\partial F}{\partial t} - n\gamma V^{n-1} \left(\frac{\partial F}{\partial y}\right)^{n-1} \left(\frac{\partial^2 F}{\partial y^2}\right) + MH^2 F = 0 \tag{4}$$

with the boundary conditions

$$(i) \quad F(0,t) = 1, \qquad t > 0, \tag{5}$$
$$(ii) \quad F(\infty,t) = 0, \qquad t > 0 \tag{6}$$

and initial condition

$$F(y,0) = 0, \qquad y > 0. \tag{7}$$

3 Solution of the problem

Our method of solution depends on the application of a one-parameter group transformation to the partial differential equation (4). Under this transformation the two independent variables will be reduced by one and the differential equation (4) transforms into an ordinary differential equation.

3.1 The group systematic formulation

The procedure is initiated with the group G, a class of transformation of one-parameter a of the form:

$$\overline{y} = h^y(a)y + k^y, \qquad \overline{t} = h^t(a)t + k^t, \qquad \overline{F} = h^F(a)F + k^F, \qquad \overline{H} = h^H(a)H + k^H, \tag{8}$$

where h's and k's are real-valued and at least differentiable in the real argument a.

3.2 The invariance analysis

To transform the differential equation, transformations of the derivatives of F and H are obtained from G via chain-rule operations:

$$\overline{S}_i = \left[\frac{h^S}{h^i}\right] S_i, \qquad \overline{S}_{ij} = \left[\frac{h^S}{h^i h^j}\right] S_{ij}, \qquad i = y,t, \qquad j = y,t, \tag{9}$$

where S stands for F.

Equation (4) is said to be invariantly transformed, for some function $A(a)$ whenever:

$$\left\{\frac{\partial\overline{F}}{\partial\overline{t}} - n\gamma V^{n-1}\left(\frac{\partial\overline{F}}{\partial\overline{y}}\right)^{n-1}\left(\frac{\partial^2\overline{F}}{\partial\overline{y}^2}\right) + M\overline{H}^2\overline{F}\right\}$$
$$= A(a)\left\{\frac{\partial F}{\partial t} - n\gamma V^{n-1}\left(\frac{\partial F}{\partial y}\right)^{n-1}\left(\frac{\partial^2 F}{\partial y^2}\right) + MH^2F\right\}. \tag{10}$$

Substitution from (8) and (9) into (10) yields

$$\frac{h^F}{h^t}\frac{\partial F}{\partial t} - n\gamma V^{n-1}\left(\frac{h^F}{h^y}\frac{\partial F}{\partial y}\right)^{n-1}\left(\frac{h^F}{(h^y)^2}\frac{\partial^2 F}{\partial y^2}\right) + M\left(h^H H + k^H\right)^2\left(h^F F + k^F\right)$$
$$= A(a)\left\{\frac{\partial F}{\partial t} - n\gamma V^{n-1}\left(\frac{\partial F}{\partial y}\right)^{n-1}\left(\frac{\partial^2 F}{\partial y^2}\right) + MH^2F\right\}. \tag{11}$$

The invariance of (11) implies

$$k^H = k^F = 0, \qquad \text{and} \qquad \frac{h^F}{h^t} = \frac{\left(h^F\right)^n}{\left(h^y\right)^{n+1}} = \left(h^H\right)^2 h^F = A(a).$$

The invariance of the auxiliary conditions (5)–(7) implies that

$$h^F = 1, \qquad k^y = k^t = 0,$$

which yields

$$h^y = \left(h^t\right)^{\frac{1}{n+1}}, \qquad h^H = \frac{1}{\sqrt{h^t}}.$$

Finally, we get the one-parameter group G, which transforms invariantly the differential equation (4) and the auxiliary conditions (5)–(7).

The group G is of the form:

$$\overline{y} = \left(h^t\right)^{\frac{1}{n+1}} y, \qquad \overline{t} = h^t t, \qquad \overline{F} = F, \qquad \overline{H} = \left(\frac{1}{\sqrt{h^t}}\right) H. \tag{12}$$

3.3 The complete set of absolute invariants

Our aim is to make use of group methods to represent the problem in the form of an ordinary differential equation.Then we have to proceed in our analysis to obtain a complete set of absolute invariants. If $\eta \equiv \eta(y,t)$ is the absolute invariant of the independent variables then,

$$g_j(y,t,F,H) = \Psi_j\left[\eta(y,t)\right], \qquad j = 1,2,$$

are the two absolute invariants corresponding to F and H.

The application of a basic theorem in group theory, see Moran and Gaggioli [9], states that: a function $g(y,t,F,H)$ is an absolute invariant of a one-parameter group if it satisfies the following first-order linear differential equation:

$$\sum_{i=1}^{4}(\alpha_i S_i + \beta_i)\frac{\partial g}{\partial S_i} = 0, \qquad S_i \equiv y,t,F,H, \tag{13}$$

where

$$\alpha_i = \frac{\partial h^{S_i}}{\partial a}\left(a^0\right) \qquad \text{and} \qquad \beta_i = \frac{\partial k^{S_i}}{\partial a}\left(a^0\right), \qquad i = 1,2,3,4 \tag{14}$$

and a^0 denotes the value of "a" which yields the identity element of the group G.

At first, we seek the absolute invariant of the independent variables. Owing to equation (13), $\eta(y,t)$ is an absolute invariant if it satisfies the following first-order linear differential equation

$$(\alpha_1 y+\beta_1)\frac{\partial \eta}{\partial y}+(\alpha_2 t+\beta_2)\frac{\partial \eta}{\partial t}=0. \tag{15}$$

Since $k^y=k^t=0$, and according to the definition of the β's then $\beta_1=\beta_2=0$.
Now, equation (15) may be rewritten in the form,

$$\alpha_1 y\frac{\partial \eta}{\partial y}+\alpha_2 t\frac{\partial \eta}{\partial t}=0.$$

Applying separation of variables method, one can be obtain a solution in the form,

$$\eta=yt^{-\beta},\qquad \text{where}\qquad \beta=\frac{\alpha_1}{\alpha_2}. \tag{16}$$

The second step is to obtain the absolute invariants of the dependent variables F and H.
By a similar analysis, using equations (12), (13) and (14), we get

$$F(y,t)=\phi(\eta), \tag{17}$$

and the second absolute invariant is

$$H(t)=q(t). \tag{18}$$

4 The reduction to an ordinary differential equation

Using a substitution from (16)–(18) into equation (4), we get

$$\left[-\beta y t^{-(\beta+1)}\right]\frac{d\phi}{d\eta}-n\gamma V^{n-1}\left(t^{-\beta}\frac{d\phi}{d\eta}\right)^{n-1}\left(t^{-2\beta}\frac{d^2\phi}{d\eta^2}\right)+Mq^2\phi=0,$$

from which we get

$$n\gamma V^{n-1}\left(\frac{d^2\phi}{d\eta^2}\right)\left(\frac{d\phi}{d\eta}\right)^{n-1}\left[t^{1-\beta(n+1)}\right]+\beta\eta\frac{d\phi}{d\eta}-Mtq^2\phi=0. \tag{19}$$

For (19) to be reduced to an ordinary differential equation in one variable η, it is necessary that the coefficients should be constants or functions of η only. Thus

$$q(t)=\frac{E}{\sqrt{t}},\qquad \beta=\frac{1}{n+1}. \tag{20}$$

Hence, equation (19) will be,

$$nw\left(\frac{d^2\phi}{d\eta^2}\right)\left(\frac{d\phi}{d\eta}\right)^{n-1}+\beta\eta\frac{d\phi}{d\eta}-N\phi=0, \tag{21}$$

where $N\left(=E^2M\right)$ and $w\left(=\gamma V^{n-1}\right)$ are constants.
Under the similarity variable η, the boundary conditions (5)–(7) are

$$\phi(0)=1,\qquad \phi(\infty)=0.$$

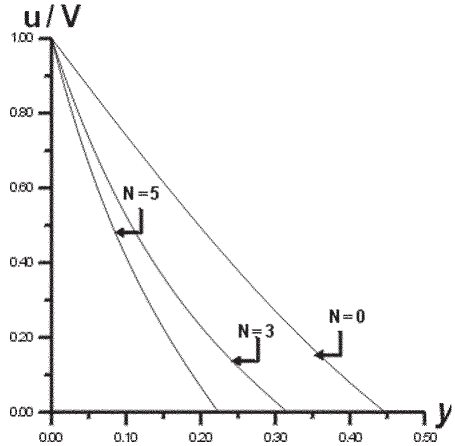

Figure 1. Effect of N on the normalized velocity for $n = 1$, $w = 0.1$ and $t = 1$.

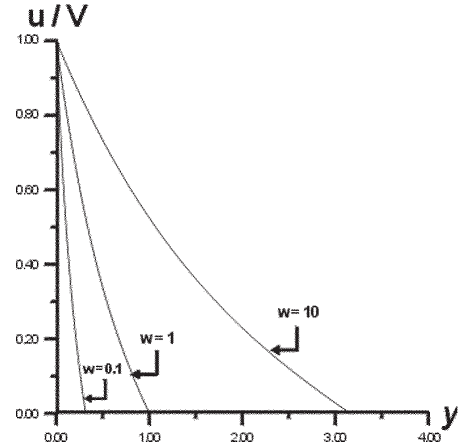

Figure 2. Effect of w on the normalized velocity for $n = 1$, $N = 3$ and $t = 1$.

5 Numerical solution

5.1 Study the effect of N

Consider $n = 1$, $w = 0.1$ and $t = 1$. From equation (20) $\beta = \frac{1}{2}$, which yields $\eta = \frac{y}{\sqrt{t}}$.

Equation (21) will be

$$\left(\frac{d^2\phi}{d\eta^2}\right) + 5\eta\frac{d\phi}{d\eta} - 10N\phi = 0.$$

The result for different values of N is plotted in Fig. 1.

5.2 Study the effect of w

Consider $n = 1$, $N = 3$ and $t = 1$. From equation (20) $\beta = \frac{1}{2}$ which yields $\eta = \frac{y}{\sqrt{t}}$.

Equation (21) will be

$$w\left(\frac{d^2\phi}{d\eta^2}\right) + \frac{\eta}{2}\frac{d\phi}{d\eta} - 3\phi = 0.$$

The result for different values of w is plotted in Fig. 2.

5.3 Study the effect of t

Consider $n = 1$, $N = 3$ and $w = 0.1$. From equation (20) $\beta = \frac{1}{2}$ which yields $\eta = \frac{y}{\sqrt{t}}$.

Equation (21) will be

$$\left(\frac{d^2\phi}{d\eta^2}\right) + 5\eta\frac{d\phi}{d\eta} - 30\phi = 0.$$

The result for different values of t is plotted in Fig. 3.

5.4 Study the effect of n

Consider $N = 3$, $t = 1$ and $w = 0.1$. Equation (21) will be

$$n\left(\frac{d^2\phi}{d\eta^2}\right) + 10\beta\eta\frac{d\phi}{d\eta} - 30\phi = 0.$$

The result for different values of n is plotted in Fig. 4.

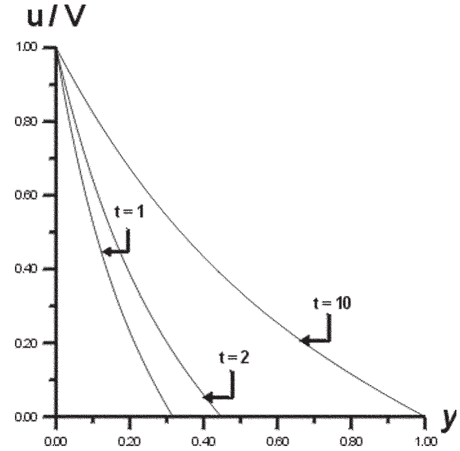

Figure 3. Effect of time t on the normalized velocity for $n = 1$, $N = 3$ and $w = 0.1$.

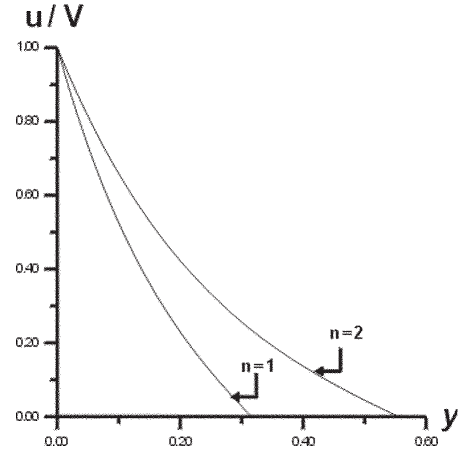

Figure 4. Effect of n on the normalized velocity for $N = 3$, $t = 1$ and $w = 0.1$.

6 Results and discussion

The methods for obtaining similarity transformation were classified into (a) direct methods and (b) group-theoretic methods. The direct methods such as separation of variables do not invoke group invariance. It is fairy straightforward and simple to apply. Group-theoretic methods on the other hand are mathematically more elegant, and the important concept of invariance under a group of transformations is always invoked. In some group-theoretic procedures such as the Birkhoff–Morgan method and the Hellums–Churchill, method the specific form of the group is assumed a priori. On the other hand, procedure such as the finite group method of Moran–Gaggioli is deductive. In this procedure, a general group of transformations is defined and similarity solutions are systematically deduced. The Rayleigh problem for a power law non-Newtonian conducting fluid, which is given by equation (1), is solved using group-theoretic method.

According to Fig. 1, the velocity of the fluid flow increases as the constant N decreases. The constant N is a property of the fluid, it depends on the density of the fluid, the magnetic conductivity and the magnetic permeability. According to Fig. 2, the velocity of the fluid flow increases as $\gamma \left(= \frac{k}{\rho}\right)$ increases, where γ is constant for the same fluid, since the studied fluid is assumed to be incompressible, where k is the coefficient of consistency and ρ is the density of the fluid. According to Fig. 3, the velocity of the fluid flow increases with increase of time.

We studied two cases for constant n, as shown in Fig. 4.

For $n = 1$, the behavior of the fluid is Newtonian. For $n > 1$, the behavior of the fluid is dilatant.

Conditional symmetries, contact symmetries and the classical Lie approach will lead better reductions and more solutions to the differential equations only but not for the initial and boundary value problems, since the given conditions limit the reduction.

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Using Group Theoretic Method to Solve Multi-Dimensional Diffusion Equation

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The nonlinear diffusion equation arises in many important areas of science and technology such as modelling of dopant diffusion in semiconductors. We give analytical solution to N-dimensional radially symmetric nonlinear diffusion equation. The transformation group theoretic approach is applied to analysis of this equation. The one-parameter group transformation reduces the number of independent variables by one, and the governing partial differential equation with the boundary conditions reduce to an ordinary differential equation with the appropriate boundary conditions. Effect of the time t on the concentration diffusion function $C(r,t)$ has been studied and the results are plotted.

1 Introduction

The problem of m-dimensional radially symmetric nonlinear diffusion equation, was treated by King [9] in 1988. He introduced an approximate similarity solution to the porous-medium equation in one and two dimensions. He studied the case $(m = 1)$ and assumed $D(C) = D_0C^n$. The problems considered arise in the modelling of dopant diffusion in semiconductors. He studied the cases $n = 1$ for arsenic and boron in silicon; $n = 2$ for phosphorus in silicon; $n = 2$ or 3 for zinc in gallium arsenide.

Also, Hill [10] in 1989 studied the case $(m = 1)$ and assumed $D(C) = C^n$ but he introduced a new exact solution for the power law diffusivity of index $n = -4/3$ using one-parameter continuous group of transformations. D. Hill and J. Hill [11] in 1990 extended the results given in [10] for particular power law diffusivities C^n (such as $n = -1/2, -1, -3/2$ and -2) using one-parameter continuous group of transformations. King [12] in 1990 gave a new closed-form similarity solutions to N-dimensional radially symmetric nonlinear diffusion equation. He studied two cases. First $D(C) = C^n$ (power-law diffusivities) for both $n > 0$ (slow diffusion), and $n < 0$ (fast diffusion), second $D(C) = e^C$ (exponential diffusivities). The mathematical technique used in the present analysis is the one-parameter group transformation. The group methods, as a class of methods, which lead to reduction of the number of independent variables, were first introduced by Birkhoff [13] in 1948, where he made use of one-parameter transformation groups. In 1952, Morgan [7] presented a theory, which has led to improvements over earlier similarity methods. The method has been applied intensively by Abd-el-Malek et al. [1, 2], Ames [3, 4, 5], Moran and Gaggioli [6] and A.J.A. Morgan [7]. In this work, we present a general procedure for applying a one-parameter group transformation to the multi-dimensional diffusion equation. Under the transformation, the partial differential equation with boundary conditions, is reduced to an ordinary differential equation with the appropriate corresponding conditions. The equation is then solved numerically using non-linear finite difference method applied to the non-linear second order boundary value problem [14], see appendix.

2 Formulation of the problem and the governing equation

Consider a multi-dimensional diffusion equation of the form:

$$\frac{\partial}{\partial r}\left[D(C)\frac{\partial C}{\partial r}\right]+\frac{m-1}{r}D(C)\frac{\partial C}{\partial r}=\frac{\partial C}{\partial t}, \tag{1}$$

with the boundary conditions

$$\begin{aligned}(i)\quad & C(0,t)=F(t),\\ (ii)\quad & C(\infty,t)=0,\end{aligned}$$

and initial condition

$$C(r,0)=0,$$

where $C(r,t)$ is the concentration and $D(C)$ is diffusion coefficient.

The functions $D(C)$ and $F(t)$ are unknown functions and their proper forms will be determined later on; and m is an arbitrary constant.

Assume

$$C(r,t)=F(t)q(r,t), \tag{2}$$

and

$$D(C)=Z(r,t), \tag{3}$$

where $q(r,t)$ is unknown function and its proper form will be determined later on.

Substitution from (2) and (3) into (1) yields

$$\frac{\partial}{\partial r}\left(Z(r,t)\frac{\partial}{\partial r}\left[F(t)q(r,t)\right]\right)+\frac{m-1}{r}Z(r,t)\frac{\partial}{\partial r}\left[F(t)q(r,t)\right]=\frac{\partial}{\partial t}\left[F(t)q(r,t)\right]. \tag{4}$$

Equation (4) can be rewritten in the form:

$$F\left[\frac{\partial Z}{\partial r}\frac{\partial q}{\partial r}+Z\frac{\partial^2 q}{\partial r^2}\right]+FZ\frac{m-1}{r}\frac{\partial q}{\partial r}-F\frac{\partial q}{\partial t}-q\frac{dF}{dt}=0 \tag{5}$$

with the boundary conditions

$$(i)\quad q(0,t)=1, \tag{6}$$

$$(ii)\quad q(\infty,t)=0, \tag{7}$$

and initial condition

$$q(r,0)=0. \tag{8}$$

3 Solution of the problem

Our method of solution depends on the application of a one-parameter group transformation to the partial differential equation (5). Under this transformation the two independent variables will be reduced by one and the differential equation (5) transforms into an ordinary differential equation.

3.1 The group systematic formulation

The procedure is initiated with the group G, a class of transformation of one-parameter a of the form:

$$\begin{aligned}&\bar{r} = h^r(a)r + k^r, \qquad \bar{t} = h^t(a)t + k^t, \qquad \bar{F} = h^F(a)F + k^F,\\ &\bar{q} = h^q(a)q + k^q, \qquad \bar{Z} = h^Z(a)Z + k^Z,\end{aligned} \tag{9}$$

where h's and k's are real-valued and at least differentiable in the real argument "a".

3.2 The invariance analysis

To transform the differential equation, transformations of the derivatives of F, q and Z are obtained from G via chain-rule operations:

$$\overline{S}_i = \left[\frac{h^S}{h^i}\right] S_i, \qquad \overline{S}_{ij} = \left[\frac{h^S}{h^i h^j}\right] S_{ij}, \qquad i, j = r, t, \tag{10}$$

where S stands for F, q and Z.

Equation (5) is said to be invariantly transformed, for some function $A(a)$ whenever:

$$\begin{aligned}&\bar{F}\left[\bar{Z}_r\bar{q}_r + \bar{Z}\bar{q}_{rr}\right] + \bar{F}\bar{Z}\frac{m-1}{\bar{r}}\bar{q}_r - \bar{F}\bar{q}_t - \bar{q}\bar{F}_t\\ &\quad = A(a)\left(F\left[Z_r q_r + Z q_{rr}\right] + FZ\frac{m-1}{r}q_r - Fq_t - qF_t\right).\end{aligned} \tag{11}$$

Substitution from (9) and (10) into (11) yields

$$\begin{aligned}&\left(h^F F + k^F\right)\left[\frac{h^Z h^q}{(h^r)^2}Z_r q_r + \left(h^Z Z + k^Z\right)\frac{h^q}{(h^r)^2}q_{rr}\right]\\ &\quad + \left(h^F F + k^F\right)\left(h^Z Z + k^Z\right)\frac{m-1}{h^r r + k^r}\frac{h^q}{h^r}q_r - \left(h^F F + k^F\right)\frac{h^q}{h^t}q_t - \left(h^q q + k^q\right)\frac{h^F}{h^t}F_t\\ &\quad = A(a)\left(F\left[Z_r q_r + Z q_{rr}\right] + FZ\frac{m-1}{r}q_r - Fq_t - qF_t\right).\end{aligned} \tag{12}$$

The invariance of (12) implies

$$k^F = k^Z = k^q = k^r = 0, \qquad \text{and} \qquad \frac{h^F h^Z h^q}{(h^r)^2} = \frac{h^F h^q}{h^t} = A(a).$$

which yields

$$h^t = \frac{(h^r)^2}{h^Z}.$$

The invariance of the auxiliary conditions (6)–(8) implies that $h^q = 1$, $k^t = 0$.

Finally, we get the one-parameter group G, which transforms invariantly the differential equation (5) and the auxiliary conditions (6)–(8).

The group G is of the form:

$$\bar{r} = h^r r, \qquad \bar{t} = \frac{(h^r)^2}{h^Z}t, \qquad \bar{F} = h^F F, \qquad \bar{q} = q, \qquad \bar{Z} = h^Z Z. \tag{13}$$

3.3 The complete set of absolute invariants

Our aim is to make use of group methods to represent the problem in the form of an ordinary differential equation. Then we have to proceed in our analysis to obtain a complete set of absolute invariants.

If $\eta \equiv \eta(r,t)$ is the absolute invariant of the independent variables, then

$$g_j(r,t;F,q,Z) = \Psi_j\left[\eta(r,t)\right], \qquad j = 1,2,3$$

are the three absolute invariants corresponding to F, q and Z represented by g_j. The application of a basic theorem in group theory, see Moran and Gaggioli [6], states that: *a function* $g(r,t;F,q,Z)$ *is an absolute invariant of a one-parameter group if it satisfies the following first-order linear differential equation:*

$$\sum_{i=1}^{5}(\alpha_i S_i + \beta_i)\frac{\partial g}{\partial S_i} = 0, \qquad S_i \equiv r,t,F,q,Z, \tag{14}$$

where

$$\alpha_i = \frac{\partial h^{S_i}}{\partial a}\left(a^0\right) \qquad \text{and} \qquad \beta_i = \frac{\partial k^{S_i}}{\partial a}\left(a^0\right), \qquad i = 1,2,3,4,5 \tag{15}$$

and a^0 denotes the value of a which yields the identity element of the group G.

The group method applied to the given partial differential equation with the specific boundary conditions yields a unique solution as the condition (14) is used.

At first, we seek the absolute invariant of the independent variables. Owing to equation (14), $\eta(r,t)$ is an absolute invariant if it satisfies the following first-order linear differential equation,

$$(\alpha_1 r + \beta_1)\frac{\partial \eta}{\partial r} + (\alpha_2 t + \beta_2)\frac{\partial \eta}{\partial t} = 0. \tag{16}$$

Since $k^r = k^t = 0$, and according to the definition of the β's then $\beta_1 = \beta_2 = 0$.

Now, equation (16) may be rewritten in the form,

$$\alpha_1 r\frac{\partial \eta}{\partial r} + \alpha_2 t\frac{\partial \eta}{\partial t} = 0.$$

Applying separation of variables method, one can obtain a solution in the form,

$$\eta = rt^{-B}, \qquad \text{where} \qquad B = \frac{\alpha_1}{\alpha_2}. \tag{17}$$

The second step is to obtain the absolute invariants of the dependent variables F, q and Z. By a similar analysis, using equations (13), (14) and (15), we get

$$F(t) = R(t)\phi(\eta), \tag{18}$$

Since $F(t)$ and $R(t)$ are independent of r, while η is a function of r and t, then $\phi(\eta)$ must be constant, say $\phi(\eta) = 1$, and from which

$$F(t) = R(t), \tag{19}$$

and the second absolute invariant is

$$q(r,t) = \theta(\eta). \tag{20}$$

Also, the last absolute invariant is

$$Z(r,t) = \Gamma(t)W(\eta). \tag{21}$$

4 The reduction to an ordinary differential equation

By means of substitution from (17)–(21) into equation (5), we get

$$R\Gamma t^{-2B}W'\theta' + R\Gamma t^{-2B}W\theta'' + R(m-1)\Gamma W\theta'\frac{t^{-B}}{r} + \frac{RB\theta'}{t}\eta - \theta R' = 0.$$

Dividing by $R\Gamma t^{-2B}$

$$W'\theta' + W\theta'' + \frac{m-1}{\eta}W\theta' + \frac{\eta t^{2B-1}}{\Gamma}\theta' - \frac{R' t^{2B}}{R\Gamma}\theta = 0. \tag{22}$$

For (22) to be reduced to an ordinary differential equation in one variable η, it is necessary that the coefficients should be constants or functions of η only. Thus

$$C_1 = \frac{Bt^{2B-1}}{\Gamma}, \qquad C_2 = \frac{R't^{2B}}{R\Gamma}. \tag{23}$$

Using (23) we get,

$$\Gamma(t) = \frac{Bt^{2B-1}}{C_1}, \qquad R(t) = t^{\frac{BC_2}{C_1}}.$$

Hence, equation (22) will be,

$$W\theta'' + W'\theta' + \frac{m-1}{\eta}W\theta' + C_1\eta\theta' - C_2\theta = 0. \tag{24}$$

Under the similarity variable η, the boundary conditions are

$$\theta(0) = 1, \qquad \theta(\infty) = 0.$$

5 Numerical solution

Consider $W = \eta$.

Case 1. C_1=1 and $C_2 = 1$. Equation (24) will be

$$\eta\theta'' + m\theta' + \eta\theta' - \theta = 0.$$

To know the final value of η, using order of Magnitude Analysis [8]

$$\frac{d\theta}{d\eta} \cong \frac{\Delta\theta}{\eta_{\max}} \cong \frac{1}{\eta_{\max}}, \qquad \frac{d^2\theta}{d\eta^2} = \frac{d}{d\eta}\left[\frac{d\theta}{d\eta}\right] \cong \frac{1}{\eta^2_{\max}}.$$

Subcase 1a. Take $B = \frac{1}{2}$, then

$$\eta = \frac{r}{\sqrt{t}}, \qquad \Gamma(t) = \frac{1}{2}, \qquad D(C) = \frac{r}{2\sqrt{t}} = \frac{\eta}{2}, \qquad F(t) = \sqrt{t}, \qquad C(r,t) = \sqrt{t}\theta(\eta).$$

The result for different values of time t is plotted in Fig. 1.

Subcase 1b. Take $B = 1$, then

$$\eta = \frac{r}{t}, \qquad \Gamma(t) = t, \qquad D(C) = r, \qquad F(t) = t, \qquad C(r,t) = t\theta(\eta).$$

The result for different values of time t is plotted in Fig. 2.

Case 2. $C_1 = 1$ and $C_2 = 2$. Equation (24) will be

$$\eta\theta'' + m\theta' + \eta\theta' - 2\theta = 0.$$

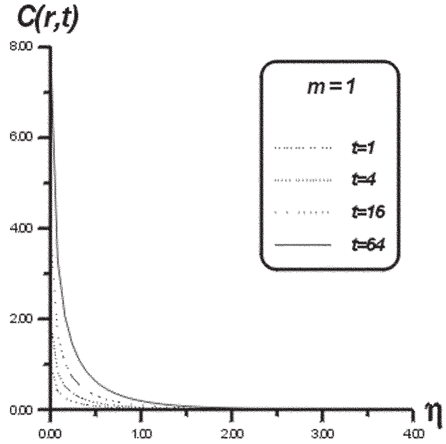

Figure 1. Effect of time t on the concentration function $C(r,t)$ for $C_1 = 1$, $C_2 = 1$ and $B = 1/2$ at $m = 1$.

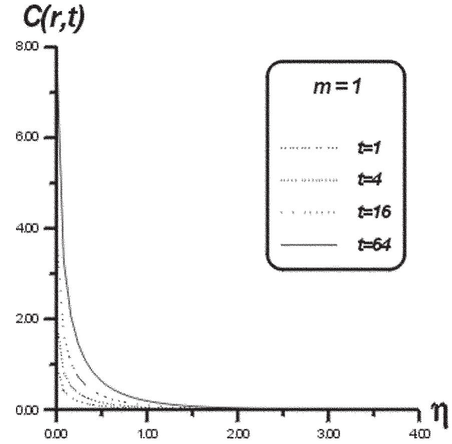

Figure 2. Effect of time t on the concentration function $C(r,t)$ for $C_1 = 1$, $C_2 = 1$ and $B = 1$ at $m = 1$.

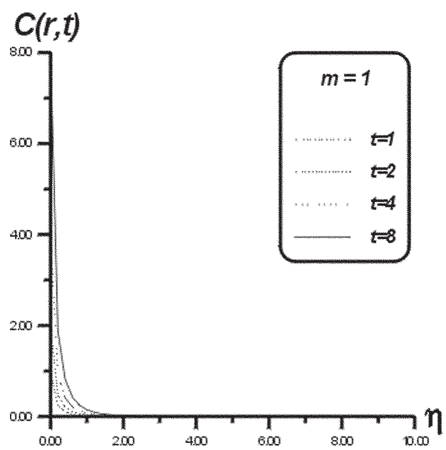

Figure 3. Effect of time t on the concentration function $C(r,t)$ for $C_1 = 1$, $C_2 = 2$ and $B = 1/2$ at $m = 1$.

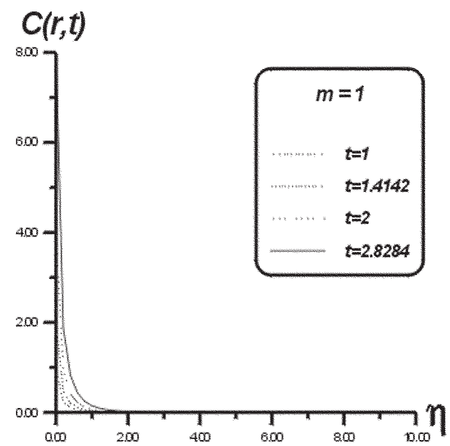

Figure 4. Effect of time t on the concentration function $C(r,t)$ for $C_1 = 1$, $C_2 = 2$ and $B = 1$ at $m = 1$.

Subcase 2a. Take $B = \frac{1}{2}$, then

$$\eta = \frac{r}{\sqrt{t}}, \qquad \Gamma(t) = \frac{1}{2}, \qquad D(C) = \frac{r}{2\sqrt{t}} = \frac{\eta}{2}, \qquad F(t) = t, \qquad C(r,t) = t\theta(\eta).$$

The result for different values of time t is plotted in Fig. 3.

Subcase 2b. Take $B = 1$, then

$$\eta = \frac{r}{t}, \qquad \Gamma(t) = t, \qquad D(C) = r, \qquad F(t) = t^2, \qquad C(r,t) = t^2\theta(\eta).$$

The result for different values of time t is plotted in Fig. 4.

Case 3. $C_1 = 3$ and $C_2 = 2$. Equation (24) will be

$$\eta\theta'' + m\theta' + 3\eta\theta' - 2\theta = 0.$$

Subcase 3a. Take $B = \frac{1}{2}$, then

$$\eta = \frac{r}{\sqrt{t}}, \qquad \Gamma(t) = \frac{1}{6}, \qquad D(C) = \frac{r}{6\sqrt{t}} = \frac{\eta}{6}, \qquad F(t) = t^{\frac{1}{3}}, \qquad C(r,t) = t^{\frac{1}{3}}\theta(\eta).$$

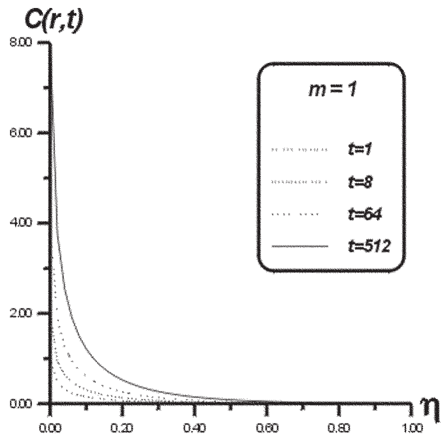

Figure 5. Effect of time t on the concentration function $C(r,t)$ for $C_1 = 3$, $C_2 = 2$ and $B = 1/2$ at $m = 1$.

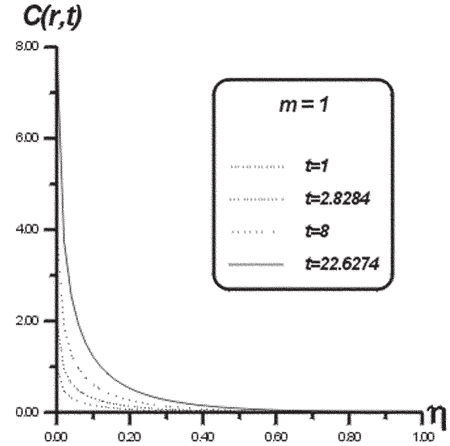

Figure 6. Effect of time t on the concentration function $C(r,t)$ for $C_1 = 3$, $C_2 = 2$ and $B = 1$ at $m = 1$.

The result for different values of time t is plotted in Fig. 5.

Subcase 3b. Take $B = 1$, then

$$\eta = \frac{r}{t}, \qquad \Gamma(t) = \frac{t}{3}, \qquad D(C) = \frac{r}{3}, \qquad F(t) = t^{\frac{2}{3}}, \qquad C(r,t) = t^{\frac{2}{3}}\theta(\eta).$$

The result for different values of time t is plotted in Fig. 6.

6 Results and discussion

The methods for obtaining similarity transformation were classified into (a) direct methods and (b) group-theoretic methods. The direct methods such as separation of variables do not invoke group invariance. It is fairy straightforward and simple to apply. Group-theoretic methods on the other hand are mathematically more elegant and the important concept of invariance under a group of transformations is always invoked. In some group-theoretic procedures such as the Birkhoff–Morgan method and the Hellums–Churchill, method the specific form of the group is assumed a priori. On the other hand, procedure such as the finite group method of Moran–Gaggioli is deductive. In this procedure, a general group of transformations is defined and similarity solutions are systematically deduced.

The N-dimensional radially symmetric nonlinear diffusion equation, which is given by equation (1) is solved without made any assumption for the function $D(C)$ and the constant m. It is found that no numerical results could be obtained for equation (24) if we take $W = \eta^2$, $W = e^{\eta}$, $W = e^{-\eta}$, $W = \frac{1}{\eta}$ and $W = \frac{1}{\eta^2+1}$. The only value of W, to obtain results that $W = \eta$. Studying different cases for values of C_1 and C_2 show that, for constant value of m ($m = 1$), $C(r,t)$ is exponential increasing as t increases.

7 Appendix

Assume $\theta'' = f(\eta, \theta, \theta')$, $\theta(0) = \alpha$ and $\theta(\infty) = \beta$. Let W_i be the numerical solution for $\theta(\eta_i)$. Substituting for the derivatives θ', θ'' with their approximations in finite difference; we get

$$\frac{W_{i-1} - 2W_i + W_{i+1}}{h^2} = f\left(\eta_i, W_i, \frac{W_{i+1} - W_{i-1}}{2h}\right), \tag{25}$$

where h is the step size in η.

Equation (25) can be written in the form

$$W_{i-1} - 2W_i + W_{i+1} = h^2 f\left(\eta_i, W_i, \frac{W_{i+1} - W_{i-1}}{2h}\right),$$

which can be rewritten as,

$$F\left(W_{i-1}, W_i, W_{i+1}\right) = 0. \tag{26}$$

Writing this equation for $i = 1, 2, 3, \ldots, n$. Taking into consideration that the space domain $\eta \in [0, \infty)$ is subdivided into the computational mesh $\eta_0 < \eta_1 < \eta_2 < \cdots < \eta_n < \eta_{n+1}$, where η_{n+1} will be at a far away distance from the initial point η_0 to represent our artificial boundary at ∞. The result is a system of nonlinear equation in the unknowns $W_1, W_2, \ldots, W_n$. The system is solved iteratively using the Newton method for such problem, which leads to,

$$J^{(K)}\left[\overline{W}^{(K+1)} - \overline{W}^{(K)}\right] = -\overline{F}^{(K)}.$$

where $J^{(K)}$ denotes the Jacobian of the system evaluate at the iterative step K. $\overline{W}^{(K)}$ and $\overline{W}^{(K+1)}$ represent the unknown vector at step K and $K+1$, respectively. $\overline{F}^{(K)}$ is the vector representing the expression in (26) above evaluated at the iterative step K. The Jacobian matrix J is obtained which is three-diagonal matrix. The resulting system is solved using the Lower-Upper decomposition.

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Hopf Bifurcations in Problems with $O(2)$ Symmetry: Canonical Coordinates Transformation

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Hopf bifurcations in problems with $O(2)$ symmetry are considered. In these problems, the Jacobian matrix is always singular at the circle of $\mathbb{Z}_2$ symmetric steady state solutions. While a couple of imaginary eigenvalue cross the imaginary axis, the Hopf bifurcation is not of standard type. The canonical coordinates transformation is used for removing the zero eigenvalue and converting the problem into the standard form. The method is applied to a system of ordinary differential equations on $\mathbb{C}^3$ with many parameters and the stable solutions are obtained using the centre manifold reduction. Further symmetry breaking bifurcation is obtained on periodic solutions, leading to modulated travelling waves solutions.

1 Introduction

We consider bifurcations which occur in systems with $O(2)$ symmetry. In particular we consider the Hopf bifurcation from a non-trivial steady state solutions giving rise to a branch of direction reversing wave (RW) solutions. Further bifurcation from these time periodic solutions lead to a branch of modulated travelling (MTW) solutions. The standard Hopf theorem [1] cannot be applied in this situation since there is a zero eigenvalue of the Jacobian at every nontrivial steady state solution, due to the group orbit of solutions.

Krupa [2] considers the related, but more general problem of bifurcation from group orbits for problems which are equivariant with respect to subgroups of $O(n)$. In this case, the degeneracy is dealt with by splitting the vector field into two parts, one tangent to the group orbit and one normal to it. A standard bifurcation analysis can then be performed on the normal vector field and the results are then interpreted for the whole vector field.

Barkley [3] considers Hopf bifurcation on the branch of travelling wave solutions, in the study of reaction diffusion system. He presented a low-dimensional ordinary differential equations model which has travelling wave solutions which undergo a Hopf bifurcation giving rise to MTW solutions. They decoupled some of the variables involved in the system by a simple change of coordinates to facilitate the analysis.

Landsberg and Knobeloch [4] studied the problem and showed that in problems with $O(2)$ symmetry a codimension-one symmetry breaking Hopf bifurcation from a circle of non-trivial steady states gives rise to periodic motions. These periodic solutions reverse their direction of propagation in a periodic manner. In another paper [5], they also refer to the modulated travelling waves which can bifurcate from the RW solutions. However, they did not perform any analysis of the bifurcations involved. We address these problems and related issues in this paper.

In this paper we first consider the method of canonical coordinates in more detail to give a clearer understanding of the type of solution which occurs and we analyse a further possible bifurcation from the branch of time periodic solutions, to modulated travelling wave solutions. In Section 2 we obtain the reduced equations to analyse the bifurcations and establish the relationship between different coordinate system employed. Section 3 is devoted to a numerical example to illustrate the method and to clarify the issues involved.

2 Setting the system

Consider the system of equations $\dot{z} = g(z, \lambda)$, where $z = (z_1, z_2, z_3) \in \mathbb{C}^3 =: X$ and $\lambda \in \mathbb{R}$ is the bifurcation parameter. Let $z_j = x_j + iy_j$, $j = 1, 2, 3$ and assume that g is equivariant with respect to the diagonal action of $O(2)$ defined by

$$\begin{aligned} r_\alpha(z_1, z_2, z_3) &= \left(e^{i\alpha} z_1, e^{i\alpha} z_2, e^{i\alpha} z_3\right), \\ s(z_1, z_2, z_3) &= (\bar{z}_1, \bar{z}_2, \bar{z}_3). \end{aligned} \tag{1}$$

Due to the reflection s the space X can be decomposed as $X = X^s \oplus X^a$, where X^s and X^a are the symmetric and anti-symmetric spaces with respect to the reflection s, respectively. Let us assume that non-trivial solutions $z_s = z_s(\lambda)$ bifurcate from trivial solutions at $\lambda = 0$. For these non-trivial steady states at least one of the variables, say z_1, is non-zero. There is a corresponding group orbit of solutions which are generated by the rotation. These solutions are contained in $\mathrm{Fix}(\mathbb{Z}_2) \times \mathbb{R}$, where $\mathbb{Z}_2 = \{I, s\}$. The reflection s implies that $y_1 = y_2 = y_3 = 0$. We now write the original equations in real form as

$$\dot{x} = f(x, \lambda), \qquad x = (x_1, x_2, x_3, y_1, y_2, y_3), \tag{2}$$

where $f = (f_1(x, \lambda), f_2(x, \lambda), f_3(x, \lambda), g_1(x, \lambda), g_2(x, \lambda), g_3(x, \lambda))$. The reflection s implies that $g_z(z_s, \lambda) = \mathrm{diag}\,(g_z^s(z_s, \lambda) : g_z^a(z_s, \lambda))$, where g_z^s and g_z^a are associated with symmetric and anti symmetric spaces, respectively (see [6]). Clearly in real form these blocks take the form $g_z^s = [f_{ij}]$, $g_z^a = [g_{ij}]$, where $f_{ij} = \frac{\partial f_i}{\partial x_j}$ and $g_{ij} = \frac{\partial g_i}{\partial y_j}$, $i, j = 1, 2, 3$. All of the derivatives are evaluated at (z_s, λ).

The rotation symmetry implies that $g_z(z_s, \lambda) A z_s = 0$ for all λ, where the linear operator A is defined by $Az = \frac{d}{d\alpha}(r_\alpha z)|_{\alpha=0}$. In this case $Az = (iz_1, iz_2, iz_3)$ and so $Az_s \in X^a$. Thus the anti-symmetric block is singular. We now assume that $g_z(z_s, \lambda)$ also has eigenvalues $\pm i\omega_0$ at (z_0, λ_0), where $z_0 = z_s(\lambda_0)$. Since we are interested in symmetry breaking Hopf bifurcation then we assume that these eigenvalues occur in an anti-symmetric block. A necessary condition for this bifurcation is that the anti-symmetric block g_z^a has a minimum dimension three. We then show that a branch of periodic solutions bifurcates from the steady state branch at (z_0, λ_0) with a spatio-temporal symmetry (s, π). Further bifurcation can be obtained by breaking this symmetry. However, due to the zero eigenvalue the Hopf bifurcation is not of standard type. We thus use canonical coordinates [4, 7] in order to decouple one of the variables and then use the standard theory.

2.1 Reduced equations

We introduce the canonical coordinates transformation

$$w_1 = \frac{z_2}{z_1}, \qquad w_2 = \frac{z_3}{z_1}, \qquad r = |z_1|, \qquad \theta = \arg(z_1), \tag{3}$$

where $w_j = u_j + iv_j \in \mathbb{C}$, $j = 1, 2$ and $r \in \mathbb{R}$ are all invariant under the rotation and $\theta \to \theta + \alpha$. This enable us to decouple the θ variable from the others, when the system (2) is written in terms of these new variables, with the result

$$\dot{U} = G(U, \lambda), \tag{4}$$

$$\dot{\theta} = G_\theta(U, \lambda), \tag{5}$$

where $U = (u_1, u_2, r_1, v_1, v_2)$ and is invariant under the rotation. The reflection s acts as $s(u_1, u_2, r, v_1, v_2, \theta) = (u_1, u_2, r, -v_1, -v_2, -\theta)$. Thus, (4) has only a reflection symmetry. We note that, for the steady state problem $G(U,\lambda)=0$, the symmetric space is characterized by $v_1=v_2=0$ and the anti-symmetric space by $u_1=u_2=r=0$. Thus for steady state solutions one can restrict the problem to the symmetric space and seek the solutions there. Now, let us $U_s=(u_1^s,u_2^s,r_1^s,v_1^s,v_2^s)$ be a steady solution of (4), then due to the reflection we can write $G_U=\operatorname{diag}(G_U^s(U_s,\lambda):G_U^a(U_s,\lambda))$, where G_U^s is 3×3 matrix and G_U^a is 2×2. Expanding equation (4) explicitly, using the definition of canonical coordinates, system (2), all the steady state assumptions, and the computing algebra package MATHEMATICA we can be shown that G_U^a is given by

$$G_U^a=\begin{bmatrix} g_{22}-u_1^s g_{12} & g_{23}-u_1^s g_{13}\\ g_{32}-u_2^s g_{12} & g_{33}-u_2^s g_{13}\end{bmatrix}.$$

We will show that if $g_z^a(z_s,\lambda)$ has a pair of imaginary eigenvalues then G_U^a also does (see Theorem 1). Since G_U has eigenvalues $\pm i\omega_0$ and no zero eigenvalue therefore we can apply the standard theory, which implies that there exists a bifurcating branch of periodic solutions with $(s,\pi)\in\mathbb{Z}_2\times S^1$ symmetry, since s and π both act as $-I$ on the eigenspace. It is possible to rescale time in order to have 2π-periodic solutions. This symmetry then implies that $-v_1(t+\pi)=v_1(t)$, $-v_2(t+\pi)=v_2(t)$, and $u_1(t+\pi)=u_1(t)$, $u_2(t+\pi)=u_2(t)$, $r(t+\pi)=r(t)$. As $s\theta=-\theta$, the equivariance condition related to equation (5) is

$$G_\theta(u_1,u_2,r,-v_1,-v_2,\lambda)=-G_\theta(u_1,u_2,r,v_1,v_2\lambda).$$

Thus, for the time periodic solutions, with $\tau=t+\pi$, we have

$$\begin{aligned}G_\theta(u_1(\tau),u_2(\tau),r(\tau),v_1(\tau),v_2(\tau),\lambda)&=G_\theta(u_1(t),u_2(t),r(t),-v_1(t),-v_2(t),\lambda)\\&=-G_\theta(u_1(t),u_2(t),r(t),v_1(t),v_2(t),\lambda).\end{aligned}$$

Integrating the above equation over the interval $[0,2\pi]$, considering 2π-periodicity of the functions u_1, u_2, r, v_1 and v_2, we obtain G_θ, hence $\dot\theta$ has zero mean, which implies that θ is periodic. Therefore, w_1 and w_2 are periodic and then the original variables z_1, z_2 and z_3 are also time periodic. A further bifurcation could occur from these periodic solutions which breaks the (s,π) symmetry. The theory related to this bifurcation is well developed [1] and this is a simple bifurcation on time periodic solutions which occurs in the reduced system (4). However, breaking this symmetry implies that $\dot\theta(t)$ has no longer zero mean and so we can write $\dot\theta(t)=c+\dot\theta_0(t)$ where $\dot\theta_0(t)$ has zero mean and c is constant. Hence $\theta(t)=ct+\theta_0(t)+k$, where k is the constant of integration that we set to zero. Since $\dot\theta_0$ has zero mean, hence θ_0 is periodic. Clearly, on the periodic solutions, due to (s,π) symmetry, $c=0$ and therefore $\theta(t)=\theta_0(t)$ is periodic. However, if this symmetry is broken, then $c\neq 0$ and so θ is not periodic but is composed of a constant drift with velocity c superimposed on a periodic motion. This bifurcation arises as a simple symmetry breaking bifurcation in system (4). The solution in the original coordinates is then given by $z_1(t)=r(t)e^{i\theta(t)}=r(t)e^{i(ct+\theta_0(t))}=e^{i(ct)}\tilde z_1(t)$, where $\tilde z_1(t)=r(t)e^{i\theta_0(t)}$ is periodic. The first equation of (3) implies that $z_2(t)=e^{i(ct)}w_1(t)\tilde z_1(t)=e^{i(ct)}\tilde z_2(t)$, where $\tilde z_2(t)=w_1(t)\tilde z_1(t)$ is periodic. Finally, the second equation of (3) implies that $z_3(t)=e^{i(ct)}\tilde z_3(t)$, where $\tilde z_3(t)$ is periodic. Hence we have the solutions of the form

$$Z(t)=r_{ct}z(t), \tag{6}$$

where $z(t)$ is a periodic function of time. Thus $c=0$ corresponds to a branch of periodic solutions while $c\neq 0$ corresponds to MTW solutions that consist of time periodic solutions drifting with constant velocity c along the group orbits. Note that the constant $k\neq 0$ simply gives rise to a one-parameter family of conjugate solutions, obtained by a constant rotation. Initially therefore, the solutions are oscillating with only a very small amount of drift and so the rotational motion, characterised by the variable θ, continues to oscillate. However, as the branch is followed further from the bifurcation point, the drift increases which could result in θ increasing (or decreasing) monotonically.

Now we show G_U^a also has a purely imaginary eigenvalues on the branch of non trivial steady state solutions. To see this, we first establish the relationship between two different coordinate systems.

2.2 Eigenvalues of the reduced system

We now consider the linearisation of equations (4) and (5) on the anti-symmetric space and obtain a connection between the two sets of coordinates in order to discuss about the eigenvalues of the reduced system. Since we have a standard Hopf bifurcation in (4) then the bifurcating solution near to the bifurcation point is given by $U(t) = U_s + \alpha\Phi(t) + O\left(\alpha^2\right)$, where $\Phi(t)$ is a solution of the linearisation of (4) about the steady state, i.e. $\Phi(t) = [0, 0, 0, V_1(t), V_2(t)]^T$, since it is the anti-symmetric component of G_U which has the imaginary eigenvalues. Again, near to the bifurcation point, we have $\theta = \theta^s + \alpha\Theta + O\left(\alpha^2\right)$, where Θ is the solution of the linearisation of (5) given by $\dot{\Theta} = \frac{\partial G_\theta}{\partial v_1}V_1 + \frac{\partial G_\theta}{\partial v_2}V_2$. It is easily shown that $\frac{\partial G_\theta}{\partial v_1} = g_{12}$ and $\frac{\partial G_\theta}{\partial v_2} = g_{13}$, evaluated at a symmetric steady state solution. The solution of this equation is $\Theta(t) = \Theta_0(t) + k$, where $\Theta_0(t)$ is periodic with zero mean and k is an arbitrary constant of integration. Thus, the linearisation of (4) and (5) about the symmetric steady state on the anti-symmetric space is given by

$$\dot{V}(t) = BV(t), \tag{7}$$

where B is a (3×3) matrix, constructed by the augmenting a third column and a third row to G_U^a. This consists of augmenting a column vector $[0, 0, 0]^T$ and a row vector $[g_{12}, g_{13}, 0]$. Note that eigenvalues of B are $\pm i\omega_0$ and zero, therefore the solution of (7) is $V_k(t) = [V_1(t), V_2(t), \Theta_0(t)]^T + k[0, 0, 1]^T = \tilde{V}(t) + ke_3$, where $k \in \mathbb{R}$ is arbitrary constant. Note that $\tilde{V}(t)$ is constructed from the complex eigenvectors of B corresponding to the eigenvalues $\pm i\omega_0$ and e_3 is the eigenvector corresponding to the zero eigenvalue. Converting back to the original coordinates, we have

$$z_1(t) = r(t)e^{i\theta(t)} = \left(r^s + O\left(\alpha^2\right)\right) e^{i\left(\theta^s + \alpha\Theta_0(t) + \alpha k + O\left(\alpha^2\right)\right)}.$$

Since $\sin\theta^s = 0$, this implies that

$$\begin{aligned} x_1(t) &= r^s\cos\theta^s + O\left(\alpha^2\right) = x_1^s + O\left(\alpha^2\right), \\ y_1(t) &= \alpha\left(r^s\cos\theta^s\right)\left(\Theta_0(t) + k\right) + O\left(\alpha^2\right) = \alpha x_1^s(\Theta_0(t) + k) + O\left(\alpha^2\right). \end{aligned}$$

Similarly, it can be shown, by using definition of canonical coordinates, that

$$\begin{aligned} x_2(t) &= x_2^s + O\left(\alpha^2\right), \\ x_3(t) &= x_3^s + O\left(\alpha^2\right), \\ y_2(t) &= \alpha\left[x_2^s(\Theta_0(t) + k) + x_1^s V_1(t)\right] + O\left(\alpha^2\right), \\ y_3(t) &= \alpha\left[x_3^s(\Theta_0(t) + k) + x_1^s V_2(t)\right] + O\left(\alpha^2\right). \end{aligned}$$

Hence, on the anti-symmetric space, the linearisation of the original equations given by

$$\dot{Y}(t) = g_z^a(z_0, \lambda_0)Y(t), \tag{8}$$

has solutions of the form

$$\mathrm{Y}_k(t) = \begin{bmatrix} x_1^s(\Theta_0(t) + k) \\ x_2^s(\Theta_0(t) + k) + x_1^s V_1(t) \\ x_3^s(\Theta_0(t) + k) + x_1^s V_2(t) \end{bmatrix} = T(\tilde{V}(t) + ke_3) = TV_k(t),$$

where $T = \begin{bmatrix} 0 & 0 & x_1^s \\ x_1^s & 0 & x_2^s \\ 0 & x_1^s & x_3^s \end{bmatrix}$. Note that $Te_3 = [x_1^s, x_2^s, x_3^s]^T = Az_0^a$, where Az_0^a is the anti-symmetric part of Az_0, which is a solution of (8) since it is independent of time and $g_z^a(z_0, \lambda_0)Az_0^a = 0$. A more precise result is the following.

Theorem 1. *If g_z^a is the anti-symmetric block with the eigenvalues $\pm i\omega_0$ and 0 then*

(i) $g_z^a(z_0, \lambda_0) = TBT^{-1}$,

(ii) $V_k(t)$ is a solution of (7) if and only if $Y_k(t) = TV_k(t)$ is a solution of (8).

Proof. (i) Note that the last column of T is $[x_1^s, x_2^s, x_3^s]^T = Az_0^a$. Thus

$$g_z^a(z_0, \lambda_0)T = x_1^s \begin{bmatrix} g_{12} & g_{13} & 0 \\ g_{22} & g_{23} & 0 \\ g_{32} & g_{33} & 0 \end{bmatrix},$$

and then it is easily verified that $T^{-1}g_z^a(z_0, \lambda_0)T = B$.

(ii) follows immediately from (i). ■

The first of these results show more clearly that when g_z^a has eigenvalues $\pm i\omega_0$, then so does G_U^a. The second presents the relationship between the eigenvectors.

3 A numerical example

3.1 An equation on $\mathbb{C}^3$

Consider the system [4],

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= z_3, \\ \dot{z}_3 &= \lambda z_1 + \nu z_2 + \eta z_3 + a|z_1|^2 z_1 + b|z_2|^2 z_1 + c|z_1|^2 z_2 + d|z_1|^2 z_3 \\ &\quad + e z_1^2 \bar{z}_2 + f z_1^2 \bar{z}_3 + g z_2^2 \bar{z}_1 + h|z_2|^2 z_2 + j z_1 \bar{z}_2 z_3 + k z_1 z_2 \bar{z}_3 + l|z_2|^2 z_3 + m z_2^2 \bar{z}_3. \end{aligned} \tag{9}$$

These equations are the normal form for a triple zero bifurcation with group $O(2)$ symmetry and such have a number of applications, particularly in fluid dynamics [9]. We write these equations as $\dot{z} = g(z, \lambda)$, where $z = (z_1, z_2, z_3) \in \mathbb{C}^3$ and λ is regarded as the bifurcation parameter. It is easily verified that this system is equivariant with respect to the diagonal action of $O(2)$ defined by (1) (see [11]). System (9) has trivial solution $z = 0$ for $\forall\ \lambda$; a bifurcating branch of solutions occurs at $\lambda = 0$, and is given by $x_1^2 = -\lambda/a$, $y_1 = y_2 = y_3 = 0$. As these solutions are invariant under the reflection symmetry s, conjugate solutions are obtained by applying the rotational operator r_α, giving rise to a circle of steady state solutions for each λ. The trivial solutions will be stable for $\lambda < 0$ and unstable for $\lambda > 0$, if $\eta, \nu < 0$. The bifurcating branch will then be stable if it is supercritical. This occurs if $a < 0$. Therefore we choose $a = -4.0$, $\eta = -2.5$, $\mu = -10$ so that a supercritical bifurcation occurs at $\lambda = 0$. In order to have a couple of imaginary eignvalue in anti-symmetric block, evaluated at a non-trivial steady state, we choose $d = 0$, $f = -16$, $c = -1$, $e = 0$. Therefore a symmetry breaking Hopf bifurcation occur at $\lambda = 2.5$, giving rise to a branch of RW's. We choose the rest of the parameters so that periodic orbits are stable. This can be carried out using the canonical coordinates transformation followed by the centre manifold reduction [10]. We omit the discussion and only introduce the rest of the parameters as: $b = -10$, $g = 30$, $h = 1$, $j = 10$, $k = -30$, $l = 20$, $m = -30$. For these

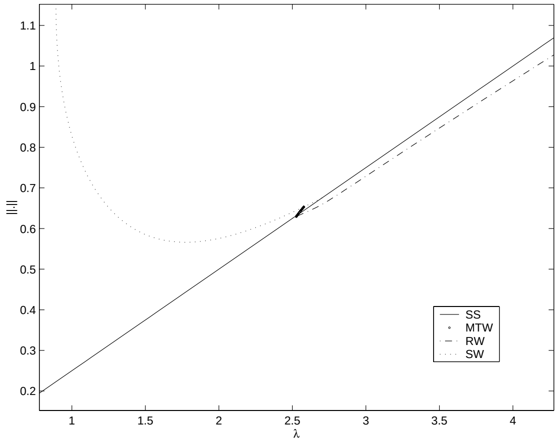

Figure 1. Bifurcation diagram of the equations (9). With $\eta = -2.5$ a branch of RW solutions appear at $\lambda = 2.5$, and the SW solutions occur at $\lambda = 2.7$. A secondary bifurcation occurs on the branch of SW solutions at $\lambda = 2.577$ giving rise to a branch of MTW solutions which connects to RW solutions at $\lambda = 2.526$.

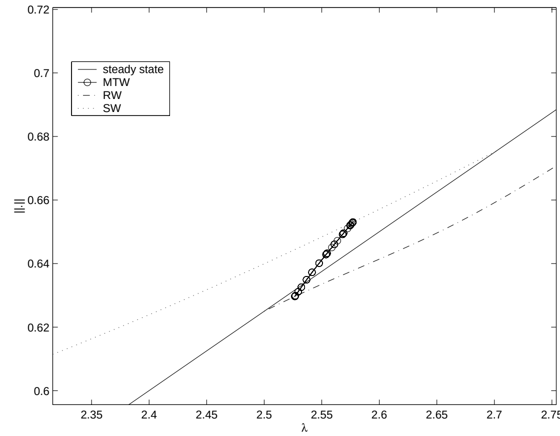

Figure 2. Enlarged bifurcation diagram around the bifurcation points. All bifurcation points stated in Fig. 1 can be seen clearly. A branch of MTW solutions connects two branches of periodic orbits. A torus bifurcation obtained on this branch at $\eta = -2.5$ and $\lambda = 2.547$.

values of the parameters we applied the numerical method described in the previous section and the following bifurcations are obtained: on the branch of non-trivial steady states a symmetry preserving Hopf bifurcation leading to a branch of periodic orbits occurs at $\lambda = 2.70$, these are standing waves (SW) which lie in symmetric space. On the branch of RW solutions a secondary bifurcation, giving rise to a branch of MTW solutions, occurs at $\lambda = 2.526$. This branch connects with the branch of SW's at $\lambda = 2.577$. On the branch of the MTW solutions a torus bifurcation is also obtained at $\lambda = 2.547$. A bifurcation diagram of these solutions is shown in Fig. 1. This diagram is enlarged in Fig. 2 to give a clear picture of the bifurcation points involved in the problem. For $\lambda = 2.595$ and $\eta = -2.50$ a RW is given in Fig. 3, this is a time periodic solution and reverses its direction of propagation in a periodic manner [4]. In Fig. 4 and Fig. 5, the MTW solutions are represented for different values of λ and η. All of these solutions were obtained using the package AUTO [8]. The MTW's were reconstructed using equation (6).

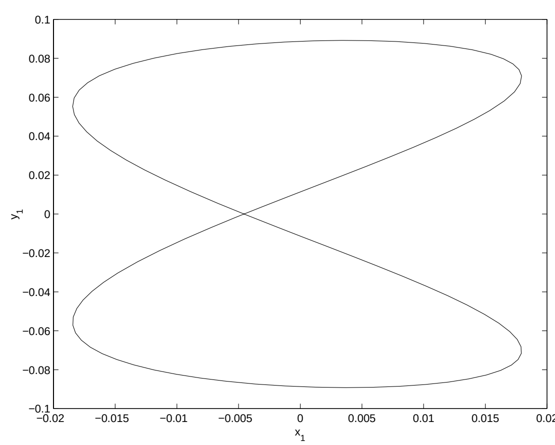

Figure 3. A RW solution at $\lambda = 2.595$ and $\eta = -2.50$. This is a time periodic solution with the spatio-temporal symmetry.

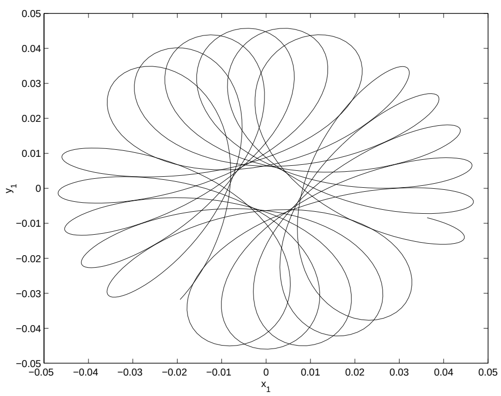

Figure 4. A MTW solution at $\lambda = 2.531$ and $\eta = -2.50$. There is a 4-petal flower (2 small and two large) repeating itself with time progression.

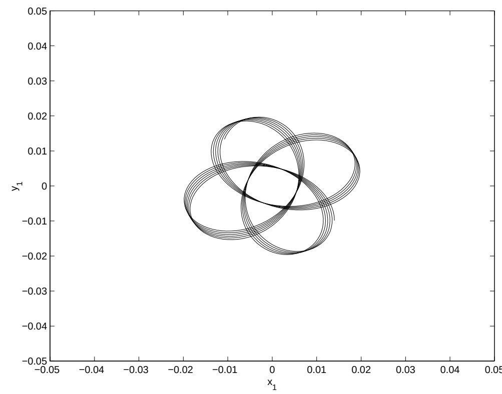

Figure 5. A MTW solution at $\lambda = 2.0886$ and $\eta = -2.0800$. There is a 4-petal flower (each petal has same amplitude) repeating itself with time progression.

4 Conclusions

The Hopf bifurcation in problems with $O(2)$ symmetry is considered. The canonical coordinates transformation were used in order to analyse the problem using standard theory, and also to convert the solutions back into the original coordinates in order to obtain a correct interpretation of the results.

We obtained time periodic solutions with spatio-temporal symmetry. Further bifurcation is obtained by breaking this symmetry resulting in MTW solutions, which is due to the fact that one of the variables in canonical coordinates drift with constant velocity. In addition an example on $\mathbb{C}^3$ with many parameters [4] was considered to clarify the analysis and centre manifold reduction is used to obtain stable solutions. Two Hopf bifurcations leading to the SW's and the RW's were obtained on steady state solutions. The occurrence of a second Hopf bifurcation indicates that if a second parameter was varied there may be a Hopf/Hopf mode interaction. This is the case that considered by Amdjadi [11] who introduced a numerical method for such mode interactions.

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Classification of Systems of Nonlinear Evolution Equations Admitting Higher-Order Conditional Symmetries

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Algorithm for construction of conditionally invariant systems of evolution equations and their subsequent reduction to the systems of ordinary differential equations is suggested. Classification and reduction theorems are formulated for n-order evolution equations and for systems of two evolution equations. Two classes of conditionally invariant second order systems of evolution equations are given, and their reduction to the systems of four ordinary differential equations is carried out.

1 Introduction

Modelling of dynamic processes in physics, chemistry and other fields of science requires solving evolution equations. Provided equations under study are linear, the methodology of constructing exact solutions is developed quite well. In the case of nonlinear equations, there are no general methods for finding their solutions. Among the most efficient methods for constructing exact solutions of nonlinear evolution equations are those based on their conditional symmetries [1, 2].

A number of Galaktionov's papers are devoted to constructing exact solutions of equations

$$u_t = F\left(u, u_x, u_{xx}\right), \qquad u_t = \frac{\partial u}{\partial t}, \qquad u_x = \frac{\partial u}{\partial x}, \qquad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

with quadratic nonlinearities. To this end the technique based on the concept of the invariant subspace [3] is employed. New approach to reduction of nonlinear evolution equations (1) using their higher symmetries was suggested in [4]. With the help of this approach, classification of evolution equations [5] and in accordance with results presented in [6] reduction of initial-value problem for them to Cauchy problem for system of ordinary differential equations (ODEs) [7] was carried out. A number of exact solutions of equation (1) with quadratic nonlinearities were obtained in [8] with the aid of ansatzes, being solutions of third-order linear ODEs.

In all the above mentioned papers the right-hand sides of equation (1) are quadratic polynomials or can be transformed to them by a certain change of variables. Classes of systems

$$u_t = u_{xx} + F\left(u, v, u_x, v_x\right), \qquad v_t = -v_{xx} + G\left(u, v, u_x, v_x\right),$$

admitting fourth-order symmetries, were described in [9]. F, G are fifth order polynomials.

In this paper we propose algorithm for construction of classes of systems of evolution equations, admitting conditional symmetries, and formulate classification and reduction theorems for systems of evolution equations, which are analogous to theorems, proved in [4]. With help of this algorithm we classify nonlinear equations

$$u_t = F\left(t, x, u, u_x, u_{xx}\right), \tag{2}$$

which admit reduction to systems of ODEs. To this end we consider these equations together with the condition

$$u_{xxx} = f\left(t, x, u, u_x, u_{xx}\right). \tag{3}$$

Equation (3) can be considered as an ODE with parameter t.

We also give examples for constructing of classes of conditionally invariant systems

$$u_t = u_{xx} + F\left(x, u, v, u_x, v_x\right), \qquad v_t = -v_{xx} + G\left(x, u, v, u_x, v_x\right) \tag{4}$$

and carry out their reduction to systems of four first-order ODEs.

2 Classification algorithm

Let us consider a system of partial differential equations (PDEs)

$$u_{it} = F_i\left(t, x, u_1, \dots, u_n\right) \tag{5}$$

under additional conditions for functions u_i

$$u_{ix} = f_i\left(t, x, u, \dots, u_n\right). \tag{6}$$

Here and henceforth we assume, unless otherwise specified, that $i = \overline{1, n}$.

Let f_i, F_i be continuously-differentiable functions of their arguments in some open domain Ω and $\bar{\boldsymbol{f}} \neq 0$ in any point of this domain, $u_i = u_i\left(t, x\right)$ are twice continuously-differentiable functions. Differentiating (5) with respect to x, (6) with respect to t and equating right-hand sides of obtained equalities we arrive at following compatibility condition for the system (5), (6)

$$F_{ix} + u_{1x} F_{i\,u_1} + \cdots + u_{nx} F_{i\,u_n} = f_{i\,t} + u_{1t} f_{i\,u_1} + \cdots + u_{n\,t} f_{i\,u_n}.$$

Taking into account (5), (6), we rewrite it in form

$$F_{ix} + f_1 F_{i\,u_1} + \cdots + f_n F_{i\,u_n} = f_{i\,t} + f_{i\,u_1} F_1 + \cdots + f_{i\,u_n} F_n. \tag{7}$$

By change of variables $\eta = x$, $\omega_i = \omega_i\left(t, x, u_1, \dots, u_n\right)$, where ω_i are first integrals of (6):

$$L\omega_i = \omega_{ix} + f_1 \omega_{iu_1} + \cdots + f_n \omega_{iu_n} = 0,$$

(7) is transformed to system

$$F_{i\eta} = g_{i0} + g_{i1} F_1 + \cdots + g_{in} F_n, \tag{8}$$

where $g_{ij}\left(t, \omega_1, \dots, \omega_n, \eta\right) = f_{iu_j}$, $g_{i0}\left(t, \omega_1, \dots, \omega_n, \eta\right) = f_{it}$.

By assumption that functions f_i are known and system (5), (6) is compatible, F_i must satisfy of linear system (8), that can be considered as an ODE with parameters $t, \omega_1, \dots, \omega_n$. Thus

$$F_i = \sum_{j=1}^{n} G_j\left(t, \omega_1, \dots, \omega_n\right) p_{ij}\left(\eta, t, \omega_1, \dots, \omega_n\right). \tag{9}$$

Here $\left(\bar{\boldsymbol{p}}_1, \dots, \bar{\boldsymbol{p}}_n\right)$ is a fundamental system of solutions of (7) and $G_1, \dots, G_n$ are arbitrary smooth functions.

Substituting general solutions of (6) into (5), (9) we obtain system of ODEs that is equivalent

$$\dot{C}_i\left(t\right) = g_i\left(t, , C_1\left(t\right), \dots, C_n\left(t\right)\right).$$

Now we consider the case that right-hand sides of equations (6) do not depend on t explicitly:

$$u_{ix} = f_i\left(x, u_1, \dots, u_n\right). \tag{10}$$

Then system (8) is homogenous ($f_{it} = 0$ and consequently $g_{i0} = 0$).

Theorem 1. *Let* $Q = \xi(x, u_1, \ldots, u_n)\partial_x + \sum\limits_{l=1}^{n} \varphi_l(x, u_1, \ldots, u_n)\partial_{u_l}$ *be symmetry operator of system (10) and in system (5)* $F_i = \varphi_i - \xi f_i$*. Then system (5), (10) is compatible. Here and in the sequel* $\partial_x = \frac{\partial}{\partial x}$, $\partial_{u_l} = \frac{\partial}{\partial u_l}$.

Proof. Since Q is a symmetry operator of system (10), then $\mathrm{Pr}^{(1)}Q\,(u_{ix} - f_i) = 0$, for $u_{ix} = f_i$.

$$\mathrm{Pr}^{(1)}Q = Q + \sum_{l=1}^{n} \varphi^l \partial_{u_{lx}}, \qquad \varphi^l = D_x(\varphi_l - \xi u_{lx}) + \xi u_{lxx}$$

is first prolongation of Q. D_x signifies total derivative with respect to x [10].

For system (10) we have

$$\mathrm{Pr}^{(1)}Q\,(u_{ix} - f_i) = \left[\xi\partial_x + \sum_{l=1}^{n}\varphi_l\partial_{u_l} + \sum_{l=1}^{n}\left(D_x(\varphi_l - \xi u_{lx}) + \xi u_{lxx}\right)\partial_{u_{lx}}\right](u_{ix} - f_i)$$
$$= D_x(\varphi_i - \xi u_{ix}) + \xi u_{ixx} - \xi f_{ix} - \sum_{l=1}^{n}\varphi_l f_{iu_l} = D_x(\varphi_i - \xi u_{ix}) + \xi D_x f_i - \xi f_{ix} - \sum_{l=1}^{n}\varphi_l f_{iu_l}$$
$$= D_x(\varphi_i - \xi u_{ix}) + \xi\sum_{l=1}^{n} f_{iu_l}u_{lx} - \sum_{l=1}^{n}\varphi_l f_{iu_l} = D_x(\varphi_i - \xi f_i) - \sum_{l=1}^{n}(\varphi_l - \xi f_l) f_{iu_l} = 0.$$

Hence $D_x(\varphi_i - \xi f_i) = \sum\limits_{l=1}^{n}(\varphi_l - \xi f_l) f_{iu_l}$, that is equivalent to (7), that is compatibility condition for the system (5), (6) ($f_{it} = 0$). ■

Theorem 2. *Let (10) admit* n *independent symmetry operators* $Q_1, \ldots, Q_n$*, where* $Q_j = \xi_j\partial_x + \sum\limits_{l=1}^{n}\varphi_{lj}\partial_{u_l}$*. Then functions* $\bar{\boldsymbol{P}}_j = \bar{\varphi}_{\mathbf{j}} - \xi_{\mathbf{j}}\bar{\boldsymbol{f}}$ *form fundamental system of solutions of (8)* ($g_{i0} = 0$) *and its general solution (compatibility condition of (5), (10)) has a form*

$$F_i = \sum_{j=1}^{n} G_j(t, \omega_1, \ldots, \omega_n)(\varphi_{ij} - \xi_j f_i), \tag{11}$$

where $\omega_i = \omega_i(x, u_1, \ldots, u_n)$*,* $G_1, \ldots, G_n$ *are arbitrary smooth functions and substitution of solution of (10)* $u_i = U_i(x, C_1(t), \ldots, C_n(t))$ *in (5), (11) gives following system of ODEs*

$$\dot{C}_i = \sum_{j=1}^{n} G_j(t, C_1, \ldots, C_n)\, g_j(C_1, \ldots, C_n) = \sum_{j=1}^{n} G_j Q_j(\omega_i)\left.\right|_{u_i = U_i}. \tag{12}$$

Proof. The first assertion of theorem (condition (11)) follows from Theorem 1 and fact, that

$$D_x\omega_i = \omega_{ix} + \sum_{l=1}^{n}\omega_{iu_l}u_{lx} = \omega_{ix} + \sum_{l=1}^{n}\omega_{iu_l}f_l = L\omega_i = 0.$$

Let us prove (12). By assumption that right-hand side (10) does not vanish anywhere in Ω, this system has n independent first integrals, hence $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{\omega}}$ are mutually inverse functions. Thus, if we substitute $u_i = u_i(x, \omega_1, \ldots, \omega_n)$ in (5) and differentiate obtained equalities with respect to t, then we have the system

$$\sum_{j=1}^{n}\frac{\partial u_i}{\partial\omega_j}D_t\omega_j = \sum_{j=1}^{n} G_j(\varphi_{ij} - \xi_j f_i). \tag{13}$$

$\det \left\| \frac{\partial u_i}{\partial \omega_j} \right\| \neq 0$, because in the opposite case functions $u_1, \ldots, u_n$ are linearly dependent and number of independent first integrals are smaller than n. Thus, system (13) as a system of linear algebraic equations has the unique solution

$$\overline{D_t \omega} = \left\| \frac{\partial u_i}{\partial \omega_j} \right\|^{-1} \sum_{j=1}^{n} G_j \overline{P_j}.$$

According to inverse function theorem, $\left\| \frac{\partial u_i}{\partial \omega_j} \right\|^{-1} = \left\| \frac{\partial \omega_i}{\partial u_j} \right\|$ and consequently this solution can be rewritten component-wise as follows:

$$\begin{aligned} D_t \omega_i &= \sum_{j=1}^{n} G_j \sum_{l=1}^{n} \left(\varphi_{lj} - \xi_j f_l \right) \omega_{iu_l} = \sum_{j=1}^{n} G_j \sum_{l=1}^{n} \left(\varphi_{lj} \omega_{iu_l} - \xi_j f_l \omega_{iu_l} \right) \\ &= \sum_{j=1}^{n} G_j \left(\sum_{l=1}^{n} \varphi_{lj} \omega_{iu_l} + \xi_j \omega_{ix} \right) = \sum_{j=1}^{n} G_j Q_j \left(\omega_i \right). \end{aligned} \tag{14}$$

The same result we obtain by immediate differentiating $\omega_i \left(x, u_1, \ldots, u_n \right)$ with respect to t in consideration of (11). Taking into account, that $D_x D_t \omega_i = D_t D_x \omega_i = 0$, we conclude that right-hand side of (14) does not depend on x explicitly. After that, to complete proof, we change $u_1, \ldots, u_n$ for $U_1, \ldots, U_n$ taking into account, that

$$C_i \left(t \right) = \omega_i \left(x, U_1, \ldots, U_n \right), \qquad \dot{C}_i = D_t \omega_i \left(x, U_1, \ldots, U_n \right). \qquad \blacksquare$$

Thus, we formulate the following algorithm for constructing of classes of conditionally invariant systems of evolution equations and their reduction to systems of ODEs:

- calculate symmetry algebra of equation (10);
- find its first integrals;
- integrate (10) (if (10) admit n-parametric solvable symmetry algebra then it can be integrated in quadratures [10]);
- determine F_i by formula (11);
- write system of ODEs (12) for functions $C_1 \left(t \right), \ldots, C_n \left(t \right)$.

3 Classification and reduction of equations (2)

Now we go to the problem classification of equations (2), which are conditionally invariant under condition (3). First we consider auxiliary systems

$$u_t = F \left(t, x, u, v, w \right), \qquad v_t = G \left(t, x, u, v, w \right), \qquad w_t = H \left(t, x, u, v, w \right); \tag{15}$$

$$u_x = v, \qquad v_x = w, \qquad w_x = f \left(t, x, u, v, w \right). \tag{16}$$

Note, that system (16) is equivalent (3). Compatibility condition of the given system is

$$\begin{aligned} &F_x + vF_u + wF_v + fF_w = G, \qquad G_x + vG_u + wG_v + fG_w = H, \\ &H_x + vH_u + wH_v + fH_w = f_t + f_u F + f_v G + f_w H. \end{aligned}$$

First integrals of systems (16) are functionally independent solutions of the equation

$$\omega_{ix} + v\omega_{iu} + w\omega_{iv} + f\omega_{iw} = 0.$$

If (15), (16) is compatible, then F, G, H satisfy the system

$$F_\eta = G, \qquad G_\eta = H, \qquad H_\eta = f_t + f_u F + f_v G + f_w H,$$

that is equivalent equation

$$F_{\eta\eta\eta} - f_{u_{xx}} F_{\eta\eta} - f_{u_x} F_\eta - f_u F = f_t. \tag{17}$$

A solution of linear equation (17) has the form $F = F^g + F^p$, where F^p is a partial solution of equation (17) and

$$F^g = G_1 p_1 + G_2 p_2 + G_3 p_3, \qquad G_j = G_j\left(t, \omega_1, \omega_2, \omega_3\right), \qquad p_j = p_j\left(\eta, \omega_1, \omega_2, \omega_3\right)$$

is the general solution of corresponding homogeneous equation.

Thus, having solved (17) we obtain in the explicit form the function F, for which system (2), (3) is compatible. According to theorem, proved in [4], substitution of ansatz which is a solution of equation (3), into (2), reduces (2) to a system of three ODEs.

Assertion analogous to Theorem 2 can be formulated for equations

$$u_t = F\left(t, x, u, u^{(1)}, \ldots, u^{(n-1)}\right), \tag{18}$$

$$u^{(n)} = f\left(x, u, u^{(1)}, \ldots, u^{(n-1)}\right), \tag{19}$$

where $u^{(i)} = \frac{\partial^i u}{\partial x^i}$, $F \in C^{n+1}(\Omega)$, $f \in C^1(\Omega)$, $\Omega \subset \mathbb{R}^{n+1}$, $u = u(x,t) \in C^{n+1}(\Omega')$, $\Omega' \subset \mathbb{R}^2$.

Theorem 3. *Let $\omega_1, \ldots, \omega_n$ be first integrals and $Q_1, \ldots, Q_n$ be independent symmetry operators of equations (19): $Q_j = \xi_j(x,u)\,\partial_x + \varphi_j(x,u)\,\partial_u$. If*

$$F = \sum_{j=1}^{n} G_j\left(t, \omega_1, \ldots, \omega_n\right)\left(\varphi_j - \xi_j u_x\right), \tag{20}$$

G_j are arbitrary sufficiently smooth functions, then system (18), (19) is compatible and substitution of general solution (19) $u = U\left(x, C_1(t), \ldots, C_n(t)\right)$ in (18) reduces it to system of ODEs

$$\dot{C}_i = \sum_{j=1}^{n} G_j\left(t, C_1, \ldots, C_n\right) g_j\left(C_1, \ldots, C_n\right) = \sum_{j=1}^{n} G_j \mathrm{Pr}^{(n-1)} Q_j\left(\omega_i\right)|_{u=U}\ .$$

Proof follows from the fact that if Q is symmetry operator of (19), then $\mathrm{Pr}^{(n-1)}Q\,\big|_{u^{(i)}=u_i}$ is symmetry operator of system

$$u_{1x} = u_2, \quad u_{2x} = u_3, \quad \ldots, \quad u_{nx} = f\left(x, u_1, \ldots, u_n\right), \quad \text{where} \quad u_1 = u.$$

We apply obtained result for classification and reduction equations (2) under additional condition

$$u_{xxx} = f\left(x, u, u_x, u_{xx}\right). \tag{21}$$

It is well-known that solution of third order ODE admitting three-parametrical solvable symmetry group can be constructed in quadratures. Using normal forms of ODEs (21), which admit three-parametrical solvable symmetry algebrae, we constructed (by formula (20)) nine classes of evolution equations (2) that are conditionally invariant under these types of (21). We also reduced obtained classes of evolution equations to system of three ODEs. Here we do not adduce these results, as they is cumbersome. This problem will be considered in further papers.

4 Examples of reduction of systems of evolution equations

Consider system

$$u_t = F\left(t, x, u, v, u_x, v_x\right), \qquad v_t = G\left(t, x, u, v, u_x, v_x\right) \tag{22}$$

under additional conditions

$$u_{xx} = f\left(x, u, v, u_x, v_x\right), \qquad v_{xx} = g\left(x, u, v, u_x, v_x\right). \tag{23}$$

We apply described procedure and obtain following system for determining functions F, G

$$F_{\eta\eta} - f_{u_x}F_\eta - f_{v_x}G_\eta - f_uF - f_vG = 0, \qquad G_{\eta\eta} - g_{u_x}F_\eta - g_{v_x}G_\eta - g_uF - g_vG = 0.$$

Theorem 4. *Let $\omega_j = \omega_j\left(x, u, v, u_x, v_x\right)$ are first integrals and*

$$Q_j = \xi_j\left(x, u, v\right)\partial_x + \varphi_j\left(x, u, v\right)\partial_u + \psi_j\left(x, u, v\right)\partial_v$$

are independent symmetry operators of system (23), $j = \overline{1, 4}$. If

$$F = \sum_{j=1}^{4} R_j\left(t, \omega_1, \ldots, \omega_4\right)\left(\varphi_j - \xi_j u_x\right), \qquad G = \sum_{j=1}^{4} R_j\left(t, \omega_1, \ldots, \omega_4\right)\left(\psi_j - \xi_j v_x\right),$$

R_j are arbitrary twice continuously-differentiable function, then system (22), (23) is compatible and substituting of solutions of (23) $u = U\left(x, C_1\left(t\right), \ldots, C_4\left(t\right)\right)$, $v = V\left(x, C_1\left(t\right), \ldots, C_4\left(t\right)\right)$ into (22) reduces it to system of four ODE

$$\dot{C}_i = \sum_{j=1}^{4} R_j\left(t, C_1, \ldots, C_4\right) g_j\left(C_1, \ldots, C_4\right) = \sum_{j=1}^{4} R_j \mathrm{Pr}^{(1)} Q_j\left(\omega_i\right)|_{u=U, v=V}, \quad i = \overline{1, 4}.$$

Proof of Theorem 4 is analogous to proof of Theorem 3 ((23) can be changed into equivalent first-order system).

Remark 1. For a system (4), (23) compatibility condition is

$$F = \sum_{j=1}^{4} R_j\left(t, \omega_1, \ldots, \omega_4\right)\left(\varphi_j - \xi_j u_x\right) - f\left(x, u, v, u_x, v_x\right),$$

$$G = \sum_{j=1}^{4} R_j\left(t, \omega_1, \ldots, \omega_4\right)\left(\psi_j - \xi_j v_x\right) + g\left(x, u, v, u_x, v_x\right).$$

There are more than sixty nonequivalent classes of systems (23) with four dimensional solvable symmetry algebras. Here we give some examples of application of exposed algorithm to construction and reduction of classes of systems (4). We write systems (23), their symmetries, general solutions and first integrals, functions F, G and reduced systems.

1. $u_{xx} = \xi''\left(x\right)\ln v_x + f\left(x\right), \qquad v_{xx} = g'\left(x\right)v_x,$

$Q_1 = \partial_u, \quad Q_2 = \partial_v, \qquad Q_3 = x\partial_u, \qquad Q_4 = \xi\left(x\right)\partial_u + v\partial_v,$

$u = \ln C_1\left(t\right)\xi\left(x\right) + \int^x\!\!\int^z \left(\xi''\left(y\right)g\left(y\right) + f\left(y\right)\right)dydz + C_3\left(t\right)x + C_4\left(t\right),$

$$v = C_1(t)\int^x e^{g(y)}dy + C_2(t), \qquad \omega_1 = v_x e^{-g(x)}, \qquad \omega_2 = v - \omega_1 \int^x e^{g(y)}dy,$$

$$\omega_3 = u_x - \xi'(x)(\ln v_x - g(x)) - \int^x \left(\xi''(y)g(y) + f(y)\right)dy,$$

$$\omega_4 = u - x\omega_3 - \xi(x)(\ln v_x - g(x)) - \int^x\!\!\int^z \left(\xi''(y)g(y) + f(y)\right)dydz,$$

$$F = \xi(x)R_4 + xR_3 + R_1 - \xi''(x)\ln v_x - f(x), \qquad G = vR_4 + R_2 + g'(x)v_x,$$

$$\dot{C}_1 = C_1R_4, \qquad \dot{C}_2 = C_2R_4 + R_2, \qquad \dot{C}_3 = R_3, \quad \dot{C}_4 = R_1.$$

2. $u_{xx} = x^{-1}f_x(v_x) + x^{-1}\ln x g(v_x), \qquad v_{xx} = x^{-1}g(v_x),$

$$Q_1 = \partial_u, \qquad Q_2 = \partial_v, \qquad Q_3 = x\partial_u, \quad Q_4 = x\partial_x + (u+v)\partial_u + v\partial_v,$$

$$u = \frac{1}{C_1(t)}\int^{C_1(t)x}\!\!\int^z \frac{f\left(H^{-1}(y)\right) + \ln y g\left(H^{-1}(y)\right)}{y}dydz - \frac{\ln C_1(t)}{C_1(t)}\int^{C_1(t)x} H^{-1}(y)\,dy$$

$$+ C_3(t)x + C_4(t), \qquad v = \frac{1}{C_1(t)}\int^{C_1(t)x} H^{-1}(y)\,dy + C_2(t), \qquad H(y) = e^{\int^y \frac{dz}{g(z)}},$$

$$\omega_1 = \frac{H(v_x)}{x}, \qquad \omega_2 = v - \frac{1}{\omega_1}\int^{H(v_x)} H^{-1}(y)\,dy,$$

$$\omega_3 = u_x - \int^{H(v_x)} \frac{f\left(H^{-1}(y)\right) + \ln y g\left(H^{-1}(y)\right)}{y}dy + \frac{\ln(H(v_x))}{x}v_x,$$

$$\omega_4 = u - x\omega_3 - \frac{x}{H(v_x)}\int^{H(v_x)}\!\!\int^z \frac{f\left(H^{-1}(y)\right) + \ln y g\left(H^{-1}(y)\right)}{y}dydz$$

$$+ \frac{x}{F(v_x)}\ln\left(\frac{H(v_x)}{x}\right)\int^{H(v_x)} F^{-1}(y)\,dy,$$

$$F = (u + v - xu_x)R_4 + xR_3 + R_1 - x^{-1}f_x(v_x) + x^{-1}\ln x g(v_x),$$

$$G = (v - xv_x)R_4 + R_2 + x^{-1}g(v_x),$$

$$\dot{C}_1 = -C_1R_3, \qquad \dot{C}_2 = C_2R_4 + R_2, \qquad \dot{C}_3 = R_3, \qquad \dot{C}_4 = (C_2 + C_4)R_4 + R_1.$$

In conclusion we note, that Theorem 4 can be easy generalized for classification and reduction of system of evolution equations

$$u_{it} = F_i\left(t, x, u_1, \ldots, u_n, u_1^{(1)}, \ldots, u_n^1, \ldots, u_1^{(k-1)}, \ldots, u_n^{(k-1)}\right)$$

under additional conditions

$$u_i^{(k)} = f_i\left(x, u_1, \ldots, u_n, u_1^{(1)}, \ldots, u_n^1, \ldots, u_1^{(k-1)}, \ldots, u_n^{(k-1)}\right),$$

admitting kn independent symmetries. Here $u_i^{(j)} = \frac{\partial^j u_i}{\partial x^j}$.

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Symmetry and Exact Solutions for Systems of Nonlinear Reaction-Diffusion Equations

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Lie symmetry reduction of systems of nonlinear reaction-diffusion equation with respect to one-dimensional algebras is carried out. Some classes of exact solutions of the investigated equations are found.

1 Introduction

Nonlinear reaction-diffusion equations are widely used in mathematical physics, chemistry and biology. In the present paper we consider the system of nonlinear diffusion equations of the following general form

$$\begin{aligned}
&\frac{\partial u_1}{\partial t} - \frac{\partial^2}{\partial x^2}(a_{11}u_1 + a_{12}u_2) = f^1(u_1, u_2),\\
&\frac{\partial u_2}{\partial t} - \frac{\partial^2}{\partial x^2}(a_{21}u_1 + a_{22}u_2) = f^2(u_1, u_2),
\end{aligned} \tag{1}$$

where u_1 and u_2 are functions dependent on t and x; a_{11}, a_{12}, a_{21}, a_{22} are constant parameters and $a_{11}a_{22} - a_{21}a_{12} \neq 0$.

In [1] a constructive algorithm was proposed for investigation of conditional and classical Lie symmetries of partial differential equations and classical symmetries of systems of two nonlinear diffusion equations with $1 + m$ independent variables $t, x_1, \ldots, x_m$ were described. Namely, all possible non-linearities f^1, f^2 and the corresponding group generators were found. We notice that symmetry properties of nonlinear multidimensional systems of reaction-diffusion equations were also investigated in papers [2, 3]. In the present paper using the results obtained in [1] we carry out symmetry reduction of equation (1) with respect to one-dimensional symmetry algebras. We restrict ourselves to such non-linearities f^1 and f^2 found in [1] which are defined up to arbitrary functions.

2 Symmetry reduction of equation (1)

We will not give the detailed calculations but present the operators, ansatzes and corresponding reduced systems for some nonlinearities f^1, f^2 found in [1, 3]. We use the following notation:

$$X_0 = \alpha\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial x}, \qquad D_1 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \frac{2}{k}\hat{B}, \qquad \hat{B} = B^{ab}u_b\frac{\partial}{\partial u_a},$$
$$D_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \frac{2}{k}\left(\frac{\partial}{\partial u_1} - 2nu_1\frac{\partial}{\partial u_2}\right), \qquad D_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \frac{2}{k}p_\alpha\frac{\partial}{\partial u_\alpha},$$

where α and β are arbitrary real coefficients, B^{ab} are elements of the 2×2 matrix B which will be specified in the following.

1. Consider the following system of type (1)

$$\frac{\partial u_1}{\partial t} - a\frac{\partial^2 u_1}{\partial x^2} = \exp\left(k\frac{u_2}{u_1}\right)\varphi_1 u_1,$$
$$\frac{\partial u_2}{\partial t} - b\frac{\partial^2 u_1}{\partial x^2} - a\frac{\partial^2 u_2}{\partial x^2} = \exp\left(k\frac{u_2}{u_1}\right)(\varphi_1 u_2 + \varphi_2), \tag{2}$$

where φ_1 and φ_2 are arbitrary (but fixed) functions of u_1, $a_{11} = a_{22} = a$, $a_{12} = 0$, $a_{21} = b$.

This system admits the symmetry operator

$$X = X_0 + \nu D_1, \qquad \text{where} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The corresponding ansatz be obtained using the Lie algorithms is

$$u_1 = \omega_1(z), \qquad u_2 = -\frac{2}{k}\ln(\nu x + \beta)\omega_1(z) + \omega_2(z), \qquad z = \frac{2(\nu x + \beta)^2}{2\nu t + \alpha}. \tag{3}$$

Substituting the ansatz (3) into (2) we come to the following reduced equations

$$2\nu z^2\dot{\omega}_1 + 2\nu^2 az\dot{\omega}_1 + 8\nu^2 az^2\ddot{\omega}_1 = -\exp\left(k\frac{\omega_2}{\omega_1}\right)\varphi_1\omega_1,$$
$$2\nu z^2\dot{\omega}_2 + \frac{2\nu^2 a}{k}\dot{\omega}_1 - \frac{8\nu^2 a}{k}z\dot{\omega}_1 + 2\nu^2 bz\dot{\omega}_1 + 2\nu^2 az\dot{\omega}_2 + 8\nu^2 bz^2\ddot{\omega}_1 + 8\nu^2 az^2\ddot{\omega}_2$$
$$= -\exp\left(k\frac{\omega_2}{\omega_1}\right)(\varphi_1\omega_2 + \varphi_2).$$

In other words the ansatz (3) reduces (2) to the system of ordinary differential equations.

The following results (related to equations found in [1]) are presented more briefly.

2. Equations:

$$\frac{\partial u_1}{\partial t} - a\frac{\partial^2 u_1}{\partial x^2} + b\frac{\partial^2 u_2}{\partial x^2} = \varphi_1 u_2 + \varphi_2 u_1, \qquad \frac{\partial u_2}{\partial t} - b\frac{\partial^2 u_1}{\partial x^2} - a\frac{\partial^2 u_2}{\partial x^2} = -\varphi_1 u_1 + \varphi_2 u_2,$$

where φ_1 and φ_2 are arbitrary functions of $\sqrt{u_1^2 + u_2^2}$, $a_{11} = a_{22} = a$, $a_{21} = -a_{12} = b$.

Symmetry:

$$X = X_0 + \mu\hat{B}, \qquad \text{where} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Ansatz:

$$u_1 = \cos\left(\frac{\mu}{\alpha}t\right)\omega_1(z) - \sin\left(\frac{\mu}{\alpha}t\right)\omega_2(z), \qquad u_2 = \sin\left(\frac{\mu}{\alpha}t\right)\omega_1(z) + \cos\left(\frac{\mu}{\alpha}t\right)\omega_2(z),$$
$$z = \beta t - \alpha x.$$

Reduced equations:

$$-\frac{\mu}{\alpha}\omega_2 + \beta(a\dot{\omega}_1 - b\dot{\omega}_2) - \alpha^2(a\ddot{\omega}_1 - b\ddot{\omega}_2) = \varphi_1\omega_2 + \varphi_2\omega_1,$$
$$\frac{\mu}{\alpha}\omega_1 + \beta(b\dot{\omega}_1 + a\dot{\omega}_2) - \alpha^2(b\ddot{\omega}_1 + a\ddot{\omega}_2) = -\varphi_1\omega_1 + \varphi_2\omega_2,$$

where φ_1 and φ_2 are functions of $\omega_1^2 + \omega_2^2$.

3. Equations:

$$\frac{\partial u_1}{\partial t} - a\frac{\partial^2 u_1}{\partial x^2} = u_1\varphi_1, \qquad \frac{\partial u_2}{\partial t} - b\frac{\partial^2 u_2}{\partial x^2} = u_2\varphi_2,$$

where φ_1 and φ_2 are arbitrary functions of $\frac{u_2}{u_1^d}$, $a_{11} = a$, $a_{12} = a_{21} = 0$, $a_{22} = b$.

Symmetry:

$$X = X_0 + \mu\hat{B}, \qquad \text{where} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}.$$

Ansatz:

$$u_1 = \exp\left(\frac{\mu}{\beta}x\right)\omega_1(z), \qquad u_2 = \exp\left(\frac{\mu d}{\beta}x\right)\omega_2(z), \qquad z = \beta t - \alpha x.$$

Reduced equations:

$$\beta\dot{\omega}_1 - a\left(\frac{\mu}{\beta}\right)^2\omega_1 + 2\alpha a\frac{\mu}{\beta}\dot{\omega}_1 - \alpha^2 a\ddot{\omega}_1 = \omega_1\varphi_1,$$
$$\beta\dot{\omega}_2 - b\left(\frac{\mu d}{\beta}\right)^2\omega_2 + 2\alpha b\frac{\mu d}{\beta}\dot{\omega}_2 - \alpha^2 b\ddot{\omega}_2 = \omega_2\varphi_2,$$

where φ_1 and φ_2 are functions of $\frac{\omega_2}{\omega_1^d}$.

4. Equation:

$$\frac{\partial u_1}{\partial t} - a\frac{\partial^2 u_1}{\partial x^2} = \varphi_1, \qquad \frac{\partial u_2}{\partial t} - b\frac{\partial^2 u_1}{\partial x^2} - a\frac{\partial^2 u_2}{\partial x^2} = \frac{u_2}{u_1}\varphi_1 + nu_2 + \varphi_2,$$

where φ_1 and φ_2 are arbitrary functions of u_1, $a_{11} = a_{22} = a$, $a_{12} = 0$, $a_{21} = b$.

Symmetry:

$$X = X_0 + \mu\exp(nt)\hat{B}, \qquad \text{where} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Ansatz:

$$u_1 = \omega_1(z), \qquad u_2 = \frac{\mu}{\alpha n}\omega_1(z)\exp(nt) + \omega_2(z), \qquad z = \beta t - \alpha x.$$

Reduced equations:

$$\beta\dot{\omega}_1 - \alpha^2 a\ddot{\omega}_1 = \varphi_1,$$
$$\beta\dot{\omega}_2 - \alpha^2 b\ddot{\omega}_1 - \alpha a\ddot{\omega}_2 = \frac{\omega_2}{\omega_1}\varphi_1 + n\omega_2 + \varphi_2,$$

where φ_1 and φ_2 are functions of ω_1.

5. Equation:

$$\frac{\partial u_1}{\partial t} - a\frac{\partial^2 u_1}{\partial x^2} = \varphi_1 u_1^{k+1}, \qquad \frac{\partial u_2}{\partial t} - b\frac{\partial^2 u_1}{\partial x^2} - a\frac{\partial^2 u_2}{\partial x^2} = (\varphi_1\ln u_1 + \varphi_2)u_1^{k+1},$$

where φ_1 and φ_2 are arbitrary functions of $u_1\exp\left(-\frac{u_2}{u_1}\right)$, $a_{11} = a_{22} = a$, $a_{12} = 0$, $a_{21} = b$.

Symmetry:

$$X = X_0 + \nu D_1, \qquad \text{where} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Ansatz:

$$u_1 = (2\nu t+\alpha)^{-\frac{1}{k}}\omega_1(z), \qquad u_2 = (2\nu t+\alpha)^{-\frac{1}{k}}\left(\omega_2(z) - \frac{1}{k}\ln(2\nu t+\alpha)\omega_1(z)\right),$$
$$z = \frac{2(\nu x+\beta)^2}{2\nu t+\alpha}.$$

Reduced equations:

$$\frac{2\nu}{k}\omega_1 + 2\nu z\dot{\omega}_1 + 8\nu^2 az\ddot{\omega}_1 = -\omega_1^{k+1}\varphi_1,$$
$$\frac{2\nu}{k}\omega_2 + 2\nu z\dot{\omega}_2 + \frac{2\nu}{k}\omega_1 + 8\nu^2 bz\ddot{\omega}_1 + 8\nu^2 az\ddot{\omega}_2 = -(\varphi_1\ln\omega_1+\varphi_2)\omega_1^{k+1},$$

where φ_1 and φ_2 are functions of $\omega_1\exp\left(-\frac{\omega_2}{\omega_1}\right)$.

3 Conditional symmetry and exact solutions

Thus we presented reductions of equations (1) using their classical symmetry found in [1]. In this section we present exact solutions of equations (1) found by conditional symmetry reduction. We use the same scheme of presentation as in Section 2.

1. Equation:

$$\frac{\partial u_1}{\partial t} - \frac{\partial^2 u_1}{\partial x^2} = u_1^3\varphi_1, \qquad \frac{\partial u_2}{\partial t} - \frac{\partial^2 u_2}{\partial x^2} = u_2^3\varphi_2, \tag{4}$$

where φ_1 and φ_2 are arbitrary functions of $\frac{u_2}{u_1}$.

Conditional symmetry:

$$X = \frac{\partial}{\partial t} - \frac{3}{x+k_1}\frac{\partial}{\partial x} - \frac{3}{(x+k_1)^2}\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right).$$

The ansatz

$$u = (x+k_1)\omega(z), \qquad z = \frac{1}{2}x^2 + k_1x + 3t$$

reduces equation (4) to the system:

$$\ddot{\omega}_1 + \varphi_1\omega_1^3 = 0, \qquad \ddot{\omega}_2 + \varphi_2\omega_2^3 = 0,$$

where φ_1 and φ_2 are functions of $\frac{\omega_2}{\omega_1}$.

Depending on the form of the functions φ_1, φ_2, we receive different solutions of the system.

1) $\varphi_1 = a > 0$, $\varphi_2 = b < 0$, where a and b are constants:

$$u_1(x,t) = \frac{\sqrt{2a}}{2a}(x+k_1)\,\mathrm{sd}\left(\frac{1}{2}x^2+k_1x+3t; \frac{1}{2}\sqrt{2}\right),$$
$$u_2(x,t) = -\frac{\sqrt{-2b}}{b}(x+k_1)\,\mathrm{ds}\left(\frac{1}{2}x^2+k_1x+3t; \frac{1}{2}\sqrt{2}\right).$$

2) $\varphi_1 = a > 0$, $\varphi_2 = 0$:

$$u_1(x,t) = \frac{\sqrt{2a}}{2a}(x+k_1)\,\mathrm{sd}\left(\frac{1}{2}x^2+k_1x+3t; \frac{1}{2}\sqrt{2}\right),$$
$$u_2(x,t) = (x+k_1)\left[\left(\frac{1}{2}x^2+k_1x+3t\right)C_1 + C_2\right].$$

2. Equation:

$$\frac{\partial u_1}{\partial t} - \frac{\partial^2 u_1}{\partial x^2} = u_1^3\varphi_1 - 2\mu^2 u_1, \qquad \frac{\partial u_2}{\partial t} - \frac{\partial^2 u_2}{\partial x^2} = u_2^3\varphi_2 - 2\mu^2 u_2, \tag{5}$$

where φ_1 and φ_2 are arbitrary functions of $\frac{u_2}{u_1}$.

Conditional symmetry:

$$X = \frac{\partial}{\partial t} + 3\mu\tan(\mu x + k_1)\frac{\partial}{\partial x} - 3\mu^2\sec^2(\mu x + k_1)\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right).$$

The ansatz

$$u = \cos(\mu x + k_1)\exp\left(-3\mu^2 t\right)\omega(z), \qquad z = \sin(\mu x + k_1)\exp(-3\mu^2 t)$$

reduces equation (5) to the system:

$$\mu^2\ddot{\omega}_1 + \omega_1^3\varphi_1 = 0, \qquad \mu^2\ddot{\omega}_2 + \omega_2^3\varphi_2 = 0,$$

where φ_1 and φ_2 are functions of $\frac{\omega_2}{\omega_1}$.

Setting more particular form for the functions φ_1, φ_2, we get the following solutions of the reduced system.

1) $\varphi_1 = a > 0$, $\varphi_2 = b > 0$:

$$u_1(x,t) = \frac{\mu\sqrt{2a}}{2a}\cos(\mu x + k_1)\exp\left(-3\mu^2 t\right)\operatorname{sd}\left[\sin(\mu x + k_1)\exp\left(-3\mu^2 t\right); \frac{1}{2}\sqrt{2}\right],$$

$$u_2(x,t) = \frac{\mu\sqrt{2b}}{2b}\cos(\mu x + k_1)\exp\left(-3\mu^2 t\right)\operatorname{sd}\left[\sin(\mu x + k_1)\exp\left(-3\mu^2 t\right); \frac{1}{2}\sqrt{2}\right].$$

2) $\varphi_1 = a < 0$, $\varphi_2 = b > 0$:

$$u_1(x,t) = -\frac{\mu\sqrt{-2a}}{a}\cos(\mu x + k_1)\exp\left(-3\mu^2 t\right)\operatorname{ds}\left[\sin(\mu x + k_1)\exp\left(-3\mu^2 t\right); \frac{1}{2}\sqrt{2}\right],$$

$$u_2(x,t) = \frac{\mu\sqrt{2b}}{2b}\cos(\mu x + k_1)\exp\left(-3\mu^2 t\right)\operatorname{sd}\left[\sin(\mu x + k_1)\exp\left(-3\mu^2 t\right); \frac{1}{2}\sqrt{2}\right].$$

3. Equation:

$$\frac{\partial u_1}{\partial t} - \frac{\partial^2 u_1}{\partial x^2} = u_1^3\varphi_1 + 2\mu^2 u_1, \qquad \frac{\partial u_2}{\partial t} - \frac{\partial^2 u_2}{\partial x^2} = u_2^3\varphi_2 + 2\mu^2 u_2, \tag{6}$$

where φ_1 and φ_2 are arbitrary functions of $\frac{u_2}{u_1}$.

Conditional symmetry:

$$X = \frac{\partial}{\partial t} - 3\mu\coth(\mu x + k_1)\frac{\partial}{\partial x} - 3\mu^2\csc h^2(\mu x + k_1)\left(u_1\frac{\partial}{\partial u_1} + u_2\frac{\partial}{\partial u_2}\right).$$

The ansatz

$$u = \sinh(\mu x + k_1)\exp\left(3\mu^2 t\right)\omega(z), \qquad z = \cosh(\mu x + k_1)\exp\left(3\mu^2 t\right)$$

reduces equation (6) to the system:

$$\mu^2\ddot{\omega}_1 + \omega_1^3\varphi_1 = 0, \qquad \mu^2\ddot{\omega}_2 + \omega_2^3\varphi_2 = 0,$$

where φ_1 and φ_2 are functions of $\frac{\omega_2}{\omega_1}$.

We present the obtained results for some functions φ_1 and φ_2.

1) $\varphi_1 = a < 0$, $\varphi_2 = b < 0$:

$$u_1(x,t) = -\frac{\mu\sqrt{-2a}}{a}\sinh(\mu x + k_1)\exp\left(3\mu^2 t\right)\mathrm{ds}\left[\cosh(\mu x + k_1)\exp\left(3\mu^2 t\right); \frac{1}{2}\sqrt{2}\right],$$
$$u_2(x,t) = -\frac{\mu\sqrt{-2b}}{b}\sinh(\mu x + k_1)\exp\left(3\mu^2 t\right)\mathrm{ds}\left[\cosh(\mu x + k_1)\exp\left(3\mu^2 t\right); \frac{1}{2}\sqrt{2}\right].$$

2) $\varphi_1 = 0$, $\varphi_2 = b > 0$:

$$u_1(x,t) = \sinh(\mu x + k_1)\exp\left(3\mu^2 t\right)\left[C_1\cosh(\mu x + k_1)\exp\left(3\mu^2 t\right) + C_2\right],$$
$$u_2(x,t) = \frac{\mu\sqrt{2b}}{2b}\sinh(\mu x + k_1)\exp\left(3\mu^2 t\right)\mathrm{sd}\left[\cosh(\mu x + k_1)\exp\left(3\mu^2 t\right); \frac{1}{2}\sqrt{2}\right].$$

Besides for equation

$$u_t - u_{xx} = -u^2,$$

we got the following solutions

$$u = \frac{(48 - 12\sqrt{6})x^2 + (48 - 12\sqrt{6})k_1 x + 40(36 - 15\sqrt{6})t + (24 - 12\sqrt{6})k_2 + 6k_1^2}{[x^2 + k_1 x + 2(15 - 5\sqrt{6})t + k_2]^2},$$

and

$$u = \frac{(48 + 12\sqrt{6})x^2 + (48 + 12\sqrt{6})k_1 x + 40(36 + 15\sqrt{6})t + (24 + 12\sqrt{6})k_2 + 6k_1^2}{[x^2 + k_1 x + 2(15 + 5\sqrt{6})t + k_2]^2}.$$

Thus we presented reduced equations and exact solutions for some of nonlinear reaction-diffusion equations whose symmetry was studied in [1, 3]. We plan to extend our results to all systems described in [3].

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Group Classification of Nonlinear Partial Differential Equations: a New Approach to Resolving the Problem

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We describe a systematic procedure for classifying partial differential equations which are invariant with respect to low-dimensional Lie algerbas. This procedure is a synthesis of the infinitesimal Lie method, the technique of equivalence transformations and the theory of classification of abstract low-dimensional Lie algebras. By way of illustration, we consider three examples of group classification of partial differential equations in new approach.

1 Introduction

This article is based on two talks (one given by each of the authors) at the Fourth International Conference "Symmetry in Nonlinear Mathematical Physics" (9–14 July, 2001, Kyiv, Ukraine). More details can be found in [9] and [16], and a short description is given in [17].

The analysis and classification of differential equations using group theory goes back to Sophus Lie. The first systematic investigation of the problem of group classification was done by L.V. Ovsiannikov [1] in 1959 for nonlinear heat equation

$$u_t = [f(u)u_x]_x,$$

where $f(u)$ is an arbitrary nonlinearity. His approach is based on the concept of the equivalence group, which is the Lie transformation group (acting in the space whose local coordinates are independent variables, the functions and their derivatives) preserving the class of particular differential equations under study. It is possible to modify Lie's algorithm in order to make it applicable for the computation of this group (see, e.g., [2]). Having obtained the equivalence group one constructs the optimal system of subgroups of the equivalence group. The last step uses Lie's algorithm for obtaining specific partial differential equations that (a) belong to the class under study, and (b) are invariant with respect to these subgroups.

This approach has been applied to a number of equations of mathematical physics. Here we mention just a few of the papers in which the group classification of nonlinear heat equations has been studied:

Akhatov, Gazizov, Ibragimov (1987, [3])

$$u_t = G(u_x)u_{xx};$$

Dorodnitsyn (1982, [4])

$$u_t = G(u)u_{xx} + \frac{dG(u)}{du}u_x^2 + g(u);$$

Oron, Rosenau (1986, [5]), Edwards (1994, [6])

$$u_t = G(u)u_{xx} + \frac{dG(u)}{du}u_x^2 + f(u)u_x;$$

Cherniha and Serov (1998, [7])

$$u_t = G(u)u_{xx} + \frac{dG(u)}{du}u_x^2 + f(u)u_x + g(u);$$

Gandarias (1996, [8])

$$u_t = u^n u_{xx} + nu^{n-1}u_x^2 + g(x)u^m u_x + f(x)u^s.$$

However, the possibility of implementing Ovsiannikov's approach in its full generality presupposes that we are able to construct the optimal system of subgroups of the equivalence group. However, even for the case when the equivalence group is finite-parameter, there arise major algebraic difficulties, since the classification problem for all finite-parameter Lie groups has not yet been solved (to say nothing about infinite-parameter Lie groups, where this problem is completely open). Consequently, there is an evident need for Ovsiannikov's approach to be modified so as to be applicable to the case of infinite-parameter equivalence groups.

Here we turn out attention to a new approach, proposed by R. Zhdanov and V. Lahno in [9], that enables us to solve efficiently the symmetry classification problem for partial differential equations even for the case of infinite-dimensional equivalence groups. It is based mainly on the following facts:

- If the partial differential equation possesses non-trivial symmetry, then it is invariant under some finite-dimensional Lie algebra of differential operators. In the event that the maximal algebra of invariance is infinite-dimensional, then it contains, as a rule, some finite-dimensional Lie algebra.
- If there are local non-singular changes of variables which transform a given differential equation into another, then the finite-dimensional Lie algebras of invariance of these equations are isomorphic, and in the group-theoretic analysis of differential equations such equations are considered to be equivalent.
- Abstract Lie algebras of up to six dimensions have already been classified [10, 11, 12, 13].

What we have in [9] is a preliminary classification of inequivalent realizations of low-dimensional Lie algebras within some specific class of first-order linear differential operators. This class is determined by the structure of the equation under study. Its elements form a representation space for realizations of Lie algebras of symmetry groups admitted by the equations belonging to the class of partial differential equations under study. A natural equivalence relation is introduced on the set of all possible realizations. Namely, two realizations are called equivalent if they are transformed into each other by the action of the equivalence group. In other words, solving the problem of symmetry classification of partial differential equations having some prescribed form, is equivalent to constructing a representation theory of Lie transformation groups (or Lie algebras of first-order partial differential operators) realized as symmetry groups (algebras) of the equations in question.

2 Description of the method

The new approach to the classification of partial differential equations is a synthesis of Lie's infinitesimal method, the use of equivalence transformations and the theory of classification of abstract finite-dimensional Lie algebras. It provides a constructive solution of the problem of the group classification of partial differential equations possessing arbitrary elements and admitting *non-trivial finite-dimensional* invariance algebras.

The group classification in the approach described here is implementation of the following algorithm:

I. The first step involves finding the form of the infinitesimal operators which generate the symmetry group of the equation under consideration, and the construction of the equivalence group of this equation. To find the form of the infinitesimal operators one uses the usual Lie algorithm. In doing this we obtain a system of linear partial differential equations of first order which connect the coefficients of the infinitesimal operators with the arbitrary term of the equation. In the following we call this system *the characterizing system of the equation*. In order to construct the equivalence group $\mathcal{E}$ of the equation one can use the infinitesimal method as well as the direct method.

II. In the second step one carries out the group classification of those equations of the given form which admit finite-dimensional Lie algebras of invariance.

    For this, one carries out a step-by-step classification of finite-dimensional Lie algebras within the specified class of infinitesimal operators, up to equivalence under transformations of the group $\mathcal{E}$. In doing this, one has to check that each algebra obtained in this way can be an invariance algebra of the equation at hand before proceeding from the realization of Lie algebras of lower dimension to the realization of Lie algebras of higher dimension. This eliminates superfluous realizations of Lie algebras. Also, those realizations of Lie algebras which are invariance algebras of the equation will, as their dimension increases, correspond to greater fixing of the arbitrary term.

    This procedure is continued until the arbitrary term in the equation is completely determined or until it is no longer possible to extend the realization of Lie algebras beyond a given dimension within the specified class of infinitesimal operators.

III. The third step is then to exploit the characterizing system or the infinitesimal method of Lie in order to find, for each of the particular choices of the arbitrary term, the maximal invariance algebra of the equation under consideration. Moreover, the equivalence of the equations obtained in this manner is determined. We note that, in as much as equivalent equations have isomorphic invariance algebras, we may test the realizations of the invariance algebras for equivalence rather than test the equations themselves.

3 Examples of the group classification

Here we give some examples illustrating how the method works.

Example 1 ([14]). Group classification of

$$u_{tx} + A(t,x)u_t + B(t,x)u_x + C(t,x)u = 0. \tag{1}$$

Ovsiannikov [15] gave a group classification of (1), using Laplace invariants

$$h = A_t + AB - C, \qquad k = B_x + AB - C.$$

His results can be formulated as follows:

Theorem 1. *Equation (1) admits a Lie symmetry algebra of dimension greater than 1 if and only the functions p, q given by*

$$p = \frac{k}{h}, \qquad q = \frac{1}{h}\partial_x\partial_y(\ln h)$$

are constant. In this case, equation (1) is equivalent either to the Euler–Poisson equation

$$u_{tx} - \frac{2u_t}{q(t+x)} - \frac{2pu_x}{q(t+x)} + \frac{4pu}{q^2(t+x)^2} = 0$$

when $q \neq 0$*, or to the equation*

$$u_{tx} + tu_t + pxu_x + ptxu = 0$$

when $q = 0$.

We have carried out the group classification of equation (1) using our method.

First, we find (by standard methods) that the infinitesimal generator of symmetries is given by

$$X = f(t)\partial_t + q(x)\partial_x + h(t,x)u\partial_u,$$

where the functions f, g, h satisfy

$$\begin{aligned} &h_t + B\dot{f} + fB_t + gB_x = 0,\\ &h_x + Ag' + gA_x + fA_t = 0,\\ &h_{tx} + C\dot{f} + fC_t + Cg' + gC_x + Ah_t + Bh_x = 0 \end{aligned} \tag{2}$$

(we omit the trivial symmetry $X = \omega(t,x)\partial_u$, where ω is an arbitrary solution of (1)).

A direct analysis of (2) is not possible. The **equivalence group** of (1) is given by transformations of the two following types:

$$\begin{aligned} &(a)\quad r = \alpha(t), \quad \xi = \beta(x), \quad v = \theta(t,x)u + \rho(t,x);\\ &(b)\quad r = \alpha(x), \quad \xi = \beta(t), \quad v = \theta(t,x)u + \rho(t,x), \end{aligned}$$

where α, β are arbitrary smooth functions and θ, ρ satisfy

$$\theta_t\rho_x + \rho_t\theta_x - \theta\rho_{tx} + \rho\theta_{tx} - 2\frac{\rho}{\theta}\theta_t\theta_x + A[\theta_t\rho - \theta\rho_t] + B[\theta_x\rho - \theta\rho_x] - C\theta\rho = 0.$$

We note that equation (1) is invariant under the operator $u\partial_u$ and that $[X, u\partial_u] = 0$. So, X and $u\partial_u$ form a two-dimensional Lie algebra. There are only two canonical forms for a two-dimensional Lie algebra

$$\begin{aligned} &A_{21} = \langle e_1, e_2\rangle \qquad \text{with} \quad [e_1, e_2] = 0,\\ &A_{22} = \langle e_1, e_2\rangle \qquad \text{with} \quad [e_1, e_2] = e_2, \end{aligned}$$

and we clearly see that only A_{21} is suitable for our purposes.

We now need to find a canonical form for the operator X. We have the following result:

Proposition 1. *Let* A_{21} *be the invariance algebra of equation (1). There are two inequivalent canonical realizations of* $A_{21} = \langle u\partial_u, X\rangle$*:*

$$A_{21}^1 = \langle u\partial_u, \partial_t\rangle, \qquad A_{21}^2 = \langle u\partial_u, \partial_t + \partial_x\rangle,$$

and the corresponding canonical forms for equation (1) are

$$A_{21}^1: \ u_{tx} + B(x)u_x + u = 0, \tag{3}$$

$$A_{21}^2: \ u_{tx} + B(z)u_x + C(z)u = 0 \tag{4}$$

with $z = t - x$.

The system (2) for equation (3) then becomes

$$h_t + B\dot{f} + gB_x = 0, \qquad h_x = 0, \qquad \dot{f} + g' = 0, \tag{5}$$

where $B = B(x)$.

We easily integrate (5) and we find $B = mx$, where $m = \text{const} \neq 0$, and equation (3) takes on the form

$$u_{tx} + mxu_x + u = 0.$$

The invariance algebra of this equation is

$$\langle u\partial_u, \partial_t, t\partial_t - x\partial_x, \partial_x - mtu\partial_u \rangle.$$

For equation (4) we find (using the same procedure) that the corresponding canonical form for equation (1) is

$$u_{tx} + \frac{m}{z}u_x + \frac{k}{z^2}u = 0,$$

where m, k are constants with $k \neq 0$, and $z = t - x$.

The invariance algebra of this equation is

$$\langle u\partial_u, \partial_t + \partial_x, t\partial_t + x\partial_x + \frac{1}{2}mu\partial_u, t^2\partial_t + x^2\partial_x + mtu\partial_u \rangle.$$

These results are equivalent to the ones obtained by Ovsiannikov.

Example 2 ([9]). Group classification of nonlinear equation of the form

$$u_t = u_{xx} + F(t, x, u, u_x). \tag{6}$$

First, we find that the infinitesimal generator of symmetries is given by

$$X = 2a(t)\partial_t + (\dot{a}(t)x + b(t))\partial_x + f(t, x, u)\partial_u,$$

where functions a, b, f, F fulfil relation

$$\begin{aligned} f_t = u_x(\ddot{a}x + \dot{b}) + (f_u - 2\dot{a})F = f_{xx} + 2u_x f_{xu} + u_x^2 f_{uu} + 2aF_t \\ + (\dot{a}x + b)F_x + fF_u + f_x F_{u_x} + u_x(f_u - \dot{a})F_{u_x}. \end{aligned} \tag{7}$$

A direct analysis of (7) is not possible.

Using our approach we have established that there are three classes of equations (6) invariant with respect to one-parameter groups, seven classes of equations (6) invariant with respect to two-parameter groups, 28 classes of equations (6) invariant with respect to three-parameter groups and 11 classes of equation (6) invariant with respect to four-parameter groups.

Here we present all representatives of 11 classes of equations (6) invariant with respect to four-parameter groups only:

1. $u_t = u_{xx} + \dfrac{\lambda\epsilon u_x}{4\sqrt{|t|}} \ln|tu_x^2| + \dfrac{\beta u_x}{\sqrt{|t|}},$
   $\epsilon = 1$ for $t > 0$, $\epsilon = -1$ for $t < 0$, $\beta \in \mathbb{R}$, $\lambda \neq 0$;
2. $u_t = u_{xx} - \lambda u_x(x + \ln|u_x|), \quad \lambda \neq 0;$
3. $u_t = u_{xx} + \lambda \exp(-u_x), \quad \lambda \neq 0;$
4. $u_t = u_{xx} + 2\ln|u_x|;$
5. $u_t = u_{xx} - u_x \ln|u_x| + \lambda u_x, \quad \lambda \in \mathbb{R};$

6. $u_t = u_{xx} + \lambda u_x^{\frac{2k-2}{2k-1}}, \quad \lambda \neq 0, \quad k = 0, \dfrac{1}{2}, 1;$
7. $u_t = u_{xx} + \dfrac{1}{4t} u_x^2;$
8. $u_t = u_{xx} - u u_{xx} + \lambda |u_x|^{\frac{3}{2}}, \quad \lambda \neq 0;$
9. $u_t = u_{xx} + \lambda^{-1} x + m\sqrt{|u_x|}, \quad \lambda > 0, \quad m \neq 0;$
10. $u_t = u_{xx} - \dfrac{1}{4}\lambda\epsilon(1-q)|t|^{-\frac{1}{2}(1+q)} u_x^2,$
    $\lambda \neq 0, \quad |q| \neq 1, \quad \epsilon = 1 \ \text{ for } \ t > 0, \quad \epsilon = -1 \ \text{ for } \ t < 0;$
11. $u_t = u_{xx} - \dfrac{1}{2}\dot{\alpha} u_x^2 (\lambda - \alpha)\left(1 + \alpha^2\right)^{-1}, \quad \lambda \in \mathbb{R}.$

Note that case 8) with $\lambda = 0$ gives rise to the Burgers equation

$$u_t = u_{xx} - u u_x,$$

which is invariant under a five-parameter group.

Example 3 ([16]). Group classification of nonlinear equations of the form

$$u_t = F(t, x, u, u_x) u_{xx} + G(t, x, u, u_x). \tag{8}$$

In [16] we find that the infinitesimal generator of symetries is given by

$$X = a(t)\partial_t + b(t, x, u)\partial_x + c(t, x, u)\partial_u,$$

where a, b, c are real-valued functions that satisfy the system of particular differential equations

$$\begin{aligned}
&(2b_x + 2u_x bu - \dot{a})F = aF_t + bF_x + cF_u + \left(c_x + u_x c_u - u_x b_x - u_x^2 bu\right) F_{u_x}, \\
&c_t - u_x b_t + (c_u - \dot{a} - u_x b_u)G + (u_x b_{xx} - c_{xx} - 2u_x c_{xu} - u_x^2 c_{uu} \\
&\qquad + 2u_x^2 b_{xu} + u_x^3 b_{uu})F = aG_t + bG_x + cG_u + \left(c_x + u_x c_u - u_x b_x - u_x^2 b_u\right) G_{u_x}.
\end{aligned} \tag{9}$$

A direct analysis of system (9) is also not possible.

The principal result [16] of group classification of equations (8) is the following:

Proposition 2. *Equation (8) admits a Lie symmetry algebra of dimension greater than 4 if it is equivalent to one of the following equations:*

1. $u_t = u^{-4} u_{xx} - 2u^{-5} u_x^2;$
2. $u_t = u_{xx} + x^{-1} u u_x - x^{-2} u^2 - 2x^{-2} u;$
3. $u_t = \exp(u_x) u_{xx};$
4. $u_t = u_x^n u_{xx}, \quad n \geq -1, \quad n \neq 0;$
5. $u_t = \exp(n \arctan u_x)\left(1 + u_x^2\right)^{-1} u_{xx}, \quad n \geq 0.$

These equations are invariant under five-dimensional Lie algebras.

Note that equation 1) is equivalent to the equation obtained by Ovsiannikov [1], equation 2) is equivalent to Burgers equation, and equations 3)–5) was obtained by Akhatov, Gazizov and Ibragimov [3].

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Expanded Lie Group Transformations and Similarity Reductions of Differential Equations

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Continuous groups of transformations acting on the expanded space of variables, which includes the equation parameters in addition to independent and dependent variables, are considered. It is shown that the use of the expanded transformations enables one to enrich the concept of similarity reductions of PDEs. The expanded similarity reductions of differential equations may be used as a tool for finding changes of variables, which convert the original PDE into another (presumably simpler) PDE. A new view on the common similarity reductions as the singular expanded group transformations may be used for defining reductions of a PDE to a specific target ODE.

1 Introduction

By an *expanded* Lie group transformation of a partial differential equation (PDE) we mean a continuous group of transformations acting on the expanded space of variables which includes the equation parameters in addition to independent and dependent variables. We consider the transformations that can be found using the Lie infinitesimal criterion with the properly expanded infinitesimal group generators. In this paper, we are only concerned with groups of point transformations, leaving aside problems involving generalized (Lie–Bäcklund) symmetry groups.

An expanded group of transformations represents a particular case of the equivalence group that preserves the class of PDEs under study – roughly speaking, having the same differential structure but with arbitrary functions having different forms. The approach to finding these equivalence transformation groups with the use of the Lie infinitesimal technique was introduced by Ovsiannikov (see, e.g., [1]) who suggested using the Lie infinitesimal criterion in the properly extended space of variables including dependent and independent variables, arbitrary functions and their derivatives. The original Ovsiannikov method was further developed by Akhatov *et al.* [2]. Their ideas have also been generalized in several papers (see, e.g., [3] and references therein). The transformations in the extended space of variables obtained by adding parameters to the list of independent variables were also used in the context of the renormalization group (RG) symmetries [4, 5].

The main purpose of this paper is to show that the use of the Lie groups of transformations in the expanded space of variables including equation parameters enables one to enrich the concept of similarity reductions as applied to PDEs. In addition, we wish to draw attention to a possibility of using these groups for finding changes of variables that remove some terms from the original equation. Although such a possibility is excluded neither in the framework of the equivalence group approach nor in the context of the RG symmetries, this aspect is obscure in those theories.

The rest of this paper is organized as follows. In Section 2 we consider some illustrative examples of expanded group of transformations that can be used for removing terms from the original differential equations. A comparison with the RG symmetry approach is made. In

Sections 3 and 4 we show how the concept of similarity reductions of PDEs can be enriched in the framework of the expanded transformation groups. Finally, in Section 5 we make some remarks and suggest possible further work.

2 Examples of application of the expanded groups

2.1 Application to ODEs: a simple linear example

We will start with a simple example that applies the technique to the well known ODE of the linear damped oscillator. Of course, this example is only of illustrative value but it is interesting from the methodological point of view since this equation is frequently used to discuss some aspects of asymptotic methods. The equation of linear oscillations with linear damping is

$$u_{tt} + au_t + u = 0 \tag{1}$$

or

$$u_{tt} + u_t + bu = 0, \tag{2}$$

where the subscripts on u denote derivatives. In the context of perturbation methods, when the parameter a or b are assumed to be small, equations (1) and (2) acquire somewhat different physical meanings – an oscillator with a weak resistance for (1) and an overdamped oscillator for (2). An example of application of the expanded transformations to the ODE (1) of the linear oscillator appeared first (to the author's knowledge) in [7]. We will discuss equation (2), which was used in [5] and [6] to illustrate the approaches to the asymptotic analysis of solutions of differential equations based on the renormalization group concept.

We consider the one-parameter (ϵ) Lie group of transformations in (t, u, b):

$$\tilde{t} = g(t,u,b,\epsilon), \qquad \tilde{u} = h(t,u,b,\epsilon), \qquad \tilde{b} = \phi(b,\epsilon) \tag{3}$$

with an infinitesimal generator of the form

$$X = \tau(t,u,b)\frac{\partial}{\partial t} + \eta(t,u,b)\frac{\partial}{\partial u} + \beta(b)\frac{\partial}{\partial b}$$

which leaves (2) invariant. The invariance requirement yields the following determining equations

$$\begin{aligned} &\tau_{uu} = 0, \qquad \eta_{uu} - 2\tau_{tu} + 2\tau_u = 0, \qquad 2\eta_{tu} - \tau_{tt} + \tau_t + 3bu\tau_u = 0,\\ &\eta_{tt} - bu\eta_u + \eta_t + 2bu\tau_t + b\eta + u\beta = 0. \end{aligned} \tag{4}$$

If one is aimed at reducing a given equation to an equation with a known general solution, there is no need in defining the most general form of the group from the determining equations. It is sufficient to define a minimal subgroup arising solely due to the presence of the generator β in the equations. Such a subgroup is found from (4) as

$$\tau = \frac{2\beta t}{1-4b}, \qquad \eta = -\frac{\beta ut}{1-4b}.$$

The finite transformations (3) are defined by solving the problem

$$\begin{aligned} &\frac{dg}{d\epsilon} = \frac{2\beta g}{1-4\phi}, \qquad \frac{dh}{d\epsilon} = -\frac{\beta gh}{1-4\phi}, \qquad \frac{d\phi}{d\epsilon} = \beta,\\ &g = t, \qquad h = u, \qquad \phi = b \qquad \text{at} \quad \epsilon = 0. \end{aligned} \tag{5}$$

Using the third equation of (5) one can go over to derivatives with respect to ϕ in the first two equations and obtain solutions in the forms

$$\tilde{t} = t\left(\frac{1-4b}{1-4\tilde{b}}\right)^{1/2}, \qquad \tilde{u} = u\exp\left\{\frac{t-\tilde{t}}{2}\right\}, \qquad \tilde{b} = b+\epsilon. \tag{6}$$

We have set $\beta = 1$ in the last equation. It can be checked that applying these transformations to the equation $\tilde{u}_{\tilde{t}\tilde{t}} + \tilde{u}_{\tilde{t}} + \tilde{b}\tilde{u} = 0$ yields equation (2).

To find the transformations that remove the last term from the equation (2) by converting it into

$$\tilde{u}_{\tilde{t}\tilde{t}} + \tilde{u}_{\tilde{t}} = 0 \tag{7}$$

we specify the formulae (6) by setting $\tilde{b} = 0$, which corresponds to the specific choice of the group parameter $\epsilon = -b$. Thus, we arrive at the transformations

$$u = \tilde{u}\exp\left\{\frac{\tilde{t}-t}{2}\right\}, \qquad \tilde{t} = t\,(1-4b)^{1/2} \tag{8}$$

expressing the solution u of the original equation (2) through the solution $\tilde{u}$ of equation (7). Substituting the general solution of (7) $\tilde{u} = C_1 + C_2\exp\{-\tilde{t}\}$ into (8) yields a general solution of (2).

Next we will compare our approach, that is based merely on the expanded transformation group, with the approaches of [5] and [6] using the RG concept. The methods of [5] and [6] were designed to improve approximate solutions of the boundary value problems for equations depending on a small parameter. Correspondingly, the initial-value problem for equation (2) with a small parameter b is considered. In [6] the renormalization technique is developed to construct the uniformly valid asymptotics using a straightforward naive perturbation expansion as a starting point. The approach of [5] treats the RG as the Lie group of transformations for a renormgroup manifold constructed in a special way (including parameters of the equations in the list of independent variables is considered as one of the possibilities). The authors consider the initial-value problem for a system of two first order ODEs replacing (2) and find that the use of the modified RG technique permits to improve a perturbation expansion solution up to the exact solution. Thus, as a matter of fact, the expanded transformations are applied in [5] with the same result as that described above in this section. However, in the method of [5], the issue of the *exact* transformations, that would reduce the equation to the simpler one by removing some terms, is obscured by the underlying asymptotic concept, and this may conceal the possibility of finding such transformations. In addition, use of an approximate solution as a starting point and embedding the initial conditions in the framework of the method make the procedure more complicated.

2.2 Application to PDEs: two-dimensional steady-state nonlinear diffusion equation

This example applies the technique to the nonlinear equation

$$u_{xx} + u_{yy} + u_x^2 + u_y^2 + bu_x + a\left(1+pe^{-u}\right) = 0, \tag{9}$$

where a, b and p are constants. This equation can be obtained from the steady-state nonlinear diffusion (heat conduction) equation with the source and the gradient term in the case where the coefficient of thermal conductivity and the source have exponential dependence on the concentration (temperature): $K(u) = K_0e^u$ and $Q(u) = Q_0 + Q_1e^u$. The equation including both the source and the gradient terms has been chosen to show how the technique works in the case when two equation parameters a and b are involved into transformations.

We consider again the one-parameter (ϵ) Lie group of transformations in (x, y, u, a, b) space:

$$X = \xi(x,y,u,a,b)\frac{\partial}{\partial x} + \zeta(x,y,u,a,b)\frac{\partial}{\partial y} + \eta(x,y,u,a,b)\frac{\partial}{\partial u} + \alpha(a,b)\frac{\partial}{\partial a} + \beta(a,b)\frac{\partial}{\partial b}$$

which leaves equation (9) invariant. Solving the determining equations yields

$$\xi = C_1 + \frac{2\alpha - \beta b}{b^2 - 4a}x + ky, \qquad \zeta = C_2 - kx + \frac{2\alpha - \beta b}{b^2 - 4a}y,$$
$$\eta = \left(1 + pe^{-u}\right)\left(C_3 + \frac{2\beta a - \alpha b}{b^2 - 4a}x - \frac{k}{2}by\right),$$

where C_1, C_2, C_3 and k are constants. Here considering groups more general than the minimal subgroup can be useful for constructing different solutions of the initial equation from a special solution of a simplified equation. We will show the finite transformations for the case of $C_1 = C_2 = 0$, as follows

$$\tilde{x} = S(\epsilon)\big(x\cos A(\epsilon) - y\sin A(\epsilon)\big), \qquad \tilde{y} = S(\epsilon)\big(x\sin A(\epsilon) + y\cos A(\epsilon)\big),$$
$$\tilde{u} = \ln\left\{-p + (p + e^u)\exp\left[C_3\epsilon + \frac{bx}{2} - \frac{b + \epsilon\beta}{2}S(\epsilon)\big(x\cos A(\epsilon) - y\sin A(\epsilon)\big)\right]\right\},$$
$$\tilde{a} = a + \epsilon\alpha, \qquad \tilde{b} = b + \epsilon\beta, \tag{10}$$

where

$$S = \left[\frac{b^2 - 4a}{b^2 - 4a + \epsilon(2\beta b - 4\alpha) + \epsilon^2\beta^2}\right]^{1/2},$$
$$A = -k\epsilon, \qquad \alpha = 1, \qquad b = 1 \qquad \text{for} \quad \beta = 0,$$
$$A = \frac{k}{\beta}\left\{\left[\left(b - 2\frac{\alpha}{\beta}\right)^2 + \epsilon(2\beta b - 4\alpha) + \epsilon^2\beta^2\right]^{1/2} - \left(b - 2\frac{\alpha}{\beta}\right)\right\} \qquad \text{for} \quad \beta \neq 0.$$

These transformations map equation (9) into the equation with parameters $\tilde{a}$ and $\tilde{b}$ calculated according to (10). In particular, it is possible to transform (9) into an equation without either the last source term or the term bu_x by setting respectively either $\beta = 0$ and $\epsilon = -a/\alpha$ or $\alpha = 0$ and $\epsilon = -b/\beta$ in the formulae (10).

3 Expanded similarity reductions of PDEs

In this section we will show that the symmetry reduction procedure implemented in the expanded space can lead to discovering transformations between equations that cannot be obtained by applying the technique described in the previous section. We will take as an example the Fokker–Planck equation

$$u_t = u_{xx} + xu_x + u \tag{11}$$

which is used in statistical physics to describe the evolution of probability distribution functions (see, for example, [8]). This equation does not include any parameters – two physical parameters in the original equation corresponding to the problem of a free particle in Brownian motion can be removed by the scaling of x and t. Nevertheless, to apply the technique, a parameter may be introduced into the equation "artificially", for example, as the coefficient in front of the last term of the equation

$$u_t - u_{xx} - xu_x - au = 0. \tag{12}$$

This coefficient cannot change as a result of rescaling the time and space coordinates without appearance of coefficients in front of other terms of the equation. It may seem that introducing this coefficient spoils the physics of the problem (the last two terms in the equation must have equal coefficients), but one may always set $a = 1$ in the final formulae.

We consider the one-parameter (ϵ) Lie group of infinitesimal transformations in (x, t, u, a) defined by

$$X = \xi(x,t,u,a)\frac{\partial}{\partial x} + \tau(x,t,u,a)\frac{\partial}{\partial t} + \eta(x,t,u,a)\frac{\partial}{\partial u} + \alpha(a)\frac{\partial}{\partial a}.$$

Applying the invariance criterion to equation (12) and solving the determining equations yields

$$\begin{aligned} &\xi = \left(C_1e^{2t} - C_2e^{-2t}\right)x + C_3e^t + C_4e^{-t}, \qquad \tau = C_0 + C_1e^{2t} + C_2e^{-2t}, \\ &\eta = \left[-C_1e^{2t}x^2 - C_3e^tx - C_1e^{2t} + a\left(C_0 + C_1e^{2t} + C_2e^{-2t}\right) + \alpha t + C_5\right]u + \Phi(x,t), \end{aligned} \tag{13}$$

where C_0, C_1, C_2, C_3, C_4 and C_5 are constants and $\Phi(x,t)$ satisfies the equation $\Phi_{xx} + x\Phi_x - \Phi_t + a\Phi = 0$.

To illustrate the approach we will take the C_1-subgroup of the full group (13): $C_0 = C_2 = C_3 = C_4 = C_5 = \Phi = 0$. We will start with defining classical (not expanded) similarity reductions corresponding to this subgroup – one should set $\alpha = 0$ and $a = 1$ to obtain a classical group from (13). With such group generators, from the invariant surface condition

$$\xi u_x + \tau u_t - \eta = 0$$

we derive the functional form of the similarity reduction, which being substituted into equation (11) eventually produces the following

$$u(x,t) = e^{-x^2/2}w(z), \quad z = xe^{-t}, \qquad w'' = 0. \tag{14}$$

Now we will apply the same procedure, but for the expanded similarity reductions in (x, t, u, a) (instead of (x, t, u)) space. The generators of the same subgroup are now given by

$$\xi = C_1xe^{2t}, \qquad \tau = C_1e^{2t}, \qquad \eta = \left[-C_1x^2e^{2t} + (a-1)C_1e^{2t} + \alpha t\right]u.$$

Integrating the characteristic system for the invariant surface condition of the form

$$\xi u_x + \tau u_t + \alpha u_a - \eta = 0$$

yields three similarity variables

$$z = xe^{-t}, \qquad \varphi = -\frac{C_1}{\alpha}a - \frac{1}{2}e^{-2t}, \qquad w = ue^{(1-a)t + x^2/2}.$$

We choose w as a new dependent variable and take z and φ as new independent variables to arrive at the following similarity reduction

$$u(x,t,a) = w(z,\varphi)e^{(a-1)t - x^2/2}, \qquad z = xe^{-t}, \qquad \varphi = -\frac{C_1}{\alpha}a - \frac{1}{2}e^{-2t}. \tag{15}$$

Although in terms of the expanded transformations such a similarity reduction reduces the number of variables, it seems that we gain nothing for solving the original PDE since, upon substituting the reduction into the equation, it will be reduced again to a PDE for a function $w(z, \varphi)$. However, this new equation will have a different form, which may allow new possibilities for solution. In particular, substituting (15) into (12) yields

$$w_\varphi - w_{zz} = 0. \tag{16}$$

Thus, the expanded similarity reduction (15) taken for $a = 1$ (we may also set $\alpha = 1$ without loss of generality) provides a transformation of the original equation (11) to the linear heat equation.

Such a transformation cannot arise as the result of application of the classical Lie group method – it is seen that the similarity solution (14) provided by the classical method corresponds to taking only a particular solution of (16), namely, $w = w_0 + w_1 z$ where w_0 and w_1 are arbitrary constants. The technique described in the previous section can produce it neither. It enables one to define only the transformations eliminating the last term of (12).

4 Similarity reductions as the singular expanded group transformations

In this section, we will show that including the equation parameters into the transformations allows one to treat the classical similarity reductions of a PDE as the expanded group transformations which are singular in some variables. This can also enable one to define reductions of the PDE to specific target ODEs.

We will take as an example the Generalized Boussinesq (GBQ) equation

$$u_{xxxx} + pu_t u_{xx} + qu_x u_{xt} + ru_x^2 u_{xx} + u_{tt} = 0 \tag{17}$$

which has a number of equations, arising in different physical applications, as special cases. Similarity reductions for equation (17) obtained from the classical Lie group method and from the Clarkson–Kruskal direct method have been considered in [9].

To apply the method we will introduce an artificial coefficient a in front of the last term, as follows

$$u_{xxxx} + pu_t u_{xx} + qu_x u_{xt} + ru_x^2 u_{xx} + au_{tt} = 0 \tag{18}$$

and consider the one-parameter Lie group of infinitesimal transformations in the expanded (x, t, u, a, p, q, r) space defined by

$$X = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + A\frac{\partial}{\partial a} + P\frac{\partial}{\partial p} + Q\frac{\partial}{\partial q} + R\frac{\partial}{\partial r}.$$

Several different families of solutions of the determining equations obtained from the invariance requirement may arise depending on the relations between the coefficients of (18). We will restrict ourselves to the case

$$p = q, \qquad r = \frac{q^2}{2a} \qquad \Longrightarrow \qquad P = Q, \qquad \frac{R}{r} = 2\frac{Q}{q} - \frac{A}{a}. \tag{19}$$

Then solving the determining equations yields

$$\xi = C_1 xt + C_2 x + C_3 t + C_4, \qquad \tau = C_1 t^2 + \left(2C_2 + \frac{A}{2a}\right) t + C_0,$$
$$\eta = \left(\frac{A}{2a} - \frac{Q}{q}\right) u + \frac{a}{q} C_1 x^2 + \frac{2a}{q} C_3 x + C_5. \tag{20}$$

Here not only the generators A and Q may depend on a and q but also the $C_0, C_1, \ldots, C_5$ are allowed to be functions of a and q due to the fact that the determining equations do not include the corresponding derivatives.

To show the idea it is sufficient to consider the subgroup of (20) defined by the conditions $C_0 = C_2 = C_3 = C_4 = C_5 = 0$ and the following

$$\frac{A}{2a} = \frac{Q}{q} \qquad \Longrightarrow \qquad R = 0 \qquad (r = \text{const}), \qquad q = \sqrt{2r}a^{1/2}, \tag{21}$$

where (19) has been used. In this special case, the finite group transformations are found by solving the problem

$$\begin{aligned} &\frac{d\tilde{x}}{d\epsilon} = C_1(\tilde{a})\tilde{x}\tilde{t}, \qquad \frac{d\tilde{t}}{d\epsilon} = C_1(\tilde{a})\tilde{t}^2 + \frac{A(\tilde{a})}{2\tilde{a}}\tilde{t}, \qquad \frac{d\tilde{u}}{d\epsilon} = \frac{1}{\sqrt{2r}}\tilde{a}^{1/2}C_1(\tilde{a})\tilde{x}^2, \qquad \frac{d\tilde{a}}{d\epsilon} = A(\tilde{a}), \\ &\tilde{x} = x, \qquad \tilde{t} = t, \qquad \tilde{u} = u, \qquad \tilde{a} = a \qquad \text{at} \quad \epsilon = 0. \end{aligned} \tag{22}$$

The solutions of the problem (22) may be represented as

$$\begin{aligned} &\tilde{t} = \frac{t\sqrt{\tilde{a}}}{\sqrt{a}[1 + tN(\tilde{a}, a)]}, \qquad \tilde{x} = \frac{x}{1 + tN(\tilde{a}, a)}, \qquad \tilde{u} = u - \frac{\sqrt{a}}{\sqrt{2r}}\left[\frac{x^2 N(\tilde{a}, a)}{1 + tN(\tilde{a}, a)}\right], \\ &N(\tilde{a}, a) = -\frac{1}{\sqrt{a}}\int_a^{\tilde{a}} \frac{C_1(\varphi)\sqrt{\varphi}}{A(\varphi)} d\varphi. \end{aligned} \tag{23}$$

In the formulae (23), $\tilde{a}(a, \epsilon)$ is a solution of the last equation of (22) with the corresponding initial condition.

We will look for a transformation removing the terms with derivatives with respect to t from (18) which requires $\tilde{a} = 0$, $\tilde{p} = 0$ and $\tilde{q} = 0$ – in view of the assumptions (19) and (21), it is sufficient to set $\tilde{a} = 0$. It is immediately seen that the transformation obtained in such a way is singular in the variable t: setting $\tilde{a} = 0$ in (23) yields $\tilde{t} = 0$. However, since the equation resulting from this transformation does not include derivatives with respect to t, one can treat it as an equation for the function $\tilde{u}(\tilde{x})$ of one variable, and then it does not matter what happens with the discarded variable t. The transformation obtained by setting $\tilde{a} = 0$ in (23) (having in mind the original equation (17) we may also set $a = 1$ now) is

$$u = \tilde{u}(\tilde{x}, 0) + \frac{\lambda}{\sqrt{2r}}\left(\frac{x^2}{1 + t\lambda}\right), \qquad \tilde{x} = \frac{x}{1 + t\lambda}, \qquad \lambda = N(0, 1). \tag{24}$$

This transformation reduces (17) to the following equation (it is readily checked by the direct substitution):

$$\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + r\tilde{u}_{\tilde{x}}^2\tilde{u}_{\tilde{x}\tilde{x}} = 0 \tag{25}$$

in which a number of independent variables is reduced as compared with the original equation. Thus, the expanded group transformations given by (23) with $\tilde{a} = 0$, which are singular in t, provide the following similarity reduction of the GBQ equation:

$$u = w(z) + \frac{\lambda}{\sqrt{2r}}\left(\frac{x^2}{1 + t\lambda}\right), \qquad z = \frac{x}{1 + t\lambda}, \tag{26}$$

where $w(z)$ satisfies

$$w'''' + rw'^2 w'' = 0. \tag{27}$$

The forms of the functions $C_1(a)$ and $A(a)$ have not been specified in the process of derivation of the above formulae. It is evident that any function $C_1(a)$, which being substituted into (23) provides $\lambda = N(0,1) \neq 0$ (for example, $C_1 = \text{const}$), and any function $A(a)$, which permits a transformation to $\tilde{a} = 0$ (for example, $A = 1$, $\tilde{a} = a + \epsilon$), are suitable.

To obtain a similarity reduction, which is more general than (26) but reduces (17) to the same ODE (27), one should consider a more general subgroup of the group (20).

The singular expanded transformations considered above, which remove terms with derivatives with respect to t from the original equation (17), can produce only the reductions to the ODE (27). To define a reduction of the original equation to another ODE (or, at least, to check whether such a reduction is possible) we will look for the expanded transformations that remove some terms from and simultaneously add other terms (desired in the target ODE) to the original PDE. For example, if we wish to define a reduction from (18) (we consider again the particular case defined by (19) and (21)) to the equation

$$w'''' + rw'^2 w'' + kw'' = 0, \tag{28}$$

we have to consider the expanded transformations of the equation

$$u_{xxxx} + qu_t u_{xx} + qu_x u_{xt} + (q^2/2a)u_x^2 u_{xx} + au_{tt} + bu_{xx} = 0 \tag{29}$$

in the (x, t, u, a, q, b) space with a requirement for $\tilde{a} = \phi(a, q, b)$, and $\tilde{q} = \kappa(a, q, b)$ to transform respectively from a and q to zero values, and for $\tilde{b} = \mu(a, q, b)$ to transform from zero to k. We will take the subgroup which for $b = \tilde{b} = 0$ would coincide with that defined by (22) and (23). Omitting the details of calculations we will show only the resulting finite transformations

$$\begin{aligned}
&\tilde{t} = \frac{t\sqrt{\tilde{a}}}{\sqrt{a}(1+tN)}, \qquad \tilde{x} = \frac{x}{1+tN},\\
&\tilde{u} = u - \frac{\sqrt{a}}{\sqrt{2r}}\left(\frac{x^2 N}{1+tN}\right) + \frac{b}{\sqrt{2ra}}t - \frac{\tilde{b}}{\sqrt{2ra}}\left(\frac{t}{1+tN}\right),\\
&\tilde{a} = a + \epsilon, \qquad \tilde{q} = \tilde{a}^2\frac{q}{a^2}, \qquad \tilde{b} = \mu(a, r, b, \tilde{a}).
\end{aligned} \tag{30}$$

The reduction from (17) to (28) is obtained from (30) by setting $a = 1$, $\tilde{a} = 0$, $b = 0$, $\tilde{b} = k$, as follows

$$u = \tilde{u}(\tilde{x}, 0) + \frac{\lambda}{\sqrt{2r}}\left(\frac{x^2}{1+t\lambda}\right) + \frac{k}{\sqrt{2ra}}\left(\frac{t}{1+t\lambda}\right), \qquad \tilde{x} = \frac{x}{1+t\lambda}, \qquad \lambda = N(0,1)$$

with a subsequent change of notation $\tilde{x} \to z$, $\tilde{u}(\tilde{x}, 0) \to w(z)$.

Other similarity reductions of equation (17) can be derived in a similar way. It is worth noting that, within this framework, the reduction to equation (27) (or (25)), which includes only the terms from the original equation, is singled out from the variety of possible reductions. The fact that it possesses some special properties was marked in [9] without addressing its special nature.

5 Concluding remarks

In this paper we have demonstrated that some new applications of the Lie group method to differential equations may arise due to the use of expanded transformation groups.

Simplifying the original equation by means of eliminating some terms from the equation via expanded symmetry groups, discussed in Section 2, cannot be considered as a quite new method – as a matter of fact, it represents a particular case of the equivalence group approach. Here our purpose was to stimulate an interest in the fact that, in this pure form, the approach offers considerable promise for applications, so that it would be helpful to introduce the corresponding options into existing computer algebra packages (it does not require significant modifications). Even if this approach does not guarantee simplifying the equation it provides a way of checking whether the simplifications are possible.

The expanded similarity reductions of differential equations considered as a tool for finding changes of variables, which convert the original PDE into another (presumably simpler) PDE, represent a new method that may be applied in a promising way. Also a new view on the common similarity reductions as the singular expanded group transformations may be used for developing a technique for defining reductions to specific target ODEs.

We will also mention a possible use of the expanded transformations (not discussed here) in the context of perturbation methods. It may enable one to introduce a small parameter into a problem in the case when this cannot be done by a rescaling procedure.

The extensions of the formalism to contact and Lie–Bäclund groups are straightforward. Other generalizations of the described approaches – for example, in the spirit of the non-classical method – are also possible.

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Nonlinear Diffusion-Convection Systems: Lie and Q-Conditional Symmetries

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A class of nonlinear diffusion-convection systems containing two Burgers-type equations is considered. New results of finding Lie and Q-conditional symmetries are presented. Moreover, examples of Lie and non-Lie ansätze and exact solutions of a diffusion-convection system are constructed.

1 Introduction

Nonlinear diffusion-convection (DC) equations of the form

$$U_t = (A(U)U_x)_x + B(U)U_x, \tag{1}$$

where $U = U(t,x)$ is the unknown function, $A(U)$ and $B(U)$ are arbitrary smooth functions and the indices t and x denote differentiation with respect to these variables, generalizes a number of the well known nonlinear second-order evolution equations, describing various processes in physics [1], chemistry [2], biology [3]. The most popular among them is the Burgers equation (BEq)

$$U_t = U_{xx} + \lambda U U_x, \qquad \lambda \in \mathbb{R} \tag{2}$$

arising in several application [4]. Lie symmetry of BEq was found in [5], while the Q-conditional symmetry (i.e., non-classical symmetry [6]) was described in [7] and [8].

In the general case a wide list of Lie symmetries for DC equations of the form (1) is presented in [9]. *A complete description of Lie symmetries,* i.e., group classification of (1) has been done in [10]. The Q-conditional symmetry was also investigated in that paper.

A natural generalization of (1) on several components is the following system of DC equations:

$$\bar{U}_t = (A(\bar{U})\bar{U}_x)_x + B(\bar{U})\bar{U}_x, \tag{3}$$

where $\bar{U} = (U_1,\ldots,U_n)$ is the unknown vector function, $A(\bar{U})$ and $B(\bar{U})$ are matrixes $n\times n$ with the elements $a_{ij}(\bar{U})$ and $b_{ij}(\bar{U})$, $i,j = 1,2,\ldots,n$ being arbitrary smooth functions. Here we deal with a particular case of (3) at $n = 2$, namely:

$$\begin{aligned} U_t &= \lambda_1 U_{xx} + UU_x + F_1(U,V)V_x, \\ V_t &= \lambda_2 V_{xx} + VV_x + F_2(U,V)U_x, \end{aligned} \tag{4}$$

where $U = U(t,x)$ and $V = V(t,x)$ are unknown functions, while λ_1 and λ_2 are arbitrary constants, F_1 and F_2 are arbitrary smooth functions assumed to be known. It is easily seen that DC system (1) is a coupled system of two Burgers-type equations.

Having in mind a *complete description* of the Lie and Q-conditional symmetries of system (1), which is a very difficult problem in the general case, we now summarize the main results obtained for some subclasses of (1). In Section 2, the complete description of the Lie symmetry of system (1) at $\lambda_1 \neq \lambda_2$ are presented. In the case $\lambda_1 = \lambda_2$ all possible pairs (F_1, F_2) are found when DC system (1) is invariant under the Galilei algebra and its standard extentions. Note that the relevant results for reaction-diffusion systems were obtained in [11, 12, 13].

In Section 3, the determining equations to find the Q-conditional symmetry of system (1) are derived. Furthermore those equations are solved under some assumptions. We have established that system (1) at $F_1 = U + m_1$, $F_2 = V + m_2$, where m_1, m_2 are some constants, admits conditional symmetry operators.

Finally (Section 4), the found symmetries are applied to construct both Lie and non-Lie ansätze of a particular DC system of the form (1). Examples of exact solutions are also presented.

2 Lie symmetry of DC system (1)

It is easily checked that the system (1) is invariant under the operators of time and space translations $P_x = \partial_x$ and $P_t = \partial_t$ for arbitrary functions F_1 and F_2. Following [10], this algebra is called *the trivial Lie algebra* of the system (1). Thus, we aim to find all pairs of functions (F_1, F_2) that lead to extensions of the trivial Lie algebra of this system. Note that we consider only *nonlinear* systems, particularly because linear equations are amenable to numerous classical methods (the Fourier method, method of Laplace transformation and so on).

Now let us formulate a theorem which gives complete information on the classical, i.e., Lie symmetry of the system (1).

Theorem 1. *All possible maximal algebras of invariance (MAI) of the system (1) for any fixed pair (F_1, F_2) and $\lambda_1 \neq \lambda_2$, $\lambda_1\lambda_2 \neq 0$ are presented in Table 1. Any other system of the form (1) with non-trivial Lie symmetry is reduced by the local substitution*

$$x^* = x - mt, \qquad t^* = t, \qquad U^* = U + m, \qquad V^* = V + m, \qquad \lambda \in \mathbb{R} \tag{5}$$

to one of those given in Table 1.

Table 1. MAI of the system (1) at $\lambda_1 \neq \lambda_2$, $\lambda_1\lambda_2 \neq 0$.

/	Nonlinearities	Restrictions	Basic operators of MAI
1.	$F_1 = Uf(\omega)$ $F_2 = Vg(\omega)$	$\omega = U/V$	P_t, P_x $D = 2tP_t + xP_x - U\partial_U - V\partial_V$
2.	$F_1 = f(\omega)$ $F_2 = g(\omega)$	$\omega = U - V$	P_t, P_x $G_x = tP_x - (\partial_U + \partial_V)$
3.	$F_1 = \alpha_1(U - V)$ $F_2 = \alpha_2(V - U)$	$\alpha_1 \neq 0$ or $\alpha_2 \neq 0$	P_t, P_x, G_x, D
4.	$F_1 = 0$ $F_2 = 0$		P_t, P_x, G_x, D $\Pi = tD - t^2P_t - x(\partial_U + \partial_V)$

The proof of Theorem 1 is based on the classical Lie scheme (see, e.g., [15, 14]) and is non-trivial because the system (1) contains two arbitrary functions of two variables. The proof of this and following theorems will be published in [16]).

Remark 1. Cases 3 and 4 in Table 1 are natural prolongations of case 2, because the extended Galilei algebra $AG_1^0(1,1) = \langle P_t, P_x, G_x^0, D\rangle$ and the generalized Galilei algebra $AG_2^0(1,1) = \langle P_t, P_x, G_x^0, D, \Pi\rangle$ are known to be standard extensions of the Galilei algebra $AG^0(1,1) = \langle P_t, P_x, G_x^0\rangle$ with zero mass (for details see [11, 12, 15]).

It turns out that the case $\lambda_1 = \lambda_2 \neq 0$ (without losing generality we can put $\lambda_1 = 1$) is more complicated than the case considered above and its complete description will be done in [16]. Here the most interesting cases are only presented.

Theorem 2. *In the case $\lambda_1 = \lambda_2 = 1$, DC system (1) for $F^1_V \neq 0$ or $F^2_U \neq 0$ is invariant under the Galilei algebra if and only if*

$$F^k = \phi(\omega) - (-1)^k \frac{U-V}{2}, \qquad k = 1,2, \quad \omega = (U-V)^\gamma \exp(U+V), \quad 0 \neq \gamma \in \mathbb{R},$$

where ϕ is an arbitrary function. The corresponding basic operators of the Galilei algebra are

$$P_t, \quad P_x, \quad G^0_x = t\partial_x + \frac{U-V}{\gamma}(\partial_U - \partial_V) - (\partial_U + \partial_V).$$

Theorem 3. *In the case $m_1 \neq m_2 \in \mathbb{R}$, MAI of nonlinear DC system*

$$\begin{aligned} &U_t = U_{xx} + UU_x + (m_1 + U)V_x, \\ &V_t = V_{xx} + VV_x + (m_2 + V)U_x \end{aligned} \tag{6}$$

is the generalized Galilei algebra $AG^0_2(1,1)$ with zero mass generated by the basic operators

$$\begin{aligned} &P_t, \quad P_x, G^0_x = tP_x + Q^0_1, \quad D_0 = 2t\partial_t + x\partial_x - U\partial_u - v\partial_v + Q^0_2, \\ &\Pi_0 = tD_0 - t^2\partial_t + xQ^0_1 + \frac{2}{m_1 - m_2}(\partial_U - \partial_V). \end{aligned} \tag{7}$$

In the case $m_1 = m_2 = 0$, MAI of (6) is infinite-dimensional algebra generated by the operators

$$\begin{aligned} &P_t, \quad P_x, \quad Q_1 = \frac{1}{2}(U-V)(\partial_U - \partial_V), \quad G_x = t\partial_x + \frac{x}{2}Q_1 - Q_2, \\ &D = 2t\partial_t + x\partial_x + \frac{1}{2}Q_1 - (U\partial_U + V\partial_V), \quad \Pi = tD_1 - t^2\partial_t + \frac{x^2}{4}Q_1 - xQ_2, \end{aligned} \tag{8}$$

which form the $AG_2(1,1)$ with non-zero mass, and the operator

$$X^\infty = (MU + MV - 2M_x)(\partial_U - \partial_V), \tag{9}$$

where $M = M(t,x)$ is an arbitrary solution of the linear diffusion equation $M_t = M_{xx}$.

In formulas (7) and (8) the operators

$$\begin{aligned} &Q^0_1 = \frac{U+V+2m_1}{m_2 - m_1}\partial_U + \frac{U+V+2m_2}{m_1 - m_2}\partial_V, \\ &Q^0_2 = \frac{m_2U + m_1V + 2m_1m_2}{m_2 - m_1}(\partial_U - \partial_V) + U\partial_U + V\partial_V, \qquad Q_2 = \frac{1}{2}(\partial_U + \partial_V). \end{aligned}$$

Remark 2. In the case $m_1 = m_2 \neq 0$, system (6) is reduced to the same with $m_1 = m_2 = 0$ by the local substitution (5).

3 Q-conditional symmetry of DC system (1)

In this section we study Q-conditional symmetry of nonlinear DC system (1). Nevertheless the main idea of the notion of Q-conditional symmetry (non-classical symmetry) is very simple and was introduced by Bluman and Cole more than 30 years ago [6], it is a very non-trivial problem to find new operators of Q-conditional symmetry for nonlinear equations arising in applications.

Moreover, to our knowledge there are even no examples of operators of Q-conditional symmetry in the case of *DC systems* of the form (3).

We remind the reader that every operator of Lie symmetry is also a Q-conditional symmetry operator therefore hereinafter we will find only purely conditional symmetry operators. It is worth also reminding on the following property of such operators: if the operator

$$Q = \partial_t + \xi(t,x,U,V)\partial_x + \eta^1(t,x,U,V)\partial_U + \eta^2(t,x,U,V)\partial_V, \tag{10}$$

where the ξ, η^1 and η^2 being the known functions, is one of the Q-conditional symmetry for DC system (1) then the operator $N(t,x,U,V)\,Q$ being N an arbitrary nonvanishing function is also the Q-conditional symmetry operator. Thus we will seek only operators of the canonical form (10). Of course, one can also find Q-conditional symmetry operators of the canonical form

$$Q = \partial_x + \eta^1(t,x,U,V)\partial_U + \eta^2(t,x,U,V)\partial_V,$$

however we aim to discuss such possibility elsewhere.

Using the known procedure (see, for example, [15], chapter 5) to construction of the operators Q of the form (10), where the coefficients ξ, η^1 and η^2 must be found, we have established the following theorem.

Theorem 4. *DC system (1) is Q-conditional invariant under the operator (10), if and only if the functions ξ, η^1, η^2 satisfy the following determining equations:*

$$\xi_{UU} = \xi_{VV} = \xi_{UV} = 0, \tag{11}$$

$$\lambda_1\eta_{VV} + F^1\xi_V = 0, \qquad \lambda_2\eta^2_{UU} + F^2\xi_U = 0, \tag{12}$$

$$\begin{aligned}
&\lambda_1\eta^1_{UU} - 2\lambda_1\xi_{xU} + 2(\xi+U)\xi_U + \frac{\lambda_1}{\lambda_2}F^2\xi_V = 0,\\
&\lambda_2\eta^2_{VV} - 2\lambda_2\xi_{xV} + 2(\xi+V)\xi_V + \frac{\lambda_2}{\lambda_1}F^1\xi_U = 0,
\end{aligned} \tag{13}$$

$$\begin{aligned}
&2\lambda_1\eta^1_{UV} - 2\lambda_1\xi_{xV} + \frac{1}{\lambda_2}(\lambda_2U + \lambda_1V + (\lambda_1+\lambda_2)\xi)\xi_V + 2F^1\xi_U = 0,\\
&2\lambda_2\eta^2_{UV} - 2\lambda_2\xi_{xU} + \frac{1}{\lambda_1}(\lambda_2U + \lambda_1V + (\lambda_1+\lambda_2)\xi)\xi_U + 2F^2\xi_V = 0,
\end{aligned} \tag{14}$$

$$\begin{aligned}
&\lambda_1\eta^1_{xx} - \eta^1_t - 2\xi_x\eta^1 + \left(\frac{\lambda_1}{\lambda_2} - 1\right)\eta^2\eta^1_V + U\eta^1_x + F^1\eta^2_x = 0,\\
&\lambda_2\eta^2_{xx} - \eta^2_t - 2\xi_x\eta^2 + \left(\frac{\lambda_2}{\lambda_1} - 1\right)\eta^1\eta^2_U + V\eta^2_x + F^2\eta^1_x = 0,
\end{aligned} \tag{15}$$

$$\begin{aligned}
&\lambda_1(2\eta^1_{xU} - \xi_{xx}) + (2\xi+U)\xi_x - 2\eta^1\xi_U + \left(1 - \frac{\lambda_1}{\lambda_2}\right)\eta^2\xi_V - \frac{\lambda_1}{\lambda_2}F^2\eta^1_V\\
&\qquad + F^1\eta^2_U + \xi_t + \eta^1 = 0,\\
&\lambda_2(2\eta^2_{xV} - \xi_{xx}) + (2\xi+V)\xi_x - 2\eta^2\xi_V + \left(1 - \frac{\lambda_2}{\lambda_1}\right)\eta^1\xi_U - \frac{\lambda_2}{\lambda_1}F^1\eta^2_U\\
&\qquad + F^2\eta^1_V + \xi_t + \eta^2 = 0,
\end{aligned} \tag{16}$$

$$\begin{aligned}
&2\lambda_1\eta^1_{xV} + (\xi_x - \eta^1_U + \eta^2_V)F^1 - 2\eta^1\xi_V\\
&\qquad + \frac{1}{\lambda_2}[(\lambda_2-\lambda_1)\xi + \lambda_2U - \lambda_1V]\eta^1_V + \eta^1F^1_U + \eta^2F^1_V = 0,\\
&2\lambda_2\eta^2_{xU} + (\xi_x - \eta^2_V + \eta^1_U)F^2 - 2\eta^2\xi_U\\
&\qquad + \frac{1}{\lambda_1}[(\lambda_2-\lambda_1)\xi + \lambda_2U - \lambda_1V]\eta^2_U + \eta^2F^2_V + \eta^1F^2_U = 0.
\end{aligned} \tag{17}$$

The overdetermined system of nonlinear equations (11)–(17) is very complicated and we have not constructed its general solutions. On the other hand, it is possible to construct the general solution under the additional condition $\eta^1_t = \eta^2_t = \eta^1_x = \eta^2_x = 0$, i.e., assuming $\eta^k = \eta^k(U,V)$, $k = 1,2$. Under such assumption the subsystem (15) with $\lambda_1 = \lambda_2$ is reduced to the condition $\xi_x = 0$ therefore other subsystems can be easily solved.

Theorem 5. *DC system (1) is Q-conditional invariant under the operator*

$$Q = \partial_t + \xi^1(t,x,U,V)\partial_x + \eta^1(U,V)\partial_U + \eta^2(U,V)\partial_V, \tag{18}$$

if and only if

$$\lambda_1 = \lambda_2 = 1, \qquad F^1 = m_1 + U, \qquad F^2 = m_2 + V, \qquad m_1, m_2 \in \mathbb{R} \tag{19}$$

and then the coefficients of the operator (18) have the form

$$\begin{aligned}
\xi^1 &= \frac{1}{2}(U+V) + \alpha_0,\\
\eta^1 &= -\frac{1}{4}\left[U(U+V)^2 + 2\alpha_0 U(U+V)\right] + \beta_0 U + \gamma_1,\\
\eta^2 &= -\frac{1}{4}\left[V(U+V)^2 + 2\alpha_0 V(U+V)\right] + \beta_0 V + \gamma_2,
\end{aligned} \tag{20}$$

if $m_1 = m_2 = 0$, and

$$\begin{aligned}
\xi^1 &= \frac{1}{2}(U+V),\\
\eta^1 &= -\frac{1}{4}\left[(U+m_1)(U+V)^2 + (m_2-m_1)U^2\right] + \beta_0 U + \gamma_1,\\
\eta^2 &= -\frac{1}{4}\left[(U+m_2)(U+V)^2 + (m_1-m_2)V^2\right] + \beta_0 V + \gamma_2,
\end{aligned} \tag{21}$$

if $m_1 \neq m_2$. Here α_0, β_0, γ_1, γ_2 are arbitrary constants.

One can see that the above listed additional conditions on the form of the operator Q are very strong because they lead only to the fixed nonlinearity $F_1 = U + m_1$, $F_2 = V + m_2$. The next theorem illustrates that the requirement $\lambda_1 = \lambda_2$ is very important.

Theorem 6. *DC system (1) at*

$$\lambda_1 \neq \lambda_2, \qquad F^1 = \frac{\lambda_1}{\lambda_2}(U+m), \qquad F^2 = \frac{\lambda_2}{\lambda_1}(V-m), \qquad m \in \mathbb{R} \tag{22}$$

is invariant under the trivial Lie algebra generated by the basic operators P_t and P_x while one admits the operator of the Q-conditional symmetry

$$Q = \partial_t - m\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\partial_x + \frac{U+V}{(\lambda_1-\lambda_2)t}(\lambda_1\partial_U - \lambda_2\partial_V). \tag{23}$$

4 Ansätze and exact solutions of a DC system

In this section we shall deal with the nonlinear DC system (6). It follows from Theorem 3 that MAI of (6) for $m_1 \neq m_2$ is the generalized Galilei algebra $AG_2^0(1,1)$ with the basic operators (7). It seems reasonable to construct Lie ansätze and to seek exact solutions of system (6) using operators (7). A full set of non-equivalent (non-conjugate) one-dimensional subalgebras of the $AG_2(1,1)$ algebra is well-known [14]. Taking into account the similarity of structures of the

$AG_2(1,1)$ algebra and $AG_2^0(1,1)$ algebra, a full set of non-equivalent one-dimensional subalgebras of the $AG_2^0(1,1)$ algebra was also constructed, namely:

$$k_1\partial_t + k_2\partial_x, \quad \partial_t + k_3G, \quad D, \quad \partial_t + \Pi, \tag{24}$$

where k_1, k_2, k_3 are arbitrary constants. Let us apply each of them for reduction of system (6) to systems of ordinary differential equations (ODEs).

a) The operator $k_1\partial_t + k_2\partial_x$ generates the ansatz

$$U = \varphi(\omega), \qquad V = \psi(\omega), \qquad \omega = k_2 t - k_1 x, \tag{25}$$

where φ, ψ are unknown functions. Substituting (25) into system (6), we arrive at the ODEs system

$$\begin{aligned} k_2\dot\varphi &= k_1^2\ddot\varphi - k_1\varphi\dot\varphi - k_1(\varphi + m_1)\dot\psi, \\ k_2\dot\psi &= k_1^2\ddot\psi - k_1(\psi + m_2)\dot\varphi - k_1\psi\dot\psi, \end{aligned} \tag{26}$$

(hereinafter $\dot\varphi = \frac{d\varphi}{d\omega}$, $\ddot\varphi = \frac{d^2\varphi}{d\omega^2}$).

b) The operator $\partial_t + k_3G$ generates the ansatz

$$\begin{aligned} U &= \frac{(k_3t - m_1)\varphi(\omega) + \psi(\omega) + (k_3t - m_1)^2}{m_1 - m_2} - m_1, \\ V &= \frac{(k_3t - m_2)\varphi(\omega) + \psi(\omega) + (k_3t - m_2)^2}{m_2 - m_1} - m_2, \qquad \omega = x - \frac{k_3}{2}t^2, \end{aligned} \tag{27}$$

which reduces system (6) the ODEs system

$$\begin{aligned} &\ddot\varphi - \varphi\dot\varphi + \dot\psi - 2k_3 = 0, \\ &\ddot\psi - \psi\dot\varphi - k_3\varphi = 0. \end{aligned} \tag{28}$$

c) The operator D generates the ansatz

$$\begin{aligned} U &= \frac{m_1t^{-1/2}\varphi(\omega) + t^{-1}\psi(\omega) + m_1m_2}{m_1 - m_2}, \\ V &= \frac{m_2t^{-1/2}\varphi(\omega) + t^{-1}\psi(\omega) + m_1m_2}{m_2 - m_1}, \qquad \omega = t^{-1/2}x. \end{aligned} \tag{29}$$

which reduces system (6) the ODEs system

$$\begin{aligned} &\ddot\varphi + \varphi\dot\varphi + \frac{1}{2}(\omega\dot\varphi + \varphi) - \dot\psi = 0, \\ &\ddot\psi + \psi\dot\varphi + \frac{1}{2}\omega\dot\psi + \psi = 0. \end{aligned} \tag{30}$$

d) Finally, the operator $\partial_0 + \Pi$ generates the ansatz

$$\begin{aligned} U = \frac{1}{m_1 - m_2}\Big\{ &\left(t^2+1\right)^{-1/2} m_1(\varphi(\omega) - 2t\omega) \\ &- \left(t^2+1\right)^{-1} (\psi(\omega) + t\omega\varphi(\omega) - 2t) + \omega^2 + m_1m_2\Big\}, \\ V = \frac{1}{m_2 - m_1}\Big\{ &\left(t^2+1\right)^{-1/2} m_2(\varphi(\omega) - 2t\omega) \\ &- \left(t^2+1\right)^{-1} (\psi(\omega) + t\omega\varphi(\omega) - 2t) + \omega^2 + m_1m_2\Big\}, \qquad \omega = \left(t^2+1\right)^{-1/2} x, \end{aligned} \tag{31}$$

which reduces system (6) the ODEs system

$$\begin{aligned} &\ddot{\varphi} + \varphi\dot{\varphi} + \dot{\psi} = 0, \\ &\ddot{\psi} + \psi\dot{\varphi} = \omega(\omega\dot{\varphi} + \varphi). \end{aligned} \tag{32}$$

Having solutions of the ODEs systems (26), (28), (30), (32) and using the relevant ansätze one easily constructs solutions of the original nonlinear DC system (6). For example, a particular solution of system (28) leads to the following exact solution of system (6):

$$\begin{aligned} U &= \frac{1}{m_1 - m_2}\left(\frac{x^2 - 2m_1tx - 4t}{t^2+1} + \frac{12}{x^2} + \frac{6m_1}{x} + m_1m_2\right), \\ V &= \frac{1}{m_2 - m_1}\left(\frac{x^2 - 2m_2tx - 4t}{t^2+1} + \frac{12}{x^2} + \frac{6m_2}{x} + m_1m_2\right). \end{aligned} \tag{33}$$

By means of the known technique (see for details [11, 15]) for the continuous transformations generated by the basic operators (7), solution (33) can be multiplied to a five-parameter family of solutions. Such multiplication is possible for any given solution of system (6). In particular case, using transformations generated by the Galilei operator G_x^0, any *time-independent (stationary) solution* $(U_0(x), V_0(x))$ is converted to the following one-parameter family of solutions of system (6)

$$\begin{aligned} U &= U_0(x+\epsilon t) - \epsilon\frac{U_0(x+\epsilon t) + V_0(x+\epsilon t) + 2m_1}{m_2 - m_1}, \\ V &= V_0(x+\epsilon t) + \epsilon\frac{U_0(x+\epsilon t) + V_0(x+\epsilon t) + 2m_2}{m_2 - m_1}, \end{aligned} \tag{34}$$

where ϵ is an arbitrary real parameter.

Let us apply the Q-conditional symmetry operators for the construction of ansätze and exact solutions of system (6). It follows from Theorem 5 that system (6) for $m_1 \neq m_2$ is Q-conditional invariant with respect to the operator

$$\begin{aligned} Q = \partial_t + \frac{U+V}{2}\partial_x - \frac{1}{4}\Big\{&(U+V)^2(U+m_1)\partial_U \\ &+ (U+V)^2(V+m_2)\partial_V - (m_1-m_2)\left(U^2\partial_U - V^2\partial_V\right)\Big\}. \end{aligned} \tag{35}$$

To construct the relevant solutions of system (6), it is necessary to integrate the Lagrange system

$$\begin{aligned} \frac{dt}{-4} = \frac{dx}{-2(U+V)} &= \frac{dU}{(U+m_1)(U+V)^2 + (m_2-m_1)U^2} \\ &= \frac{dV}{(V+m_2)(U+V)^2 + (m_1-m_2)V^2}. \end{aligned} \tag{36}$$

In contrast to the analogous systems for Lie operators (24), system (36) is *nonlinear* with respect to the unknown functions U and V, therefore there is a problem to construct its general solution. It turns out that this system can be essentially simplified by the substitution

$$t = t, \qquad x = x, \qquad w = U + V, \qquad z = \frac{m_1 - m_2}{2}\left(\frac{U+V}{U-V} - \frac{m_1+m_2}{m_1-m_2}\right). \tag{37}$$

Indeed, the relevant calculations show that system (36) takes the form

$$\frac{dt}{-4} = \frac{dx}{-2w} = \frac{dw}{w^2(w-2z)} = \frac{dz}{w\left(z^2 - m_1m_2\right)}. \tag{38}$$

The first integrals J_1, J_2, J_3 of system (38) depend on the sign of the term m_1m_2, i.e., there are three different cases: $m_1m_2 = 0$, $m_1m_2 > 0$ and $m_1m_2 < 0$. Considering the first of them (other two cases see in [16]), we obtain

$$J_1 = t + \frac{4}{wz} - \frac{2}{z^2}, \qquad J_2 = x - \frac{2}{z}, \qquad J_3 = \frac{3}{wz^2} - \frac{1}{z^3}. \tag{39}$$

Thus, we construct the non-Lie ansatz (6)

$$J_1 = \varphi(J_2), \qquad J_3 = \psi(J_2), \tag{40}$$

being φ and ψ new unknown functions, for finding solutions of the original nonlinear DC system (6). Substituting ansatz (40) into (6) in the case $m_2 = 0$, $m_1 = 1$ (this system for $m_1 \neq 1$ is reduced to the same with $m_1 = 1$ by the substitution $t \to m_1^{-2}t$, $x \to m_1^{-1}x$, $U \to m_1U$, $V \to m_1V$), we arrive at the ODEs system

$$\begin{aligned} &\ddot{\varphi} + 1 = 0, \\ &4\dddot{\psi} + \dot{\varphi} = 0. \end{aligned} \tag{41}$$

Since (41) is the linear system, its general solution can be easily found. Thus, substituting one into ansatz (39), (40), we obtain the two-parameter family of solutions of system (6) with $m_2 = 0$, $m_1 = 1$:

$$U = \frac{\frac{2}{3}x^3 + 2x^2 + 4C_1(x+2) + 4(C_2 - t)}{W}, \qquad V = \frac{4(t - C_2) - 2x^2}{W}, \tag{42}$$

where $W = \frac{1}{12}x^4 + t^2 + C_1(x^2 - 2t) + 2C_2x$ and C_1, C_2 are arbitrary parameters.

Some other non-Lie ansätze and exact solutions are presented in [16].

5 Conclusions

In this paper, Theorem 1 is presented that gives a complete description of Lie symmetries of the nonlinear diffusion-convection system (1) for $\lambda_1 \neq \lambda_2$, $\lambda_1\lambda_2 \neq 0$. In contrast to reaction-diffusion systems (a complete description of Lie symmetries of those systems was done in [13]), we have established only four non-equivalent cases when system (1) is invariant with respect to the non-trivial Lie algebras. Obviously, the nonlinear fixed terms UU_x and VV_x (see (1)) play a role of the strong restrictions of Lie symmetry for system (1).

The nonlinear DC system (6) with unique symmetry properties has been also found. This system is invariant under the generalized Galilei algebras $AG_2^0(1,1)$ in the case $m_1 \neq m_2$ and $AG_2(1,1)$ in the case $m_1 = m_2$ (see Theorem 3). On the other hand, system (6) admits the operators of Q-conditional symmetry with the cubic nonlinearities on the dependent variables U and V (see Theorem 5). To our knowledge, such operators for *system of nonlinear evolution equations* are found for the first time. Analogous operators were found before for single reaction-diffusion equations [17, 15, 18] and single reaction-diffusion-convection equations [10]. Finally, it should be stressed that the process of reduction of (6) is very non-trivial if one uses the Q-conditional symmetry operators (18), (20)–(21). However, the relevant reduction leads to very simple ODEs systems (see, for example, (41) that were easily solved therefore exact solutions of the nonlinear DC system (6) were obtained.

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Some Invariant Solutions of the Savage–Hutter Model for Granular Avalanches

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Consider the spatially one-dimensional time dependent system of equations, obtained by Savage and Hutter, which describes the gravity-driven free surface flow of granular avalanches. All the similarity solutions of this system are found by means of the group analysis. The results of computing experiments are reduced and their physical treatment is considered.

1 Introduction

The most general non-dimensional form of the Savage–Hutter theory, which includes both the [1] and [2] formulations, can be obtained by reducing the theory of Gray *et al.* [3] to one-dimension. The time-dependent depth integrated mass and downslope momentum balance equations are

$$\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} = 0, \tag{1}$$

$$\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(Q^2 h^{-1}\right) + \frac{\partial}{\partial x}\left(\beta h^2/2\right) = hs, \tag{2}$$

where h is the avalanche thickness, Q is the depth averaged downslope volume flux and x is the curvilinear downslope coordinate. The source term s on the right-hand side of (2) is composed of the downslope component of gravitational acceleration, the Coulomb sliding friction with basal angle of friction δ and gradients of the basal topography height, b, above the curvilinear coordinate system. It takes the form

$$s = \sin\zeta - Q|Q|^{-1}\tan\delta\left(\cos\zeta + \lambda\kappa Q^2 h^{-2}\right) - \varepsilon\cos\zeta\frac{\partial b}{\partial x}, \tag{3}$$

where ζ is inclination angle of the curvilinear coordinate to the horizontal and $\kappa = -\partial\zeta/\partial x$ is the curvature. In a typical avalanche the thickness magnitude H^* is much smaller than its length L^*, which is reflected in the small non-dimensional parameter $\varepsilon = H^*/L^*$. The shallowness assumption also requires that typical avalanche lengths are shorter than the radius of curvature R^* of the curvilinear coordinate, which introduces the second non-dimensional parameter $\lambda = L^*/R^*$. The function

$$\beta = \varepsilon K\cos\zeta, \tag{4}$$

where the earth pressure coefficient K was proposed to take two limiting states $K_{\rm act}$ and $K_{\rm pas}$ associated with extensive ($\partial u/\partial x \geq 0$) and compressive ($\partial u/\partial x < 0$) motions, respectively. Savage & Hutter [1] showed that

$$K_{\rm act/pas} = 2\sec^2\phi\left(1 \mp \left\{1 - \cos^2\phi\sec^2\delta\right\}^{1/2}\right) - 1, \tag{5}$$

where ϕ is the internal angle of friction of the granular material.

Recent experiments suggest that the jump in K at $\partial u/\partial x = 0$ is unrealistic and that a slowly varying function or a constant earth pressure coefficient is more realistic. In this paper it is therefore assumed that K is constant. It shall also be assumed that the slope is flat, $b \equiv 0$, and inclined at a constant angle ζ to the horizontal. This implies that the curvature $\kappa = 0$ and that β is constant. In addition placing the restriction that the volume flux $Q > 0$ implies that the source term $s = s_0$ is also constant.

Three exact solutions to the system of equations (1)–(2) are currently known. These are the parabolic cap similarity solution and the 'M'-wave solutions, derived by Savage & Hutter [1], and the travelling shock wave solution [4] on a non-accelerative slope. In this paper we seek to find further simple solutions of physical interest.

2 Results of the group analysis and construction of the invariant solutions

Consider a moving coordinate system

$$\eta = x - s_0 t^2/2. \tag{6}$$

The relative flux $\widehat{Q}$ in the moving coordinate system is then given by

$$\widehat{Q} = Q - h s_0 t. \tag{7}$$

In the new variable the system (1), (2) may be written in the form

$$\frac{\partial h}{\partial t} + \frac{\partial \widehat{Q}}{\partial \eta} = 0, \tag{8}$$

$$\frac{\partial \widehat{Q}}{\partial t} + \frac{\partial \left(\widehat{Q}^2 h^{-1}\right)}{\partial \eta} = -\frac{\partial \left(0.5 \beta h^2\right)}{\partial \eta}. \tag{9}$$

If to introduce the relative rate $\widehat{u}$ by the relation $\widehat{u} = \widehat{Q}/h$ then the system (8), (9) takes the form

$$\frac{\partial h}{\partial t} + \frac{\partial \widehat{u} h}{\partial \eta} = 0, \tag{10}$$

$$\frac{\partial \widehat{u}}{\partial t} + \widehat{u}\frac{\partial \widehat{u}}{\partial \eta} + \beta \frac{\partial h}{\partial \eta} = 0. \tag{11}$$

These equations coincide with the shallow-water equations considered by Ibragimov [5]. The system (10), (11) admits the symmetry Lie algebra $L_5 \bigoplus L_\infty$ [6]. Used this algebra and the relation $\widehat{u} = \widehat{Q}/h$ we find the following basis of Lie algebra for (8), (9)

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial \eta}, \qquad X_3 = t\frac{\partial}{\partial t} + \eta \frac{\partial}{\partial \eta}, \qquad X_4 = \eta \frac{\partial}{\partial \eta} + 2h\frac{\partial}{\partial h} + 3\widehat{Q}\frac{\partial}{\partial \widehat{Q}},$$
$$X_5 = t\frac{\partial}{\partial \eta} + h\frac{\partial}{\partial \widehat{Q}}, \qquad X_\infty = Z(\widehat{u}, h)\frac{\partial}{\partial \eta} + T(\widehat{u}, h)\frac{\partial}{\partial t},$$

where the functions $Z(\widehat{u}, h)$, $T(\widehat{u}, h)$ are defined by the linear equations

$$Z_{\widehat{u}} - \widehat{u} T_{\widehat{u}} + h T_h = 0, \qquad Z_h - \widehat{u} T_h + \beta T_{\widehat{u}} = 0.$$

Here $\widehat{u} = \widehat{Q}/h$. In this paper we will consider only the invariant solutions with respect to the stretching transformations of η and t. Evidently, the stretching transformations of η and t are generated by the infinitesimal operator (or generator) $X \in L_5$

$$X = (\mu - 1)X_3 + X_4. \tag{12}$$

Any invariant function f solves the partial differential equation $Xf = 0$ [7]. Therefore the basis of invariants is furnished by

$$J_1 = \eta t^{-\mu/(\mu-1)}, \qquad J_2 = ht^{-2/(\mu-1)}, \qquad J_3 = \widehat{Q}t^{-3/(\mu-1)}, \qquad \mu \neq 1,$$
$$J_1 = t, \qquad J_2 = h\eta^{-2}, \qquad J_3 = \widehat{Q}\eta^{-3}, \qquad \mu = 1.$$

Consequently, the invariant solution of the system (8), (9) with respect to the group, generated by (12), are defined in the form

$$h = \eta^2 F(t), \qquad \widehat{Q} = \eta^3 \Phi(t), \qquad \mu = 1, \tag{13}$$

where the functions $F(t)$ and $\Phi(t)$ are found by the equations

$$\dot{\Phi} + 4\Phi^2 F^{-1} + 2\beta F^2 = 0, \qquad \dot{F} + 3\Phi = 0, \tag{14}$$

where the dot denotes differentiation with respect to t, and

$$h = t^{2/(\mu-1)} f(z), \qquad \widehat{Q} = t^{3/(\mu-1)} q(z), \qquad \mu \neq 1, \tag{15}$$

where $z = J_1 = \eta t^{-\mu/(\mu-1)}$, and the functions $f(z)$, $q(z)$ satisfy the following system

$$\frac{3}{\mu-1}q - \frac{\mu}{\mu-1}q'z + \left(q^2 f^{-1} + 0.5\beta f^2\right)' = 0, \tag{16}$$
$$\frac{2}{\mu-1}f - \frac{\mu}{\mu-1}zf' + q' = 0. \tag{17}$$

Here, the stroke denotes differentiation with respect to z. Note that the system (16), (17), and (14) represent ordinary differential equations, and the relative velocity $\widehat{u} = \eta\Phi(t)/F(t)$ for $\mu = 1$ and $\widehat{u} = t^{1/(\mu-1)}q(z)/f(z)$ for $\mu \neq 1$.

3 Qualitative analysis of the family of the self-similar solution

3.1 The case $\mu = 1$

The system (14) may be solved exactly. It easy to obtain

$$\pm t = \begin{cases} \dfrac{3a}{2b^{3/2}} \ln\left[\left(\dfrac{\theta+1}{\theta_0+1}\right)\left(\dfrac{\theta_0-1}{\theta-1}\right)\right] - \dfrac{3}{\sqrt{b}}\left(\dfrac{\theta}{F^{1/3}} - \dfrac{\theta_0}{F_0^{1/3}}\right), & b > 0, \\ \dfrac{2}{\sqrt{a}}\left(F_0^{-1/2} - F^{-1/2}\right), & b = 0, \\ \dfrac{3a}{(-b)^{3/2}}\left(\tan^{-1}\psi - \tan^{-1}\psi_0\right) + \dfrac{3}{\sqrt{-b}}\left(\dfrac{\psi}{F^{1/3}} - \dfrac{\psi_0}{F_0^{1/3}}\right), & b < 0, \end{cases} \tag{18}$$

where $a = 36\beta > 0$, $\theta = \left(\left(aF^{1/3} + b\right)/b\right)^{1/2}$ and $\psi = \left(\left(aF^{1/3} + b\right)/(-b)\right)^{1/2}$, and θ_0 and ψ_0 are the same functions evaluated at $F = F_0$; b, F_0 are the constants of integration. For $\mu = 1$ the exact solution of equations (1) and (2) is of the form

$$h = \left(x - s_0 t^2/2\right)^2 F(t), \qquad Q = hs_0 t + \left(x - s_0 t^2/2\right)^3 \Phi(t). \tag{19}$$

where $\Phi(t) = -\dot{F}(t)/3$, and F is defined by (18). Evidently, the function $F(t)$ has a growing and decaying branch corresponding to the positive and negative roots in (18). The growing branches are particularly interesting as they imply that the avalanche thickness can increase without bound within a finite period of time for all choices of the parameter b. In the case $b < 0$ the solution degenerates for $F < F^* = (-b/a)^3$.

3.2 The case $\mu = -2$

Equation (17) can be integrated directly when $\mu = -2$ to give

$$q = 2fz/3 + c_1, \tag{20}$$

where c_1 is an arbitrary constant. Substituting the volume flux from (20) the momentum balance (16) reduces to

$$f' = f^2(c_1/3 - 2zf/9)\left(c_1^2 - \beta f^3\right)^{-1}. \tag{21}$$

In the case $c_1 = 0$ equation (21) can be integrated to give

$$f(z) = (9\beta)^{-1}z^2 + c_2, \tag{22}$$

where c_2 is an arbitrary constant. The exact solution of (1), (2) in this case is

$$h = t^{-2/3}\left[(9\beta)^{-1}\left(x - s_0t^2/2\right)^2 t^{-4/3} + c_2\right], \tag{23}$$

$$Q = hs_0t + 2t^{-1}\left[(9\beta)^{-1}\left(x - s_0t^2/2\right)^3 t^{-2} + c_2\left(x - s_0t^2/2\right)t^{-2/3}\right]/3. \tag{24}$$

Let us now consider more general solutions for the case $\mu = -2$ when $c_1 \neq 0$ in equation (21). It is convenient to introduce new variables y, p and ζ for f, q and z by the scalings

$$f = c_1y(c_1\beta)^{-1/3}, \qquad q = c_1p, \qquad z = 3(c_1\beta)^{1/3}\zeta, \tag{25}$$

which transform (21) into a parameter independent form

$$\frac{\partial y}{\partial \zeta} = y^2(1 - 2\zeta y)\left(1 - y^3\right)^{-1}. \tag{26}$$

The avalanche thickness is non-negative and we therefore restrict attention to the domain $y \geq 0$, $-\infty < \zeta < \infty$. The solutions of equation (26) are illustrated in Fig. 1. On the line $y = 1$, the gradient $\partial y/\partial\zeta \to \pm\infty$ for all points except one, where $1 - 2\zeta y = 0$. Consequently, the point $\zeta = 1/2$, $y = 1$ is a singular point. The asymptotic behaviour of the solution in the vicinity of the line $y = 1$ is described by formula, which can be obtain from (26),

$$y = 1 \pm \sqrt{2[(\zeta - \zeta_0)(\zeta + \zeta_0 - 1)]/3}, \qquad \zeta \to \zeta_0, \qquad y \to 1. \tag{27}$$

If $\zeta_0 = 1/2$ then

$$y = 1 \pm \sqrt{2/3}\,(\zeta - 1/2), \qquad \zeta \to 1/2, \qquad y \to 1. \tag{28}$$

The singular point $(1/2, 1)$ is a saddle point. On the line $y = 1/(2\zeta)$ the gradient $\partial y/\partial\zeta = 0$. Therefore, in the points of this line the function $y(\zeta)$ assumes an extremum. In the region $y < 1$ $y(\zeta)$ has maximum and when $y > 1$, $y(\zeta)$ has a minimum. When $\zeta \to \pm\infty$, two asymptotic formulas emerge, first

$$\zeta \to \infty, \qquad y \to 1/\zeta; \qquad \zeta \to -\infty, \qquad y \to -1/(2\zeta) \tag{29}$$

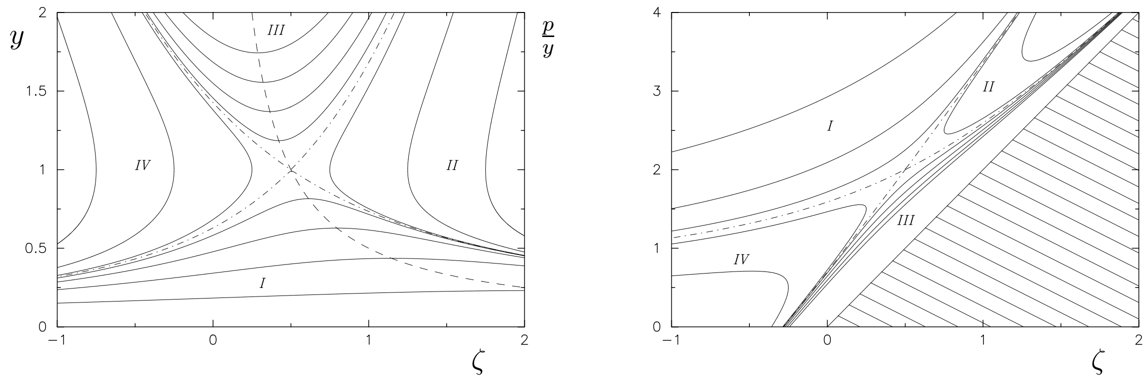

Figure 1. Four classes of asymmetric thickness, y, and velocity, p/y, solutions are illustrated as a function of ζ and labelled *I–IV*. The seperatrix between the solution domains is indicated with the dot-dash curve and the dashed line indicates where the first derivative of the thickness $\partial y/\partial\zeta = 0$. There are no solutions in the hashed region.

for the region $y > 1$, and second

$$\zeta \to \pm\infty, \qquad y \to \zeta^2, \tag{30}$$

for the region $y < 1$.

The curves, which pass through the singular point $(1/2, 1)$, divide the domain into the four parts. Each part has its own family of solutions of (26), which are denoted by I, II, III, IV (see Fig. 1). The families I–IV together with relations (6), (7), (15), (20), (25) define the various solutions of the system (1), (2). From a physical point of view family I is interesting for the motion of avalanches. Using this family we can construct the solution for various concrete situations.

Let us write the considered invariant solution for $\mu = -2$ and $c_1 \neq 0$ in the form

$$h(x,t) = (t+t_0)^{-2/3}\frac{c_1}{(c_1\beta)^{1/3}}\, y\left[\frac{x - 0.5s_0(t+t_0)^2}{3(c_1\beta)^{1/3}(t+t_0)^{2/3}}\right], \tag{31}$$

$$Q(x,t) = hs_0(t+t_0) + (t+t_0)^{-1}[c_1 + 2(x - 0.5s_0(t+t_0)^2)h(x,t)/3], \tag{32}$$

where $y(\zeta) \in I, II, III, IV$ and t_0, c_1 are arbitrary constants.

4 Results of the calculations, physical interpretation

4.1 Example 1. M-waves

The solutions (23), (24) was found using a separable variable approach by Savage & Hutter [1], who called it an "M"-wave. The name arose from the shape of a truncated solution

$$h(z,t) = \begin{cases} t^{-2/3} f(z), & |z| \le 1, \\ 0, & |z| > 1, \end{cases} \tag{33}$$

which connected a finite part of the solution (22) with regions of zero thickness and flux on either side. At $z = \pm 1$ there are jump discontinuities in both the thickness and the flux, which should satisfy the mass and momentum jump conditions. They are obtained from the system (1), (2)

$$x_j'[\![h]\!] - [\![Q]\!] = 0, \qquad x_j'[\![Q]\!] - [\![Q^2h^{-1} + \beta h^2/2]\!] = 0, \tag{34}$$

where x_j is the position and x'_j is the normal velocity of the discontinuity, the jump bracket $[\![f]\!] = f^+ - f^-$ and $f^\pm = f(x_j \pm 0)$. As we shall now show this is not the case. Assuming that on the positive side of the discontinuity $h^+ \neq 0$ and $Q^+ \neq 0$ and that on the negative side $h^- = Q^- = 0$ the jump conditions 34 imply

$$x'_j h^+ - Q^+ = 0, \qquad x'_j Q^+ - (Q^+)^2/h^+ - \beta(h^+)^2/2 = 0. \tag{35}$$

Mass balance therefore requires that $x'_j = Q^+/h^+$ and if this is substituted in the momentum jump condition we find that $h^+ = 0$ contradicting our original assumption that $h^+ \neq 0$. The jump conditions are therefore not satisfied by the truncated M-wave solution (33), and expansion fans develop at the jumps.

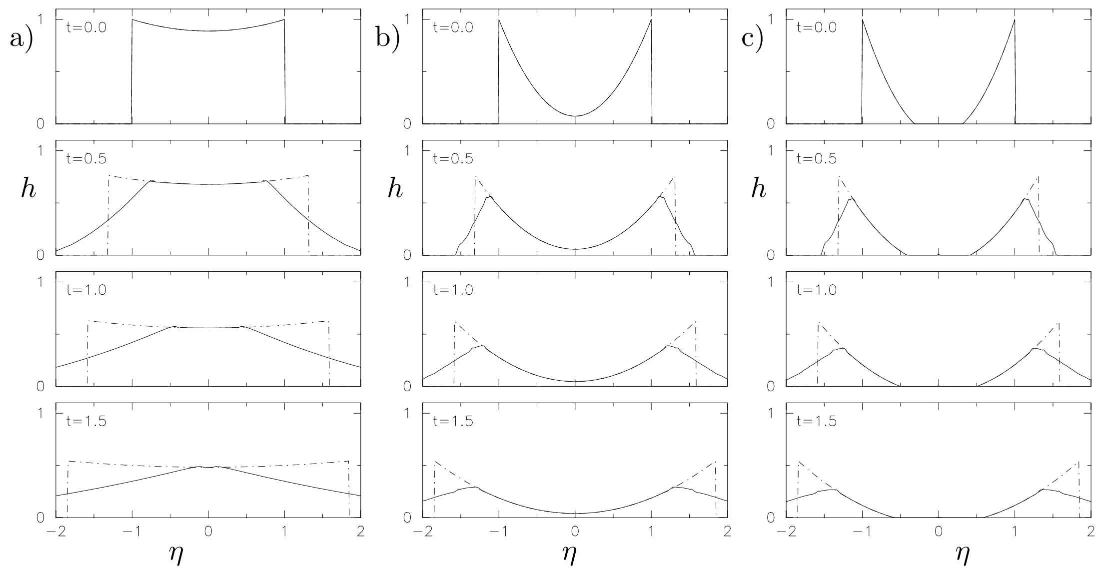

Figure 2. The temporal and spatial evolution of the avalanche thickness for the cases (a) $\beta = 1.0$, (b) $\beta = 0.12$ and (c) $\beta = 0.1$ are illustrated. The constant $c_2 = 1 - (9\beta)^{-1}$ to ensure that $h(\pm 1) = 1$. The solid line shows the computed solution in the accelerating coordinate system η and the dot-dash line shows the exact solution. Time is measured relative to initial conditions and the M-wave is evaluated at a finite time $t_0 = 1$.

To understand the collapse of the truncated M-wave in greater detail a series of numerical simulations have been performed using a Total Variational Diminishing Lax Friedrich's scheme [4]. This is a shock capturing method that has been extensively tested against the parabolic cap solution and the travelling shock solution. Fig. 2 shows the M-wave (33) and the numerical solutions for the avalanche thickness for various β at a sequence of time-steps in the accelerated coordinate system, η. The constant $c_2 = 1 - (9\beta)^{-1}$ to ensure that $h(\pm 1) = 1$. In each case the M-wave spreads out laterally, diminishing in height, and close to the discontinuities the shock expands as expected. The overlap domain, where the M-wave (33) and the computed solution are in close agreement, can either expand or contract in the physical domain. For $\beta = 1$ the overlap region decreases with time and the M-wave is destroyed in finite time. However, for thin avalanches where the aspect ratio $\varepsilon \ll 1$, and hence $\beta \ll 1$, the overlap domain expands in the physical domain despite the collapse close to the discontinuities. This is because the stretching of the solution is faster than the inward propagation speed of the disturbance from the discontinuities.

For $c_2 < 0$ the M-wave solution (33) contains regions of negative thickness. In this case the M-wave can be linked to the trivial solution $h \equiv 0$ using the jump conditions (34). This time, however, because the thicknesses and fluxes are zero on both sides of the discontinuity, the jump conditions are trivially satisfied. Case (c) in Fig. 2 shows the evolution of such an M-wave for the case $\beta = 0.1$.

These numerical simulations demonstrate that for shallow avalanches, in which $\beta \ll 1$, the invariant stretching solutions may exist over an expanding region in the physical domain even though they may not satisfy boundary conditions at the ends.

4.2 Example 2. Evolution of the wave

Let us consider a second example: it assumes that the initial instant the avalanche thickness distribution is described by the formula

$$h_0 = \frac{s_0}{2\beta x_0}\left(\frac{H_0}{3}\right)^2 y\left(\frac{x - x_0}{H_0}\right), \tag{36}$$

where H_0, x_0 are know the parameters and the function $y(\zeta)$ is one of the family I (Fig. 3). This profile has a maximum at $x = x_{\max}$. For example, $y(\zeta)$ is a function which has the maximum in the point $\zeta_{\max} = 1$ ($y_{\max} = 0.5$). Therefore, $x_{\max} = x_0 + H_0$ and $h_{0\max} = s_0 H_0^2/(16\beta x_0)$. $h_{0\max}$ is the initial amplitude of the wave. The mathematical model for this problem is the Cauchy problem for the system (1), (2) with the initial condition (36).

The solution of this problem is described by the formulas (31), (32) with the constants c_1, t_0 and function $y(\zeta)$ which are defined by comparison of expression (31) with (36) at $t = 0$. Fig. 3 a) shows the evolution of the wave with time; the calculations were performed with the following values of the parameters: $\zeta = 45^o$, $\delta = 30^o$, $s_0 = 0.2989$, $\beta = 0.1$, $x_0 = 0.1$, $H_0 = 1$. Fig. 3 b) displays the corresponding behaviour of the mass flux.

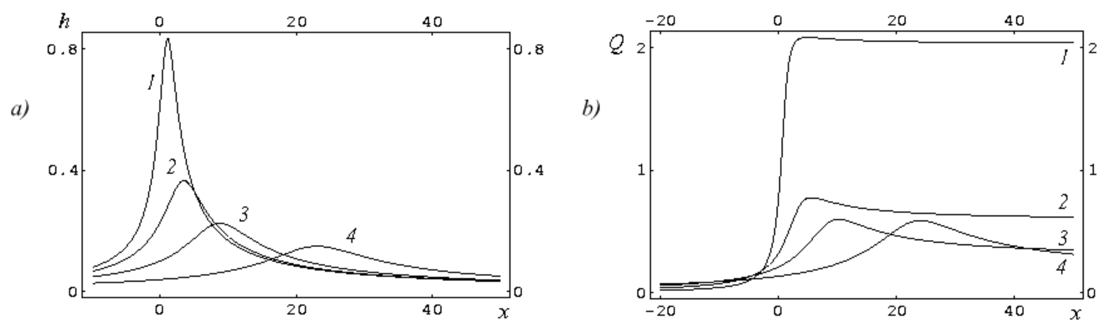

Figure 3. Evolution of the wave a) and the mass flux b) for (31), (32) ($1 - t = 0$, $2 - t = 2$, $3 - t = 5$, $4 - t = 10$).

4.3 Example 3. Evolution of the wave during the "time of the life"

Let us consider the situation when the initial wave begins to grow and moves contrary to the downslope direction. It is possible when the initial profile of the thickness is described by the expression

$$h_0 = AY\left(\frac{x - x_0}{\lambda_0}\right), \qquad Y(\zeta) = \frac{1}{y_{\max}} y(\zeta), \qquad y \in I, \tag{37}$$

where A is an amplitude, λ_0 is a length of the wave, x_0 is a parameter, and the mass flux is $Q_\infty = -q_\infty(t_0 - t)^{-1} < 0$ in the right side of the wave when $x \to \infty$, and $Q_{-\infty} = 0$ when

$x \to -\infty$. In order to construct the solution for this situation we again take the invariant solution (31), (32). It is easy to check that the system (1), (2) admits the transformations

$$Q = -\bar{Q}, \qquad h = \bar{h}, \qquad t = -\bar{t}, \qquad x = \bar{x}; \tag{38}$$
$$Q = \bar{Q}, \qquad h = \bar{h}, \qquad t = \bar{t} + t_0, \qquad x = \bar{x}. \tag{39}$$

Consequently, the any solution of the equations (1), (2) which was transformed by the (38), (39) again is a solution of the system (1), (2). Therefore from (31), (32) we have

$$\begin{aligned} h(x,t) &= (t_0 - t)^{-2/3} \frac{c_1}{(c_1\beta)^{1/3}} y\left(\frac{x - \hat{x}_0(t)}{\lambda(t)}\right), \\ Q(x,t) &= -h(x,t)\left[s_0(t_0 - t) + (t_0 - t)^{-1}\left(\frac{c_1}{h(x,t)} + \frac{2}{3}(x - \hat{x}_0)\right)\right], \end{aligned} \tag{40}$$

where $\bar{x}_0(t) = 0.5s_0(t_0 - t)^2$, $\lambda(t) = 3(c_1\beta)^{1/3}(t_0 - t)^{2/3}$, $y \in I$.

The boundary condition $Q_\infty = -q_\infty(t_0 - t)^{-1}$ define c_1: $c_1 = \frac{1}{3}q_\infty$. Comparing the first expression of (40) with (37), we find

$$y_{\max} = \frac{A\lambda_0}{q_\infty} < 1, \qquad t_0 = \frac{1}{3}\sqrt{\frac{\lambda_0^3}{q_\infty \beta}}, \qquad \zeta_{\max} = \frac{1}{2y_{\max}}, \qquad x_0 = 0.5s_0t_0^2. \tag{41}$$

The time t_0 can be named "Time of the life of the wave", because for $t > t_0$ the wave does not exist. The physical meaning is clear. In the infinity we have the mass sources. The mass of the avalanche grow and moves contrary to the downslope direction. But we have on the basal surface the contrary flux that produces the wave which grows. The increase of the mass increases the capacity of the avalanche to move to the downslope direction. But the power of the mass flux growed too. The struggle of these factors causes the infinite increase of the amplitude of the wave. The law of the motion of the wave is obtained from (40)

$$x_{\max}(t) = 0.5s_0(t_0 - t)^2 + 3\zeta_{\max}\left(\frac{\beta q_\infty}{3}\right)^{1/3}(t_0 - t)^{2/3}.$$

The numerical results are obtained for following values of parameters: $s_0 = 0.2989$, $A = 0.083$, $\beta = 0.1$, $\lambda_0 = 1$, $q_\infty = 0.166$ and are illustrated by the Fig. 4. From (41) we found $y_{\max} = 0.5$, $t_0 = 2.587$, $x_0 = 1$.

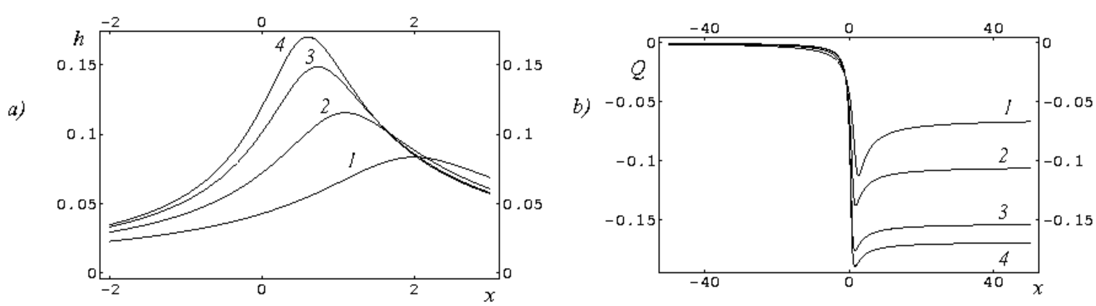

Figure 4. Behaviour of thickness a) and the mass flux b) during the "time of the life".

4.4 Example 4 ($\mu = 2$ in the general system (16), (17))

Let us consider the situation which is characterised by $\mu = 2$. This situation is interesting, because it allows us to model realistic case for the avalanches. For example, let us assume that

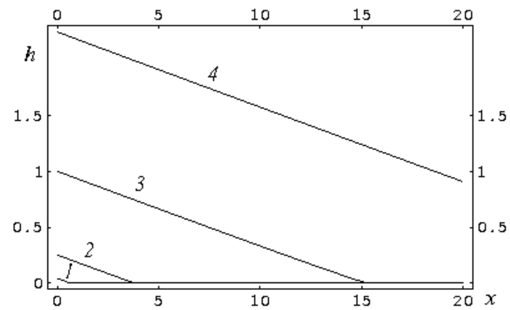

Figure 5. Dynamics of the thickness of the avalanche for $s_0 = 0.2989$, $h_0 = 0.01$ ($1 - t = 0$, $2 - t = 5$, $3 - t = 10$, $4 - t = 15$).

at the initial time the thickness of the avalanche is zero in the region $x > 0$ (there is a wall at point $x = 0$). The large mass of the granular material is in the region $x < 0$. When $t > 0$ the screen is lifted and suddenly removed, such that the thickness of the orifice is $h_0 t^2$. It is necessary to describe the evolution of free boundary of the avalanche for $x > 0$. For this purpose we take the solution (15) where $\mu = 2$

$$h = t^2 f(z), \qquad \hat{Q} = t^3 q(z), \qquad z = \left(x - 0.5 s_0 t^2\right) t^{-2}. \tag{42}$$

The expression for z from (42) can be written in the form

$$z = x t^{-2} - 0.5 s_0.$$

The latter relation shows that the region $z \le -0.5 s_0$ corresponds to the domain $x \le 0$. Now, in order to resolve the problem we must find the solution of the system (16), (17) when $\mu = 2$ and with the boundary conditions

$$z = -0.5 s_0, \qquad f = h_0, \tag{43}$$

$$f = 0, \qquad q = 0. \tag{44}$$

Fig. 5 illustrates the dynamics of the thickness of the avalanche for $s_0 = 0.2989$, $h_0 = 0.01$. Curve 1 corresponds to the moment $t = 3$; curve 2 is for $t = 5$; and curves 3, 4 are for $t = 10$, $t = 15$.

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Symmetric Sets of Solutions to Differential Problems

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The presence of a (Lie-point) symmetry for a differential equation leads naturally to the useful notions of symmetric sets of solutions, i.e. of sets which are mapped into themselves by the symmetry, and of orbits of solutions. We introduce the definition of *partial* symmetry, and show that the above notions may be preserved, although the symmetry is not exact. We consider the quite exceptional case of the Liouville equation, which admits an extremely large algebra of symmetries (the conformal symmetry algebra), and we shall see that any modification of this equation destroys this situation, but leaves the possibility of the existence of partial symmetries. Other simple examples are also considered, including a case of generalized (or Lie–Bäcklund) symmetry.

1 Introduction

It is certainly well known that symmetry principles may offer several useful tools and many different implications in the analysis of differential equations (see, e.g., [1–10] and references therein), but probably the most obvious and direct consequence is the fact that any symmetry of a given equation transforms solutions into (generally, different) solutions of the same equation. In other words, given a differential equation, say $\Delta = 0$, with a set of solutions $\mathcal{S}_\Delta$, a symmetry T of this equation is an invertible transformation such that $T(\mathcal{S}_\Delta) = \mathcal{S}_\Delta$; in this sense, we can say that $\mathcal{S}_\Delta$ is a *symmetric set of solutions under* T.

For the sake of concreteness and simplicity, we will be concerned here only with the case of partial differential equations, written in the usual form [3]

$$\Delta := \Delta\left(x, u^{(m)}\right) = 0, \tag{1}$$

where Δ is a smooth function (or possibly a system of ℓ functions) of the p "independent" variables $x := (x_1, \ldots, x_p) \in \mathbb{R}^p$ and of the q "dependent" variables $u := (u_1, \ldots, u_q) \in \mathbb{R}^q$, together with the derivatives of the u_α with respect to the x_i ($\alpha = 1, \ldots, q$; $i = 1, \ldots, p$) up to some order m. Also, we will consider here mainly continuous Lie-point symmetries, in the usual sense and under the usual assumptions (see [3]), although our arguments (in Section 3) could be easily extended e.g. to generalized or Lie–Bäcklund symmetries (as we will briefly show by means of an example in Section 4), or also to discrete symmetries. Denoting by

$$X = \xi_i(x,u)\frac{\partial}{\partial x_i} + \varphi_\alpha(x,u)\frac{\partial}{\partial u_\alpha} \tag{2}$$

the infinitesimal Lie generator of a symmetry of the given problem (1), we can also say that $\mathcal{S}_\Delta$ is a symmetric set under X.

A strictly related fact to the presence of a symmetry, is that, given any solution u_0 of $\Delta = 0$, then there is an *orbit* $u^{[\lambda]}$ of solutions obtained under the application to u_0 of the (finite) transformations $T = T^{[\lambda]} = \exp(\lambda X)$ generated by X (here λ is the real Lie parameter, and – as usual – we have generally only a *local* group of transformations $T^{[\lambda]}$, i.e. λ runs only in some interval.) Clearly, orbits provide examples of symmetric sets of solutions under X. Apart from the trivial case of solutions u_0 invariant under T, any orbit $u^{[\lambda]}$ can be naturally parametrized by the Lie parameter λ, and satisfies the differential equation (in "evolutionary form" [3])

$$Q\,u^{[\lambda]} = \frac{\mathrm{d}u^{[\lambda]}}{\mathrm{d}\lambda}, \tag{3}$$

where

$$Q = -\xi_i(x,u)\frac{\partial}{\partial x_i} + \varphi_\alpha(x,u)\frac{\partial}{\partial u_a}. \tag{4}$$

Many examples of this situation are well known. In Section 2, we shall consider a rather exceptional case, which is provided by the Liouville equation (both in the "Galilean" or in the "Minkowski" case, see below (5), (6)), which has an enormous relevance in mathematical physics, and which exhibits the quite singular and peculiar (i.e., unique in its class) property of admitting an extremely large algebra of symmetries, the conformal symmetry algebra.

However, the case of Liouville equation is certainly exceptional. In fact, the examples of PDE's admitting nontrivial symmetries (Lie-point or generalized) are relatively rare. Therefore, one is urged to extend the concept of symmetry. Notions of conditional, nonclassical or similar notions of symmetries are also well known (see e.g. [2, 6, 11, 12, 13, 14, 15]). In Section 3, we shall introduce the notion of *partial symmetry* (see [16]), which is in some sense intermediate between that of exact and of conditional symmetry; we shall show in particular the existence also in this case, although the partial symmetry T is *not* exact, of proper subsets $\mathcal{P} \subset \mathcal{S}_\Delta$ of solutions of the given equation, which are *symmetric sets*, i.e. such that $T(\mathcal{P}) = \mathcal{P}$, meaning that $\mathcal{P}$ is a subset of solutions which are transformed into one another by T. Similarly, the notion of orbit of solutions under the partial symmetry T remains valid, together with its characteristic property expressed by equations (3), (4).

2 The symmetry properties of the Liouville equation

The equation

$$u_{xx} + u_{yy} = \exp(u), \qquad u = u(x,y) \tag{5}$$

has a long history. It was introduced by Liouville, studied by Poincaré, Picard, and many others in the past, and reconsidered in recent years. Actually, it enters in many areas of applied mathematics and physics, including fluid vortex theory, problems concerning electric charge distribution round a glowing wire, surface singularities, instantons and solitons theory, whereas the recent interest is concerned mainly with $(2+1)$-dimensional quantum gravity (see e.g. [17, 18, 19]). The modern applications in classical and quantum field theory deal not only with the "Galilean" version of the Liouville equation (5), but also with its "Minkowski" form

$$u_{xx} - u_{yy} = \exp(u) \tag{6}$$

but, for simplicity, we will consider only the equation (5) (actually, all our conclusions can be suitably extended to the case (6)).

We start considering, instead of (5), the following general equation

$$u_{xx} + u_{yy} = F(u), \tag{7}$$

where $F = F(u)$ is a (smooth) function, and perform the "group theoretical analysis" of this equation, i.e. look for its Lie-point symmetries depending on the choice of $F(u)$ (we can exclude the completely elementary case of "linear" $F = a + bu$). According to standard and well known procedures [3], one can easily see that, in addition to the obvious translation and rotation symmetries, and apart from the special case

$$F(u) = (u+k)^{1+r}, \qquad r, k = \text{const}, \quad r \neq 0$$

admitting the symmetry

$$X = \frac{r}{2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) - (u+k)\frac{\partial}{\partial u}$$

the *unique* case admitting an "interesting" symmetry is just

$$u_{xx} + u_{yy} = \pm \exp(\pm u) \tag{8}$$

which exhibits the following family of symmetries

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \varphi(x,y)\frac{\partial}{\partial u}, \tag{9}$$

where the coefficients $\xi(x,y)$, $\eta(x,y)$ must satisfy

$$\xi_x = \eta_y, \qquad \xi_y = -\eta_x \tag{10}$$

which imply that

$$\Delta\xi = 0, \qquad \Delta\eta = 0 \tag{11}$$

(in other words, ξ and η must be harmonic conjugated functions), and where φ is given by

$$\varphi = -(\xi_x + \eta_y) = -2\xi_x. \tag{12}$$

We shall then say that (8) admits "full conformal symmetry". This result may be in some sense reversed and strengthened in the following form (the proof can be obtained by means of direct calculations)

Proposition 1. *A PDE for the function $u = u(x,y)$ of the form $\Delta := \Delta(u, u_{xx}, u_{yy}) = 0$ admits full conformal symmetry if and only if Δ depends on u, u_{xx}, u_{yy} only through the combination $\widetilde{u} := (u_{xx} + u_{yy})\exp(\pm u)$.*

Starting from any solution of the Liouville equation, and using its symmetries, one can write down many different orbits of solutions. Precisely, let $u_0 = u(x_0, y_0)$ be any solution, expressed in terms of the "initial" variables denoted here for convenience by x_0, y_0; let us perform a conformal (finite) transformation into the new variables $x = x(\lambda)$, $y = y(\lambda)$, with the infinitesimal generators defined by the harmonic conjugated functions ξ, η, i.e. a transformation satisfying

$$\frac{\partial x}{\partial\lambda} = \xi(x,y), \qquad \frac{\partial y}{\partial\lambda} = \eta(x,y) \tag{13}$$

with the "initial conditions"

$$x(0) = x_0, \qquad y(0) = y_0. \tag{14}$$

Let us denote by

$$x_0 \to x \equiv x(\lambda) = p(x_0, y_0, \lambda), \qquad y_0 \to y \equiv y(\lambda) = q(x_0, y_0, \lambda) \tag{15}$$

this transformation, and by

$$x_0 = P\big(x(\lambda), y(\lambda), \lambda\big), \qquad y_0 = Q\big(x(\lambda), y(\lambda), \lambda\big) \tag{16}$$

its inverse, then the orbit of new solutions is given by

$$u^{[\lambda]} := u_0\Big(P(x(\lambda), y(\lambda), \lambda), Q(x(\lambda), y(\lambda), \lambda)\Big) + w(x, y; \lambda), \tag{17}$$

where

$$\begin{aligned} w(x, y; \lambda) &= -\int_0^\lambda \Big(\xi_x\big(P(x(\lambda'), y(\lambda'), \lambda'), Q(x(\lambda'), y(\lambda'), \lambda')\big) + \eta_y\big(\dots\big)\Big)\mathrm{d}\lambda' \\ &= -\ln(\nabla P \cdot \nabla P) = \ln(\nabla Q \cdot \nabla Q). \end{aligned} \tag{18}$$

For instance, an orbit of solutions to the Liouville equation is the following

$$u^{[\lambda])} = -\ln\left(\frac{\left(1 + 2\lambda x + \lambda^2\left(x^2 + y^2\right)\right)^2}{2}\sin^2\left(\frac{x + \lambda\left(x^2 + y^2\right)}{1 + 2\lambda x + \lambda^2\left(x^2 + y^2\right)}\right)\right).$$

It has been obtained from (18) choosing in (9) $\xi = x^2 - y^2$, $\eta = 2xy$ and starting from a known solution to the Liouville equation (which can be recognized putting $\lambda = 0$ in the above expression).

3 Partial symmetries

Let us consider a general differential problem, given in the form of a system of ℓ partial differential equations, and shortly denoted, as usual, as in (1). Let

$$X = \xi_i(x, y)\frac{\partial}{\partial x_i} + \varphi_\alpha(x, y)\frac{\partial}{\partial u_\alpha} \tag{19}$$

be a given vector field, where ξ_i and φ_α are respectively p and q smooth functions. We will shortly denote by X^* the "suitable" prolongation of X, i.e. the prolongation which is needed when one has to consider its application to the differential problem in consideration. Alternatively, we may consider X^* as the infinite prolongation of X, it is clear indeed that only a finite number of terms are required and will appear in all the actual computations. The vector field X is (the Lie generator of) an exact symmetry of the differential problem (1) if and only if

$$X^*\Delta\Big|_{\Delta=0} = 0, \tag{20}$$

i.e. if and only if the prolongation X^* (here obviously, $X^* = \mathrm{pr}^{(m)}(X)$, the m-th prolongation of X), applied to the differential operator Δ defined by (1) vanishes once restricted to the set $S^{(0)} := \mathcal{S}_\Delta$ of the solutions to the problem $\Delta = 0$.

We now assume that the vector field X is *not* a symmetry of (1), hence $X^*\Delta\Big|_{S^{(0)}} \neq 0$: let us then put

$$\Delta^{(1)} := X^*\Delta. \tag{21}$$

This defines a differential operator $\Delta^{(1)}$, of order m' not greater than the order m of the initial operator Δ. Assume now that the set of the simultaneous solutions of the two problems $\Delta = 0$ and $\Delta^{(1)} = 0$ is not empty, and let us denote by $S^{(1)}$ the set of these solutions. It can happen that this set is mapped into itself by the transformations generated by X: *this situation is characterized precisely by the property*

$$X^*\Delta^{(1)}\Big|_{S^{(1)}} = 0.$$

Then, in this case, we can conclude that, although X is not a symmetry for the full problem (1), it generates anyway a transformation which leaves globally invariant a family of solutions of (1): this family is precisely $S^{(1)}$.

But it can also happen that $X^*\Delta^{(1)}|_{S^{(1)}} \neq 0$, we then put

$$\Delta^{(2)} := X^*\Delta^{(1)} \tag{22}$$

and look for the solutions of the system

$$\Delta = \Delta^{(1)} = \Delta^{(2)} = 0$$

and repeat the argument as before: if the set $S^{(2)}$ of the solutions of this system is not empty and satisfies in addition the condition

$$X^*\Delta^{(2)}\Big|_{S^{(2)}} = 0$$

then X is a symmetry for the subset $S^{(2)}$ of solutions of the initial problem (1), exactly as before.

Clearly, the procedure can be iterated, and we can say:

Proposition 2. *Given the general differential problem (1) and a vector field (19), define, with* $\Delta^{(0)} := \Delta$,

$$\Delta^{(r+1)} := X^*\Delta^{(r)}. \tag{23}$$

Denote by $S^{(r)}$ *the set of the simultaneous solutions of the system*

$$\Delta^{(0)} = \Delta^{(1)} = \cdots = \Delta^{(r)} = 0 \tag{24}$$

and assume that there is an integer s *such that* $S^{(r)}$ *is not empty for* $r \leq s$, *and*

$$X^*\Delta^{(r)}\Big|_{S^{(r)}} \neq 0 \qquad \text{for} \quad r = 0, 1, \ldots, s-1, \tag{25}$$

$$X^*\Delta^{(s)}\Big|_{S^{(s)}} = 0. \tag{26}$$

Then the set $S^{(s)}$ *provides a family of solutions to the initial problem (1) which is mapped into itself by the transformations generated by* X.

It is clear that, given a differential problem and a vector field X, it can happen that the above procedure gives no result, i.e. that at some k-th step the set $S^{(k)}$ turns out to be empty. Assume instead that a nonempty subset $S^{(s)}$ of solutions has been found according to the above procedure: we shall then say that X is a *partial symmetry* for the problem (1), and the subset of solutions $\mathcal{P} := S^{(s)}$ obtained in this way is globally invariant under X and therefore a symmetric set.

Alternatively, one may also say that this vector field X is an *exact* symmetry for the system

$$\begin{aligned} &\Delta = 0, \\ &\Delta^{(1)} = 0, \\ &\quad\vdots \\ &\Delta^{(s)} = 0. \end{aligned} \tag{27}$$

It must be emphasized that the solutions in this set are, in general, *not* invariant under the action of X: only the set $S^{(s)}$ is globally invariant, while the solutions are transformed into one another under the X action. As in the case of exact symmetries, the set of solutions in $S^{(s)}$ will be constituted by one or more *orbits* under the action of the one-parameter Lie group $T^{[\lambda]} = \exp(\lambda X)$, and each family $u^{[\lambda]}$ satisfies the same differential equation (3), (4). It may happen that the set $S^{(s)}$ contains also solutions u_0 which are *invariant* under $T^{[\lambda]}$, i.e. $T^{[\lambda]}u_0 = u_0$, which can be considered as trivial orbits: if this is the case, then the partial symmetry X is also a *conditional symmetry* (see [2, 12, 13]) for the problem at hand. In this sense, we can say that partial symmetries extend the notion of conditional symmetries.

4 Partial symmetries and symmetric sets: examples

We will briefly propose here some quite simple examples of PDE's admitting partial symmetries and of symmetric sets of solutions under these symmetries. More elaborate examples, including e.g. Boussinesq and Korteweg-de Vries equations, can be found in [16]. The idea can be suitably extended also to ordinary differential equations and to dynamical systems, with an application to Mel'nikov theory for the appearance of chaotic homoclinic (or heteroclinic) motion [20], or to discrete symmetries as well [16].

Example 1. It has been shown in Section 2 that the 2-dimensional Laplace equation with nonlinear additional terms $F(u)$ admits quite exceptionally some symmetry; the same is true in the presence of terms containing higher order derivatives. But partial symmetries may be allowed. Consider e.g. equations of the form

$$u_{xx} + u_{yy} = G(u, u_{x^m}), \tag{28}$$

where u_{x^m} stands for the m-th order derivative $\partial^m u/\partial x^m$, $m > 2$, and with $\partial G/\partial u_{x^m} \neq 0$. Now, the vector field

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \tag{29}$$

generating rotations in the plane x, y is clearly not a symmetry for (28), but it is a partial symmetry. Indeed, applying our procedure, one gets at the first step

$$X^*\Delta = \Delta^{(1)} = m\,\frac{\partial G}{\partial u_{x^m}}\,\frac{\partial^m u}{\partial x^{m-1}\partial y} = 0. \tag{30}$$

But applying the convenient prolongation X^* to this equation, one obtains $X^*(u_{x^{m-1}y}) \neq 0$ (indeed, (30) does not admit rotation symmetry), and therefore other steps are necessary in order to reach the condition $X^*\Delta^{(s)} = 0$, as requested by Proposition 2. One finds finally that the symmetric set $S^{(s)}$ of solutions must satisfy, together with the initial equation (28), the system of the $m+1$ equations

$$\frac{\partial^m u}{\partial x^n \partial y^{m-n}} = 0, \qquad n = 0, \ldots, m$$

(i.e., all the m-th order derivatives must vanish). For instance, if $G = G(u_{xxxx})$, the set $S^{(s)}$ has the form

$$\begin{aligned} S^{(s)} := \Big\{ u = A_0 + \frac{c}{4}\left(x^2+y^2\right) + A_1 x + B_1 y \\ + A_2\left(x^2-y^2\right) + B_2 xy + A_3\left(x^3-3xy^2\right) + B_3\left(3x^2y-y^3\right) \Big\}, \end{aligned}$$

where $c = G(0)$, and it is easy to recognize that this set contains a set of rotationally *invariant* solutions, and different families of orbits of solutions which are transformed into themselves under rotations. The presence in this set of rotationally invariant solutions shows that the rotation symmetry is in this example also a conditional symmetry for the equation (28), but the notion of partial symmetry provides clearly a larger set of solutions. Let us emphasize that it should be *not* sufficient to impose only the vanishing of the "symmetry breaking term" in the initial equation (28), or only the first condition obtained above (30), i.e. one or both of the conditions

$$\frac{\partial^m u}{\partial x^m} = 0, \qquad \frac{\partial^m u}{\partial x^{m-1}\partial y} = 0$$

indeed, a generic solution of these equations and of the initial one would be transformed by rotations into a $v(x,y)$ which is *not* a solution!

Example 2. As another example, consider vector fields of the form

$$X = \varphi_\alpha(x)\frac{\partial}{\partial u_\alpha}. \tag{31}$$

If this is an exact symmetry of some equation $\Delta = 0$, one has that — given any solution u_0 of this equation — then $u_0 + \lambda\varphi$ is also a solution. But if X is only a partial symmetry, then this is true only for some special u_0: this gives rise to a "partial linear superposition principle". For instance, for the equation

$$\Delta := u_x^2 - u_y^2 - u_x - 2u_y - u + x = 0 \tag{32}$$

one can verify that the vector field

$$X = \exp(-x-y)\frac{\partial}{\partial u} \tag{33}$$

is a partial (not exact) symmetry, and in fact

$$u^{[\lambda]}(x,y) = x + \lambda\exp(-x-y)$$

is a symmetric set of solutions to (32). Notice that this set contains just a single orbit, and that there are no invariant solutions under the above (33) in this set: this means that in this example the partial symmetry X is *not* a conditional symmetry.

Example 3. Our final example deals with generalized (or Lie–Bäcklund) symmetries, and illustrates that our method is also applicable to these symmetries. We consider an equation for $u = u(t,x)$ of the form (Burgers, Fisher, Fitzhugh–Nagumo equations are of this form)

$$u_t = u_{xx} + R(u, u_x) \tag{34}$$

and the generalized vector field

$$X = (u_{xx} - 2u)\frac{\partial}{\partial u}. \tag{35}$$

It has been shown by Zhdanov [21] that (35) is a conditional Bäcklund symmetry for equations of the form (34) if and only if the nonlinear term R satisfies a special equation (see [21]). We now choose

$$R = u_x^2 - u^2$$

which does *not* satisfy Zhdanov equation. However, repeating word for word our above procedure, it can be seen that (35) is a partial Bäcklund symmetry for this equation, and in fact

$$u_{\pm}^{[\lambda]} = \exp(t \pm x + \lambda)$$

are two families of solutions to the above equation. As expected, no invariant solution under (35) is contained in this set, and therefore (35) is not a conditional Bäcklund symmetry.

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Symmetries, Singularities and Integrability in Nonlinear Mathematical Physics and Cosmology

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An overview is given of the interplay between symmetries, singularities and integrability and their uses in nonlinear problems arising in Mathematical Physics and Cosmology. A particularly important aspect is the role of nonlocal symmetries in deciding about integrability of complex nonlinear problems which do not apparently admit solutions in closed form. The need for a new approach to the evolution of symmetries themselves is also discussed.

1 Concepts of integrability

In all the areas of Mathematical Modelling which give rise to differential equations the modelling process includes the solution of those differential equations, be they (systems of) ordinary differential equations or partial differential equations. If this be possible in some sense, the system of differential equations is said to be integrable. (Note that we exclude numerical integration since this requires merely the existence of a continuous solution and that property can even be found in chaotic/turbulent systems.) A critical question is the meaning of "in some sense". There are four possible ways to prescribe integrability. They are

(i) the ability to display a nonlocal functional equation involving the dependent and independent variables; this need not be explicit and, should the equation be implicit, the inversion by means of the Implicit Function Theorem need be no more than local,

(ii) the existence of a number of functionally independent first integrals/invariants equal to the order of the system in general and half that for a Lagrangian system as a consequence of Liouville's Theorem [1],

(iii) the existence of a sufficient number of Lie symmetries to reduce the differential equation (or system; unless otherwise obviously the singular implies the plural) to an algebraic equation and

(iv) the possession of the Painlevé Property.

These concepts are not entirely equivalent. In particular (*iv*) requires that the solution be analytic or possess no more than algebraic branch points in the complex plane (planes for more than one independent variable) and this is not demanded by (*i*), (*ii*) and (*iii*) although, of course, the idea that a solution must be analytic to be considered as a solution has been with us since the days of Poincaré. Even (*i*) and (*ii*) are not equivalent since it is not always possible to eliminate nonlocally the derivatives from the functionally independent first integrals/invariants. Case (*iii*) differs from (*i*) and (*ii*) since the final algebraic equation is in terms of the invariants of the symmetries used in the reduction of order and the reversal of the process – on the assumption that a nonlocal solution of the algebraic equation exists – requires a series of quadratures which one may not be able to perform in closed form. In the case of Lagrangian systems the celebrated theorem of Noether [2] allows the identification of (*ii*) and (*iii*). The precise nature of the relationship between (*iii*) and (*iv*) has yet to be revealed although some recent work points to a subtler relation than previously expected [3–7].

2 Evolution of symmetry

When Lie introduced his ideas of symmetry based upon the geometry of infinitesimal transformations [8], the symmetries were naturally in the variables of the extended configuration space. With his introduction of contact transformations [9] the variables of the transformation became those of the extended phase space. For Lie both point and contact symmetries were seen in the context of the geometry of a space of finite dimensions. The adoption of generalised symmetries by Noether removed this constraint, particularly in the case of partial differential equations. (The order of the equation for an ordinary differential equation provides an effective bound in that case.) The inclusion of nonlocal symmetries was necessitated by the observation of the so-called "hidden symmetries" in which "regular symmetries" seemed to appear from nowhere on the lowering or raising of the order of an equation. To take a trivial example, in the change of order

$$Y''' = 0 \quad \Leftrightarrow \quad y'' = 0; \quad x = X, \quad y = Y'$$

the point symmetries

$$\gamma_1 = x^2\partial_x + xy\partial_y, \qquad \Gamma_2 = X^2\partial_X + 2XY\partial_Y$$

of the latter and former equations respectively come from the nonlocal symmetries

$$\Gamma_1 = X^2\partial_X + 3\left(XY - \int Y\mathrm{d}X\right)\partial_Y, \qquad \gamma_2 = x^2\partial_x + 2\left(\int y\mathrm{d}x\right)\partial_y$$

of the former and latter respectively.

When one accepts the generality of form implied by a nonlocal symmetry, there is as little need for the imputed esoterica of 'hidden' as there is to distinguish between geometrical and dynamical symmetries in Mechanics.

A feature of the Lie symmetries of a differential equation is that they constitute an algebra, a representation of a group, and the algebra is used to place a given differential equation in an equivalence class. As a trivial example all scalar second order ordinary differential equations have eight point symmetries with the algebra $sl(3,\mathbb{R})$ and so belong to the equivalence class of $y'' = 0$. In the case of $y'' = 0$ not all of those eight symmetries are required to specify it completely. There is, as it were, an oversupply of symmetry for the specification just as there is for the integrability, for, if we require the equation

$$y'' = f(x, y, y')$$

to possess the three symmetries, just three of the eight point symmetries constituting the elements of $sl(3,\mathbb{R})$,

$$\gamma_1 = \partial_x, \qquad \gamma_2 = \partial_y, \qquad \gamma_3 = x\partial_y,$$

the right hand side is constrained to be zero. Any scalar second order ordinary differential equation is completely specified by three symmetries [11].

When Krause introduced the concept of a complete symmetry group [12], the vehicle for his exposition was the Kepler problem with the equation of motion

$$\ddot{\boldsymbol{r}} + \frac{\mu\hat{\boldsymbol{r}}}{r^2} = 0, \qquad r^2 = x^2 + y^2 + z^2 \tag{1}$$

which possesses the five Lie point symmetries

$$\Gamma_1 = y\partial_z - z\partial_y, \qquad \Gamma_2 = z\partial_x - x\partial_z, \qquad \Gamma_3 = x\partial_y - y\partial_x,$$
$$\Gamma_4 = \partial_t, \qquad \Gamma_5 = 3t\partial_t + 2r\partial_r$$

with the algebra $A_2 \oplus so(3)$. These five symmetries are insufficient to specify completely (1) and Krause found it necessary to find the three nonlocal symmetries

$$\Gamma_6 = \left(\int x\,\mathrm{d}t\right)\partial_t + xr\partial_r, \qquad \Gamma_7 = \left(\int y\,\mathrm{d}t\right)\partial_t + yr\partial_r, \qquad \Gamma_8 = \left(\int z\,\mathrm{d}t\right)\partial_t + zr\partial_r,$$

a type of generalised conformal symmetry, to complete the task. Subsequently Nucci [13] obtained these nonlocal symmetries by standard local methods. Nucci and Leach [14] added an additional six nonlocal symmetries obtained by means of a reduction for the Kepler Problem based on the Ermanno–Bernoulli components of the Laplace–Runge–Lenz vector and showed that similar results were obtained for other systems possessing a conserved vector analogous to the Laplace–Runge–Lenz vector.

In the gradual evolution of the concept of a symmetry – a process of over a century – there have been both gains and losses. The gains have included an increased variety of systems that can be integrated using symmetries and a greater understanding of the rôle played by symmetry in integrability. For example the generalisation of the Hénon–Heiles problem [15] with Hamiltonian

$$H = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + Ax^2 + By^2\right) + D^2y - \frac{1}{3}Cy^3$$

is known to be integrable in the cases that $C = -2D$, $C = -6D$ and $C = -D$. Clearly the existence of one first integral, the Hamiltonian, is due to the symmetry ∂_t. The existence of a second first integral is due to the existence of another point symmetry in the first two cases. For the third the responsible symmetry is the nonlocal symmetry [16]

$$\Gamma = y\partial_t + \dot{y}\left(2x - F\right)\partial_x + y\partial_y,$$

where, in the coefficient function of ∂_x, F is the nonlocal term given by

$$F = \int \frac{\dot{x}\dot{y} + xy(1+2x)}{\dot{y}^2}\,\mathrm{d}t.$$

In the computation using the Lie method of the first integral

$$I = \dot{x}\dot{y} + xy + \frac{1}{3}x^3 + xy^2$$

that coefficient is not used.

An even more dramatic example is found in the trivially integrated

$$yy'' - y'^2 = f'y^{n+2} + nfy'y^{n+1}, \tag{2}$$

which was advanced as an integrable equation devoid of symmetry [17, 18]. By means of the simple, albeit nonlocal, transformation

$$X = x, \qquad Y = -\int nfy^n\,\mathrm{d}x + \log\left[-\int nfy^n\,\mathrm{d}x\right] - \log f$$

(2) becomes

$$\frac{\mathrm{d}^2Y}{\mathrm{d}X^2} = 0$$

which possesses eight Lie point symmetries with the algebra $sl(3,\mathbb{R})$. These Lie point symmetries find expression as nonlocal symmetries for (2).

There are two areas of loss. In the first instance the ease of calculation of Lie point symmetries and its algorithmic implementation in symbolic manipulation codes is lost when one seeks nonlocal symmetries and somewhat diminished in the cases of contact and generalised symmetries. This very practical problem is likely to maintain the popularity of point, contact and generalised symmetries for many years to come. At a more elevated mathematical level is the problem of deciding between those symmetries which are useful and those which are useless. How does one decide if a nonlocal symmetry is useful or not? Exponential nonlocal symmetries are fine for determining invariants [19] but not for reduction of order since the reverse procedure is not a matter of quadratures [20]. We have instanced above examples in which one would not credit the nonlocal symmetry as having more than curiosity value and yet integrability results. The resolution of this question is one of the more difficult theoretical problems in the study of symmetry. For the nonce one's choice of the type of symmetry to use is more than likely to be based upon utilitarianism than generality [21].

3 Putting symmetry to work

We illustrate the uses of symmetry in resolving some classes of problems which arise in Mathematical Physics and Cosmology.

There exist hierarchies of integrable partial differential equations which have attracted considerable attention over the last forty years. One of these of more recent interest is the hierarchy of evolution equations

$$u_t = R^m[u]\left(u^{-2}u_x\right)_x, \tag{3}$$

where the recursion operator

$$R[u] = D_x^2 u^{-1} D_x^{-1}$$

generates the hierarchy. This hierarchy has been shown to be linearisable, to possess an infinite number of symmetries and autohodograph transformations [22, 23]. The class (3) possesses four Lie point symmetries [24], *videlicet*

$$\Gamma_1 = \partial_t, \qquad \Gamma_2 = \partial_x, \qquad \Gamma_3 = (m+2)t\partial_t + u\partial_u, \qquad \Gamma_4 = -x\partial_x + u\partial_u$$

and these may be used to reduce the $1+1$ evolution equation to a nonlinear ordinary differential equation. A suitable choice for the reduction is [25]

$$\Gamma = \frac{1}{m+2}\Gamma_3 + (m+1)\Gamma_4 = t\partial_t - \frac{m+1}{m+2}x\partial_x + u\partial_u$$

(Γ_1 and Γ_2 could be included to allow for translation in t and x, but here we are illustrating a point and not essaying an exhaustive study.) and, since the reduced equation inherits a scaling symmetry, a further transformation based on that symmetry leads to the autonomous equation

$$\mathrm{e}^{-T}R[X]^m\mathrm{e}^T\left[-\left(\frac{1}{X}\right)' + \frac{1}{X}\right]' + \frac{m+1}{m+2}\dot{X} - \frac{1}{m+2}X = 0,$$

where

$$R[X] = -\left(\mathrm{e}^T\right)^2 X^{-1}\mathrm{e}^{-T}D_T^{-1}\mathrm{e}^{-T}$$

and the prime represents differentiation with respect to the new independent variable T, in which one notes that there is a preservation of the recursion property. The overall transformation is

$$T = -\log xt^{\frac{m+1}{m+2}}, \qquad X = uxt^{\frac{2m+3}{m+2}}.$$

We conclude an example taken directly from Cosmology [26]. The general Lagrangian leading to the full Bianchi-scalarfield dynamics (that is Einstein equations for an homogeneous but anisotropic spacetime with a scalarfield matter source with a self-interacting potential $V(\phi)$) has the form

$$\mathcal{L} = \mathrm{e}^{3\lambda}\left[R^* + 6\dot{\lambda}^2 - \frac{3}{2}\left(\dot{\beta}_1^2 + \dot{\beta}_2^2\right) - \dot{\phi}^2 + 2V(\phi)\right], \tag{4}$$

where R^* is the Ricci scalar playing the role of a potential term, β_1 and β_2 are suitable variables describing the anisotropy and derivatives are taken with respect to proper time t. The Euler–Lagrange equations for (4) are

$$\ddot{\lambda} + \frac{3}{2}\dot{\lambda}^2 + \frac{3}{8}\left(\dot{\beta}_1^2 + \dot{\beta}_2^2\right) + \frac{1}{4}\dot{\phi}^2 - \frac{1}{12}\mathrm{e}^{-3\lambda}\left(\mathrm{e}^{3\lambda}R^*\right)_\lambda - \frac{1}{2}V(\phi) = 0,$$
$$\ddot{\beta}_1 + 3\dot{\beta}_1\dot{\lambda} + \frac{1}{3}\frac{\partial R^*}{\partial \beta_1} = 0,$$
$$\ddot{\beta}_2 + 3\dot{\beta}_2\dot{\lambda} + \frac{1}{3}\frac{\partial R^*}{\partial \beta_2 0} = 0,$$
$$\ddot{\phi} + 3\dot{\phi}\dot{\lambda} + V' = 0.$$

For homogeneous Bianchi Class A models the Ricci scalar R^* has the explicit form

$$R^* = -\frac{1}{2}\mathrm{e}^{-2\lambda}\left[N_1^2\mathrm{e}^{4\beta_1} + \mathrm{e}^{-2\beta_1}\left(N_2\mathrm{e}^{\sqrt{3}\beta_2} - N_3\mathrm{e}^{-\sqrt{3}\beta_2}\right)^2\right.$$
$$\left. -2N_1\mathrm{e}^{2\beta_1}\left(N_2\mathrm{e}^{\sqrt{3}\beta_2} + N_3\mathrm{e}^{-\sqrt{3}\beta_2}\right)\right] + \frac{1}{2}N_1N_2N_3(1 + N_1N_2N_3)$$

and for Class B universes

$$R^* = 2a^2\mathrm{e}^{-2\lambda}\left(3 - \frac{N_2N_3}{a^2}\right)\mathrm{e}^{\beta}$$

with

$$\beta = \frac{2}{3a^2 - N_2N_3}\left(N_2N_3\beta_1 + \sqrt{-3a^2N_2N_2}\beta_2\right),$$

where a, N_1, N_2 and N_3 are the usual classification constants. For the symmetry analysis it is convenient to make the substitutions

$$u = \mathrm{e}^{\lambda}, \qquad v = \mathrm{e}^{\beta_1}, \qquad w = \mathrm{e}^{\sqrt{3}\beta_2}.$$

We illustrate the results for the simplest Bianchi Type I models and for the open Bianchi Type V family in the case of a constant scalar field potential, *i.e.* $V(\phi) = C$.

In the case of Bianchi Type I with a constant potential the Noether point symmetries are

$$\partial_t, \qquad v\partial_v, \qquad w\partial_w, \qquad \partial_\phi, \qquad v\log w\partial_v - 3w\log v\partial_w,$$
$$v\phi\partial_v - \frac{3}{2}\log v\partial_\phi, \qquad w\phi\partial_w - \frac{1}{2}\log w\partial_\phi.$$

In addition there are the three Lie point symmetries

$$u\partial_u, \qquad \mathrm{e}^{\sqrt{3}Ct}\left\{\partial_t + u\partial_u\right\}, \qquad \mathrm{e}^{\sqrt{3}Ct}\left\{\partial_t - u\partial_u\right\}.$$

We find the first integrals/invariants (listed against the corresponding symmetry)

$$\begin{aligned}
&v\partial_v, && I_1 = u^3\dot{v}/v,\\
&w\partial_w, && I_2 = u^3\dot{w}/w,\\
&\partial_\phi, && I_3 = u^3\dot{\phi},\\
&v\log w\partial_v - 3w\log v\partial_w, && I_4 = \frac{\dot{u}^2}{u\dot{\phi}} - \frac{u^3\dot{v}}{4v\dot{\phi}} - \frac{\dot{\phi}}{6},\\
& && I_5 = t - \alpha\,\mathrm{arcsinh}\,\frac{u^3+M}{\beta},
\end{aligned}$$

where

$$\alpha = \frac{2}{3\sqrt{K}}, \qquad \beta = \left[\frac{2I_3^2}{3K^2}\left(K - 6I_4^2\right) - \frac{C}{2K}\right]^{1/2}, \qquad M = \frac{2I_3I_4}{K}, \qquad K = \frac{I_1^2+I_2^2}{I_3^2}.$$

By inverting the invariant I_5 we obtain $u(t)$ and hence $v(t)$, $w(t)$ and $\phi(t)$ from the quadrature of the first three integrals. Thus we have an explicit solution for this model.

For the Bianchi Type V in the case of a constant potential we obtain the Noether point symmetries

$$\partial_t, \qquad v\partial_v, \qquad w\partial_w, \qquad \partial_\phi$$

and the additional Lie point symmetries

$$v\log w\partial_v - 3w\log v\partial_w, \qquad v\phi\partial_v - \frac{3}{2}\log v\partial_\phi, \qquad w\phi\partial_w - \frac{1}{2}\log w\partial_\phi.$$

We obtain the integrals

$$\begin{aligned}
&\partial_t, && I_1 = u^3\dot{v}/v,\\
& && I_2 = u^3\dot{w}/w,\\
& && I_3 = u^3\dot{\phi},\\
&v\partial_v, && I_4 = \frac{1}{2}(\log u)^3\frac{\dot{u}^2}{u} - f(u),
\end{aligned}$$

where

$$\begin{aligned}
f(u) = {} & \frac{1}{16u^2}\left[4(\log u)^3 + 6(\log u)^2 + 6\log u + 3\right]\\
& + \frac{C}{8}(\log u)^4\frac{1}{48u^6}\left(3I_1^2 + I_2^2 + 2I_3^2\right)\left[(\log u)^3 + \frac{1}{2}(\log u)^2 + \frac{1}{6}\log u + \frac{1}{36}\right].
\end{aligned}$$

In contrast to Type I one is left with the quadrature of I_4 and inversion of the result to obtain $u(t)$. This is not possible in closed form and so we have a system which is integrable but for which an explicit global solution is not available.

4 Discussion

In this paper we provided an overview of the interplay between three fundamental notions of dynamics, namely, symmetry (local and nonlocal) singularities and integrability. There are many questions that remain open in this field some of which come about from considerations arising when one tries to apply the results obtained from the calculations of symmetries to decide about the integrability of the given family of nonlinear systems. For example we know that the cosmological solutions discussed above evolve to other solutions in the limit of large times. This evolution is usually one from a complex (for instance anisotropic) early time state to a simpler late time, isotropic one. It is also true that in such cases an originally nonintegrable system evolves asymptotically to an integrable one. This fact raises an interesting point regarding symmetries and integrability: If symmetry is indeed needed as a fundamental ingredient of the integrability properties of an arbitrary nonlinear system, this has to somehow show in its long term evolution. How do the calculated symmetries of a system evolve as the system changes in time? Almost all work on symmetry and integrability to date has been concerned, in some sense, only with the "statics" of the problem. We believe that only a theory of the dynamical evolution of the symmetries themselves as a given system evolves in time will be needed to provide the means to understand and explain why particular systems of differential equations have the complicated symmetry properties they appear to have. As such a theory is completely lacking at present, examples that show in a clear way the road to proceed will be most welcome.

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On New Exact Solutions of the Eikonal Equation

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Some new classes of exact solutions of the investigated equation have been found.

The relativistic eikonal equation is fundamental in theoretical and mathematical physics. Here we consider the equation

$$\frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu} \equiv \left(\frac{\partial u}{\partial x_0}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2 - \left(\frac{\partial u}{\partial x_2}\right)^2 - \left(\frac{\partial u}{\partial x_3}\right)^2 = 1. \tag{1}$$

In [1] it has been shown that the maximal local invariance group of the equation (1) is the conformal group $C(1,4)$ of the 5-dimensional Poincaré–Minkowski space. Using special ansatzes multiparameter families of exact solutions of the eikonal equation were constructed [1, 2, 3, 4].

It is well known that the conformal group $C(1,4)$ contains the generalized Poincaré group $P(1,4)$ as a subgroup. The group $P(1,4)$ is the group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. For the investigation of the equation (1) we have used the continuous subgroups [5, 6, 7, 8, 9] of the group $P(1,4)$. Earlier using the subgroup structure of the group $P(1,4)$, we have constructed ansatzes which reduce the equation (1) to differential equations with fewer independent variables. The corresponding symmetry reduction has been done. Among the reduced equations there are one-, two-, and three-dimensional ones. For some of the reduced equations we have found its exact solutions. On this base some classes of exact solutions of the eikonal equation have been constructed. The part of the results obtained can be found in [10, 11, 12].

The present paper is devoted to the construction of new exact solutions of the investigated equation. In order to find these solutions we have solved some other reduced equations. Using the solutions of these reduced equations, we have obtained some new classes of exact solutions of the eikonal equation.

At first, we present some new exact solutions of the investigated equation which have been obtained on the base of solutions of one-dimensional reduced equations.

1. $$\alpha \ln\left(\left(\alpha^2 + x_0^2 - x_1^2 - x_2^2 - u^2\right)^{1/2} + \varepsilon\alpha\right) - \varepsilon\left(\alpha^2 + x_0^2 - x_1^2 - x_2^2 - u^2\right)^{1/2} - x_3 - \alpha\ln(x_0+u) = 0, \quad \varepsilon = \pm 1;$$
2. $$\alpha \ln\left(\left(\alpha^2 + x_0^2 - x_3^2 - u^2\right) + \varepsilon\alpha\right) = \varepsilon\left(\alpha^2 + x_0^2 - x_3^2 - u^2\right) + x_2 + \alpha\ln(x_0+u) + c, \quad \varepsilon = \pm 1;$$
3. $$\frac{1}{2}(x_0+u)^4 - \left(x_0 + \frac{1}{2}(\beta-1) - c\right)(x_0+u)^3 + \left[\frac{x_1^2}{2} + \frac{x_2^2}{2} + \frac{x_3^2}{2} + (\beta-1)x_0 - \left(c+\frac{1}{2}\right)\beta + c\right](x_0+u)^2 + \left[\beta\left(x_0 - \frac{x_1^2}{2} - \frac{x_2^2}{2}\right) + \frac{x_1^2}{2} + \frac{x_3^2}{2} - c\beta\right](x_0+u) - \beta\frac{x_1^2}{2} = 0;$$
4. $$\frac{1}{2}(x_0+u)^4 - \left(x_0 + \frac{1}{2}(\beta+1) - c\right)(x_0+u)^3 + \left[\frac{x_1^2}{2} + \frac{x_2^2}{2} + \frac{x_3^2}{2} + (\beta+1)x_0 - \left(c-\frac{1}{2}\right)\beta - c\right](x_0+u)^2 - \left[\beta\left(x_0 + \frac{x_1^2}{2} + \frac{x_2^2}{2}\right) + \frac{x_1^2}{2} + \frac{x_3^2}{2} - c\beta\right](x_0+u) + \beta\frac{x_1^2}{2} = 0;$$

5. $\frac{1}{2}(x_0+u)^3-\left(x_0+\frac{k}{2}+c\right)(x_0+u)^2+\left(\frac{x_1^2}{2}+\frac{x_2^2}{2}+\frac{x_3^2}{2}+kx_0+ck\right)(x_0+u)-k\frac{x_3^2}{2}=0.$

Now, we give some new exact solutions of the eikonal equation which have been constructed on the base of solutions of two-dimensional reduced equations.

1. $u=i\varepsilon\sqrt{c_1^2+1}\sqrt{x_1^2+x_2^2}+c_1x_3+c_2, \quad \varepsilon=\pm1;$
2. $u=\varepsilon x_0\sqrt{c_1^2+1}+c_1x_3+c_2, \quad \varepsilon=\pm1;$
3. $u=\varepsilon x_0\sqrt{c_1^2+1}+c_1\sqrt{x_1^2+x_2^2}+c_2, \quad \varepsilon=\pm1;$
4. $u=\varepsilon x_0\sqrt{c_1^2+1}+c_1\sqrt{x_1^2+x_2^2+x_3^2}+c_2, \quad \varepsilon=\pm1;$
5. $u=i\varepsilon x_2\sqrt{c_1^2+1}+c_1x_3+c_2, \quad \varepsilon=\pm1;$
6. $u^2=x_0^2-\left(\left(x_1^2+x_2^2\right)^{1/2}+c_1\right)^2-(x_3+c_2)^2;$
7. $u^2=x_0^2-(x_1+c_1)^2-(x_2+c_2)^2-x_3^2;$
8. $u^2=(x_0+c_1)^2-\left(\left(x_1^2+x_2^2\right)^{1/2}+c_2\right)^2-x_3^2;$
9. $(x_0-u+c_1)(x_0+u)=(x_2+c_2)^2+x_3^2;$
10. $(x_0-u+c_1)(x_0+u)=(x_3+c_2)^2+x_1^2+x_2^2;$
11. $(x_0-u+c_1)(x_0+u)=\left(\left(x_1^2+x_2^2\right)^{1/2}+c_2\right)^2+x_3^2;$
12. $$\arcsin\frac{x_2}{\sqrt{x_1^2+x_2^2}}+\frac{x_3}{\varepsilon(x_0+u)}=f(x_0+u)$$
$$+i\varepsilon\left[\sqrt{1+\frac{x_1^2+x_2^2}{(x_0+u)^2}}+\ln\left(\frac{\sqrt{x_1^2+x_2^2}}{x_0+u+\sqrt{(x_0+u)^2+x_1^2+x_2^2}}\right)\right], \quad \varepsilon=\pm1,$$

where f is an arbitrary smooth function;

13. $$\frac{1}{2}\arcsin\frac{x_3}{\sqrt{x_3^2+u^2}}+\frac{1}{e}\arcsin\frac{x_2}{\sqrt{x_1^2+x_2^2}}$$
$$=\frac{1}{e}\left[\sqrt{c_1e^2\left(x_1^2+x_2^2\right)^2-1}+\arctan\left(\frac{1}{\sqrt{c_1e^2\left(x_1^2+x_2^2\right)^2-1}}\right)\right]$$
$$+i\frac{\varepsilon}{2}\left[\sqrt{4c_1\left(x_3^2+u^2\right)+1}+\ln\left(\frac{\sqrt{x_3^2+u^2}}{1+\sqrt{4c_1\left(x_3^2+u^2\right)^2+1}}\right)\right]+c_2, \quad \varepsilon=\pm1, \quad e\neq0;$$
14. $$\arcsin\frac{x_2}{\sqrt{x_1^2+x_2^2}}-\frac{1}{e}\operatorname{arccosh}\frac{x_0}{\sqrt{x_0^2-u^2}}$$
$$=\frac{\varepsilon}{e}\left[\sqrt{c_1e^2\left(x_0^2-u^2\right)+1}+\ln\left(\frac{\sqrt{c_1}e\sqrt{x_0^2-u^2}}{1+\sqrt{c_1e^2\left(x_0^2-u^2\right)+1}}\right)\right]$$
$$-\sqrt{c_1\left(x_1^2+x_2^2\right)-1}-\arctan\left(\frac{1}{\sqrt{c_1\left(x_1^2+x_2^2\right)-1}}\right)+c_2, \quad \varepsilon=\pm1, \quad e\neq0;$$

$$15.\quad \arcsin\frac{x_2}{\sqrt{x_1^2+x_2^2}}-\frac{1}{d}\ln(x_0+u)=\sqrt{c_1\left(x_1^2+x_2^2\right)^2-1}$$
$$+\arctan\left(\frac{1}{\sqrt{c_1\left(x_1^2+x_2^2\right)^2-1}}\right)+\frac{\varepsilon}{d}\left[\sqrt{1-d^2\left(u^2+x_3^2-x_0^2\right)c_1}\right.$$
$$\left.-\operatorname{arctanh}\left(\frac{1}{\sqrt{1-d^2\left(u^2+x_3^2-x_0^2\right)c_1}}\right)-\frac{\varepsilon}{2}\ln\left(u^2+x_3^2-x_0^2\right)\right]+c_2,\ \varepsilon=\pm1,\ d\neq0;$$

$$16.\quad u=c_1x_3+i\left[\sqrt{c_1^2\left(x_1^2+x_2^2\right)+1}-\operatorname{arctanh}\left(\frac{1}{\sqrt{c_1^2\left(x_1^2+x_2^2\right)+1}}\right)\right]$$
$$-\left(\varepsilon\arcsin\frac{x_1}{\sqrt{x_1^2+x_2^2}}+x_0\right)+c_2,\quad \varepsilon=\pm1;$$

$$17.\quad u=c_1x_3+\frac{i}{2}\left[\sqrt{4\left(c_1^2+1\right)\left(x_1^2+x_2^2\right)+d_4^2}\right.$$
$$\left.-d_4\operatorname{arctanh}\left(\frac{d_4}{\sqrt{4\left(c_1^2+1\right)\left(x_1^2+x_2^2\right)+d_4^2}}\right)\right]-\frac{d_4}{2}\arcsin\frac{x_1}{\sqrt{x_1^2+x_2^2}}+c_2;$$

$$18.\quad \varepsilon x_1(x_0+u)=i\varepsilon x_2\sqrt{(x_0+u)^2+1}-x_3+f(x_0+u),\quad \varepsilon=\pm1,$$

where f is an arbitrary smooth function;

$$19.\quad \frac{\alpha}{\mu}\operatorname{arccosh}\frac{x_0}{\sqrt{x_0^2-u^2}}-\arcsin\frac{x_1}{\sqrt{x_1^2+x_2^2}}=\sqrt{c_1\left(x_1^2+x_2^2\right)-1}$$
$$-\arctan\left(\sqrt{c_1\left(x_1^2+x_2^2\right)-1}\right)+\frac{1}{\mu}\sqrt{\left(c_1\mu^2+1\right)\left(x_0^2-u^2\right)+\alpha^2}$$
$$-\frac{\alpha}{\mu}\operatorname{arctanh}\left(\sqrt{\frac{\left(c_1\mu^2+1\right)\left(x_0^2-u^2\right)}{\alpha^2}+1}\right)-\frac{x_3}{\mu}+c_2,\quad \alpha,\mu\neq0;$$

$$20.\quad \frac{\alpha}{\mu}\arcsin\frac{x_2}{\sqrt{x_1^2+x_2^2}}-\arcsin\frac{x_3}{\sqrt{x_3^2+u^2}}=\sqrt{c_1\left(x_1^2+x_2^2\right)-\frac{\alpha^2}{\mu^2}}$$
$$-\frac{\alpha}{\mu}\arctan\left(\sqrt{\frac{c_1\mu^2\left(x_1^2+x_2^2\right)}{\alpha^2}-1}\right)+\sqrt{\frac{\left(1-c_1\mu^2\right)\left(x_3^2+u^2\right)}{\mu^2}-1}$$
$$-\arctan\left(\sqrt{\frac{\left(1-c_1\mu^2\right)\left(x_3^2+u^2\right)}{\mu^2}-1}\right)+\frac{x_0}{\mu}-c_2,\quad \alpha,\mu\neq0.$$

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On Differential Invariants of First- and Second-Order of the Splitting Subgroups of the Generalized Poincaré Group $P(1,4)$

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Functional bases of differential invariants of the first-order of the splitting subgroups of the group $P(1,4)$ have been constructed. For majority of these subgroups functional bases of differential invariants of the second-order have also been described.

The differential invariants of the local Lie groups of the point transformations play an important role in the group-analysis of differential equations (see, for example [1–10]). In particular, with the help of these invariants we can construct differential equations with non-trivial symmetry groups. Differential invariants have been studied in many works (see, for example [1–10]).

The present paper is devoted to construction of functional bases of differential invariants of the first- and second-order for the splitting subgroups of generalized Poincaré group $P(1,4)$. The group $P(1,4)$ is the group of rotations and translations of five-dimensional Minkowski space $M(1,4)$. This group has many applications in theoretical and mathematical physics [11, 12, 13]. In order to present some of the results obtained we have to consider the Lie algebra of the group $P(1,4)$.

1 The Lie algebra of the group $P(1,4)$ and its continuous subalgebras.

The Lie algebra of the group $P(1,4)$ is given by the 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$ and P'_μ $(\mu,\nu = 0,1,2,3,4)$, satisfying the commutation relations

$$\left[P'_\mu, P'_\nu\right] = 0, \qquad \left[M'_{\mu\nu}, P'_\sigma\right] = g_{\mu\sigma}P'_\nu - g_{\nu\sigma}P'_\mu,$$
$$\left[M'_{\mu\nu}, M'_{\rho\sigma}\right] = g_{\mu\rho}M'_{\nu\sigma} + g_{\nu\sigma}M'_{\mu\rho} - g_{\nu\rho}M'_{\mu\sigma} - g_{\mu\sigma}M'_{\nu\rho},$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$. Here, and in what follows, $M'_{\mu\nu} = iM_{\mu\nu}$.

Further we will use following basis elements:

$$G = M'_{40}, \qquad L_1 = M'_{32}, \qquad L_2 = -M'_{31}, \qquad L_3 = M'_{21},$$
$$P_a = M'_{4a} - M'_{a0}, \qquad C_a = M'_{4a} + M'_{a0} \quad (a = 1,2,3),$$
$$X_0 = \frac{1}{2}\left(P'_0 - P'_4\right), \qquad X_k = P'_k \quad (k = 1,2,3), \qquad X_4 = \frac{1}{2}\left(P'_0 + P'_4\right).$$

In order to study the subgroup structure of the group $P(1,4)$ we used the method proposed in [14]. Continuous subgroups of the group $P(1,4)$ have been found in [15–19].

One of the important consequences of the description of the continuous subalgebras of the Lie algebra of the group $P(1,4)$ is that the Lie algebra of the group $P(1,4)$ contains as subalgebras the Lie algebra of the Poincaré group $P(1,3)$ and the Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$ [13], i.e. it naturally unites the Lie algebras of the symmetry groups of relativistic and nonrelativistic quantum mechanics.

2 The differential invariants of the first-order for splitting subgroups of the group $P(1,4)$

For all splitting subgroups of the group $P(1,4)$ the functional bases of differential invariants of the first-order have been constructed. Below, we present some of the results obtained.

At first, let us consider the following representation of the Lie algebra of the group $P(1,4)$:

$$P'_0 = \frac{\partial}{\partial x_0}, \qquad P'_1 = -\frac{\partial}{\partial x_1}, \qquad P'_2 = -\frac{\partial}{\partial x_2}, \qquad P'_3 = -\frac{\partial}{\partial x_3},$$
$$P'_4 = -\frac{\partial}{\partial x_4}, \qquad M'_{\mu\nu} = -\left(x_\mu P'_\nu - x_\nu P'_\mu\right). \tag{1}$$

For this representation of the considered Lie algebra we have obtained the functional bases of differential invariants of the first-order for all its splitting subalgebras.

Below, for some of the splitting subalgebras of the Lie algebra of the group $P(1,4)$ we write its basis elements and corresponding functional basis of differential invariants.

1. $\langle L_3 + eG, X_3,\ e > 0\rangle$,
$$\left(x_0^2 - x_4^2\right)^{1/2}, \quad (x_1^2 + x_2^2)^{1/2}, \quad \ln(x_0 + x_4) + e\arctan\frac{x_1}{x_2}, \quad u, \quad x_1u_2 - x_2u_1,$$
$$(x_0 + x_4)(u_0 + u_4), \quad u_3, \quad u_0^2 - u_4^2, \quad u_1^2 + u_2^2, \quad u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \quad \mu = 0, 1, 2, 3, 4;$$
2. $\langle L_3 + dG, P_3, X_4,\ d > 0\rangle$,
$$\left(x_1^2 + x_2^2\right)^{1/2}, \quad u, \quad x_1u_2 - x_2u_1, \quad \frac{x_0 + x_4}{u_0 - u_4}, \quad \frac{u_0 - u_4}{x_0 + x_4}x_3 + u_3,$$
$$d\arctan\frac{u_1}{u_2} + \ln(x_0 + x_4), \quad u_1^2 + u_2^2, \quad u_0^2 - u_3^2 - u_4^2;$$
3. $\langle P_1, P_2, P_3, X_4\rangle$,
$$x_0 + x_4, \quad u, \quad \frac{x_1}{x_0 + x_4} + \frac{u_1}{u_0 - u_4}, \quad \frac{x_2}{x_0 + x_4} + \frac{u_2}{u_0 - u_4}, \quad \frac{x_3}{x_0 + x_4} + \frac{u_3}{u_0 - u_4},$$
$$u_0 - u_4, \quad u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$
4. $\langle G, L_1, L_2, L_3, X_4\rangle$,
$$\left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}, \quad u, \quad (x_0 + x_4)(u_0 + u_4), \quad x_1u_1 + x_2u_2 + x_3u_3,$$
$$u_0^2 - u_4^2, \quad u_1^2 + u_2^2 + u_3^2;$$
5. $\langle G, P_1, P_2, X_1, X_2, X_4\rangle$,
$$x_3, \quad u, \quad \frac{x_0 + x_4}{u_0 - u_4}, \quad u_3, \quad u_0^2 - u_1^2 - u_2^2 - u_4^2;$$
6. $\langle L_3, P_1, P_2, P_3, X_1, X_2, X_4\rangle$,
$$x_0 + x_4, \quad u, \quad \frac{x_3}{x_0 + x_4} + \frac{u_3}{u_0 - u_4}, \quad u_0 - u_4, \quad u_0^2 - u_1^2 - u_2^2 - u_3^2 - u_4^2;$$

7. $\langle G, L_3, P_1, P_2, X_1, X_2, X_3, X_4\rangle$,

$$u, \quad \frac{x_0+x_4}{u_0-u_4}, \quad u_3, \quad u_0^2-u_1^2-u_2^2-u_4^2;$$

8. $\langle L_3+bG, P_1, P_2, P_3, X_0, X_1, X_2, X_3, X_4,\ b>0\rangle$,

$$u, \quad u_0^2-u_1^2-u_2^2-u_3^2-u_4^2.$$

Now, let us consider an other representation of the Lie algebra of the group $P(1,4)$

$$P_0' = \frac{\partial}{\partial x_0}, \qquad P_1' = -\frac{\partial}{\partial x_1}, \qquad P_2' = -\frac{\partial}{\partial x_2}, \qquad P_3' = -\frac{\partial}{\partial x_3},$$
$$P_4' = -\frac{\partial}{\partial u}, \qquad M_{\mu\nu}' = -\left(x_\mu P_\nu' - x_\nu P_\mu'\right), \qquad x_4 \equiv u. \tag{2}$$

More details about this representation can be found in [20].

Taking into account this representation of the considered Lie algebra we have constructed the functional bases of differential invariants of the first-order for all its splitting subalgebras.

Here, for some of the splitting subalgebras of the Lie algebra of the group $P(1,4)$ we give its basis elements and corresponding functional basis of differential invariants.

1. $\langle L_3+\varepsilon P_3, \varepsilon=\pm1\rangle$,

$$x_0+u, \quad \left(x_1^2+x_2^2\right)^{1/2}, \quad \left(x_0^2-x_3^2-u^2\right)^{1/2}, \quad \frac{x_3}{x_0+u}+\frac{u_3}{u_0+1},$$
$$\varepsilon\arctan\frac{x_1}{x_2}-\frac{x_3}{x_0+u}, \quad \frac{u_3^2}{(u_0+1)^2}+\frac{2}{u_0+1}, \quad \frac{x_1u_2-x_2u_1}{x_1u_1+x_2u_2}, \quad \frac{u_1^2+u_2^2}{(u_0+1)^2},$$
$$u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \quad \mu=0,1,2,3;$$

2. $\langle G, L_3\rangle$,

$$x_3, \quad \left(x_1^2+x_2^2\right)^{1/2}, \quad \left(x_0^2-u^2\right)^{1/2}, \quad (x_0+u)^2\frac{u_0-1}{u_0+1}, \quad \frac{x_1u_2-x_2u_1}{x_1u_1+x_2u_2},$$
$$\frac{u_0^2-1}{u_3^2}, \qquad \frac{u_1^2+u_2^2}{u_3^2};$$

3. $\langle G, P_1, P_2\rangle$,

$$x_3, \quad \left(x_0^2-x_1^2-x_2^2-u^2\right)^{1/2}, \quad \frac{x_0+u}{u_0+1}u_3, \quad x_1+\frac{x_0+u}{u_0+1}u_1, \quad x_2+\frac{x_0+u}{u_0+1}u_2,$$
$$\frac{u_0^2-u_1^2-u_2^2-1}{u_3^2};$$

4. $\langle G, L_3, P_1, P_2\rangle$,

$$x_3, \quad \left(x_0^2-x_1^2-x_2^2-u^2\right)^{1/2}, \quad \frac{x_0+u}{u_0+1}u_3, \quad \frac{u_0^2-u_1^2-u_2^2-1}{u_3^2},$$
$$\left(x_1+\frac{x_0+u}{u_0+1}u_1\right)^2+\left(x_2+\frac{x_0+u}{u_0+1}u_2\right)^2;$$

5. $\langle G, P_3, L_3, X_1, X_2\rangle$,

$$\left(x_0^2-x_3^2-u^2\right)^{1/2}, \quad x_3+\frac{x_0+u}{u_0+1}u_3, \quad \left(u_1^2+u_2^2\right)\left(\frac{x_0+u}{u_0+1}\right)^2, \quad \frac{u_0^2-u_3^2-1}{u_1^2+u_2^2};$$

6. $\langle G, L_1, L_2, L_3, X_0, X_4\rangle$,

$$\left(x_1^2+x_2^2+x_3^2\right)^{1/2}, \quad \frac{(x_1u_1+x_2u_2+x_3u_3)^2}{u_0^2-1}, \quad \frac{u_1^2+u_2^2+u_3^2}{u_0^2-1};$$

7. $\langle G, P_1, P_2, P_3, X_1, X_2, X_4\rangle$,

$$x_3 + \frac{x_0+u}{u_0+1}u_3, \quad \left(u_0^2 - u_1^2 - u_2^2 - u_3^2 - 1\right)\left(\frac{x_0+u}{u_0+1}\right)^2;$$

8. $\langle G, L_3, P_1, P_2, P_3, X_1, X_2, X_4\rangle$,

$$x_3 + \frac{x_0+u}{u_0+1}u_3, \quad \left(u_0^2 - u_1^2 - u_2^2 - u_3^2 - 1\right)\left(\frac{x_0+u}{u_0+1}\right)^2.$$

3 On differential invariants of the second-order for splitting subgroups of the group $P(1,4)$

We have constructed functional bases of differential invariants of the second-order for some splitting subgroups of the group $P(1,4)$. Now, we present some of the results obtained.

Let us consider the representation (1) of the Lie algebra of the group $P(1,4)$. For this representation of the considered Lie algebra we have constructed the functional bases of differential invariants of the second-order for some its splitting subalgebras. Below, for some of the splitting subalgebra of the Lie algebra of the group $P(1,4)$ we write its basis elements and corresponding functional basis of differential invariants.

$$\langle L_3 + eG,\ e>0\rangle,$$

$$x_3, \quad \left(x_0^2 - x_4^2\right)^{1/2}, \quad \left(x_1^2 + x_2^2\right)^{1/2}, \quad e\arctan\frac{x_1}{x_2} + \ln(x_0+x_4), \quad u,$$

$$(x_0+x_4)(u_0+u_4), \qquad x_1u_2 - x_2u_1, \quad u_3, \quad u_0^2 - u_4^2, \quad u_1^2 + u_2^2,$$

$$e\arctan\frac{u_{13}}{u_{23}} + 2\ln(x_0+x_4), \quad \ln(u_{00}+u_{44}+\sqrt{2}u_{04}) - 2\sqrt{2}e\arctan\frac{x_1}{x_2},$$

$$\arctan\left(\frac{u_{02}+u_{24}}{u_{01}+u_{14}}\right) + 2\arctan\frac{x_1}{x_2}, \quad 4e\arctan\frac{u_1}{u_2} - \ln\left((u_{01}+u_{14})^2 + (u_{02}+u_{24})^2\right),$$

$$\frac{u_{03}+u_{34}}{(u_0+u_4)^2}, \quad \arctan\left(\frac{\sqrt{2}u_{12}}{u_{11}-u_{22}}\right) + 2\sqrt{2}\arctan\frac{u_1}{u_2}, \quad u_{33}, \quad u_{00}-u_{44}, \quad u_{11}+u_{22},$$

$$u_{03}^2 - u_{34}^2, \quad u_{13}^2 + u_{23}^2, \quad u_{11}^2 + u_{12}^2 + u_{22}^2, \quad u_{00}^2 - u_{04}^2 + u_{44}^2, \quad u_{01}^2 + u_{02}^2 - u_{14}^2 - u_{24}^2,$$

$$u_{01}u_{24} - u_{02}u_{14}, \quad u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \quad u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}, \quad \mu,\nu = 0,1,2,3,4.$$

Now, let us consider the representation (2) of the Lie algebra of the group $P(1,4)$. Taking into account this representation of the considered Lie algebra we have obtained the functional bases of differential invariants of the second-order for some its splitting subalgebras.

Here, for some of the splitting subalgebra of the Lie algebra of the group $P(1,4)$ we give its basis elements and corresponding functional basis of differential invariants.

$$\langle L_3\rangle,$$

$$x_0, \quad x_3, \quad \left(x_1^2 + x_2^2\right)^{1/2}, \quad u, \quad x_1u_1 + x_2u_2, \quad u_0, \quad u_3, \quad u_1^2 + u_2^2,$$

$$\left(x_1^2 - x_2^2\right)u_{01} + 2x_1x_2u_{02}, \quad 2\sqrt{2}\arctan\frac{x_1}{x_2} - \arctan\left(\frac{u_{11}-u_{22}}{\sqrt{2}u_{12}}\right), \quad u_{00}, \quad u_{03}, \quad u_{33},$$

$$u_{11}+u_{22}, \quad u_{01}^2 + u_{02}^2, \quad u_{13}^2 + u_{23}^2, \quad u_{11}^2 + u_{12}^2 + u_{22}^2, \quad u_{02}u_{13} - u_{01}u_{23},$$

$$u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \quad u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}, \quad \mu,\nu = 0,1,2,3.$$

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On Differential Equations of First- and Second-Order in the Space $M(1,3) \times R(u)$ with Nontrivial Symmetry Groups

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The differential equations of the first order in the space $M(1,3) \times R(u)$ which are invariant under splitting subgroups of the group $P(1,4)$ have been constructed. For majority of these subgroups the differential equations of the second-order in the same space have also been described.

The differential equations with nontrivial symmetry groups play an important role in theoretical and mathematical physics, mechanics, gas dynamics (see, for example, [1–8]).

In many cases these equations can be written in the following form:

$$F(J_1, J_2, \ldots, J_t) = 0, \tag{1}$$

where F is an arbitrary enough smooth function of its arguments, $\{J_1, J_2, \ldots, J_t\}$ are functional bases of differential invariants of the corresponding symmetry groups.

Differential invariants of the local Lie groups of the point transformations have been studied in many works (see, for example, [1, 4, 9–12]).

The present work is devoted to the construction of differential equations of the first- and second-order in the space $M(1,3) \times R(u)$, which are invariant under splitting subgroups of the generalized Poincaré group $P(1,4)$.

In order to give some of the results obtained we must consider the Lie algebra of the group $P(1,4)$.

1 The Lie algebra of the group $P(1,4)$ and its continuous subalgebras

The Lie algebra of the group $P(1,4)$ is given by the 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$ and P'_μ $(\mu, \nu = 0, 1, 2, 3, 4)$, satisfying the commutation relations

$$\left[P'_\mu, P'_\nu\right] = 0, \qquad \left[M'_{\mu\nu}, P'_\sigma\right] = g_{\mu\sigma} P'_\nu - g_{\nu\sigma} P'_\mu,$$
$$\left[M'_{\mu\nu}, M'_{\rho\sigma}\right] = g_{\mu\rho} M'_{\nu\sigma} + g_{\nu\sigma} M'_{\mu\rho} - g_{\nu\rho} M'_{\mu\sigma} - g_{\mu\sigma} M'_{\nu\rho},$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$. Here, and in what follows, $M'_{\mu\nu} = i M_{\mu\nu}$.

Let us consider following representation of the Lie algebra of the group $P(1,4)$

$$P'_0 = \frac{\partial}{\partial x_0}, \qquad P'_1 = -\frac{\partial}{\partial x_1}, \qquad P'_2 = -\frac{\partial}{\partial x_2}, \qquad P'_3 = -\frac{\partial}{\partial x_3},$$
$$P'_4 = -\frac{\partial}{\partial u}, \qquad M'_{\mu\nu} = -\left(x_\mu P'_\nu - x_\nu P'_\mu\right), \qquad x_4 \equiv u.$$

More details about this representation can be found in [8].

Further we will use following basis elements:

$$G = M'_{40}, \qquad L_1 = M'_{32}, \qquad L_2 = -M'_{31}, \qquad L_3 = M'_{21},$$
$$P_a = M'_{4a} - M'_{a0}, \qquad C_a = M'_{4a} + M'_{a0}, \qquad (a = 1, 2, 3),$$
$$X_0 = \frac{1}{2}\left(P'_0 - P'_4\right), \qquad X_k = P'_k \qquad (k = 1, 2, 3), \qquad X_4 = \frac{1}{2}\left(P'_0 + P'_4\right).$$

In order to study the subgroup structure of the group $P(1,4)$ we used the method proposed in [13]. Splitting subgroups of the group $P(1,4)$ have been found in [14, 15].

2 The differential equations of the first-order in the space $M(1,3) \times R(u)$

The differential equations of the first-order in the space $M(1,3) \times R(u)$, which are invariant under splitting subgroups of the group $P(1,4)$ have been constructed. These equations can be written in the form (1), where $\{J_1, J_2, \ldots, J_t\}$ are functional bases of differential invariants of the first-order of the splitting subgroups of the group $P(1,4)$.

Below, for some splitting subgroups of the group $P(1,4)$, we write the basis elements of its Lie algebras and corresponding arguments $J_1, J_2, \ldots, J_t$ of the function F.

1. $\langle L_3 \rangle$,

$$J_1 = x_0, \quad J_2 = x_3, \quad J_3 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_4 = u, \quad J_5 = x_1 u_2 - x_2 u_1,$$
$$J_6 = u_0, \quad J_7 = u_3, \quad J_8 = u_1^2 + u_2^2, \quad u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \quad \mu = 0, 1, 2, 3;$$

2. $\langle P_3 + C_3, L_3 \rangle$,

$$J_1 = x_0, \quad J_2 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_3 = \left(x_3^2 + u^2\right)^{1/2}, \quad J_4 = \frac{u_3 u + x_3}{u - x_3 u_3},$$
$$J_5 = \frac{x_1 u_2 - x_2 u_1}{x_1 u_1 + x_2 u_2}, \quad J_6 = \frac{u_1^2 + u_2^2}{u_0^2}, \quad J_7 = \frac{u_3^2 + 1}{u_0^2};$$

3. $\langle P_1, P_2, X_3 \rangle$,

$$J_1 = x_0 + u, \quad J_2 = \left(x_0^2 - x_1^2 - x_2^2 - u^2\right)^{1/2}, \quad J_3 = \frac{x_1}{x_0 + u} + \frac{u_1}{u_0 + 1},$$
$$J_4 = \frac{x_2}{x_0 + u} + \frac{u_2}{u_0 + 1}, \quad J_5 = \frac{u_3}{u_0 + 1}, \quad J_6 = \frac{u_1^2 + u_2^2 + 2(u_0 + 1)}{(u_0 + 1)^2};$$

4. $\langle G, P_1, P_2, P_3 \rangle$,

$$J_1 = \left(x_0^2 - x_1^2 - x_2^2 - x_3^2 - u^2\right)^{1/2}, \quad J_2 = x_1 + \frac{x_0 + u}{u_0 + 1} u_1, \quad J_3 = x_2 + \frac{x_0 + u}{u_0 + 1} u_2,$$
$$J_4 = x_3 + \frac{x_0 + u}{u_0 + 1} u_3, \quad J_5 = \left(u_0^2 - u_1^2 - u_2^2 - u_3^2 - 1\right)\left(\frac{x_0 + u}{u_0 + 1}\right)^2;$$

5. $\langle L_3, P_1, P_2, P_3, X_4 \rangle$,

$$J_1 = x_0 + u, \quad J_2 = \frac{x_3}{x_0 + u} + \frac{u_3}{u_0 + 1},$$
$$J_3 = \left(\frac{x_1}{x_0 + u} + \frac{u_1}{u_0 + 1}\right)^2 + \left(\frac{x_2}{x_0 + u} + \frac{u_2}{u_0 + 1}\right)^2, \quad J_4 = \frac{u_1^2 + u_2^2 + u_3^2 + 2(u_0 + 1)}{(u_0 + 1)^2};$$

6. $\langle G, P_3, L_3, X_1, X_2, X_4 \rangle$,

$$J_1 = x_3 + \frac{x_0 + u}{u_0 + 1} u_3, \quad J_2 = \left(u_1^2 + u_2^2\right)\left(\frac{x_0 + u}{u_0 + 1}\right)^2, \quad J_3 = \frac{u_0^2 - u_3^2 - 1}{u_1^2 + u_2^2};$$

7. $\langle G, L_3, P_1, P_2, P_3, X_3, X_4\rangle$,

$$J_1 = \left(x_1 + \frac{x_0+u}{u_0+1}u_1\right)^2 + \left(x_2 + \frac{x_0+u}{u_0+1}u_2\right)^2,$$

$$J_2 = \left(u_0^2 - u_1^2 - u_2^2 - u_3^2 - 1\right)\left(\frac{x_0+u}{u_0+1}\right)^2;$$

8. $\langle G, L_3, P_1, P_2, X_1, X_2, X_3, X_4\rangle$,

$$J_1 = \frac{x_0+u}{u_0+1}u_3, \quad J_2 = \frac{u_0^2 - u_1^2 - u_2^2 - 1}{u_3^2};$$

9. $\langle G, L_1, L_2, L_3, X_0, X_1, X_2, X_3, X_4\rangle$,

$$J_1 = \frac{u_1^2 + u_2^2 + u_3^2}{u_0^2 - 1};$$

10. $\langle L_1, L_2, L_3, P_1 - C_1, P_2 - C_2, P_3 - C_3, X_1, X_2, X_3, X_0 + X_4\rangle$,

$$J_1 = u, \quad J_2 = u_0^2 - u_1^2 - u_2^2 - u_3^2;$$

11. $\langle L_1, L_2, L_3, P_1 + C_1, P_2 + C_2, P_3 + C_3, X_0, X_1, X_2, X_3, X_4\rangle$,

$$J_1 = \frac{u_1^2 + u_2^2 + u_3^2 + 1}{u_0^2}.$$

3 On differential equations of the second-order in the space $M(1,3) \times R(u)$

Some of the differential equations of the second-order in the space $M(1,3) \times R(u)$, which are invariant under splitting subgroups of the group $P(1,4)$ have been described. The equations obtained have the form (1), where $\{J_1, J_2, \ldots, J_t\}$ are functional bases of differential invariants of the second-order of corresponding splitting subgroups of the group $P(1,4)$.

In the following, for some splitting subgroup of the group $P(1,4)$, we give the basis elements of its Lie algebra and corresponding arguments $J_1, J_2, \ldots, J_t$ of the function F.

$\langle L_3, X_0\rangle$,

$$J_1 = x_3, \quad J_2 = x_0 - u, \quad J_3 = \left(x_1^2 + x_2^2\right)^{1/2}, \quad J_4 = \frac{x_1u_2 - x_2u_1}{x_1u_1 + x_2u_2}, \quad J_5 = u_0,$$

$$J_6 = u_3, \quad J_7 = u_1^2 + u_2^2, \quad J_8 = (x_1u_1 - x_2u_2)u_{01} + (x_1u_2 + x_2u_1)u_{02},$$

$$J_9 = 2\sqrt{2}\arctan\frac{u_1}{u_2} - \arctan\left(\frac{u_{11} - u_{22}}{\sqrt{2}u_{12}}\right), \quad J_{10} = u_{00}, \quad J_{11} = u_{03}, \quad J_{12} = u_{33},$$

$$J_{13} = u_{11} + u_{22}, \quad J_{14} = u_{01}^2 + u_{02}^2, \quad J_{15} = u_{13}^2 + u_{23}^2, \quad J_{16} = u_{11}^2 + u_{12}^2 + u_{22}^2,$$

$$J_{17} = u_{02}u_{13} - u_{01}u_{23}, \quad u_\mu \equiv \frac{\partial u}{\partial x_\mu}, \quad u_{\mu\nu} \equiv \frac{\partial^2 u}{\partial x_\mu \partial x_\nu}, \quad \mu, \nu = 0, 1, 2, 3.$$

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Symmetry of Nonlinear Schrödinger Equations with Harmonic Oscillator Type Potential

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The group classification in the class of nonlinear Schrödinger equations of the form $i\,\psi_t + \Delta\psi + k|x|^2\psi - f\,(|\psi|)\,\psi = 0$ was carried out. The maximal Lie invariance algebra of such equations was calculated.

The study of nonlinear Schrödinger equations using symmetry methods began in 1972 by Niederer's article [1], in which maximal Lie invariance algebra (MIA) of the free Schrödinger equation was calculated for the first time. In 1973 U. Niederer [2] calculated the MIA of the linear Schrödinger equation with the harmonic oscillator type potential. In article [4] nonlinear Schrödinger equations of the form

$$i\,\psi_t + \Delta\psi + F\,(t, x, \psi, \psi^*, \psi_t, \psi_t^*) = 0$$

are considered, and classification of one-dimensional equations which admit the MIA of dimension n ($n \leq 3$) is carried out.

In this article the group classification of the nonlinear generalized Schrödinger equations

$$i\,\psi_t + \Delta\psi + k|x|^2\psi - f\,(|\psi|)\,\psi = 0 \tag{1}$$

in the n-dimensional space is performed. The differentiable function $f = f(|\psi|)$ and the constant k are arbitrary elements. As a particular case of the problem the invariance algebra of the free Schrödinger equations and the Schrödinger equations with the harmonic oscillator type potential are found. The linear equation (the case $f = 0$) was considered in papers [1, 2] and given here for the completeness of results only. The results of this article for $k = 0$ coincide with results of paper [3], which is devoted to group classification of the nonlinear Schrödinger equations of the form

$$i\,\psi_t + \Delta\psi + F\,(\psi, \psi^*) = 0.$$

Theorem 1. *The Lie algebra of the kernel of principal groups of equation (1) is*

$$A^{\mathrm{ker}} = \langle \partial_t, J_{ab}, M\rangle.$$

The Lie algebra of the kernel of principal groups of equation (1) for fixed k is

1) $A_0^{\mathrm{ker}} = \langle \partial_t, \partial_a, J_{ab}, G, M\rangle$ *in the case* $k = 0$;
2) $A_-^{\mathrm{ker}} = \langle M, \partial_t, J_{ab}, e^{2\varkappa t}(\partial_a + \varkappa x^a M), e^{-2\varkappa t}(\partial_a - \varkappa x^a M)\rangle$
in the case $k = -\varkappa^2$, $\varkappa > 0$;
3) $A_+^{\mathrm{ker}} = \langle M, \partial_t, J_{ab}, \sin 2\varkappa t\,\partial_a + \varkappa x^a \cos 2\varkappa t\, M, \cos 2\varkappa t\,\partial_a - \varkappa x^a \sin 2\varkappa t\, M\rangle$
in the case $k = \varkappa^2$, $\varkappa > 0$.

Here $M = i\,(\psi\,\partial_\psi - \psi^*\partial_{\psi^*})$, $G = t\partial_a - \frac{x^a}{2}\,M$, $J_{ab} = x^b\partial_a - x^a\partial_b$.

Theorem 2. *The class of equations (1) admits the following equivalence transformations:*

$$\begin{aligned}&1)\ \tilde t = t,\ \tilde x = x,\ \tilde\psi = e^{i\alpha t}\psi,\ \tilde f = f-\alpha,\ \tilde k = k;\\ &2)\ \tilde t = \varepsilon^2 t,\ \tilde x = \varepsilon x,\ \tilde\psi = \psi,\ \tilde f = \varepsilon^{-2} f,\ \tilde k = \varepsilon^{-4} k;\\ &3)\ \tilde t = t,\ \tilde x = x,\ \tilde\psi = \varepsilon\psi,\ \tilde f = f,\ \tilde k = k.\end{aligned} \qquad (2)$$

Here $\alpha, \varepsilon \in \mathbb{R}$, $\varepsilon \neq 0$.

The classification of extension of the MIA will be carried out accurate to transformations (2).

Theorem 3. *The complete set of nonequivalent cases of extension of the MIA of equations (1) are exhausted by the following (we adduce only operators from extensions of algebra* $A_0^{\rm ker}$, $A_-^{\rm ker}$, $A_+^{\rm ker}$ *for the cases* $k=0$, $k=-\varkappa^2$, $k=\varkappa^2$ *correspondingly)*

$$\begin{aligned}
&1)\ f=(\delta_1+i\delta_2)|\psi|^\gamma,\ \gamma\neq 0, 4/n: && I-\gamma D;\\
&2)\ f=(\delta_1+i\delta_2)|\psi|^{4/n}: && I-\frac{4}{n}D,\ \Pi;\\
&3)\ f=i\delta_2\ln|\psi|: && I+\delta_2 tM;\\
&4)\ f=(\delta_1+i\delta_2)\ln|\psi|,\ \delta_1\neq 0: && e^{\delta_1 t}(\delta_1 I+\delta_2 M);\\
&5)\ f=0,\ k=0: && I,\ D,\ \Pi;\\
&6)\ f=0,\ k=-\varkappa^2,\ \varkappa>0: && I,\ e^{4\varkappa t}\left(\partial_t+2\varkappa x^a\partial_a+4\varkappa^2|x|^2M-n\varkappa I\right),\\
& && e^{-4\varkappa t}\left(\partial_t-2\varkappa x^a\partial_a+4\varkappa^2|x|^2M+n\varkappa I\right);\\
&7)\ f=0,\ k=\varkappa^2,\ \varkappa>0: && I,\ \cos 4\varkappa t\left(\partial_t+2\varkappa^2|x|^2M\right)-\sin 4\varkappa t\,(2\varkappa x^a\partial_a-\varkappa nI),\\
& && \sin 4\varkappa t\left(\partial_t+2\varkappa^2|x|^2M\right)+\cos 4\varkappa t\,(2\varkappa x^a\partial_a-\varkappa nI).
\end{aligned}$$

Here $I=\psi\partial_\psi+\psi^*\partial\psi^*$, $D=2t\partial_t+x^a\partial_a-\frac{n}{2}I$, $\Pi=t^2\partial_t+tx^a\partial_a+\frac{|x|^2}{4}M-\frac{nt}{2}I$, $\{\delta_1,\ \delta_2,\ \gamma\}\subset\mathbb{R}$.

The results of group classification of generalized nonlinear Schrödinger equations obtained in this article can be used for the construction of exact solutions of these equations. These results can be considered as the basis for further analysis of generalized nonlinear Schrödinger equations. We plan to finish complete group classification for the case $f=f(\psi,\psi^*)$ and an arbitrary potential $V=V(t,x)$ and to construct exact solutions of such generalized nonlinear Schrödinger equations.

Because of the extensions of the MIA in the cases $k=0$ and $k\neq 0$ are equal we hope to build the equivalence transformations between these classes.

We are going to use the results of present paper for the investigation of Q-conditional (non-classical) symmetries of Schrödinger equations.

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The Use of p-adic Numbers in Calculating Symmetries of Evolution Equations

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There exist equations with generalized symmetries that do not have infinitely many generalized symmetries. We explain how to prove such a fact using p-adic numbers and calculate examples using symbolic calculus.

1 Introduction

The title of this text is the same as the title of the talk I gave at the conference "Symmetry in Nonlinear Mathematical Physics 2001". It is a misleading title. P-adic numbers are not used in calculating symmetries. They are used to prove that certain (infinitely many) symmetries do not exist. The material presented here is not new, it can be found in [8, 9], but the exposition is.

It was observed and conjectured, cf. [6, 5, 7], that the existence of one (or a few) symmetries implies the existence of infinitely many symmetries. This turned out not to be the case. The first equation with finitely many symmetries was found by Bakirov [1]:

$$u_t = 5u_4 + v_0^2, \qquad v_t = v_4$$

has a sixth order symmetry

$$u_t = 11u_6 + 5v_0v_2 + 4v_1^2, \qquad v_t = v_6,$$

where the i^{th} x-derivative of v_0 is denoted v_i. It was shown (with extensive computer algebra computations) that there are no other symmetries up to order 53. The authors of [2] proved using p-adic numbers that the system of Bakirov does not possess another symmetry at any higher order.

Have a look at the following points in the complex plane, see Fig. 1. You see 2745 points inside the upper half unit circle. Let us associate to every such a point r a new evolution equation

$$u_t = \left(1 + r^4\right) u_4 + v_0^2, \qquad v_t = (1 + r)^4 v_4. \tag{1}$$

We show that all these equations have *one* higher order generalized symmetry.

2 The symmetry condition

Let $K(v)$, $S(v)$ be polynomials that are quadratic in v_0 and its x-derivatives v_i. The Lie-bracket, see [10], between

$$u_t = a_1u_n + K(v), \qquad v_t = a_2v_n$$

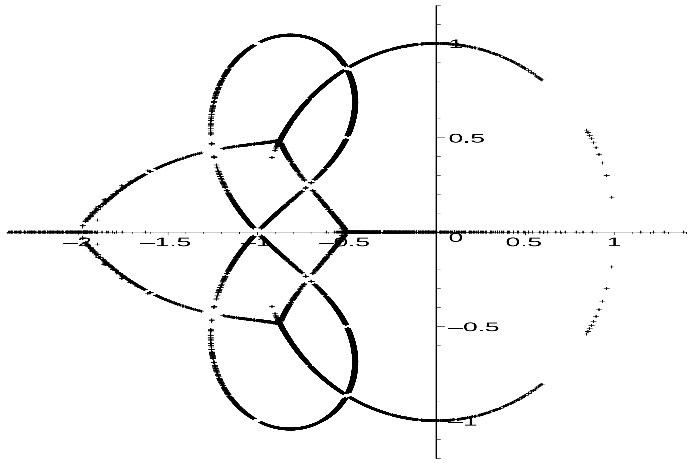

Figure 1. Roots of G-functions that correspond to almost integrable fourth order Bakirov like equations.

and

$$u_t = b_1 u_m + S(v), \qquad v_t = b_2 v_m$$

vanishes when

$$a_1 D^n S(v) - a_2 D_{S(v)} v_n = b_1 D^m K(v) - b_2 D_{K(v)} v_m, \tag{2}$$

where total differentiation is done by

$$D = \partial_x + \sum_{i=0}^{\infty} v_{i+1} \partial_{v_i}$$

and the Fréchet derivative is given by the operator

$$D_{K(v)} = \sum_{i=0}^{\infty} \partial_{v_i} K(v) D^i.$$

We will solve this equation (2) using the symbolic calculus, which was first developed in [4]. The Gel'fand–Dikiĭ transformation

$$v_i v_j \mapsto \frac{\xi_1^i \xi_2^j + \xi_1^j \xi_2^i}{2}$$

maps every quadratic polynomial $P(v)$ to $P(\xi_1, \xi_2)$. It has the properties

- $DP(v) \mapsto (\xi_1 + \xi_2) P(\xi_1, \xi_2)$,
- $D_{P(v)} v_n \mapsto (\xi_1^n + \xi_2^n) P(\xi_1, \xi_2)$.

Therefore equation (2) reads symbolically

$$G_n[a](\xi_1, \xi_2) S(\xi_1, \xi_2) = G_m[b](\xi_1, \xi_2) K(\xi_1, \xi_2),$$

where the so called G-functions are given by the polynomials

$$G_n[a](\xi_1, \xi_2) = a_1(\xi_1 + \xi_2)^n - a_2(\xi_1^n + \xi_2^n)$$

which can easily be solved

$$S = \frac{G_m[b](\xi_1, \xi_2)}{G_n[a](\xi_1, \xi_2)} K$$

if $G_n[a](\xi_1, \xi_2)$ divides $G_m[b](\xi_1, \xi_2)$.

3 Common roots

We call r a root of $f(\xi_1, \xi_2)$ if $(\xi_1 - r\xi_2)$ divides $f(\xi_1, \xi_2)$. If r is a root of $G_n[a](\xi_1, \xi_2)$ then

$$\frac{a_1}{a_2} = \frac{1+r^n}{(1+r)^n} = \frac{1+(1/r)^n}{(1+1/r)^n}$$

and hence $1/r$ is a root as well. A point s is another root if

$$U_n(r,s) = G_n[1+r^n, (1+r)^n](s,1)$$

vanishes, i.e.

$$(1+r)^n + (r+rs)^n - (1+s)^n - (s+rs)^n = 0. \tag{3}$$

The functions $G_n[1+r^n, (1+r)^n](\xi_1, \xi_2)$ and $G_m[1+r^m, (1+r)^m](\xi_1, \xi_2)$ have a common set of roots $\left\{r, \frac{1}{r}, s, \frac{1}{s}\right\}$ if the resultant of $U_n(r,s)$ and $U_m(r,s)$ with respect to s vanishes. This gives a very effective way to find equations with symmetries.

Example 1. We tread the Bakirov system. The resultant of $U_4(r,s)$ and $U_6(r,s)$ is

$$R = 2r^4 + 10r^3 + 15r^2 + 10r + 2.$$

The ratio of eigenvalues of the system is

$$\frac{1+r^4}{(1+r)^4} \text{ modulo } R = 5.$$

The ratio of eigenvalues of the symmetry is

$$\frac{1+r^6}{(1+r)^6} \text{ modulo } R = 11.$$

The quadratic part of the system is chosen $K(v) = v_0^2 \mapsto 1$, the quadratic part of the symmetry is calculated

$$S = \frac{G_6[11,1](\xi_1,\xi_2)}{G_4[5,1](\xi_1,\xi_2)} 1 = 5\frac{\xi_1^2+\xi_2^2}{2} + 4\xi_1\xi_2 \mapsto 5v_2v_0 + 4v_0^2.$$

Remark that we could have chosen any function $K(v)$.

We have calculated all resultants between $U_4(r,s)$ and $U_m(r,s)$, where $4 < m < 155$. We added their degrees and divided by four to obtain 2745, the number of fourth order equations with a symmetry of order less than 155. All zero points are numerically calculated and plotted in Fig. 1. The points on the curve throught -1, together with the points on the real line and the unit circle, are mapped to real values by

$$r \rightarrow \frac{1+r^4}{(1+r)^4}.$$

For the other we get complex eigenvalue ratios. The curve throught -1 is the set of zeropoints of

$$x^4 + 3x^3 + 4x^2 + 3x + 1 + \left(3x + 2x^2\right) y^2 + y^4$$

which appears as a factor of $U_4(x+iy, x-iy)$. A big question here is where the other curve comes from or at least how to describe it.

The resultants between $U_4(r,s)$ and $U_m(r,s)$ with respect to s, where $8 < m < 12$,

$$r^4 + 8r^3 + 12r^2 + 8r + 1,$$
$$14r^4 + 58r^3 + 87r^2 + 58r + 14,$$
$$3r^8 + 22r^7 + 69r^6 + 130r^5 + 159r^4 + 130r^3 + 69r^2 + 22r + 3.$$

You do not want to see the rest of the list. To indicate the size of the expressions involved, the resultant between $U_4(r,s)$ and $U_{154}(r,s)$ has degree 148 and coefficients that have 63 digets.

4 No more symmetry

We now ask the question whether a given equation has more than one symmetry. A p-adic method allows us to conclude that there exist only a finite number of symmetries. It is extremely powerful in our context. The method is based on the fact that if some equation does not have a solution in some p-adic field then it can not have a solution in $\mathbb{C}$. Moreover the method reduces the number of orders that need to be checked to a finite number.

P-adic numbers are represented by formal power series in a prime p

$$a = \sum_{n \geq 0} a_n p^n$$

with $a_n \in \mathbb{Z}/p$. The field of p-adic numbers is called $\mathbb{Z}_p$. The invertible elements are in $\mathbb{Z}_p^\times$, they have $a_0 \neq 0$.

Not all (complex) numbers are in every p-adic field. The following lemma of Hensel can be used to check whether for example $\sqrt{2}i$ is in $\mathbb{Z}_7$.

Lemma 1 (Hensel). *A polynomial*

$$f(x) = \sum_{i=0}^{n} a_i x^i \qquad \textit{with} \quad a_i \in \mathbb{Z}_p$$

has a root α in $\mathbb{Z}_p^\times$ if $\exists \alpha_1 \in \mathbb{Z}/p$ such that

- $f(\alpha_1) \equiv 0 \mod p$,
- $f'(\alpha_1) \not\equiv 0 \mod p$.

We now formulate the lemma of Skolem that form the basis of the method.

Lemma 2 (Skolem). *If $x_i \in \mathbb{Z}_p^\times$ then by the Fermat little theorem*

$$\exists\, y_i \in \mathbb{Z}_p : \; x_i^{p-1} = 1 + y_i p.$$

Let $U_n^m = \sum_{i=1} c_i y_i^m x_i^n$ for $m = 0, 1$.

- *If $U_k^0 \not\equiv 0 \mod p$ then $\forall\, r$ $U^0_{k+r(p-1)} \neq 0$,*
- *If $U_k^0 = 0$ and $U_k^1 \not\equiv 0 \mod p$ then $\forall\, r > 0$ $U^0_{k+r(p-1)} \neq 0$.*

Notice that equation (3) has the form $U_n^0 = 0$ with $i = 4$, $c_i = (-1)^i$ and

$$x_1 = 1 + s, \qquad x_2 = 1 + r, \qquad x_3 = s(1 + r), \qquad x_4 = r(1 + s).$$

Example 2. We tread the Bakirov system. With the lemma of Hensel one can show that $2r^4 + 10r^3 + 15r^2 + 10r + 2$ has two roots in $\mathbb{Z}_{181}$. Take $r \equiv 66 + 13p,\ s \equiv 139 + 29p$. Calculate modulo p^2

$$x_1 \equiv 140 + 29p, \qquad x_2 \equiv 67 + 13p, \qquad x_3 \equiv 82, \qquad x_4 \equiv 9 + 165p$$

and modulo p

$$y_1 \equiv 40, \qquad y_2 \equiv 33, \qquad y_3 \equiv 46, \qquad y_4 \equiv 140.$$

We have that $m = 0, 1, 4, 6$ are the only values less than $p - 1$ such that $U^0_m \equiv 0$ modulo p and that

$$U^1_0 \equiv 78, \qquad U^1_1 \equiv 173, \qquad U^1_4 \equiv 169, \qquad U^1_6 \equiv 162.$$

With the lemma of Skolem we may now conclude that if there is a symmetry it has be at order 6.

It is verified that all fourth order systems (1) with a symmetry of order less than 155 have one symmetry. The proof is done automatically by a computer using the lemma of Skolem in MAPLE [3]. The hard part is finding a good prime p. Once you know p, the conditions are very easily checked. We list some modulo p solutions of the resultants between $U_4(r, s)$ and $U_m(r, s)$ for $8 < m < 12$ in the specific fields

$$\begin{array}{lll} 71, & 72 & \in \mathbb{Z}/293, \\ 79, & 175 & \in \mathbb{Z}/491, \\ 26, & 44 & \in \mathbb{Z}/53. \end{array}$$

5 Conclusion

More results in this direction can be found in [8, 9], as well as the proofs of the relevant lemmas. It is proven that there exist infinitely many evolution equations with finitely many symmetries. All systems of order n with $4 < n < 11$ with symmetries of order m with $n < m < n+150$ have been calculated. Some improvements on the p-adic method have been made. These made it possible to show that among all the calculated systems there are only 3 equations with 2 symmetries, counter examples to the conjecture stated in [7, p. 255]. These systems have order 7 and their symmetries appear at order 11 and 29.

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Method of Replacing the Variables for Generalized Symmetry of D'Alembert Equation

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It is shown that by generalized understanding of symmetry the d'Alembert equation for one component field is invariant with respect to arbitrary reversible coordinate transformations.

1 Introduction

Symmetries play an important role in particle physics and quantum field theory [1], nuclear physics [2], mathematical physics [3]. It is proposed some receptions for finding the symmetries, for example, the method of replacing the variables [4], the Lie algorithm [3], the theoretical-algebraic approach [5]. The purpose of the present work is the generalization of the method of replacing the variables. We start from the following Definition of symmetry.

2 Definition of symmetry. Examples

Definition 1. Let some partial differential equation $\hat{L}'\phi'(x') = 0$ be given. By symmetry of this equation with respect to the variables replacement $x' = x'(x)$, $\phi' = \phi'(\Phi\phi)$ we shall understand the compatibility of the engaging equations system $\hat{A}\phi'(\Phi\phi) = 0$, $\hat{L}\phi(x) = 0$, where $\hat{A}\phi'(\Phi\phi) = 0$ is obtained from the initial equation by replacing the variables, $\hat{L}' = \hat{L}$, $\Phi(x)$ is some weight function. If the equation $\hat{A}\phi'(\Phi\phi) = 0$ can be transformed into the form $\hat{L}(\Psi\phi) = 0$, the symmetry will be named the standard Lie symmetry, otherwise the generalized symmetry.

Elements of this Definition were used to study the Maxwell equations symmetries [6, 7, 8]. In the present work we shall apply Definition 1 for investigation of symmetries of the one-component d'Alembert equation:

$$\Box'\phi'(x') = \frac{\partial^2\phi'}{\partial {x_1}'^2} + \frac{\partial^2\phi'}{\partial {x_2}'^2} + \frac{\partial^2\phi'}{\partial {x_3}'^2} + \frac{\partial^2\phi'}{\partial {x_4}'^2} = 0.$$

Let us introduce arbitrary reversible coordinate transformations $x' = x'(x)$ and a transformation of the field variable $\phi' = \phi(\Phi\phi)$, where $\Phi(x)$ is some unknown function, as well as take into account

$$\frac{\partial\phi'}{\partial x_i'} = \sum_j \frac{\partial\phi'}{\partial\xi}\frac{\partial\Phi\phi}{\partial x_j}\frac{\partial x_j}{\partial x_i'},$$
$$\frac{\partial^2\phi'}{\partial x_i'^2} = \sum_j \frac{\partial^2 x_j}{\partial x_i'^2}\frac{\partial\phi'}{\partial\xi}\frac{\partial\Phi\phi}{\partial x_j} + \sum_{jk}\frac{\partial^2\Phi\phi}{\partial x_j\partial x_k}\frac{\partial x_j}{\partial x_i'}\frac{\partial x_k}{\partial x_i'}\frac{\partial\phi'}{\partial\xi} + \sum_{jk}\frac{\partial^2\phi'}{\partial\xi^2}\frac{\partial\Phi\phi}{\partial x_j}\frac{\partial\Phi\phi}{\partial x_k}\frac{\partial x_j}{\partial x_i'}\frac{\partial x_k}{\partial x_i'},$$

where $\xi = \Phi\phi$. After replacing the variables we find that the equation $\Box'\phi' = 0$ transforms into itself, if the system of the engaging equations is fulfilled:

$$\sum_i \sum_j \frac{\partial^2 x_j}{\partial {x'_i}^2} \frac{\partial \phi'}{\partial \xi} \frac{\partial \Phi\phi}{\partial x_j} + \sum_i \sum_{j=k} \left(\frac{\partial x_j}{\partial x'_i}\right)^2 \frac{\partial \phi'}{\partial \xi} \frac{\partial^2 \Phi\phi}{\partial {x_j}^2} + \sum_i \sum_{j<k} \sum_k 2 \frac{\partial x_j}{\partial x'_i} \frac{\partial x_k}{\partial x'_i} \frac{\partial \phi'}{\partial \xi} \frac{\partial^2 \Phi\phi}{\partial x_j \partial x_k}$$
$$+ \sum_i \sum_{j=k} \left(\frac{\partial x_j}{\partial x_i}\right)^2 \frac{\partial^2 \phi'}{\partial \xi^2} \left(\frac{\partial \Phi\phi}{\partial x_j}\right)^2 + \sum_i \sum_{j<k} \sum_k 2 \frac{\partial x_j}{\partial x'_i} \frac{\partial x_k}{\partial x'_i} \frac{\partial^2 \phi'}{\partial \xi^2} \frac{\partial \Phi\phi}{\partial x_j} \frac{\partial \Phi\phi}{\partial x_k} = 0,$$
$$\Box\phi = 0. \tag{1}$$

Here $x = (x_1, x_2, x_3, x_4)$, $x_4 = ict$, c is the speed of light, t is the time. Let us put the solution of d'Alembert equation ϕ into the first equation of the set (1). If the obtained equation has a solution, then the set (1) will be compatible. According to Definition 1 it will mean that the arbitrary reversible transformations $x' = x'(x)$ are the symmetry transformations of the initial equation $\Box'\phi' = 0$. Owing to presence of the expressions $(\partial\Phi\phi/\partial x_j)^2$ and $(\partial\Phi\phi/\partial x_j)(\partial\Phi\phi/\partial x_k)$ in the first equation from the set (1), the latter has non-linear character. Since the analysis of non-linear systems is difficult we suppose that

$$\frac{\partial^2 \phi'}{\partial \xi^2} = 0. \tag{2}$$

In this case the non-linear components in the set (1) turn to zero and the system will be linear. As result we find the field transformation law by integrating the equation (2)

$$\phi' = C_1 \Phi\phi + C_2 \to \phi' = \Phi\phi. \tag{3}$$

Here we suppose for simplicity that the constants of integration are $C_1 = 1$, $C_2 = 0$. It is this law of field transformation that was used within the algorithm [7, 8]. It marks the position of the algorithm in the generalized variables replacement method. Taking into account the formulae (2) and (3), we find the following form for the system (1):

$$\frac{\partial^2 \phi'}{\partial \xi^2} = 0, \qquad \phi' = \Phi\phi,$$
$$\sum_j \Box' x_j \frac{\partial \Phi\phi}{\partial x_j} + \sum_i \sum_j \left(\frac{\partial x_j}{\partial x'_i}\right)^2 \frac{\partial^2 \Phi\phi}{\partial {x_j}^2} + \sum_i \sum_{j<k} \sum_k 2 \frac{\partial x_j}{\partial x'_i} \frac{\partial x_k}{\partial x'_i} \frac{\partial^2 \Phi\phi}{\partial x_j \partial x_k} = 0,$$
$$\Box\phi = 0. \tag{4}$$

Since here $\Phi(x) = \phi'(x' \to x)/\phi(x)$, where $\phi'(x')$ and $\phi(x)$ are the solutions of d'Alembert equation, the system (4) has a common solution and consequently is compatible. This means that the arbitrary reversible transformations of coordinates $x' = x'(x)$ are symmetry transformations for the one-component d'Alembert equation if the field is transformed with the help of weight function $\Phi(x)$ according to the law (3). The form of this function depends on d'Alembert equation solutions and the law of the coordinate transformations $x' = x'(x)$.

Next we shall consider the following examples.

Let the coordinate transformations belong to the *Poincaré group* P_{10}:

$$x'_j = L_{jk} x_k + a_j,$$

where L_{jk} is the matrix of the Lorentz group L_6, a_j are the parameters of the translation group T_4. In this case we have

$$\Box' x_j = \sum_k L'_{jk} \Box' x'_k = 0, \qquad \sum_i \frac{\partial x_j}{\partial x'_i} \frac{\partial x_k}{\partial x'_i} = \sum_i L'_{ji} L'_{ki} = \delta_{jk}.$$

The last term in the second equation (4) turns to zero. The set reduces to the form

$$\Box\Phi\phi = 0, \qquad \Box\phi = 0. \tag{5}$$

According to Definition 1 this is a sign of the Lie symmetry. The weight function belongs to the set in [8]:

$$\Phi_{P_{10}}(x) = \frac{\phi'(x)}{\phi(x)} \in \left\{1; \frac{1}{\phi(x)}; \frac{P_j\phi(x)}{\phi(x)}; \frac{M_{jk}\phi(x)}{\phi(x)}; \frac{P_jP_k\phi(x)}{\phi(x)}; \frac{P_jM_{kl}\phi(x)}{\phi(x)}; \cdots\right\},$$

where P_j, M_{jk} are the generators of Poincaré group, $j, k, l = 1, 2, 3, 4$. In the space of d'Alembert equation solutions the set defines a rule of the change from a solution to solution. The weight function $\Phi(x) = 1 \in \Phi_{P_{10}}(x)$ determines transformational properties of the solutions $\phi' = \phi$, which means the well-known relativistic symmetry of d'Alembert equation [9, 10].

Let the transformations of coordinates belong to the *Weyl group* W_{11}:

$$x'_j = \rho L_{jk}x_k + a_j,$$

where $\rho = \text{const}$ is the parameter of the scale transformations of the group Δ_1. In this case we have

$$\Box' x_j = \rho' \sum_k L'_{jk}\Box' x'_k = 0, \qquad \sum_i \frac{\partial x_j}{\partial x'_i}\frac{\partial x_k}{\partial x'_i} = \sum_i {\rho'}^2 L'_{ji}L'_{ki} = {\rho'}^2\delta_{jk} = \rho^{-2}\delta_{jk}.$$

The set (4) reduces to the set (5) and has the solution $\Phi_{W_{11}} = C\Phi_{P_{10}}$, where $C = \text{const}$. The weight function $\Phi(x) = C$ and the law $\phi' = C\phi$ means the well-known Weyl symmetry of d'Alembert equation [9, 10]. Let here C be equal ρ^l, where l is the conformal dimension[1] of the field $\phi(x)$. Consequently, d'Alembert equation is W_{11}-invariant for the field ϕ with arbitrary conformal dimension l. This property is essential for the Voigt [4] and Umov [12] works as will be shown just below.

Let the coordinate transformations belong to the *Inversion group* I:

$$x'_j = -\frac{x_j}{x^2}, \qquad x^2 = {x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2, \qquad x^2{x'}^2 = 1.$$

In this case we have

$$\Box' x_j = \frac{4x'_j}{{x'}^4} = -4x_jx^2, \qquad \sum_i \frac{\partial x_j}{\partial x'_i}\frac{\partial x_k}{\partial x'_i} = {\rho'}^2(x')\delta_{jk} = \frac{1}{{x'}^4}\delta_{jk} = x^4\delta_{jk}.$$

The set (4) reduces to the set:

$$-4x_j\frac{\partial\Phi\phi}{\partial x_j} + x^2\Box\Phi\phi = 0, \qquad \Box\phi = 0. \tag{6}$$

The substitution of $\Phi(x) = x^2\Psi(x)$ transforms the equation (6) for $\Phi(x)$ into the equation $\Box\Psi\phi = 0$ for $\Psi(x)$. It is a sign of the Lie symmetry. The equation has the solution $\Psi = 1$. The result is $\Phi(x) = x^2$. Consequently, the field transforms according to the law $\phi' = x^2\phi(x) = \rho^{-1}(x)\phi(x)$. This means the conformal dimension $l = -1$ of the field $\phi(x)$ in the case of d'Alembert equation symmetry with respect to the Inversion group I in agreement with [5, 10]. In a general case the weight function belongs to the set:

$$\Phi_I(x) = x^2\Psi(x) \in \left\{x^2; \frac{x^2}{\phi(x)}; x^2\frac{P_j\phi(x)}{\phi(x)}; x^2\frac{M_{jk}\phi(x)}{\phi(x)}; x^2\frac{P_jP_k\phi(x)}{\phi(x)}; \cdots\right\}. \tag{7}$$

[1]The conformal dimension is the number l characterizing the behavior of the field under scale transformations $x' = \rho x$, $\phi'(x') = \rho^l\phi(x)$ [11].

Let the coordinate transformations belong to the *Special Conformal Group* C_4:

$$x'_j = \frac{x_j - a_j x^2}{\sigma(x)}, \qquad \sigma(x) = 1 - 2a\cdot x + a^2 x^2, \qquad \sigma\sigma' = 1.$$

In this case we have

$$\Box' x_j = 4\left(a_j - a^2 x_j\right)\sigma(x), \qquad \sum_i \frac{\partial x_j}{\partial x'_i}\frac{\partial x_k}{\partial x'_i} = {\rho'}^2(x')\delta_{jk} = \sigma^2(x)\delta_{jk}.$$

The set (4) reduces to the set:

$$4\sigma(x)\left(a_j - a^2 x_j\right)\frac{\partial \Phi\phi}{\partial x_j} + \sigma^2(x)\Box\Phi\phi = 0, \qquad \Box\phi = 0. \tag{8}$$

The substitution of $\Phi(x) = \sigma(x)\Psi(x)$ transforms the equation (8) into the equation $\Box\Psi\phi = 0$ which corresponds to the Lie symmetry. From this equation we have $\Psi = 1$, $\Phi(x) = \sigma(x)$. Therefore $\phi' = \sigma(x)\phi(x)$ and the conformal dimension of the field is $l = -1$ as above. Analogously to (7), the weight function belongs to the set:

$$\Phi_{C_4}(x) = \sigma(x)\Psi(x) \in \left\{\sigma(x); \frac{\sigma(x)}{\phi(x)}; \sigma(x)\frac{P_j\phi(x)}{\phi(x)}; \sigma(x)\frac{M_{jk}\phi(x)}{\phi(x)}; \cdots\right\}.$$

From here we can see that $\phi(x) = 1/\sigma(x)$ is the solution of d'Alembert equation. Combination of W_{11}, I and C_4 symmetries means the well-known d'Alembert equation conformal C_{15}-symmetry [5, 9, 10].

Let the coordinate transformations belong to the *Galilei group* G_1:

$$x'_1 = x_1 + i\beta x_4, \qquad x'_2 = x_2, \qquad x'_3 = x_3, \qquad x'_4 = \gamma x_4, \qquad c' = \gamma c,$$

where $\beta' = -\beta/\gamma$, $\gamma' = 1/\gamma$, $\beta = V/c$, $\gamma = (1 - 2\beta n_x + \beta^2)^{1/2}$. In this case we have

$$\Box' x_j = 0, \qquad \sum_i \left(\frac{\partial x_1}{\partial x'_i}\right)^2 = 1 - {\beta'}^2, \qquad \sum_i \left(\frac{\partial x_2}{\partial x'_i}\right)^2 = \sum_i \left(\frac{\partial x_3}{\partial x'_i}\right)^2 = 1,$$

$$\sum_i \left(\frac{\partial x_4}{\partial x'_i}\right)^2 = {\gamma'}^2, \qquad \sum_i \frac{\partial x_1}{\partial x'_i}\frac{\partial x_2}{\partial x'_i} = \sum_i \frac{\partial x_1}{\partial x'_i}\frac{\partial x_3}{\partial x'_i} = \sum_i \frac{\partial x_2}{\partial x'_i}\frac{\partial x_3}{\partial x'_i} = \sum_i \frac{\partial x_2}{\partial x'_i}\frac{\partial x_4}{\partial x'_i} = 0,$$

$$\sum_i \frac{\partial x_1}{\partial x'_i}\frac{\partial x_4}{\partial x'_i} = i\beta'\gamma' = -i\beta/\gamma^2.$$

After putting these expressions into the set (4) we find [8]:

$$\Box\Phi\phi - \frac{\partial^2\Phi\phi}{\partial {x_4}^2} - \left(i\frac{\partial}{\partial x_4} + \beta\frac{\partial}{\partial x_1}\right)^2 \frac{\Phi\phi}{\gamma^2} = \left[\frac{(i\partial_4 + \beta\partial_1)^2}{\gamma^2} - \triangle\right]\Phi\phi = 0.$$

In accordance with Definition 1 it means that the Galilei symmetry of d'Alembert equation is the generalized symmetry. The weight function belongs to the set [7]:

$$\Phi_{G_1}(x) = \frac{\phi'(x' \to x)}{\phi(x)} \in \left\{\frac{\phi(x')}{\phi(x)}; \frac{1}{\phi(x)}; \frac{P'_j\phi(x')}{\phi(x)}; \frac{[\Box', H'_1]\phi(x')}{\phi(x)}; \cdots\right\},$$

where $H'_1 = it'\partial_{x'}$ is the generator of the Galilei transformations. For plane waves the weight function $\Phi(x)$ is [6, 7, 8]:

$$\Phi_{G_1}(x) = \frac{\phi(x' \to x)}{\phi(x)} = \exp\left\{-\frac{i}{\gamma}\left[(1-\gamma)k\cdot x - \beta\omega\left(n_x t - \frac{x}{c}\right)\right]\right\},$$

where $k = (\boldsymbol{k}, k_4)$, $\boldsymbol{k} = \omega\boldsymbol{n}/c$ is the wave vector, $\boldsymbol{n}$ is the wave front guiding vector, ω is the wave frequency, $k_4 = i\omega/c$, $k'_1 = (k_1 + i\beta k_4)/\gamma$, $k'_2 = k_2/\gamma$, $k'_3 = k_3/\gamma$, $k'_4 = k_4$, $\boldsymbol{k}'^2 = \boldsymbol{k}^2$ - inv. (For comparison, in the relativistic case we have $k'_1 = (k_1 + i\beta k_4)/(1-\beta^2)^{1/2}$, $k'_2 = k_2$, $k'_3 = k_3$, $k'_4 = (k_4 - i\beta k_1)/(1-\beta^2)^{1/2}$, $\boldsymbol{k}'^2 + {k_4'}^2 = \boldsymbol{k}^2 + {k_4}^2$ - inv as is well-known).

The results obtained above we illustrate by means of the Table 1:

Group	P_{10}	W_{11}	I	C_4	G_1
$WF\ \Phi(x)$	1	ρ^l	x^2	$\sigma(x)$	$\exp\{-i[(1-\gamma)k\cdot x - \beta\omega(n_x t - x/c)]/\gamma\}$

For the different transformations $x' = x'(x)$, the weight functions $\Phi(x)$ may be found in a similar way.

Let us note that in the symmetry theory of d'Alembert equation, the conditions (4) for transforming this equation into itself combine the requirements formulated by various authors, as can be seen in the Table 2:

Author	*Coordinates transform.*	*Group*	*Conditions of invariance*	*Fields transform.*
Voigt [4]	$x'_j = A_{jk}x_k$	$L_6 X \Delta_1$	$A'_{ji}A'_{ki} = \rho'^2\delta_{jk}$	$\phi' = \phi$
Umov [12]	$x'_j = x_j{}'(x)$	W_{11}	$\dfrac{\partial x_j}{\partial x'_i}\dfrac{\partial x_k}{\partial x'_i} = \rho'^2\delta_{jk},\ \Box' x_j = 0$	$\phi' = \phi$
Di Jorio [13]	$x'_j = L_{jk}x_k + a_j$	P_{10}	$L'_{ji}L'_{ki} = \delta_{jk},\ \dfrac{\partial^2\phi'}{\partial\phi_\alpha\partial\phi_\beta} = 0$	$\phi' = m_\alpha\phi_\alpha + m_0$
				$\alpha = 1,\dots,n$
Kotel'nikov	$x'_j = x'_j(x)$	C_4	$\dfrac{\partial x_j}{\partial x'_i}\dfrac{\partial x_k}{\partial x'_i} = \rho'^2(x')\delta_{jk}$	$\phi'_\alpha = \psi D_{\alpha\beta}\phi_\beta$
[6, 7, 8]			$\dfrac{\partial^2\phi'_\alpha}{\partial\xi_\beta\partial\xi_\gamma} = 0,\ \Box'\phi'_\alpha = 0 \to$	$\xi_\alpha = \psi\phi_\alpha$
			$\hat{A}\phi'_\alpha(\psi\phi_1,\dots,\psi\phi_6) = 0$	$\alpha,\beta = 1,\dots,6$
			$\Box\phi_\beta = 0$	
	$x'_j = x'_j(x)$	G_1	$\dfrac{\partial^2\phi'_\alpha}{\partial\xi_\beta\partial\xi_\gamma} = 0,\ \Box'\phi'_\alpha = 0 \to$	$\phi'_\alpha = \psi M_{\alpha\beta}\phi_\beta$
			$\hat{B}\phi'_\alpha(\psi\phi_1,\dots,\psi\phi_6) = 0$	$\xi_\alpha = \psi\phi_\alpha$
			$\Box\phi_\beta = 0$	$\alpha,\beta = 1,\dots,6$

Here m_α, m_0 are some numbers, $D_{\alpha\beta}$ and $M_{\alpha\beta}$ are the 6×6 numerical matrices.

According to this Table for the field $\phi' = \phi$ with conformal dimension $l = 0$ and the linear homogeneous coordinate transformations from the group $L_6 \times \triangle_1 \in W_{11}$ with $\rho = (1-\beta^2)^{1/2}$, the formulae were proposed by Voigt [4, 9]. In the plain waves case they correspond to the transformations of the 4-vector $k = (\boldsymbol{k}, k_4)$ and proper frequency ω_0 according to the law $k'_1 = (k_1 + i\beta k_4)/\rho(1-\beta^2)^{1/2}$, $k'_2 = k_2/\rho$, $k'_3 = k_3/\rho$, $k'_4 = (k_4 - i\beta k_1)/\rho(1-\beta^2)^{1/2}$, $\omega'_0 = \omega_0/\rho$, $k'x' = kx$ - inv. In the case of the W_{11}-coordinate transformations belonging to the set of arbitrary transformations $x' = x'(x)$ the requirements for the one component field with $l = 0$ were found by Umov [12]. The requirement that the second derivative $\partial^2\phi'/\partial\phi_\alpha\partial\phi_\beta = 0$ with $\Phi = 1$ be turned into zero was introduced by Di Jorio [13]. The weight function $\Phi \neq 1$ and the set (4) were proposed by the author of the present work [6, 7, 8].

By now well-studied have been only the d'Alembert equation symmetries corresponding to the linear systems of the type (5), (6), (8). These are the well-known relativistic and conformal symmetry of the equation. The investigations corresponding to the linear conditions (4) are much more scanty and presented only in the papers [6, 7, 8]. The publications corresponding to the non-linear conditions (1) are absent completely. The difficulties arising here are connected with analysis of compatibility of the set (1) containing the non-linear partial differential equation.

3 Conclusion

It is shown that under generalized understanding of the symmetry according to Definition 1, d'Alembert equation for one component field is invariant with respect to any arbitrary reversible coordinate transformations $x' = x'(x)$. In particular, they contain transformations of the conformal and Galilei groups realizing the type of standard and generalized symmetry for $\Phi(x) = \phi'(x' \to x)/\phi(x)$. The concept of partial differential equations symmetry is conventional.

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Subgroups of Extended Poincaré Group and New Exact Solutions of Maxwell Equations

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Using three-parameter subgroups of the extended Poincaré group $\tilde{P}(1,3)$ we have constructed ansatzes reducing the Maxwell equations to systems of ordinary differential equations. This enables us to construct a number of new exact solutions of the Maxwell equations.

1 Introduction

The electromagnetic field is described by the electric $\boldsymbol{E} = \boldsymbol{E}(x_0, \boldsymbol{x})$ and magnetic $\boldsymbol{H} = \boldsymbol{H}(x_0, \boldsymbol{x})$ fields. In the absence of charges, we have the system of vacuum Maxwell equations

$$\operatorname{rot} \boldsymbol{E} = -\frac{\partial \boldsymbol{H}}{\partial x_0}, \qquad \operatorname{div} \boldsymbol{H} = 0, \qquad \operatorname{rot} \boldsymbol{H} = \frac{\partial \boldsymbol{E}}{\partial x_0}, \qquad \operatorname{div} \boldsymbol{E} = 0. \tag{1}$$

As it is well-known [1, 2], the maximal point symmetry group admitted by the Maxwell equations (1) is the 16-parameter group which is the direct product of the 15-parameter conformal group $C(1,3)$ and of the one-parameter Heaviside–Larmor–Rainich group H. It contains as a subgroup the extended Poincaré group $\tilde{P}(1,3)$ generated by the following vector fields:

$$\begin{aligned}
&P_\mu = \partial_{x_\mu}, \qquad J_{0a} = x_0 \partial_{x_a} + x_a \partial_{x_0} + \varepsilon_{abc}(E_b \partial_{H_c} - H_b \partial_{E_c}),\\
&J_{ab} = x_b \partial_{x_a} - x_a \partial_{x_b} + E_b \partial_{E_a} - E_a \partial_{E_b} + H_b \partial_{H_a} - H_a \partial_{H_b},\\
&D = x_\mu \partial_{x_\mu} - 2(E_a \partial_{E_a} + H_a \partial_{H_a}).
\end{aligned} \tag{2}$$

Here $\mu = 0, 1, 2, 3$; $a, b, c = 1, 2, 3$; summation over repeated indices is understood, the index μ taking the values $0, 1, 2, 3$ and the indices a, b taking the values $1, 2, 3$; ε_{abc} is the totally antisymmetric third-order tensor, $\partial_{x_\mu} = \frac{\partial}{\partial x_\mu}$, $\partial_{E_a} = \frac{\partial}{\partial E_a}$, $\partial_{H_a} = \frac{\partial}{\partial H_a}$.

The large symmetry group admitted by the Maxwell equations allows one to construct many exact solutions by the symmetry reduction method [3, 4, 5, 6, 7, 8]. Using three-parameter subgroups of the Poincaré group $P(1,3)$ with generators P_μ, $J_{\mu\nu}$ (2) enabled us to obtain in [9, 10] a number of exact solutions of the system (1).

The aim of the present report is to give an exhaustive description of $\tilde{P}(1,3)$-invariant ansatzes for the Maxwell field $(\boldsymbol{E}, \boldsymbol{H})$ reducing equations (1) to systems of ordinary differential equations. Using them we will construct new exact solutions of the Maxwell equations.

Let $\tilde{p}(1,3)$ be the Lie algebra of the Poincaré group with the generators (2) and $\tilde{p}^{(1)}(1,3)$ be the Lie algebra having as basis elements

$$P_\mu^{(1)} = \partial_{x_\mu}, \qquad J_{\mu\nu}^{(1)} = x^\mu \partial_{x_\nu} - x^\nu \partial_{x_\mu}, \qquad D_\mu^{(1)} = x_\mu \partial_{x_\mu},$$

where $\mu, \nu = 0, 1, 2, 3$; lowering of the indices μ, ν is performed with the help of the metric tensor of the Minkowski space-time $g_{\mu\nu}$.

Next, let L be a subalgebra of the algebra $\tilde{p}(1,3)$ having rank r, and let the projection of the algebra L onto $\tilde{p}^{(1)}(1,3)$ have rank $r^{(1)}$. It follows from the general theory of invariant solutions of differential equations ([3]) that subalgebras of the algebra L satisfying the additional condition $r = r^{(1)} = 3$ give rise to ansatzes reducing (1) to systems of ordinary differential equations. It is not difficult to see that in the case $\dim L = 3$ and a basis of functionally independent invariants of the algebra L consists of seven functions $\Omega_i = \Omega_i(x_0, \boldsymbol{x}, \boldsymbol{E}, \boldsymbol{H})$ $(i = 1, 2, \ldots, 6)$ and $\omega = \omega(x_0, \boldsymbol{x})$. The structure of an invariant ansatz is completely determined by the form of the functions Ω_i.

Let us introduce the notations

$$\boldsymbol{V} = (E_1\ E_2\ E_3\ H_1\ H_2\ H_3)^T, \qquad \boldsymbol{W} = \left(\tilde{E}_1\ \tilde{E}_2\ \tilde{E}_3\ \tilde{H}_1\ \tilde{H}_2\ \tilde{H}_3\right)^T.$$

Then the general form of the basis elements of the three-dimensoinal Lie algebra $L = \langle X_a | a = 1, 2, 3\rangle$ reads as

$$X_a = \xi_{a\mu}(x_0, \boldsymbol{x})\partial_{x_\mu} + \rho_{alk} V_k \partial_{V_l}.$$

Here, and in the following, $m, n, k, l = 1, 2, \ldots, 6$; $\mu, \nu = 0, 1, 2, 3$.

As the basis elements (2) realize a linear representation of the algebra $\tilde{p}(1,3)$ and, the condition $r = r^{(1)}$ holds, the general form of an ansatz invariant with respect to a three-dimensional subalgebra $L \in \tilde{p}(1,3)$ reads [8, 9, 10]

$$\boldsymbol{V} = \Lambda \boldsymbol{W}(\omega), \tag{3}$$

where $\Lambda = \Lambda(x_0, \boldsymbol{x})$ is a 6×6 matrix nonsingular in some domain of the space $\mathbb{R}_{0,3} = \{(x_0, \boldsymbol{x}) : x_\mu \in \mathbb{R},\ \mu = 0, 1, 2, 3\}$ which, together with a smooth scalar function $\omega = \omega(x)$, satisfies the following system of partial differential equations:

$$\xi_{a\mu}\frac{\partial \Lambda_{mn}}{\partial_{x_\mu}} + f_{ml}\rho_{aln} = 0, \tag{4}$$

$$\xi_{a\mu}\frac{\partial \omega_{mn}}{\partial_{x_\mu}} = 0. \tag{5}$$

Here the symbol Λ_{mn} stands for the (m, n) entry of the matrix Λ.

Thus, the problem of symmetry reduction of the Maxwell equations by scale-invariant ansatzes contains as a subproblem integration of systems of the form (4), (5) for each inequivalent three-dimensional algebra. Remarkably, there is no need to consider all inequivalent algebras, since the following results hold:

Lemma 1 ([9]). *Let $\boldsymbol{E}$, $\boldsymbol{H}$ be functions of x_1, x_2, $\xi = \frac{1}{2}(x_0 - x_3)$ only. Then the Maxwell equations can be integrated, and their general solution is given by*

$$E_1 = \frac{1}{2}(R + R^* + T_1 + T_1^*), \qquad E_2 = \frac{1}{2}(iR - iR^* + T_2 + T_2^*), \qquad E_3 = S + S^*,$$
$$H_1 = \frac{1}{2}(iR - iR^* - T_2 - T_2^*), \qquad E_2 = \frac{1}{2}(R + R^* - T_1 - T_1^*), \qquad E_3 = iS - iS^*,$$

where $T_a = \frac{\partial^2 \sigma_a}{\partial \xi^2}$, $a = 1, 2$; $S = \frac{\partial \sigma_1}{\partial \xi} + i\frac{\partial \sigma_2}{\partial \xi} + \lambda(z)$, $R = -2\left(\frac{\partial \sigma_1}{\partial z} + i\frac{\partial \sigma_2}{\partial z}\right) + \frac{d\lambda}{dz}\xi$; $\sigma = \sigma_a(z, \xi)$, $z = x_1 + ix_2$ and $\lambda = \lambda(z)$ are arbitrary analytic functions.

Lemma 2 ([11]). *Let $\boldsymbol{E}$, $\boldsymbol{H}$ be functions of x_0, x_3 only. Then the Maxwell equations can be integrated, and their general solution is given by the formulae below*

$$E_1 = f_1(\xi) + g_1(\eta), \qquad E_2 = f_2(\xi) + g_2(\eta), \qquad E_3 = C_1,$$
$$H_1 = f_2(\xi) - g_2(\eta), \qquad H_2 = -f_1(\xi) + g_1(\eta), \qquad H_3 = C_2,$$

where f_1, f_2, g_1, g_2 are arbitrary smooth functions, $\xi = x_0 - x_3$, $\eta = x_0 + x_3$ and C_1, C_2 are arbitrary real constants.

Consequently, to obtain new solutions of the Maxwell equations it is sufficient to restrict our considerations to those three-dimensional subalgebras of $\tilde{p}(1,3)$ which are not conjugate to subalgebras of $p(1,3)$ and, in addition, fulfill the conditions

$$1)\ \ r = r^{(1)} = 3; \qquad 2)\ \ \langle P_0 \pm P_3\rangle \not\subset L, \quad \langle P_0, P_3\rangle \not\subset L; \qquad 3)\ \ \langle P_1, P_2\rangle \not\subset L.$$

Making use of the classification of inequivalent subalgebras of the algebra $\tilde{p}(1,3)$ obtained in [9, 10] we have checked that the above conditions are satisfied by the following seven subalgebras [11]:

$$\begin{aligned}
&L_1 = \langle J_{12}, D, P_0\rangle; \qquad L_2 = \langle J_{12}, D, P_3\rangle; \qquad L_3 = \langle J_{03}, D, P_1\rangle;\\
&L_4 = \langle J_{03}, J_{12}, D\rangle; \qquad L_5 = \langle G_1, J_{03} + \alpha D, P_2\rangle \ \ (0 < |\alpha| \le 1);\\
&L_6 = \langle J_{03} - D + P_0 + P_3, G_1, P_2\rangle; \qquad L_7 = \langle J_{03} + 2D, G_1 + P_0 - P_3, P_2\rangle,
\end{aligned}$$

where $G_1 = J_{01} - J_{13}$.

As direct verification shows, the basis elements of the above algebras satisfy the condition $r = r^{(1)} = 3$. Consequently, each of them gives rise to an ansatz of the type given in (3). Furthermore, these ansatzes can be represented in a unified way, namely

$$\begin{aligned}
E_1 &= \theta\{(\tilde{E}_1 \cos\theta_3 - \tilde{E}_2 \sin\theta_3)\cosh\theta_0 + (\tilde{H}_1 \sin\theta_3 + \tilde{H}_2 \cos\theta_3)\sinh\theta_0\\
&\quad + 2\theta_1 \tilde{E}_3 + 2\theta_2 \tilde{H}_3 + 4\theta_1\theta_2\Sigma_1 + 2(\theta_1^2 - \theta_2^2)\Sigma_2\},\\
E_2 &= \theta\{(\tilde{E}_2 \cos\theta_3 + \tilde{E}_1 \sin\theta_3)\cosh\theta_0 + (\tilde{H}_2 \sin\theta_3 - \tilde{H}_1 \cos\theta_3)\sinh\theta_0\\
&\quad - 2\theta_1 \tilde{H}_3 + 2\theta_2 \tilde{E}_3 + 4\theta_1\theta_2\Sigma_2 - 2(\theta_1^2 - \theta_2^2)\Sigma_1\},\\
E_3 &= \theta\{\tilde{E}_3 + 2\theta_1\Sigma_2 + 2\theta_2\Sigma_1\},\\
H_1 &= \theta\{(\tilde{H}_1 \cos\theta_3 - \tilde{H}_2 \sin\theta_3)\cosh\theta_0 - (\tilde{E}_1 \sin\theta_3 + \tilde{E}_2 \cos\theta_3)\sinh\theta_0\\
&\quad + 2\theta_1 \tilde{H}_3 - 2\theta_2 \tilde{E}_3 - 4\theta_1\theta_2\Sigma_2 + 2(\theta_1^2 - \theta_2^2)\Sigma_1\},\\
H_2 &= \theta\{(\tilde{H}_2 \cos\theta_3 + \tilde{H}_1 \sin\theta_3)\cosh\theta_0 + (\tilde{E}_1 \cos\theta_3 - \tilde{E}_2 \sin\theta_3)\sinh\theta_0\\
&\quad + 2\theta_1 \tilde{E}_3 + 2\theta_2 \tilde{H}_3 + 4\theta_1\theta_2\Sigma_1 + 2(\theta_1^2 - \theta_2^2)\Sigma_2\},\\
H_3 &= \theta\{\tilde{H}_3 + 2\theta_1\Sigma_1 - 2\theta_2\Sigma_2\},
\end{aligned}$$

where

$$\begin{aligned}
\Sigma_1 &= [(\tilde{H}_2 - \tilde{E}_1)\sin\theta_3 - (\tilde{E}_2 + \tilde{H}_1)\cos\theta_3]e^{-\theta_0},\\
\Sigma_1 &= [(\tilde{E}_2 + \tilde{H}_1)\sin\theta_3 + (\tilde{H}_2 - \tilde{E}_1)\cos\theta_3]e^{-\theta_0},
\end{aligned}$$

and the functions $\theta = \theta(x_0, \boldsymbol{x})$, $\theta_\beta = \theta_\beta(x_0, \boldsymbol{x})$ $(\beta = 0, 1, 2)$, $\omega = \omega(x_0, \boldsymbol{x})$ are ([11]):

$$\begin{aligned}
&L_1 : \theta = x_3^2, \ \ \theta_1 = \arctan\frac{x_2}{x_1}, \ \ \theta_0 = \theta_2 = 0, \ \ \omega = \frac{x_1^2 + x_2^2}{x_3^2};\\
&L_2 : \theta = x_0^2, \ \ \theta_1 = \arctan\frac{x_2}{x_1}, \ \ \theta_0 = \theta_2 = 0, \ \ \omega = \frac{x_1^2 + x_2^2}{x_0^2};\\
&L_3 : \theta = x_2^2, \ \theta_0 = \ln\left|(x_0 + x_3)x_2^{-1}\right|, \ \ \theta_1 = \theta_2 = 0, \ \ \omega = \left(x_0^2 - x_3^2\right)x_2^{-2};\\
&L_4 : \theta = x_0^2 - x_3^2, \ \ \theta_0 = \frac{1}{2}\ln\left|(x_0 + x_3)(x_0 - x_3)^{-1}\right|, \ \ \theta_1 = \arctan\frac{x_2}{x_1}, \ \ \theta_2 = 0,\\
&\qquad \omega = \left(x_1^2 + x_2^2\right)\left(x_0^2 - x_3^2\right)^{-1};
\end{aligned}$$

$$L_5: 1)\ \ \theta = x_0 - x_3, \quad \theta_0 = -\frac{1}{2}\ln|x_0 - x_3|, \quad \theta_1 = 0, \quad \theta_2 = -\frac{1}{2}x_1(x_0 - x_3)^{-1},$$
$$\omega = x_0 + x_3 - x_1^2(x_0 - x_3)^{-1} \quad \text{for} \quad \alpha = -1;$$
$$2)\ \ \theta = x_0^2 - x_1^2 - x_3^2,\ \theta_0 = \frac{1}{2\alpha}\ln\left|x_0^2 - x_1^2 - x_3^2\right|, \quad \theta_1 = 0, \quad \theta_2 = -\frac{1}{2}x_1(x_0 - x_3)^{-1},$$
$$\omega = 2\alpha\ln|x_0 - x_3| + (1-\alpha)\ln\left|x_0^2 - x_1^2 - x_3^2\right| \quad \text{for} \quad \alpha \neq -1;$$
$$L_6: \theta = x_0 - x_3, \quad \theta_0 = -\frac{1}{2}\ln|x_0 - x_3|, \quad \theta_1 = 0, \quad \theta_2 = -\frac{x_1}{2(x_0 - x_3)},$$
$$\omega = x_0 + x_3 - x_1^2(x_0 - x_3)^{-1} + \ln|x_0 - x_3|;$$
$$L_7: \theta = \left(4x_1 - (x_0 - x_3)^2\right)^2, \quad \theta_0 = \frac{1}{2}\ln\left|4x_1 - (x_0 - x_3)^2\right|, \quad \theta_1 = 0,$$
$$\theta_2 = -\frac{1}{4}(x_0 - x_3), \quad \omega = \left[x_0 + x_3 - x_1(x_0 - x_3) + \frac{1}{6}(x_0 - x_3)^3\right]\left|4x_1 - (x_0 - x_3)^2\right|^{-\frac{3}{2}}.$$

Substituting the ansatzes obtained in this way into the initial system (1) yields systems of ordinary differential equations for the unknown functions $\tilde{E}_a$, $\tilde{H}_a$ $(a = 1, 2, 3)$. If, for example, we take the ansatz invariant under the algebra L_1 and insert it into the Maxwell equations, then, after some algebraic manipulations, we obtain the following system for $\tilde{E}_a(\omega)$, $\tilde{H}_a(\omega)$ $(a = 1, 2, 3)$:

$$2\omega(1+\omega)\ddot{\tilde{E}}_3 + (7\omega + 2)\dot{\tilde{E}}_3 + 3\tilde{E}_3 = 0, \qquad 2\omega(1+\omega)\ddot{\tilde{H}}_3 + (7\omega + 2)\dot{\tilde{H}}_3 + 3\tilde{H}_3 = 0,$$
$$f = h = -2\sqrt{\omega}(\tilde{E}_3 + (1+\omega)\dot{\tilde{E}}_3), \qquad g = -\rho = 2\sqrt{\omega}(\tilde{H}_3 + (1+\omega)\dot{\tilde{H}}_3),$$

where

$$f = \tilde{E}_1 + \tilde{H}_2, \qquad g = \tilde{E}_2 - \tilde{H}_1, \qquad h = \tilde{E}_1 - \tilde{H}_2,$$
$$\rho = \tilde{E}_2 + \tilde{H}_1, \qquad \dot{\tilde{E}}_3 = \frac{d\tilde{E}_3}{d\omega}, \qquad \ddot{\tilde{E}}_3 = \frac{d^2\tilde{E}_2}{d\omega^2}.$$

Taking into account that we have $\omega \geq 0$, we represent the general solution of the above system as follows

$$\tilde{E}_3 = (1+\omega)^{-\frac{3}{2}}\left[C_1\left(\ln\left|\frac{\sqrt{1+\omega}-1}{\sqrt{1+\omega}+1}\right| + 2\sqrt{1+\omega}\right) + C_2\right],$$
$$\tilde{H}_3 = (1+\omega)^{-\frac{3}{2}}\left[C_3\left(\ln\left|\frac{\sqrt{1+\omega}-1}{\sqrt{1+\omega}+1}\right| + 2\sqrt{1+\omega}\right) + C_4\right],$$

where C_1, C_2, C_3, C_4 are integration constants, and we easily get the corresponding exact solutions of the Maxwell equations (1):

$$E_a = -\frac{2C_1 x_a}{x_3\left(x_1^2 + x_2^2\right)} + x_a\sigma^{-\frac{3}{2}}A_{12}, \qquad E_3 = x_3\sigma^{-\frac{3}{2}}A_{12},$$
$$H_a = -\frac{2C_3 x_a}{x_3\left(x_1^2 + x_2^2\right)} + x_a\sigma^{-\frac{3}{2}}A_{34}, \qquad H_3 = x_3\sigma^{-\frac{3}{2}}A_{34}.$$

Here $A_{ij} = C_i\left(\ln\left|\frac{\sqrt{\sigma}-x_3}{\sqrt{\sigma}+x_3}\right| + 2x_3^{-1}\sqrt{\sigma}\right) + C_j$, $\sigma = x_1^2 + x_x^2 + x_3^2$, $a = 1, 2$.

Let us note that the systems of ordinary differential equations obtained via reduction of the Maxwell equations by ansatzes invariant under the remaining algebras L_2–L_7 are also integrable in terms of elementary functions.

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Invariance of Quasilinear Equations of Hyperbolic Type with Respect to Three-Dimensional Lie Algebras

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We have completely solved the problem of description of quasi-linear hyperbolic differential equations in two independent variables, that are invariant under three-parameter Lie groups.

The problem of group classification of differential equations is one of the central problems of modern symmetry analysis of differential equations [1]. One of the important classes are hyperbolic equations. The problem of group classification of such equations was discussed by many authors (see for instance [2–6]). In this paper we consider the problem of the group classification of equations of the form:

$$u_{tt} = u_{xx} + F(t, x, u, u_x), \tag{1}$$

where $u = u(t, x)$ and F is an arbitrary nonlinear differentiable function, with $F_{u_x,u_x} \neq 0$ is an arbitrary nonlinear smooth function, which dependent variables u or u_x. We use the following notation $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, $F_{u_x} = \frac{\partial F}{\partial u_x}$, $u_t = \frac{\partial u}{\partial t}$, $u_{tt} = \frac{\partial^2 u}{\partial t^2}$. For the group classification of equation (1) we use the approach proposed in [7]. Here we give three main results (for details, the reader is referred to [8]).

Theorem 1. *The infinitesimal operator of the symmetry group of the equation (1) has the following form:*

$$X = (\lambda t + \lambda_1)\partial_t + (\lambda x + \lambda_2)\partial_x + (h(x)u + r(t, x))\partial_u, \tag{2}$$

where λ, λ_1, λ_2 are arbitrary real constants and $h(x)$, $r(t, x)$ are arbitrary functions which satisfy the condition

$$\begin{aligned} r_{tt} - \frac{d^2h}{dx^2}u - r_{xx} + (h - 2\lambda)F - (\lambda t + \lambda_1)F_t - (\lambda x + \lambda_2)F_x \\ - (hu + r)F_u - 2u_x\frac{dh}{dx} - u_x(h - \lambda)F_{u_x} - \frac{dh}{dx}uF_{u_x} - r_xF_{u_x} = 0. \end{aligned} \tag{3}$$

Theorem 2. *The equivalence group of the equation (1) is given by transformations of the following form:*

$$\bar{t} = \gamma t + \gamma_1, \qquad \bar{x} = \epsilon\gamma x + \gamma_2, \qquad v = \rho(x)u + \theta(t, x), \tag{4}$$

$\gamma \neq 0$, $\rho \neq 0$, $\epsilon = \pm 1$.

Theorem 3. *In the class of operators (2), there are no realizations of the algebras $so(3)$ and $sl(2, \mathbb{R})$.*

From this theorem we obtain the following:

Note 1. In the class of operators (2) there are no realizations of any real semi-simple Lie algebras.

Note 2. There are no equations (1) which have algebras of invariance, isomorphic by real semi-simple algebras, or contain those algebras as subalgebras.

The set of three-dimensional solvable Lie algebras consists of the following two decomposable Lie algebras:

$$A_{3.1} = A_1 \oplus A_1 \oplus A_1 = 3A_1; \qquad A_{3.2} = A_{2.2} \oplus A_1, \quad [e_1, e_2] = e_2,$$

and the following seven of non-decomposable Lie algebras:

$$\begin{aligned}
&A_{3.3}: \quad [e_2, e_3] = e_1;\\
&A_{3.4}: \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_1 + e_2;\\
&A_{3.5}: \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2;\\
&A_{3.6}: \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = -e_2;\\
&A_{3.7}: \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = qe_2, \quad (0 < |q| < 1);\\
&A_{3.8}: \quad [e_1,\ e_3] = -e_2, \quad [e_2,\ e_3] = e_1;\\
&A_{3.9}: \quad [e_1,\ e_3] = qe_1 - e_2, \quad [e_2,\ e_3] = e_1 + qe_2,\ (q > 0).
\end{aligned}$$

We give the realizations of the algebras $A_{3.3}$, $A_{3.4}$, $A_{3.5}$, $A_{3.9}$ and the corresponding values of the functions F in the equation (1). Here we find only equations, which are non-equivalent to equations of the form

$$u_{tt} = u_{xx} - u^{-1}u_x^2 + A(x)u_x + B(x)u \ln|u| + uD(t,x),$$

and which was classified in [8].

$$\begin{aligned}
&A_{3.3}^1 = \langle u\partial_u,\ \partial_x,\ m\partial_t + xu\partial_u\rangle,\ m \neq 0: \quad F = -u^{-1}u_x^2 + u\tilde{G}(\omega), \quad \omega = t - mu_xu^{-1};\\
&A_{3.3}^2 = \langle \partial_u,\ \partial_x,\ m\partial_t + x\partial_u\rangle,\ m \neq 0: \quad F = \tilde{G}(\omega), \quad \omega = mu_x - t;\\
&A_{3.3}^3 = \langle \partial_u,\ \partial_t,\ \partial_x + t\partial_u\rangle: \quad F = \tilde{G}(u_x);\\
&A_{3.3}^4 = \left\langle u\partial_u,\ \partial_t + k\partial_x,\ m\partial_t + \frac{1}{k}xu\partial_u\right\rangle,\ k > 0,\ m \in \mathbb{R}:\\
&\qquad F = -u^{-1}u_x^2 + u\tilde{G}(\omega), \quad \omega = x - kt + mku^{-1}u_x;\\
&A_{3.3}^5 = \langle e^{mt}\partial_u, \partial_x, \partial_t + (mu + xe^{mt})\partial_u\rangle,\ (m > 0):\\
&\qquad F = m^2u + e^{mt}\tilde{G}(\omega), \quad \omega = e^{-mt}u_x - t;\\
&A_{3.3}^6 = \langle \partial_u,\ \partial_t,\ t\partial_u\rangle: \quad F = \tilde{G}(x, u_x);\\
&A_{3.3}^7 = \langle u\partial_u,\ \partial_t - \beta^{-1}xu\partial_u,\ \partial_t + \beta\partial_x\rangle,\ \beta > 0:\\
&\qquad F = -u^{-1}u_x^2 + u\tilde{G}(\omega), \quad \omega = x - \beta t - \beta^2u_xu^{-1};\\
&A_{3.3}^8 = \langle u\partial_u,\ \partial_t - xu\partial_u,\ \partial_x\rangle: \quad F = t^2u + 2tu_x + u\tilde{G}(\omega), \quad \omega = t + u_xu^{-1};\\
&A_{3.3}^9 = \left\langle e^{kt}\partial_u,\ \partial_t + ku\partial_u,\ \beta\partial_x + te^{kt}\partial_u\right\rangle,\ \beta > 0,\ k > 0:\\
&\qquad F = k^2u + \frac{2kx}{\beta}e^{kt} + e^{kt}\tilde{G}(\omega), \quad \omega = e^{-kt}u_x;\\
&A_{3.3}^{10} = \left\langle |t|^{\frac{1}{2}}\partial_u, -|t|^{\frac{1}{2}}\ln|t|\partial_u,\ t\partial_t + x\partial_x + \frac{1}{2}u\partial_u\right\rangle:\\
&\qquad F = -\frac{u}{4}t^{-2} + u_x^3\tilde{G}(\omega, v), \quad \omega = tx^{-1},\ v = xu_x^2;\\
&A_{3.3}^{11} = \langle \partial_u, -t\partial_u,\ \partial_t + k\partial_x\rangle,\ k > 0: \quad F = \tilde{G}(\omega, u_x), \quad \omega = x - kt;
\end{aligned}$$

$$A^1_{3.4} = \left\langle \eta^{m-1}\partial_u,\ \partial_t + k\partial_x,\ t\partial_t + x\partial_x + \left(mu + t\eta^{m-1}\right)\partial_u \right\rangle,$$
$$\eta = x - kt, \quad k \geq 0, \quad m \in \mathbb{R}, \quad m \neq 2:$$
$$F = (k^2-1)(m-1)(m-2)\eta^{-2}u - \frac{2k(1-m)}{2m-4}\eta^{m-2} + \eta^{2-m}\tilde{G}(\omega),$$
$$\omega = ((1-m)u + \eta u_x)\eta^{3m-4};$$
$$A^2_{3.4} = \left\langle e^{ktx^{-1}}\partial_u,\ \partial_t + kx^{-1}u\partial_u,\ t\partial_t + x\partial_x + \left(u + te^{ktx^{-1}}\right)\partial_u \right\rangle,\ k \neq 0:$$
$$F = u\left(k^2t^2x^{-4} - 2ktx^{-3} + k^2x^{-2}\right) + 2ktu_x x^{-2} + e^{ktx^{-1}}\left(2k\ln|x|x^{-1} + x^{-1}\tilde{G}(\omega)\right),$$
$$\omega = e^{-ktx^{-1}}\left(u_x + ktux^{-2}\right);$$
$$A^3_{3.4} = \left\langle kx^{-1}u\partial_u,\ \partial_t - k\ln|x|x^{-1}u\partial_u,\ t\partial_t + x\partial_x \right\rangle,\ k > 0:$$
$$F = k^2t^2ux^{-4} - 3ktux^{-3} + 2ktu_x x^{-2} + 2ktux^{-3}\ln|u| - 2ux^{-2}\ln|u| + 2u_x x^{-1}\ln|u| + x^{-2}u\ln^2|u| + ux^{-2}\tilde{G}(\omega),$$
$$\omega = xu_x u^{-1} + \ln|u| + ktx^{-1};$$
$$A^4_{3.4} = \left\langle |t|^{\frac{1}{2}}\partial_u, -|t|^{\frac{1}{2}}\ln|t|\partial_u,\ t\partial_t + x\partial_x + \frac{3}{2}u\partial_u \right\rangle:$$
$$F = -\frac{u}{4}t^{-2} + u_x^{-1}\tilde{G}(\omega, v), \quad \omega = tx^{-1}, \quad v = x^{-1}u_x^2;$$
$$A^5_{3.4} = \langle \partial_u, -t\partial_u,\ \partial_t + k\partial_x + u\partial_u \rangle, k > 0:$$
$$F = u_x\tilde{G}(\omega, v), \quad \omega = x - kt, \quad v = \ln|u_x| - t;$$
$$A^1_{3.5} = \left\langle \eta^{m-1}\partial_u,\ \partial_t + k\partial_x,\ t\partial_t + x\partial_x + mu\partial_u \right\rangle,\ \eta = x - kt,\ k > 0,\ m \in \mathbb{R}:$$
$$F = \left(k^2-1\right)(m-1)(m-2)u\eta^{-2} + \eta^{m-2}\tilde{G}(\omega), \quad \omega = ((1-m)u + u_x\eta)\eta^{-m};$$
$$A^2_{3.5} = \left\langle \partial_x,\ |t|^{m-1}\partial_u,\ t\partial_t + x\partial_x + mu\partial_u \right\rangle,\ m \in \mathbb{R}:$$
$$F = \left(2u - 3mu - m^2u\right)t^{-2} + t^{m-2}\tilde{G}(\omega), \quad \omega = u_x t^{m-1};$$
$$A^3_{3.5} = \langle \partial_t,\ \partial_x,\ t\partial_t + x\partial_x \rangle: \quad F = u_x^2\tilde{G}(u);$$
$$A^4_{3.5} = \langle \partial_t,\ \partial_x,\ t\partial_t + x\partial_x + mu\partial_u \rangle,\ m \neq 0,1,2: \quad F = |u|^{\frac{m-2}{m}}\tilde{G}(\omega), \quad \omega = u_x^{-1}|u|^{\frac{m-1}{m}};$$
$$A^5_{3.5} = \langle \partial_t,\ \partial_x,\ t\partial_t + x\partial_x + \partial_u \rangle,\ m \neq 0: \quad F = e^{-2u}\tilde{G}(\omega), \quad \omega = e^u u_x;$$
$$A^6_{3.5} = \left\langle \partial_t,\ x^{-1}u\partial_u,\ t\partial_t + x\partial_x \right\rangle,\ k \neq 0:$$
$$F = 2u_x x^{-1}\ln|u| + u\ln^2|u|x^{-2} - 2u\ln|u|x^{-2} + x^{-2}u\tilde{G}(\omega), \quad \omega = u_x u^{-1}x + \ln|u|;$$
$$A^7_{3.5} = \left\langle \partial_t + kx^{-1}u\partial_u,\ e^{ktx^{-1}}\partial_u,\ t\partial_t + x\partial_x + u\partial_u \right\rangle,\ k \in \mathbb{R}:$$
$$F = uk\left(kt^2 - 2xt + kx^2\right)x^{-4} + 2ktu_x x^{-2} + e^{ktx^{-1}}x^{-1}\tilde{G}(\omega),$$
$$\omega = e^{-ktx^{-1}}\left(u_x + ktux^{-2}\right);$$
$$A^1_{3.9} = \langle \sin(t)\partial_u,\ \cos(t)\partial_u, \partial_t + k\partial_x + qu\partial_u \rangle,\ k \geq 0,\ q > 0:$$
$$F = -u + u_x\tilde{G}(\eta, v), \quad \eta = x - kt,\ v = e^{-qt}u_x;$$
$$A^2_{3.9} = \left\langle\ |t|^{\frac{1}{2}}\sin\left(\frac{\ln|t|}{2(k-q)}\right)\partial_u, |t|^{\frac{1}{2}}\cos\left(\frac{\ln|t|}{2(k-q)}\right)\partial_u,\ 2(k-q)(t\partial_t + x\partial_x) + ku\partial_u \right\rangle,$$
$$k \in \mathbb{R},\ q > 0,\ k \neq q: \quad F = -\frac{(k-q)^2+1}{4(k-q)^2}t^{-2}u + |t|^{\frac{4q-3k}{2(k-q)}}\tilde{G}(\omega, v),$$
$$\omega = tx^{-1}, \quad v = |t|^{k-2q}|u_x|^{2(k-q)}.$$

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New Exact Solutions of Khokhlov–Zabolotskaya–Kuznetsov Equation

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Khokhlov–Zabolotskaya–Kuznetsov equation $(\phi_t + \phi\phi_x - \alpha\phi_{xx})_x - 1/2(\phi_{yy} + \phi_{zz}) = 0$ and its solutions are analyzed. A series of complete exact analytical solutions related to the one-dimensional and vectorial inhomogeneous Burgers equation are presented. A concrete example which corresponds to a special form of the inhomogeneous term is analyzed. Reduction to the traveling wave solution is considered.

1 Introduction

The Khokhlov–Zabolotskaya–Kuznetsov equation (KhZKE) describes the evolution of the spreading of nonlinear diffraction waves whose cross-section is large compared to their length. This is one of the basic equations of nonlinear wave processes. As the generalized KhZKE usually the equation

$$\frac{\partial}{\partial\tau}\left(\frac{\partial p}{\partial z} - \lambda p\frac{\partial p}{\partial \tau} + \hat{L}p\right) = \frac{c}{2}\Delta_\perp p \tag{1}$$

is accepted, where $p = p(z,\tau)$ usually means pressure, z, t are space and time coordinates, $\lambda = \varepsilon/c_0^3\rho_0$ is a parameter characterizing nonlinearity, c is the velocity of sound in the medium; $\Delta_\perp = \Delta(x,y)$ is a two-dimensional Laplacian according to the parameters in the cross-section of the wave packet; $\hat{L}$ in the general case is an integro-differential operator determined by the frequency dependence of weak dispersion and dissipative properties of the medium. Most frequently a generalization of KhZKE containing the second derivative

$$\hat{L} = -b\frac{\partial^2}{\partial\tau^2} \tag{2}$$

is used, which describes dissipation, the finite width of the weak shock wave front in particular.

KhZKE (1) looks rather awkward, nevertheless, it is known to have the exact analytical solution [1]. The present work contains the whole series of exact KhZKE solutions with the second-order operator $\hat{L}$. A concrete solution corresponding to the traveling wave solution is considered.

2 One-dimensional case

Let us divide our search for KhZKE solution into two stages. First of all, we will write the KhZKE as an inhomogeneous Burgers equation and then try to find its exact complete solutions. For the sake of further simplification, it is feasible to represent the constant b in the expression for operator $\hat{L}$ as

$$b \to \frac{b}{2c^3\rho}, \tag{3}$$

where ρ is the density index of the medium.

Then, through substitution of variables

$$z \to \frac{1}{\lambda p_0} t, \quad \tau \to -x, \quad p \to p_0 \phi, \quad \frac{b}{2\varepsilon p_0} \to \nu, \quad x \to \frac{2\lambda p_0}{c} x, \quad y \to \frac{2\lambda p_0}{c} y, \tag{4}$$

KhZKE (1) transforms into

$$(\phi_t + \phi\phi_x - \alpha\phi_{xx})_x - \frac{1}{2}(\phi_{yy} + \phi_{zz}) = 0. \tag{5}$$

By integrating equation (5) by x variable, let us represent the KhZKE as an inhomogeneous Burgers equation (IBE):

$$\phi_t + \phi\phi_x - \alpha\phi_{xx} = \beta f, \tag{6}$$

where $f = 1/2 \int_{x_0}^{x} (\phi_{yy} + \phi_{zz})\, dx$, and β is a certain constant introduced to ensure the possibility of changing the influence of the inhomogeneous term $f = f(x, y, z, t)$. In a whole series of cases when the dependence of $\phi(y, z)$ solution is negligible, or if we are interested in the asymptotic solution resulting form the mediumization of the initial equation KhZK (zonal mediumization, Reynold's mediumization, etc.), the righthand part can be presented as $f(x, t)$.

The Hopf and Cole transformation [2, 3]

$$\phi = -2\alpha\partial_x \ln w \tag{7}$$

relates each solution of the diffusion equation (DE)

$$w_t = \alpha w_{xx} \tag{8}$$

to a corresponding solution $\phi(x, t)$ of Burgers equation (BE) [4]:

$$\phi_t + \phi\phi_x - \alpha\phi_{xx} = 0\,. \tag{9}$$

This allows a detail analysis of the formation and evolution of shock waves in a nonlinear environment.

However, upon introducing into equation (8) even a simplest inhomogenous term, the interrelation between BE and DE through Hopf and Cole transformation (7) disappears. DE (8) is a simplest parabolic equation, therefore, searching for solutions of the inhomogeneous diffusion equation

$$w_t + \alpha w_{xx} = h(x, t) \tag{10}$$

and of the corresponding inhomogeneous BE (9) generalization, approximate methods of calculation (most frequently the method of finite differences) are applied [5, 6].

To obtain a pithy inhomogeneous generalization of BE, let us consider a commutative diagram:

$$\begin{tikzcd} SE \arrow[r, "t\to -it"] \arrow[d, "h^{-1}"'] & DE \\ SENT & IBE \arrow[l, "t\leftarrow it"] \arrow[u, "h"'] \end{tikzcd} \tag{11}$$

where SE is Schrödinger equation, DE is diffusion equation, IBE is inhomogeneous Burgers equation (6) and SENT is Schrödinger equation with a nonlinear term (not to be mixed with nonlinear Schrödinger equation). The map h is the Hopf–Cole transformation (7) and h^{-1} is the inverse Hopf–Cole transformation

$$w \xrightarrow{h^{-1}} w_0 \exp\left\{-\frac{1}{2\alpha}\int \phi(x,t)\,dx\right\}. \tag{12}$$

It is important that IBE (6) can be got by the transformation (7) of a linear type diffusion equation

$$w_t = \alpha w_{xx} - \frac{\beta}{2\alpha}F(x,t)w, \tag{13}$$

where

$$F(x,t) = \int_{x_0}^{x} d\xi\, f(\xi,t) + C(t), \tag{14}$$

with x_0 as an arbitrary constant, while $C(t)$ is an arbitrary function of t.

3 The vectorial Khokhlov–Zabolotskaya–Kuznetsov equation

While studying the spread of nonlinear waves in a three-dimensional space not in one, but in all spatial directions, it is the three-dimensional vectorial Khokhlov–Zabolotskaya–Kuznetsov equation that suits the purpose best:

$$\boldsymbol{\nabla}\left[\boldsymbol{\phi}_t + (\boldsymbol{\phi}\boldsymbol{\nabla})\boldsymbol{\phi} - \alpha\boldsymbol{\nabla}(\boldsymbol{\nabla}\boldsymbol{\phi})\right] - \boldsymbol{\nabla}^2\boldsymbol{\phi} = 0, \tag{15}$$

where $\boldsymbol{\phi} = \boldsymbol{\phi}(\boldsymbol{x},t) \in \mathbb{R}^3$, $\alpha > 0$, and $\boldsymbol{\nabla}$ is the gradient operator. If the influence of the medium from the right-hand side of the equation can be reduced effectively to a function on space and time coordinates, then the corresponding vectorial inhomogeneous Burgers equation (VIBE) is

$$\boldsymbol{\phi}_t + (\boldsymbol{\phi}\boldsymbol{\nabla})\boldsymbol{\phi} - \alpha\boldsymbol{\nabla}(\boldsymbol{\nabla}\boldsymbol{\phi}) = \beta\boldsymbol{f}, \tag{16}$$

where $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x},t)$ is a function only of space–time coordinates.

In hydrodynamics, the VIBE and VBE are obtained by refusing the condition that the pressure gradient disappears in the direction perpendicular to the direction of motion of the nonlinear wave: $\boldsymbol{\nabla}_\perp p = 0$ [7]. Such equation, together with the continuity equation, was proposed to study the cosmological models of the Early Universe [8, 9]. Only comparatively recently a mathematically strict notion of the generalized solution of such a system was suggested, and it shows that the variational representation of the generalized solution in the two-dimensional case essentially differs from that of the one-dimensional case [10].

Like in the one-dimensional case, the linear type diffusion equation

$$w_t = \boldsymbol{\nabla}(\boldsymbol{\nabla}w) - \frac{\beta}{2\alpha}F(\boldsymbol{r},t)w, \tag{17}$$

by the vectorial generalization of Hopf–Cole transformation

$$\boldsymbol{\phi}(\boldsymbol{r},t) = -2\alpha\boldsymbol{\nabla}\ln w \tag{18}$$

can be mapped into an VIBE (16), where

$$F(\boldsymbol{r},t) = \int_{\boldsymbol{r}_0}^{\boldsymbol{r}} d\boldsymbol{\xi}\,\boldsymbol{f}(\boldsymbol{\xi},t) + C(t), \tag{19}$$

with $\boldsymbol{r}_0$ as an arbitrary constant, while $C(t)$ is an arbitrary function of t.

The solution of linear equation (17) is

$$w(\boldsymbol{r},t)=\int d\boldsymbol{r'}\,K(\boldsymbol{r},t,\boldsymbol{r'},0)\,w(\boldsymbol{r'},0), \tag{20}$$

where the kernel $K(\boldsymbol{r},t,\boldsymbol{r'},0)$ satisfies the heat type kernel equation,

$$K_t-\alpha\boldsymbol{\nabla}^2K+\frac{\beta}{2\alpha}F(\boldsymbol{r},t)K=0, \tag{21}$$

with the initial condition $K(\boldsymbol{r},0,\boldsymbol{r'},0)=\delta(\boldsymbol{r}-\boldsymbol{r'})$. The solution of this equation can be expressed by the Feynman–Kac path integral formula:

$$K(\boldsymbol{r},t,\boldsymbol{r'},0)=\int[D\boldsymbol{r}]\exp\left(-\frac{S}{2\alpha}\right), \tag{22}$$

where S is the related action, i.e.,

$$S[\boldsymbol{r}(t)]=\int_0^t d\tau\left[\frac{1}{2}\dot{\boldsymbol{r}}^2+\beta F(\boldsymbol{r},\tau)\right]. \tag{23}$$

In the case of the traveling wave solution the function $\boldsymbol{\phi}(\boldsymbol{\xi})$, where $\boldsymbol{\xi}=\boldsymbol{x}-\boldsymbol{u}t$, obeys the equation

$$[(\boldsymbol{\phi}-\boldsymbol{u})\boldsymbol{\nabla}]\,\boldsymbol{\phi}=\alpha\boldsymbol{\nabla}(\boldsymbol{\nabla}\boldsymbol{\phi})+\beta\boldsymbol{f}. \tag{24}$$

According to the Helmholtz theorem, the field $\boldsymbol{\phi}(\boldsymbol{\xi})$ can be split into the sum of the gradient and vortex fields

$$\boldsymbol{\phi}=\boldsymbol{\phi}_g+\boldsymbol{\phi}_v, \tag{25}$$

where $\boldsymbol{\phi}_g=\boldsymbol{\nabla}\psi$, i.e. $\boldsymbol{\nabla}\times\boldsymbol{\phi}_g=0$, and $\boldsymbol{\phi}_v=\boldsymbol{\nabla}\times\boldsymbol{\chi}$, i.e. $\boldsymbol{\nabla}\cdot\boldsymbol{\phi}_v=0$.

In the same way also the inhomogeneous term $\beta\boldsymbol{f}$ can be represented: $\boldsymbol{f}(\xi)=\boldsymbol{f}_g(\xi)+\boldsymbol{f}_v(\xi)$.

From equation (26) it follows that $\boldsymbol{\phi}_g(\boldsymbol{\xi})$ for $\boldsymbol{\phi}_v(\boldsymbol{\xi})=\boldsymbol{f}_v(\boldsymbol{\xi})=0$ must obey the equation

$$\alpha\boldsymbol{\nabla}\boldsymbol{\phi}=\frac{1}{2}\boldsymbol{\phi}^2-(\boldsymbol{u}\boldsymbol{\phi})-\beta\varphi+\frac{1}{2}C_1, \tag{26}$$

where C_1 is the integration constant independent of $\boldsymbol{\xi}$, and $\boldsymbol{f}(\boldsymbol{\xi})=\boldsymbol{\nabla}\varphi$.

Let $\psi=\psi(\boldsymbol{\xi})$ be the solution of the three-dimensional Schrödinger equation

$$\triangle\psi+(C_2+a\varphi)\psi=0. \tag{27}$$

Then the gradient part of $\phi(\xi)$

$$\boldsymbol{\phi}(\boldsymbol{\xi})=\boldsymbol{\phi}(\boldsymbol{x}-\boldsymbol{u}t)=-2\alpha\boldsymbol{\nabla}\ln\psi+\boldsymbol{u} \tag{28}$$

is the solution of equation (26) and, consequently, of the initial VIBE (26) for

$$C_1=\boldsymbol{u}^2+4\alpha^2C_2\qquad\text{and}\qquad\beta=-2\alpha^2a. \tag{29}$$

Equation (26) suggests that the vortical constituent $\boldsymbol{\phi}_v(\boldsymbol{\xi})$ of the field obeys the equation

$$[(\boldsymbol{\phi}-\boldsymbol{u})\boldsymbol{\nabla}]\,\boldsymbol{\phi}=\beta\boldsymbol{f}, \tag{30}$$

where now $\boldsymbol{f}=\boldsymbol{e}^if_i=\boldsymbol{\nabla}\times\boldsymbol{\chi}$. The solution of this equation is

$$\boldsymbol{\phi}=\boldsymbol{u}+\boldsymbol{e}^i\sqrt{2\beta F_i}, \tag{31}$$

where $\boldsymbol{e}^i$ is the unit basis vector and $\partial_iF_j=f_j\,\delta_{ij}$.

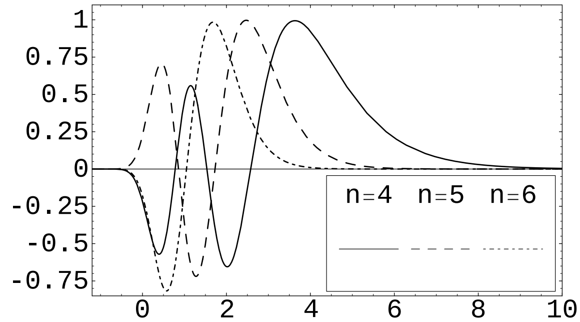

Figure 1. Solution $\phi(\xi) = -2\alpha\psi_n'/\psi_n + u$, where $\psi_n(y) = y^{(l+1)/2}e^{-\frac{1}{2}y}L_n^{l+1/2}(y)$, $y = \sqrt{\gamma/2}\xi^2$ of the VIBE (16) in the case, when $f(\xi) = \gamma\xi/2 - (l+1)/\xi$, integration constant $\alpha = \beta = 1$, $l = 7/2$ and parameter n changes from $n = 4$ to $n = 6$. All solutions are normalized to the amplitude values

Consider another potential. In the case of inhomogeneous term it looks like a three-dimensional oscillator

$$\boldsymbol{f}(\boldsymbol{\xi}) = \frac{1}{2}\gamma\boldsymbol{\xi} - \frac{l+1}{|\boldsymbol{\xi}|}\boldsymbol{e}, \qquad \boldsymbol{\xi} = \boldsymbol{r} - \boldsymbol{u}t. \tag{32}$$

Such choice of the inhomogeneous term corresponds to the potential

$$\varphi(\boldsymbol{\xi}) = \frac{1}{4}\gamma^2\boldsymbol{\xi}^2 - \frac{l(l+1)}{\boldsymbol{\xi}^2} - \gamma\left(l + \frac{3}{2}\right). \tag{33}$$

Then for the IBE

$$\boldsymbol{\phi}_t + (\boldsymbol{\phi}\boldsymbol{\nabla})\boldsymbol{\phi} - \alpha\boldsymbol{\nabla}(\boldsymbol{\nabla}\boldsymbol{\phi}) = \beta\left[\frac{1}{2}\gamma(\boldsymbol{r} - \boldsymbol{u}t) - \frac{l+1}{|\boldsymbol{r} - \boldsymbol{u}t|}\boldsymbol{e}\right] \tag{34}$$

the solution is

$$\boldsymbol{\phi}(\boldsymbol{\xi}) = -2\alpha\boldsymbol{\nabla}\psi_n/\psi_n + \boldsymbol{u}, \tag{35}$$

where $\psi_n(y) = y^{(l+1)/2}e^{-\frac{1}{2}y}L_n^{l+1/2}(y)$, $y = \sqrt{\gamma/2}\xi^2$. For a graphic representation of solution $\phi(|\boldsymbol{r} - \boldsymbol{u}t|)$, see Fig. 1.

In the case of potential (33) we have an infinite number of constants $C_2 = 2\gamma n$ and, consequently, the same infinite number of integration constants

$$C_1 = u^2 - 8\alpha^2\gamma n. \tag{36}$$

We can see that the gradient constituent $\boldsymbol{\phi}_g$ (28) of the VIBE qualitatively does not differ from the one-dimensional case (6) and has the same number of exact complete analytical solutions with the spatial variable $\boldsymbol{x} - \boldsymbol{u}t$. However, the presence of the vortical constituent $\boldsymbol{\phi}_v$ in the multi-dimensional case draws a qualitative difference between the VIBE and the one-dimensional IBE (6). Note that $\boldsymbol{\phi} = \boldsymbol{\phi}_g + \boldsymbol{\phi}_v$, because of the nonlinearity of the VIBE, is not its solution.

4 Discussion and conclusions

Exact solutions of any evolution equation are known for very limited special cases, therefore new exact solutions of KhZKE are very interesting in themselves.

Besides, the KhZKE is a limit case of a lot of mathematical models of more complicated nonlinear and dissipative systems. Exact analysis of a corresponding KhZKE provides a useful information about the behavior of such systems.

Using the known relation between the diffusion and Schrödinger equations, which is contained in diagram (11), we obtain that solution for $\boldsymbol{\nabla}\boldsymbol{\varphi}(\boldsymbol{\xi})$ expresses the solution of the Schrödinger equation in the presence of nonlinearity.

$$i\boldsymbol{\phi}_t + \boldsymbol{\phi}(\boldsymbol{\nabla}\boldsymbol{\phi}) - \alpha\boldsymbol{\nabla}(\boldsymbol{\nabla}\boldsymbol{\phi}) = \boldsymbol{\varphi}(\boldsymbol{x} - \boldsymbol{u}t), \tag{37}$$

Sometimes, when the solutions of initial equations exhibit an exotic behavior, we can speak only about solutions of the enveloping model in the neighborhood of the solutions of initial equations. For instance, the one-dimensional equation of motion of ideal gas, as is well known, has a discontinuity in the gas flow, at the same time viscous gas has no such discontinuities, and only shock transitions at low meanings of viscosity are obtained. In this sense, the heat equation

$$\boldsymbol{\phi}_t - \alpha\boldsymbol{\nabla}(\boldsymbol{\nabla}\boldsymbol{\phi}) = f(x), \tag{38}$$

describes the stationary heat distribution in a certain volume, because solutions of Poisson equation can be obtained from the heat equation in the limit of transition at $t \to \infty$. In this same sense, an IBE with the time-independent right-hand side is a covering model of a stationary nonlinear Poisson equation

$$\boldsymbol{\phi}\boldsymbol{\nabla}\boldsymbol{\phi} - \alpha\boldsymbol{\nabla}(\boldsymbol{\nabla}\boldsymbol{\phi}) = \beta\boldsymbol{f}(\boldsymbol{x}). \tag{39}$$

This is especially actual for the sign changing coefficients α and β for so-called equations with changing parabolicity [11].

The obtained KhZKE solutions, because of their general character, allow a wide range of applications. As a concrete example, it is quite appropriate to mention the Kardar–Parisi–Zhang (KPZ) equation in (1+1)-dimension systems and crystal growth [12], the nonlinear dynamics of a moving line [13], galaxy formations [14, 15, 9], behavior of magnetic flux line in superconductor [16], and spin glasses [17]. Numerous examples of the applications are presented in [18].

Finally, exact solutions can be considered as a test model for the very promising and actively developing field of computer simulations [19].

Exact solutions of the Schrödinger equation are known to be related to the internal symmetry of a corresponding Hamiltonian [20]. As follows from the considered above subject, the algebra of supersymmetry should exist also in nonlinear and inhomogeneous cases of KhZKE, which in the physical sense is far from obvious.

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Lie, Partially Invariant, and Nonclassical Submodels of Euler Equations

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The Euler equations describing motion of an incompressible ideal fluid are investigated with symmetry point of view. We review some results on Lie, partially invariant, and nonclassical submodels of these equations.

1 Introduction

Hydrodynamics partial differential equations are traditional objects of investigation by means of methods of group analysis [1]. It is well known [2, 3] that the maximal Lie invariance algebra of the Euler equations (EEs)

$$\vec{u}_t + (\vec{u} \cdot \vec{\nabla})\vec{u} + \vec{\nabla} p = \vec{0}, \qquad \operatorname{div} \vec{u} = 0, \tag{1}$$

which describe flows of an ideal incompressible fluid, is the infinite dimensional algebra $A(E)$ generated by the following basis elements:

$$\begin{aligned}
&\partial_t, \quad J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial_{u^b} - u^b\partial_{u^a} \quad (a<b),\\
&D^t = t\partial_t - u^a\partial_{u^a} - 2p\partial_p, \quad D^x = x_a\partial_a + u^a\partial_{u^a} + 2p\partial_p,\\
&R(\vec{m}) = R(\vec{m}(t)) = m^a(t)\partial_a + m^a_t(t)\partial_{u^a} - m^a_{tt}(t)x_a\partial_p,\\
&Z(\chi) = Z(\chi(t)) = \chi(t)\partial_p.
\end{aligned} \tag{2}$$

Such anomalously wide Lie invariance is typical for hydrodynamics equations of incompressible fluids, which are written in the Euler coordinates.

In the following $\vec{u} = \{u^a(t, \vec{x})\}$ denotes the velocity of the fluid, $p = p(t, \vec{x})$ denotes the pressure, $\vec{x} = \{x_a\}$, $\partial_t = \partial/\partial t$, $\partial_a = \partial/\partial x_a$, $\vec{\nabla} = \{\partial_a\}$, $\triangle = \vec{\nabla} \cdot \vec{\nabla}$ is the Laplacian, $m^a = m^a(t)$ and $\chi = \chi(t)$ are arbitrary smooth functions of t (for example, from $C^\infty((t_0, t_1), \mathbb{R})$). The fluid density is set equal to unity. Summation over repeated indices is implied, and we have $a, b = 1, 2, 3$. Subscripts of functions denote differentiation with respect to the corresponding variables.

2 Lie invariant solutions of Euler equations

A number of Lie submodels of (1) have been already constructed. For example, in [4, 5, 6, 7] EEs are reduced to partial differential equations in two and three independent variables by means of using the Lie algorithm.

Using well-known Lie symmetry group of EEs, we describe all its possible (inequivalent) Lie submodels. Namely, we find complete sets of inequivalent one-, two-, and three-dimensional subalgebras of $A(E)$. Then, we construct the corresponding ansatzes of codimension one, two, and three as well as reduced systems of partial differential equations in three and two independent variables and reduced systems of ordinary differential equations. Lie symmetry properties of the reduced systems of partial differential equations are investigated. There exists a number of reduced systems admitting Lie symmetries which are not induced by Lie symmetries of the initial Euler equations. (Existence of such symmetries was firstly proved by L.V. Kapitanskiy [8, 9] just for the axially symmetric Euler equations.) The reduced systems of ordinary differential equations are integrated or for them partial exact solutions are found. As a result, new large classes of exact solutions of EEs, which contain, in particular, arbitrary functions, are constructed. Numbers of investigated objects are the following ones:

5 families of one-dimensional inequivalent subalgebras
5 families of ansatzes of codimension one (all the families of subalgebras can be used to reduce the EEs by the standard method)
4 classes of reduced systems (two classes of reduced systems can be united)
2 classes of reduced systems that have non-induced Lie symmetries

16 families of two-dimensional inequivalent subalgebras
14 families of ansatzes of codimension two (14 subalgebras can be used to reduce EEs by the standard method)
11 classes of reduced systems (there exist 3 pairs of classes of reduced systems, which can be united)
2 classes of reduced systems are completely integrated
1 reduced system is linearized on a subset of solutions

45 families of three-dimensional inequivalent subalgebras
21 families of ansatzes of codimension three (only 21 families of subalgebras can be used to reduce EEs by the standard method)
10 classes of reduced systems solutions of which are not solutions of completely integrated reduced systems with two independent variables

Now we consider two stationary Lie submodels of codimension 3, which do not have analogs in the case of viscous fluids as their construction essentially bases on specific invariance of EEs with respect the time dilations generated by the operator D^t. Moreover, integrating of these nonlinear submodels can be reduced to solving of second order linear ODEs. Below we give the corresponding subalgebras, ansatzes, reduced systems, and their solutions.

1. $\langle\partial_t,\ J_{12}+\alpha_1 D^t,\ R(0,0,1)+\alpha_2 D^t\rangle$, where $(\alpha_1,\alpha_2)\neq(0,0)$;

$$u^1=(x_1\varphi^1-x_2\varphi^2)e^{\zeta},\quad u^2=(x_2\varphi^1+x_1\varphi^2)e^{\zeta},\quad u^3=\varphi^3e^{\zeta},\quad p=he^{2\zeta},$$

where $\zeta=-\alpha_2x_3-\alpha_1\arctan x_2/x_1$, $\omega=(x_1^2+x_2^2)^{1/2}$, and new unknown functions $\varphi^a=\varphi^a(\omega)$ and $h=h(\omega)$ satisfy the reduced system

$$\begin{aligned}
&\omega\varphi^1\varphi^1_\omega-(\alpha_1\varphi^2+\alpha_2\varphi^3)\varphi^1+(\varphi^1)^2-(\varphi^2)^2+\omega^{-1}h_\omega=0,\\
&\omega\varphi^1\varphi^2_\omega-(\alpha_1\varphi^2+\alpha_2\varphi^3)\varphi^2+2\varphi^1\varphi^2-2\alpha_1\omega^{-2}h=0,\\
&\omega\varphi^1\varphi^3_\omega-(\alpha_1\varphi^2+\alpha_2\varphi^3)\varphi^3-2\alpha_2h=0,\\
&\omega\varphi^1_\omega+2\varphi^1-(\alpha_1\varphi^2+\alpha_2\varphi^3)=0.
\end{aligned}\tag{3}$$

If $\varphi^1=0$, then $h=\varphi^2=\alpha_2\varphi^3=0$ and we obtain a trivial solution of EEs. Let $\varphi^1\neq0$. It follows from system (3) that

$$\varphi^2=\frac{\alpha_1(\omega\varphi^1_\omega+2\varphi^1)+\alpha_2\beta\omega^2\varphi^1}{\alpha_2^2\omega^2+\alpha_1^2},\qquad \varphi^3=\frac{\alpha_2(\omega\varphi^1_\omega+2\varphi^1)-\alpha_1\beta\omega^2\varphi^1}{\alpha_2^2\omega^2+\alpha_1^2},$$

$$h=\frac{\omega^2}{2}\frac{\omega^2\varphi^1\varphi^1_{\omega\omega}-(\omega\varphi^1_\omega)^2-4(\varphi^1)^2}{\alpha_2^2\omega^2+\alpha_1^2}+\frac{\omega^2}{2}\varphi^1\frac{(\alpha_1^2-\alpha_2^2\omega^2)\omega\varphi^1_\omega+2\alpha_1\varphi^1(\alpha_2\beta\omega^2+2\alpha_1)}{(\alpha_2^2\omega^2+\alpha_1^2)^2},$$

where $\varphi^1=\omega^{-2}(\alpha_2^2\omega^2+\alpha_1^2)^{1/2}\psi(\omega)$, ψ is an arbitrary solution of the second order linear ODE

$$\psi_{\omega\omega}+\frac{1}{\omega}\psi_\omega+\left(\alpha_2^2\frac{\alpha_1^2-\alpha_2^2\omega^2}{(\alpha_2^2\omega^2+\alpha_1^2)^2}+\left(\beta+\frac{\alpha_1\alpha_2}{\alpha_2^2\omega^2+\alpha_1^2}\right)^2+(\alpha_2^2\omega^2+\alpha_1^2)(\omega^{-2}+\gamma)\right)\psi=0,$$

β and γ are arbitrary constants. For some values of parameters the general solution of the last equation can be expressed via elementary or special functions. So, in the case $\alpha_2=0$

$$\begin{aligned}
&\psi=Z_\nu(\sqrt{\beta^2+\alpha_1^2\gamma}\ \omega) && \text{if}\quad \beta^2+\alpha_1^2\gamma\neq0,\quad \nu=\alpha_1\sqrt{-\gamma},\\
&\psi=C_1\omega^\beta+C_2\omega^{-\beta} && \text{if}\quad \beta^2+\alpha_1^2\gamma=0,\quad \beta\neq0,\\
&\psi=C_1\ln\omega+C_2 && \text{if}\quad \beta=\gamma=0.
\end{aligned}$$

Here and below Z_ν is the general Bessel function of order ν, W is the Whittaker functions, C_0, C_1, C_2, and C_3 are arbitrary constants. In the case $\alpha_1=0$

$$\begin{aligned}
&\psi=Z_1(\sqrt{\beta^2+\alpha_2^2}\ \omega) && \text{if}\quad \gamma=0,\\
&\psi=\frac{1}{\omega}W\left(\frac{\beta^2+\alpha_2^2}{4\alpha_2\sqrt{-\gamma}};\frac12;\alpha_2\sqrt{-\gamma}\ \omega^2\right) && \text{if}\quad \gamma\neq0.
\end{aligned}$$

2. $\langle\partial_t,\ D^x+\alpha D^t+\varkappa J_{12}+R(0,0,\mu t)+Z(\varepsilon_1),\ R(0,0,1)+Z(\varepsilon_2)\rangle$, where $\alpha\neq0$, $\mu(\alpha-1)=0$, $\varepsilon_1(\alpha-1)=\varepsilon_2(2\alpha-1)=0$;

$$\begin{aligned}
&u^1=r^{-\alpha}(x_1\varphi^3-x_2(\varphi^1+\varkappa\varphi^3)),\quad u^1=r^{-\alpha}(x_2\varphi^3+x_1(\varphi^1+\varkappa\varphi^3)),\\
&u^3=r^{1-\alpha}\varphi^2+\mu\ln r,\quad p=r^{2-2\alpha}h+\varepsilon_1\ln r+\varepsilon_2x_3
\end{aligned}$$

where $r=(x_1^2+x_2^2)^{1/2}$, $\omega=\arctan x_2/x_1-\varkappa\ln r$, and new unknown functions $\varphi^a=\varphi^a(\omega)$ and $h=h(\omega)$ satisfy the reduced system

$$\begin{aligned}
&\varphi^1\varphi^1_\omega+(1-\alpha)\varphi^3\varphi^1+((1+\varkappa^2)\varphi^3+\varkappa\varphi^1)(\varphi^1+\varkappa\varphi^3)-2(1-\alpha)\varkappa h+(1+\varkappa^2)h_\omega=\varkappa\varepsilon_1,\\
&\varphi^1\varphi^2_\omega+(1-\alpha)\varphi^3\varphi^2+\mu\varphi^3+\varepsilon_2=0,\\
&\varphi^1\varphi^3_\omega+(1-\alpha)\varphi^3\varphi^3-(\varphi^1+\varkappa\varphi^3)^2+2(1-\alpha)h-\varkappa h_\omega+\varepsilon_1=0,\\
&\varphi^1_\omega+(2-\alpha)\varphi^3=0.
\end{aligned}\tag{4}$$

There exist three different cases of integration of system (4). If $\alpha=2$ then any solution of (4) belongs to a family from the following ones

$$\begin{aligned}
&\varphi^1=\varphi^2=0,\quad \varphi^3=C_1,\quad h=-\tfrac12(1+\varkappa^2)C_1^2;\\
&\varphi^1=\varphi^3=h=0,\quad \varphi^2=\varphi^2(\omega);\\
&\varphi^1=C_1,\quad \varphi^2=C_2,\quad \varphi^3=0,\quad h=-\tfrac12C_1^2;\\
&\varphi^1=C_1,\quad \varphi^2=C_2(\omega+C_3)^{-1},\quad \varphi^3=-C_1(\omega+C_3)^{-1},\quad h=(\varkappa(\omega+C_3)^{-1}-\tfrac12)C_1^2;\\
&\varphi^1=C_1,\quad \varphi^2=C_2\cos^{-1}(C_3\omega+C_4),\quad \varphi^3=C_1C_3\tan(C_3\omega+C_4),\\
&h=\tfrac12C_1^2(C_3^2(1+\varkappa^2)-1)-\varkappa C_1\varphi^3;\\
&\varphi^1=C_1,\quad \varphi^2=\frac{C_2}{B_1e^{C_3\omega}+B_2e^{-C_3\omega}},\quad \varphi^3=-C_1C_3\frac{B_1e^{C_3\omega}-B_2e^{-C_3\omega}}{B_1e^{C_3\omega}+B_2e^{-C_3\omega}},\\
&h=-\tfrac12C_1^2(C_3^2(1+\varkappa^2)+1)-\varkappa C_1\varphi^3.
\end{aligned}$$

In the case $\alpha=1$ we obtain that $\varepsilon_2=0$, $\varphi^2=\mu\ln\varphi^1+C_0$, $\varphi^3=-\varphi^1_\omega$, $h_\omega=\varkappa(\varphi^1\varphi^1_{\omega\omega}-(\varphi^1_\omega)^2)$, and $(1+\varkappa^2)\varphi^1_{\omega\omega}-2\varkappa\varphi^1_\omega+\varphi^1=\varepsilon_1(\varphi^1)^{-1}$. If additionally $\varepsilon_1=0$ then

$$\varphi^1=\left(C_1\cos\frac{\omega}{1+\varkappa^2}+C_2\sin\frac{\omega}{1+\varkappa^2}\right)\exp\frac{\varkappa\omega}{1+\varkappa^2},\quad h=-\frac12\frac{C_1^2+C_2^2}{1+\varkappa^2}\exp\frac{2\varkappa\omega}{1+\varkappa^2}+C_3.$$

Let $\alpha \notin \{1;2\}$. Then $\varepsilon_1 = \mu = 0$, $\varepsilon_2(2\alpha-1) = 0$, $\varphi^3 = -(2-\alpha)^{-1}\varphi^1_\omega$,

$$h = -\frac{1}{2}\left(\frac{-2\varkappa}{2-\alpha}\varphi^1\varphi^1_\omega + \frac{1+\varkappa^2}{(2-\alpha)^2}(\varphi^1_\omega)^2 + (\varphi^1)^2 - \frac{C_0}{(1-\alpha)(2-\alpha)}(\varphi^1)^{2\frac{1-\alpha}{2-\alpha}}\right),$$

$$\varphi^2 = (\varphi^1)^{\frac{1-\alpha}{2-\alpha}}\left(C_3 + \varepsilon_2 \textstyle\int (\varphi^1)^{-\frac{3-2\alpha}{2-\alpha}} d\omega\right),$$

$$(1+\varkappa^2)\varphi^1_{\omega\omega} - 2\varkappa(2-\alpha)\varphi^1_\omega + (2-\alpha)^2\varphi^1 = C_0(\varphi^1)^{-\frac{\alpha}{2-\alpha}}.$$

The last equation is easy solved if $C_0 = 0$.

3 $SO(3)$-partially invariant solutions of Euler equations

The concept of partially invariant solutions was introduced by Ovsiannikov [1] as a generalization of invariant solutions, which is possible for systems of partial differential equations (PDEs). The algorithm for finding partially invariant solutions is very difficult to apply. For this reason it is used more rarely than the classical Lie algorithm for constructing invariant solutions.

In this section we describe the process of constructing $SO(3)$-partially invariant solutions of the minimal defect which is equal to 1 for the given representation of $so(3)$ generated by the operators J_{ab} from $A(E)$ (2) (see [10] for detail).

A complete set of functionally independent invariants of the group $SO(3)$ in the space of the variables $(t, \vec{x}, \vec{u}, p)$ is exhausted by the functions t, $|\vec{x}|$, $\vec{x}\cdot\vec{u}$, $|\vec{u}|$, p, so any $SO(3)$-partially invariant solution of the minimal defect has the form

$$u^R = v(t,R),\ u^\theta = w(t,R)\sin\psi(t,R,\theta,\varphi),\ u^\varphi = w(t,R)\cos\psi(t,R,\theta,\varphi),\ p = p(t,R). \quad (5)$$

Hereafter for convenience the spherical coordinates are used. Substituting (5) into EEs (1), we obtain the system of PDEs for the functions v, w, ψ, p :

$$\begin{aligned}
&v_t + vv_R - R^{-1}w^2 + p_r = 0, \qquad w_t + vw_R + R^{-1}vw = 0,\\
&w(\psi_t + v\psi_R + R^{-1}w\psi_\theta\sin\psi + R^{-1}w\cos\psi(\sin\theta)^{-1}(\psi_\varphi - \cos\theta)) = 0,\\
&Rv_r + 2v + w\psi_\theta\cos\psi - (\sin\theta)^{-1}w\sin\psi(\psi_\varphi - \cos\theta) = 0.
\end{aligned} \quad (6)$$

It follows from (6) if $w = 0$ that $v = \eta R^{-2}$, $p = \eta_t R^{-1} - \frac{1}{2}\eta^2 R^{-4} + \chi$, where η and χ are arbitrary smooth functions of t. The corresponding solution of EEs

$$u^R = \frac{\eta}{R^2}, \qquad u^\theta = u^\varphi = 0, \qquad p = \frac{\eta_t}{R} - \frac{\eta^2}{2R^4} + \chi \quad (7)$$

is invariant with respect to $SO(3)$. Note that flow (7) is a solution of the Navier–Stokes equations too, and it is the unique $SO(3)$-partially invariant solutions of the minimal defect for them.

Below $w \neq 0$. Then two last equations of (6) form an overdetermined system in the function ψ. This system can be rewritten as follows

$$\begin{aligned}
&\psi_\theta + Rw^{-1}\sin\psi(\psi_t + v\psi_R) = -G\cos\psi,\\
&\psi_\varphi + Rw^{-1}\cos\psi(\psi_t + v\psi_R)\sin\theta = G\sin\psi\sin\theta + \cos\theta,
\end{aligned} \quad (8)$$

where $G = w^{-1}(Rv_R + 2v)$. The Frobenius theorem gives the compatibility condition of (8):

$$G_t + vG_R = R^{-1}w\left(1 + G^2\right). \quad (9)$$

If condition (9) holds, system (8) is integrated implicitly and its general solution has the form

$$F(\Omega_1, \Omega_2, \Omega_3) = 0, \quad (10)$$

where F is an arbitrary function of Ω_1, Ω_2, and Ω_3,

$$\Omega_1 = \frac{\sin\theta\sin\psi - G\cos\theta}{\sqrt{1+G^2}}, \qquad \Omega_2 = \varphi + \arctan\frac{\cos\psi}{\cos\theta\sin\psi + G\sin\theta}, \qquad \Omega_3 = h(t,r),$$

$h = h(t,R)$ is a fixed solution of the equation $h_t + vh_R = 0$ such that $(h_t, h_R) \neq (0,0)$. Equation (10) can be solved with respect to ψ in a number of cases, for example, if either $F_{\Omega_1} = 0$ or $F_{\Omega_2} = 0$. Equation (9) and two first equation of (6) form the "reduced" system for the invariant functions v, w, and p. It can be represented as the union of the system

$$R^2 f_{tR} + ff_{RR} - (f_R)^2 = g, \quad R^2 g_t + fg_R = 0, \qquad f := R^2 v, \quad g := (Rw)^2, \tag{11}$$

for the functions v and w (this system can be also considered a system for the functions f and g) and the equation

$$p_R = -v_t - vv_R - R^{-1}w^2 \tag{12}$$

which is one for the function p if v and w are known. Therefore, to construct solutions for EEs, we are to carry out the following chain of actions: 1) to solve system (11); 2) to integrate (12) with respect to p; 3) to find the function ψ from (10).

Theorem 1. *The maximal Lie invariance algebra of (11) is the algebra*

$$\mathcal{A} = \langle\, \partial_t,\ D^R = R\partial_R + v\partial_v + w\partial_w,\ D^t = t\partial_t - v\partial_v - w\partial_w \,\rangle.$$

A complete set of $\mathcal{A}$-inequivalent one-dimensional subalgebras of $\mathcal{A}$ is exhausted by four algebras $\langle\, \partial_t \,\rangle$, $\langle\, D^R \,\rangle$, $\langle\, \partial_t + D^R \,\rangle$, $\langle\, D^t + \kappa D^R \,\rangle$. In [10] we constructed the corresponding ansatzes for the functions v and w as well as the reduced systems arising after substituting the ansatzes into (11). Two first reduced systems were integrated completely. We also found all the solutions of system (11), for which f and g are polynomial with respect to R.

4 Nonclassical symmetries of Euler equations

In this section we give results on Q-conditional symmetry [11, 12] of (1) with respect to single differential operator $Q = \xi^0(t,\vec{x},\vec{u},p)\partial_t + \xi^a(t,\vec{x},\vec{u},p)\partial_a + \eta^a(t,\vec{x},\vec{u},p)\partial_{u^a} + \eta^0(t,\vec{x},\vec{u},p)\partial_p$, which were firstly presented in [13].

Theorem 2. *Any operator Q of Q-conditional symmetry of the Euler equations (1) either is equivalent to a Lie symmetry operator of (1) or is equivalent* (mod $A(E)$) *to the operator*

$$\widetilde{Q} = \partial_3 + \zeta(t,x_3,u^3)\partial_{u^3} + \chi(t)x_3\partial_p, \tag{13}$$

where $\zeta_{u^3} \neq 0$, $\zeta_3 + \zeta\zeta_{u^3} = 0$, $\zeta_t + (u^3\zeta + \chi x_3)\zeta_{u^3} + (\zeta)^2 + \chi = 0$.

It follows from Theorem 2 that there exist two classes of the possible reductions on one independent variable for EEs, namely, the Lie reductions and the reductions corresponding to operators of form (13). Lie reductions of EEs (1) to systems in three independent variables were investigated in [5]. An ansatz constructed with the operator $\widetilde{Q}$ has the following form:

$$u^1 = v^1, \quad u^2 = v^2, \quad u^3 = x_3 v^3 + \psi(t,v^3), \quad p = q + \tfrac{1}{2}\chi(t)x_3^2,$$

where $v^a = v^a(t,x_1,x_2)$, $q = q(t,x_1,x_2)$, the function $\psi = \psi(t,v^3)$ is a solution of the equation $\psi_t - ((v^3)^2 + \chi)\psi_{v^3} + v^3\psi = 0$. Substituting this ansatz into (1), we obtain the corresponding reduced system $(i,j = 1,2)$:

$$v^i_t + v^j v^i_j + q_i = 0, \quad v^3_t + v^j v^3_j + (v^3)^2 + \chi = 0, \quad v^j_j + v^3 = 0.$$

The analogous problem for the Navier–Stokes equations (NSEs)

$$\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} + \nabla p - \nu\triangle\vec{u} = \vec{0}, \qquad \operatorname{div}\vec{u} = 0 \qquad (\nu \neq 0) \tag{14}$$

describing the motion of an incompressible viscous fluid was solved by Ludlow, Clarkson, and Bassom in [14]. Their result can be reformulated in the following manner: *any (real) operator Q of nonclassical symmetry of (14) is equivalent to a Lie symmetry operator of (14).* Therefore, all the possible reductions of NSEs on one independent variable are exhausted by the Lie reductions. The maximal Lie invariance algebra of NSEs (14) is similar to one of EEs (see [15, 16]):

$$A(\mathrm{NS}) = \langle \partial_t,\ J_{ab},\ D^t + \tfrac{1}{2}D^x,\ R(\vec{m}(t)),\ Z(\zeta(t))\rangle.$$

The Lie reductions of NSEs were completely described in [17].

It should be noted that non-classical invariance of hydrodynamics equation (in particular, the Euler and Navier–Stokes equations) with respect to involutive families of two and three operators have not been investigated. The complete solving of this complicated problem would allow to describe all the possible reductions of the equations under considerations to systems of PDEs in two independent variables and to systems of ODEs.

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Differential Invariants and Application to Riccati-Type Systems

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We generalize the classical Lie results on a basis of differential invariants for a one-parameter group of local transformations to the case of arbitrary number of independent and dependent variables. It is proved that if universal invariant of a one-parameter group is known then a complete set of functionally independent differential invariants can be constructed via one quadrature and differentiations. Some applications of first-order differential invariants to Riccati-type systems are also presented.

1 Introduction

The theory of differential invariants nowadays undergoes active development and is widely used for integration in quadratures and for order lowering of ordinary differential equations, and also for description of classes of invariant differential equations [1, 2]. In the theory of differential invariants a major role is played by various versions of the conjecture on the finite basis of differential invariants that could be non-rigorously formulated in the following way: *for an arbitrary group G of local transformations there exists such finite set of differential invariants that every differential invariant of the group G can be represented as a function of these invariants and their derivatives.* A statement of such type (for one-parameter group of local transformations in the space of two variables) was proved by S. Lie himself [3] (see also [4, 5]) and soon afterwards it was essentially generalized by A. Tresse [6]. The recent progress in this direction is due to the works by L.V. Ovsyannikov [7] and P. Olver [1, 8, 9, 10], where the notions of the operator of invariant differentiation, differential invariant coframe etc. are introduced, and results on rank stabilization of the prolonged group action and on the estimates for the number of differential invariants are obtained. There is also considerable number of papers devoted to the search for the differential invariants of specific groups (see e.g. [11, 12, 13, 14, 15]).

In the present paper we study the differential invariants for one-parameter group of local transformations in the space of n independent and m dependent variables ($m, n \in \mathbb{N}$). (The case $n = m = 1$ was considered in [16, 17] and in our work [18]. The results for the case $n = 1$ with arbitrary $m \in \mathbb{N}$ were published in [19].) The importance of this problem stems from its being a part of the problem of search for differential invariants of a group of arbitrary dimension [1, 5]. Our generalization of the Lie theorem on differential invariants of a one-parameter group of local transformations was done not only with respect to the number of independent and dependent variables, but also with respect to strengthening of its statements. It was proved that if a differential invariant is known, then a complete set of functionally independent differential invariants can be constructed through one quadrature and differentiations. As a side product, we also present some results on the existence of basis of differential invariants rational in higher jet coordinates. A link between differential invariants and integration of systems of Riccati-type equations was analyzed. Let us note that results of this paper can be generalized for some classes of multiparameter groups of local transformations (or Lie algebras of differential operators).

2 Generalization of Lie theorem on differential invariants

Let $Q = \xi^a(x,u)\partial_{x_a} + \eta^i(x,u)\partial_{u^i}$ be an infinitesimal operator of a one-parameter group G of local transformations which act on the set $M \subset J_{(0)} = X \times U$, where $X \simeq \mathbb{R}^n$ is the space of independent variables $x = (x_1, x_2, \dots, x_n)$ and $U \simeq \mathbb{R}^m$ is the space of dependent variables $u = (u^1, u^2, \dots, u^m)$, $G^{(r)}$ is a prolongation of the action of the group G for the subset $M_{(r)} = M \times U^{(1)} \times U^{(2)} \times \cdots \times U^{(r)}$ of the jet space $J_{(r)} = X \times U_{(r)}$ of r-th order jets over the space $X \times U$ (here $U_{(r)} = U \times U^{(1)} \times U^{(2)} \times \cdots \times U^{(r)}$, $r \geq 1$, $Q^{(r)}$ is the r-th prolongation of Q [1, 5]). A function $I\colon M_{(r)} \to \mathbb{R}$ is called a differential invariant of the order r for the group G (or for the operator Q) if it is an invariant of the prolonged action of $G^{(r)}$ (of $Q^{(r)}$). A necessary and sufficient condition for the function I to be an r-th order differential invariant of the group G is the equality $Q^{(r)}I = 0$.

Here and below, if not otherwise stated, the indices a, b, c, d run from 1 to n, indices i, j, k, l run from 1 to m. The summation over the repeated indices is understood.

Let $I = I(x,u) = \big(I^1(x,u), I^2(x,u), \dots, I^{m+n-1}(x,u)\big)$ be a complete set of functionally independent invariants (or *a universal invariant* [7]) for an operator Q, and $J(x,u)$ is a particular solution of the equation $QJ = 1$. Then the functions $I^1(x,u)$, $I^2(x,u)$, $\dots$, $I^{m+n-1}(x,u)$ and $J(x,u)$ are functionally independent. Let us make a local change of variables: $y_c = I^c(x,u)$, $c = \overline{1, n-1}$, $y_n = J(x,u)$ are new independent variables, and $v^i = I^{i+n-1}(x,u)$ are new dependent variables. In terms of variables $y = (y_1, y_2, \dots, y_n)$ and $v = (v^1, v^2, \dots, v^m)$ the operator Q has the form ∂_{y_n}. Thus for any $r \geq 1$ the form of the prolonged operator $Q^{(r)}$ coincides with $Q = \partial_{y_n}$, and therefore

$$\hat{y} = (y_1, y_2, \dots, y_{n-1}),$$
$$v_{(r)} = \left\{ v^i_\alpha = \frac{\partial^{|\alpha|} v^i}{\partial y_1^{\alpha_1} \partial y_2^{\alpha_2} \cdots \partial y_n^{\alpha_n}} \;\middle|\; \alpha_a \in \mathbb{N} \cup \{0\},\ |\alpha| = \sum_{a=1}^{n} \alpha_a \leq r \right\}$$

(here $v^i_\alpha = v^i$, if $|\alpha| = 0$) form a complete set of its functionally independent invariants, and $(\hat{y}, v_{(r)})$ is a universal invariant of the group $G^{(r)}$. (Functional independence of the components $\hat{y}$ and $v_{(r)}$ is obvious, as $(y, v_{(r)})$ is a set of variables in the space $J_{(r)}$.) This means that $(\hat{y}, v)$ is a fundamental set of differential invariants for the operator Q, i.e. any differential invariant of the operator Q can be represented as a function of $\hat{y}$ and v and of the derivatives of v with respect to operators of G-invariant differentiation. These operator coincide here with the operators $D_{y_a} = \partial_{y_a} + v^i_{y_a}\partial_{v^i} + v^i_{y_a y_b}\partial_{v^i_{y_b}} + \cdots$ of total derivatives with respect to the variables y_a.

Let us go back to the variables x, u. In terms of these variables

$$D_{y_c} = \frac{(-1)^{c+a}}{\Delta} \frac{D(I^d,\, {\scriptstyle d=\overline{1,n-1},\, d\neq c},\, J)}{D(x_b,\, {\scriptstyle b=\overline{1,n},\, b\neq a})} D_{x_a}, \qquad c = \overline{1, n-1},$$
$$D_{y_n} = \frac{(-1)^{n+a}}{\Delta} \frac{D(I^d,\, {\scriptstyle d=\overline{1,n-1}})}{D(x_b,\, {\scriptstyle b=\overline{1,n},\, b\neq a})} D_{x_a}, \tag{1}$$

where $D_{x_a} = \partial_{x_a} + u^i_{x_a}\partial_{u^i} + u^i_{x_a x_b}\partial_{u^i_{x_b}} + \cdots$ is the operator of total derivative with respect to the variable x_a, and

$$\frac{D(I^d,\, {\scriptstyle d=\overline{1,n-1},\, d\neq c},\, J)}{D(x_b,\, {\scriptstyle b=\overline{1,n},\, b\neq a})}, \quad \frac{D(I^d,\, {\scriptstyle d=\overline{1,n-1}})}{D(x_b,\, {\scriptstyle b=\overline{1,n},\, b\neq a})}, \quad \Delta = \frac{D(I^d,\, {\scriptstyle d=\overline{1,n-1}},\, J)}{D(x_b,\, {\scriptstyle b=\overline{1,n}})}$$

denote Jacobians (of total derivatives)

of the functions I^d, $d = \overline{1, n-1}$, $d \neq c$, J with respect to the variables x_b, $b = \overline{1, n}$, $b \neq a$,

of the functions I^d, $d = \overline{1, n-1}$ with respect to the variables x_b, $b = \overline{1, n}$, $b \neq a$,

of the functions I^d, $d = \overline{1, n-1}$, J with respect to the variables x_b, $b = \overline{1, n}$,

respectively.

As a result, we arrive at the following theorem.

Theorem 1. *Let $I(x,u) = \big(I^1(x,u), I^2(x,u), \dots, I^{m+n-1}(x,u)\big)$ be a universal invariant of an operator Q and $J(x,u)$ be a particular solution of the equation $QJ = 1$. Then functions*

$$I^c(x,u), \quad D_{y_1}^{\alpha_1} D_{y_2}^{\alpha_2} \cdots D_{y_n}^{\alpha_n} I^{i+n-1}(x,u),$$

where $c = \overline{1, n-1}$, $\alpha_a \in \mathbb{N} \cup \{0\}$, $\sum\limits_{a=1}^{n} \alpha_a \le r$, and operators D_{y_a} are determined by the formulae (1), form a complete set of functionally independent r-th order differential invariants (or a universal differential invariant) for the operator Q.

Corollary 1. *For any operator Q there exists a complete set of functionally independent r-th order differential invariants, where every invariant is a rational function of the variables u^i_α ($\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_a \in \mathbb{N} \cup \{0\}$, $0 < \sum\limits_{a=1}^{n} \alpha_a \le r$) of the jet space $J_{(r)}$ with coefficients depending on x_a and u^j.*

Corollary 2. *If $I = (I^1(x,u), I^2(x,u), \dots, I^{m+n-1}(x,u))$ is a universal invariant for the operator Q and $J = J(x,u)$ is a particular solution for the equation $QJ = 1$, then the functions*

$$\begin{aligned}
D_{y_c} I^{i+n-1} &= \frac{(-1)^{c+a}}{\Delta} \frac{D(I^d, {\scriptstyle d=\overline{1,n-1},\, d\neq c,}\, J\,)}{D(x_b, {\scriptstyle b=\overline{1,n},\, b\neq a})} D_{x_a} I^{i+n-1}, \qquad c = \overline{1, n-1}, \\
D_{y_n} I^{i+n-1} &= \frac{(-1)^{n+a}}{\Delta} \frac{D(I^d, {\scriptstyle d=\overline{1,n-1}}\,)}{D(x_b, {\scriptstyle b=\overline{1,n},\, b\neq a})} D_{x_a} I^{i+n-1}
\end{aligned} \tag{2}$$

form a complete set of functionally independent differential invariants having exactly order one for the operator Q.

Note that if a universal invariant I for the operator Q is known, then a particular solution for the equation $QJ = 1$ may be easily found via one quadrature. E.g. if for some fixed a $\xi^a \neq 0$, we have a particular solution

$$J(x,u) = \int dx_a / \xi^a \left(X^1 \dots, X^{a-1}, x_a, X^{a+1}, \dots, X^n, U^1, \dots, U^m\right),$$

where $x_b = X^b(x_a, C)$, $b \neq a$, $u^j = U^j(x, C)$ is the solution for the system of algebraic equations $I(x,u) = C := (C_1, C_2, \dots, C_{m+n-1})$ with respect to the variables x_b, $b \neq a$, u^j, and after the integration it is necessary to perform an inverse substitution $C = I(x,u)$ (there is no summation with respect to a in this case). Likewise, when $\eta^i \neq 0$ for some fixed i, then we can assume

$$J(x,u) = \int du^i / \eta^i \left(X^1, \dots, X^n, U^1, \dots, U^{i-1}, u^i, U^{i+1}, \dots, U^m\right)$$

(there is no summation with respect to i), where $x_b = X^b(u^i, C)$, $u^j = U^j(u^i, C)$, $j \neq i$, is the solution of the system of algebraic equations $I(x,u) = C$ with respect to the variables x_b, u^j, $j \neq i$.

Thus, the following theorem holds.

Theorem 2. *If a universal invariant is found for the operator Q, then a complete set of functionally independent differential invariants of arbitrary order may be constructed through one quadrature and differentiations.*

3 Invariant differentials

Let us introduce a notion of an invariant differential that is a particular case of a more general notion of a first-order contact invariant differential form in the jet space [9].

Definition 1. A differential $dW(x,u)$ will be called *invariant* with respect to a group G (with operator Q), if it does not change under action of transformations from the group G.

A criterion for invariance of a differential is an equality $dQW(x,u) = 0$. Two essentially different cases are possible:

1) the function $W(x,u)$ is an invariant of the operator Q, i.e. $QW(x,u) = 0$; than the differential $dW(x,u)$ is automatically invariant with respect to the operator Q (*invariant differential of the first type*);
2) the function $W(x,u)$ is not invariant under Q, while the differential $dW(x,u)$ is (*invariant differential of the second type*); then $QW(x,u)$ is a non-zero constant.

If a set of functions $I(x,u) = (I^q(x,u))_{q=\overline{1,m+n-1}}$ and $J(x,u)$, that determine a universal invariant of the operator Q and the invariant differential of the second type is type, then all such sets may be found according to the formulae

$$\hat{I}(x,u) = F(I(x,u)), \qquad \hat{J}(x,u) = J(x,u) + H(I(x,u)), \tag{3}$$

where $F = \left(F^1, F^2, \ldots, F^{m+n-1}\right)$ and H are differentiable functions of their arguments, $|\partial F/\partial I| \neq 0$. The formulae (3) determine the equivalence relation Ω on the set $\mathcal{M}$ of collections of $m+n$ smooth functions of $m+n$ variables with a non-zero Jacobian. We denote the corresponding set of equivalence classes as $\mathcal{M}/\Omega$.

Proposition 1. *There is a one-to-one correspondence between $\mathcal{M}/\Omega$ and the set of non-zero operators $\{Q\}$ in the space of variables (x,u): the set $\{(I(x,u); J(x,u))\}$ of solutions of the system $QI^q = 0$, $q = \overline{1, m+n-1}$, $QJ = 1$, where I^q are functionally independent, is an element of the set $\mathcal{M}/\Omega$, and vice versa, if $(I(x,u); J(x,u))$ is a representative of an equivalence class from $\mathcal{M}/\Omega$, then the system $QI^q = 0$, $q = \overline{1, m+n-1}$, $QJ = 1$ is a determined system of linear algebraic equations with respect to the coefficients of the corresponding operator Q.*

4 The case $n = 1$

Let us consider in more detail the case of one independent variable x ($n = 1$), for which it is possible to obtain a more compact formulation of Theorem 1 and of its corollaries, and also to obtain some additional results.

Theorem 1′. *Let $I = I(x,u) = \left(I^1(x,u), I^2(x,u), \ldots, I^m(x,u)\right)$ be a universal invariant of the operator Q and $J(x,u)$ be a particular solution of the equation $QJ = 1$. Then the function*

$$I^j(x,u), \qquad \left(\frac{1}{D_xJ}D_x\right)^s I^j(x,u), \quad s = \overline{1,r},$$

where $D_x = \partial_x + u^i_x\partial_{u^i} + u^i_{xx}\partial_{u^i_x} + \cdots$ is the operator of total derivative with respect to the variable x, form a complete set of r-th order functionally independent differential invariants (or a universal differential invariant) of the operator Q.

Corollary 1′. *For any operator Q there exists a complete set of n-th order functionally independent differential invariants, where every invariant is a rational function of the variables u^i_x, u^i_{xx}, …, $(u^i)^{(n)}$ of the jet space with coefficients depending on x and u^i.*

Corollary 2′. *If $I=\big(I^1(x,u),I^2(x,u),\dots,I^m(x,u)\big)$ is a universal invariant of the operator Q and $J=J(x,u)$ is a particular solution of the equation $QJ=1$, then the functions*

$$I^j_{(1)}=I^j_{(1)}(x,u_{(1)})=\frac{dI^j}{dJ}=\frac{D_xI^j}{D_xJ}=\frac{I^j_x+I^j_{u^i}u^i_x}{J_x+J_{u^{i'}}u^{i'}_x} \tag{4}$$

form a complete set of functionally independent differential invariants of exactly first order for the operator Q.

Corollary 3. *The components of universal differential invariants having exactly order one of the operator Q may be sought for in the form of fractional-linear functions of the variables u^i_x of the jet space with coefficients depending on x and u^i.*

Corollary 2′ may be restated using the notion of the invariant differential.

Corollary 4. *The ratio of invariant differentials of the operator Q of first and second type is its differential invariant of exactly first order. If dI^1, dI^2, …, dI^m form a complete set of independent invariant differentials of the first type for the operator Q, then its ratio with its invariant differential of the second type exhaust functionally independent differential invariants of exactly first order for the operator Q.*

Corollary 5 (Lie Theorem [3, 4, 5]). *Let $n=m=1$, $I(x,u)$ and $I_{(1)}(x,u,u_x)$ are differential invariants of zero and of exactly first order for the operator Q. Then the functions*

$$I,\quad I_{(1)},\quad \frac{d^sI_{(1)}}{dI^s}=\left(\frac{1}{D_xI}D_x\right)^s I_{(1)},\quad s=\overline{1,r-1},$$

form a complete set of n-th order functionally independent differential invariants for the operator Q.

The operators of G-invariant differentiation for the case of one independent variable are traditionally sought for in the form

$$\mathcal{D}=\frac{1}{D_xI^0}D_x,$$

where I^0 is a differential invariant for the group G (see e.g. Corollary 5). Therefore, for the construction of an arbitrary-order universal differential invariant for a one-parameter group of local transformations by means of the operator of G-invariant differentiation of this form it is necessary to know $m+1$ functionally independent differential invariants for the group G of the possibly minimal, or m functionally independent zero-order differential invariants (or simply invariants), and one exactly first-order differential invariant. The algorithm suggested in the Theorem 1′ allows to avoid direct construction of differential invariants.

Example 1. (Cf. [1, 5].) Let $n=m=1$ and $G=\mathrm{SO}(2)$ be a group of rotations acting on $X\times U\simeq\mathbb{R}^2$, with an infinitesimal operator $Q=u\partial_x-x\partial_u$. $I=\sqrt{x^2+u^2}$ is an invariant of the group G (of the operator Q), whence (in notation from the proof of Theorem 2) $U(x,C)=\pm\sqrt{C^2-x^2}$. Then

$$J=\pm\int\frac{dx}{\sqrt{C^2-x^2}}=\pm\arcsin\frac{x}{C}=\pm\arcsin\frac{x}{\sqrt{x^2+u^2}}$$

(here we put the integration constant to be zero), whence

$$I_{(1)}=\frac{I_x+I_uu_x}{J_x+J_uu_x}=\frac{x+uu_x}{-u+xu_x}\sqrt{x^2+u^2},\qquad\text{or}\qquad \widetilde{I}_{(1)}=\frac{x+uu_x}{-u+xu_x}$$

is a first-order differential invariant for the operator Q.

5 The standard approach and integration of Riccati-type systems

Within the framework of direct method the differential invariants having exactly first order are found as invariants of the first prolongation

$$Q^{(1)} = \xi^a \partial_{x_a} + \eta^i \partial_{u^i} + \left(\eta^k_c + \eta^k_{u^j} u^j_c - \xi^b_{x_c} u^k_b - \xi^b_{u^j} u^j_c u^k_b \right) \partial_{u^k_c}$$

of the operator Q, or as the first integrals of the corresponding characteristic system of ordinary differential equations

$$\frac{dx_a}{\xi^a} = \frac{du^i}{\eta^i} = \frac{du^k_c}{\eta^k_c + \eta^k_{u^j} u^j_c - \xi^b_{x_c} u^k_b - \xi^b_{u^j} u^j_c u^k_b}, \tag{5}$$

that depend not only on x and u, but also on the other the variables of the space $J_{(1)}$. (Here u^i_a is a variable of the jet space $J_{(1)}$, which corresponds to the derivative $\partial u^i / \partial x_a$; lower indices of functions stand for the derivatives with respect to the corresponding variables; there is no summation over a, c, i and k in the latter equation). Integration of the system (5) is, as a rule, a highly cumbersome task. If a universal invariant $I(x, u)$ for the operator Q is known, then it amounts to integration of Riccati-type systems of the form

$$\frac{du^k_c}{dx_a} = -\frac{\xi^b_{u^j}}{\xi^a} u^j_c u^k_b + \frac{\eta^k_{u^j}}{\xi^a} u^j_c - \frac{\xi^b_{x_c}}{\xi^a} u^k_b + \frac{\eta^k_{x_c}}{\xi^a} \Bigg|_{\substack{u=U(x_a,C) \\ x_d=X^d(x_a,C),\, d\neq a}}, \tag{6}$$

if $\xi^a \neq 0$ for some fixed a, or

$$\frac{du^k_c}{du^i} = -\frac{\xi^b_{u^j}}{\eta^i} u^j_c u^k_b + \frac{\eta^k_{u^j}}{\eta^i} u^j_c - \frac{\xi^b_{x_c}}{\eta^i} u^k_b + \frac{\eta^k_{x_c}}{\eta^i} \Bigg|_{\substack{x=X(u^i,C), \\ u^l=U^l(u^i,C),\, l\neq i}}, \tag{7}$$

if $\eta^i \neq 0$ for some fixed i. Here $x_d = X^d(x_a, C)$, $d \neq a$, $u = U(x, C)$ and $x = X(u^i, C)$, $u^l = U^l(u^i, C)$, $l \neq i$, are solutions of the system of algebraic equations $I(x, u) = C$ with respect to the variables x_d, $d \neq a$, u and x, u^l, $l \neq i$, respectively. The constants $C = (C_1, C_2, \ldots, C_{m+n-1})$ in the systems (6) and (7) are considered as parameters. The case $\eta^i \neq 0$ could be reduced to the case $\xi^a \neq 0$ by means of the locus transformation:

$$\begin{aligned} &\tilde{x}_a = u^i, \quad \tilde{x}_d = x_d, \quad \tilde{u}^i = x_a, \quad \tilde{u}^l = u^l, \quad d \neq a, \quad l \neq i, \\ &\tilde{u}^i_a = \frac{1}{u^i_a}, \quad \tilde{u}^i_d = -\frac{u^i_d}{u^i_a}, \quad \tilde{u}^l_a = \frac{u^l_a}{u^i_a}, \quad \tilde{u}^l_d = u^l_d - \frac{u^i_d}{u^i_a} u^l_a. \end{aligned}$$

For this reason we will consider in detail only the case $\xi^a \neq 0$.

The method we suggested in Corollary 2 for finding differential invariants having exactly first order, unlike the standard method, allows to avoid direct integration of systems of Riccati equations (6) or (7) and to find a solution through one quadrature and differentiation. This result means that *in the case of known universal invariant $I(x, u)$ for the operator Q the systems (6) and (7) are always integrable by means of one quadrature.* Really, the general solution for the system (6) could be given explicitly by m non-linked systems of linear algebraic equations

$$D_{x_b} \hat{I}^j \Bigg|_{\substack{u=U(x_a,C) \\ x_d=X^d(x_a,C),\, d\neq a}} = 0,$$

where $\hat{I}^j = I^{j+n-1} + \sum\limits_{d=1}^{n-1} \widetilde{C}_{jd} I^d + \widetilde{C}_{jn} J$, $\widetilde{C}_{ib}$ are arbitrary constants. To write the solution in the explicit form, we introduce some additional notation:

$$\begin{aligned}
&\bar{x} = (x_d)_{d=1,\, d\neq a}^n, \quad \bar{X} = (X^d)_{d=1,\, d\neq a}^n, \quad z = x_a,\\
&I^{\bar{x}} = (I^d)_{d=1}^{n-1}, \quad I^u = (I^{j+n-1})_{j=1}^m,\\
&C^{\bar{x}} = (C_d)_{d=1}^{n-1}, \quad C^u = (C_{j+n-1})_{j=1}^m,\\
&\widetilde{C}' = (\widetilde{C}_{jd})_{j=1}^m{}_{d=1}^{n-1}, \quad \widetilde{C}'' = (\widetilde{C}_{jn})_{j=1}^m, \quad \hat{I} = I^u + \widetilde{C}' I^{\bar{x}} + \widetilde{C}'' J.
\end{aligned}$$

Then the general solution for the system (6) is given by the formulae

$$(u_b^j)_{j=1}^m{}_{b=1}^n = -\hat{I}_u^{-1} \hat{I}_x \Big|_{\substack{u=U(x_a,C)\\ \bar{x}=\bar{X}(x_a,C)}},$$

or

$$\begin{aligned}
&(u_a^j)_{j=1}^m = U_z - U_{C^{\bar{x}}} \bar{X}_{C^{\bar{x}}}^{-1} \bar{X}_z + H((\widetilde{C}' + \widetilde{C}'' J_{C^{\bar{x}}}) \bar{X}_{C^{\bar{x}}}^{-1} \bar{X}_z - \widetilde{C}'' J_{C^{\bar{x}}}),\\
&(u_b^j)_{j=1}^m{}_{b=1,\, b\neq a}^n = U_{C^{\bar{x}}} \bar{X}_{C^{\bar{x}}}^{-1} - H(\widetilde{C}' + \widetilde{C}'' J_{C^{\bar{x}}}) \bar{X}_{C^{\bar{x}}}^{-1},
\end{aligned}$$

where $H = (U_{C^u} - U_{C^{\bar{x}}} \bar{X}_{C^{\bar{x}}}^{-1} \bar{X}_{C^u})(E + \widetilde{C}'' J_{C^u} - (\widetilde{C}' + \widetilde{C}'' J_{C^{\bar{x}}}) \bar{X}_{C^{\bar{x}}}^{-1} X_{C^u})^{-1}$, E is the $m \times m$ unit matrix; the signs of vector-functions with lower indices of the sets of variables designate the corresponding Jacobi matrices. To ensure the existence in some neighborhood of a fixed point (x^0, u^0) of all inverse matrices, that are mentioned above, it is sufficient to consider constants $\widetilde{C}_{ib}$ to be small and to perform a (preliminarily determined as non-degenerate) linear change in the set of invariants for the matrix $I_{(\bar{x},u)}(x^0, u^0)$ to be the unit matrix.

If we put $\widetilde{C}_{ib} = 0$, then we obtain a particular solution

$$(u_a^j)_{j=1}^m = U_z - U_{C^{\bar{x}}} \bar{X}_{C^{\bar{x}}}^{-1} \bar{X}_z, \qquad (u_b^j)_{j=1}^m{}_{b=1,\, b\neq a}^n = U_{C^{\bar{x}}} \bar{X}_{C^{\bar{x}}}^{-1}.$$

The solution of the system (6) in explicit form should be written separately for the case $n = 1$. As in this case $u = U(x, C)$ is the general solution of the system $du^j/dx = \eta^j(x, u)/\xi(x, u)$, then it is easy to verify that $u_x = U_x(x, C)$ is a particular solution for the system (6) (here, as in (6), C is a set of parameters). The general solution for the system (6) has the form

$$u_x = -(I_u - \widetilde{C} \otimes J_u)^{-1} (I_x + \widetilde{C} J_x) \Big|_{u=U(x,C)} = U_z - U_C (E + \widetilde{C} \otimes J_C)^{-1} \widetilde{C} J_z \tag{8}$$

(in the latter equality the substitution $x = z$, $u = U(z, C)$ was performed), where E is the unit matrix of the dimensions $m \times m$, $\widetilde{C} = (\widetilde{C}_1, \widetilde{C}_2, \ldots, \widetilde{C}_m)^{\mathrm{T}}$ is a column of arbitrary constants, $I_u = (I^i_{u^j})$, $I_x = (I^i_x)$, $U_z = (U^k_z)$, $U_C = (U^i_{C_j})$, $\widetilde{C} \otimes J_u = (\widetilde{C}^k J_{u^l})$, $\widetilde{C} \otimes J_C = (\widetilde{C}^k J_{C^l})$. The inverse matrices in (8) always exist for sufficiently small $\widetilde{C}_i$.

Example 2. Let $n = m = 1$, $Q = \exp(-x - u)(\partial_x + u\partial_u)$. $I(x, u) = u \exp(-x)$ is an invariant for the operator Q, whence $U(x, C) = C \exp(x)$. Then

$$J = \int \frac{dx}{\exp(-x - C \exp(x))} = \frac{1}{C} \exp(C \exp(x)) = \frac{\exp(x + u)}{u}$$

(here we set the integration constant to be zero). Therefore,

$$I_{(1)} = \frac{I_x + I_u u_x}{J_x + J_u u_x} = \frac{\exp(-2x - u) u^2 (u_x - u)}{u + u u_x - u_x} \qquad \text{or} \qquad \widetilde{I}_{(1)} = \frac{u_x - u}{u + u u_x - u_x} \exp(-u)$$

is a first-order differential invariant for the operator Q. System (6) for the operator Q consists of one equation which has the form

$$\frac{du_x}{dx} = u_x^2 + (2 - C\exp(x))u_x - C\exp(x).$$

The function $v = C\exp(x)$ is its particular solution. The general solution of this Riccati equation is given by the formula

$$u_x = C\exp(x) - \frac{C^2\exp(2x)}{C\exp(x) - 1 + \widehat{C}\exp(-C\exp(x))},$$

where $\widehat{C}$ is an arbitrary constant.

Example 3. Let $n = m = 1$, $Q = xu(x\partial_x + ku\partial_u)$, $k \in \mathbb{R}$. $I(x,u) = ux^{-k}$ is an invariant for the operator Q, whence $U(x,C) = Cx^k$. Then

$$J = \int \frac{dx}{Cx^{k+2}} = \begin{cases} \dfrac{\ln x}{C} = \dfrac{\ln x}{xu}, & \text{if} \quad k = -1, \\ -\dfrac{x^{-(k+1)}}{(k+1)C} = -\dfrac{1}{(k+1)xu}, & \text{if} \quad k \neq -1 \end{cases}$$

(here we set the integration constant to be zero). The corresponding Riccati equation has the form

$$\frac{du_x}{dx} = -\frac{1}{Cx^k}u_x^2 + \frac{2(k-1)}{x}u_x + kCx^{k-2}.$$

The function $u_x = kCx^{k-1}$ is its particular solution. The general solution of this Riccati equation is given by the formulae

$$u_x = -\frac{C}{x^2}\left(1 + \frac{1}{\widehat{C} - \ln x}\right), \quad \text{if } k = -1, \quad \text{or}$$
$$u_x = Cx^{k-1}\left(k - \frac{k+1}{1 + \widehat{C}x^{k+1}}\right), \quad \text{if } k \neq -1,$$

where $\widehat{C}$ is an arbitrary constant.

Remark. For well-known transformation groups on the plane (i.e. $n = m = 1$) integrability in quadratures of equations (6) and (7) as a rule obviously follows from the form of these equations. For instance, when $\xi_u = 0$ or $\eta_x = 0$, they are a linear equation or a Bernoulli equation respectively. If G is a one-parameter group of conformal transformations, then $\xi_x = \eta_u$ and $\xi_u = -\eta_x$, and therefore in equations (6) and (7) variables are separated:

$$\frac{dv}{v^2+1} = \frac{\eta_x}{\xi}dx \,\Bigg|_{u=U(x,C)} \qquad \text{and} \qquad \frac{dv}{v^2+1} = \frac{\eta_x}{\eta}du \,\Bigg|_{x=X(u,C)}.$$

Example 4. Let $n = 1$, $m = 2$, $Q = \exp(-x - u^1 - u^2)(\partial_x + u^1\partial_{u^1} + u^2\partial_{u^2})$. $I^1(x,u^1,u^2) = u^1\exp(-x)$ and $I^2(x,u^1,u^2) = u^2\exp(-x)$ are invariants for the operator Q, whence $U^1(x,C^1,C^2) = C^1\exp(x)$ and $U^1(x,C^1,C^2) = C^2\exp(x)$. Then

$$J\left(x,C^1,C^2\right) = \int \frac{dx}{\exp\left(-x - (C^1+C^2)\exp(x)\right)} = \frac{\exp\left(\left(C^1+C^2\right)\exp(x)\right)}{C^1+C^2}.$$

(here we set the integration constant to be zero). The corresponding Riccati-type system has the form

$$\frac{du^1_x}{dx} = \left(u^1_x + u^2_x\right) u^1_x + \left(2 - C^1 \exp(x)\right) u^1_x - C^1 \exp(x) u^2_x - C^1 \exp(x),$$
$$\frac{du^2_x}{dx} = \left(u^1_x + u^2_x\right) u^2_x - C^2 \exp(x) u^1_x + \left(2 - C^2 \exp(x)\right) u^2_x - C^2 \exp(x).$$

According to (8), the general solution of this system is given by the formula

$$\begin{pmatrix} u^1_x \\ u^2_x \end{pmatrix} = \exp(x) \begin{pmatrix} C^1 \\ C^2 \end{pmatrix} + \\ + \frac{\left(C^1 + C^2\right) \exp(2x) J\left(x, C^1, C^2\right)}{1 - \left(\widetilde{C}^1 + \widetilde{C}^2\right)\left(\exp(x) - (C^1 + C^2)^{-1}\right) J\left(x, C^1, C^2\right)} \begin{pmatrix} \widetilde{C}^1 \\ \widetilde{C}^2 \end{pmatrix},$$

where $\widetilde{C}^1$, $\widetilde{C}^2$ are arbitrary constants.

Example 5. Let $n = 1$, $m = 2$, $Q = \exp(u^1 + u^2)(\partial_x + u^2\partial_{u^1} - u^1\partial_{u^2})$. Then,

$$U^1\left(x, C^1, C^2\right) = C^1 \cos x + C^2 \sin x,$$
$$U^2\left(x, C^1, C^2\right) = -C^1 \sin x + C^2 \cos x,$$
$$J\left(x, C^1, C^2\right) = \int \exp\left(-\left(C^1 + C^2\right) \cos x - \left(C^2 - C^1\right) \sin x\right) dx.$$

The corresponding Riccati-type system has the form

$$\frac{du^1_x}{dx} = -\left(u^1_x + u^2_x\right) u^1_x + \left(-C^1 \sin x + C^2 \cos x\right) u^1_x + \left(-C^1 \sin x + C^2 \cos x + 1\right) u^2_x,$$
$$\frac{du^2_x}{dx} = -\left(u^1_x + u^2_x\right) u^2_x - \left(C^1 \cos x + C^2 \sin x + 1\right) u^1_x - \left(C^1 \cos x + C^2 \sin x\right) u^2_x.$$

It follows from (8) that the general solution of this system is given by the formula

$$\begin{pmatrix} u^1_x \\ u^2_x \end{pmatrix} = \begin{pmatrix} -C^1 \sin x + C^2 \cos x \\ -C^1 \cos x - C^2 \sin x \end{pmatrix} + \\ + \frac{\exp\left(-\left(C^1 + C^2\right) \cos x - \left(C^2 - C^1\right) \sin x\right)}{1 - \widetilde{C}^1 J_{C^1} - \widetilde{C}^2 J_{C^2}} \begin{pmatrix} \widetilde{C}^1 \cos x + \widetilde{C}^2 \sin x \\ -\widetilde{C}^1 \sin x + \widetilde{C}^2 \cos x \end{pmatrix},$$

where $\widetilde{C}^1$, $\widetilde{C}^2$ are an arbitrary constant.

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Heat Equation on Riemann Manifolds: Morphisms and Factorization to Smaller Dimension

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In [1, 2, 3, 4, 5] there was proposed a method of a factorization of PDE. The method is based on reduction of complicated systems to more simple ones (for example, due to dimension decrease). This concept is proposed in general case for the arbitrary PDE systems, and its concrete investigation is developing for the heat equation case. The category of second order parabolic equations posed on arbitrary manifolds is considered. In this category, for the given nonlinear heat equation we could find morphisms from it to other parabolic equations with the same or a smaller number of independent variables. This allows to receive some classes of solutions of original equation from the class of all solutions of such a reduced equation. Classification of morphisms (with the selection from every equivalence class of the simplest "canonical" representatives) is carried out. Necessary and sufficient conditions for canonical morphisms of heat equation to the parabolic equation on the other manifold are derived. These conditions are formulated in the differential geometry language. The comparison with invariant solutions classes, obtained by the Lie group methods, is carried out. It is proved that discovered solution classes are richer than invariant solution classes, even if we find any (including discontinuous) symmetry groups of original equation.

1 General equation category

Definition 1. *Task* is a pair $A = (N_A, E_A)$, where N_A is a set, E_A is a system of equations for graph $\Gamma \subset N_A = M_A \times K_A$ of a function $u : M_A \to K_A$.

Let $S(A)$ be a set of all subsets $\Gamma \subset N_A$ satisfying E_A.

Definition 2. We will say that a (ordered) pair of a tasks $A = (N_A, E_A)$, $B = (N_B, E_B)$ *admits a map* $F_{AB} : N_A \to N_B$, if for any $\Gamma \subset N_B$, $\Gamma \in S(B) \Leftrightarrow F_{AB}^{-1}(\Gamma) \in S(A)$.

Of course, these definitions are rather informal, but they will be correct when we define more exactly the notion "system of equations" and the class of assumed subsets $\Gamma \subset N_A$. Let us consider the *general equation category* $\mathcal{E}$, whose objects are tasks (with some refinement of the sense of the notion "system of equations"), and morphisms $\mathrm{Mor}(A, B)$ are admitted by the pair (A, B) maps with natural composition law.

For the given task A we could define the set $\mathrm{Mor}(A, \mathcal{A})$ of all morphisms A in a framework of some fixed subcategory $\mathcal{A}$ of the general equation category (let us call such morphisms and corresponding tasks B "factorization of A"). The tasks, which factorize A, are naturally divided into classes of isomorphic tasks, and morphisms $\mathrm{Mor}(A, \cdot)$ are divided into equivalence classes.

The proposed approach is conceptually close to the developed in [6] approach to investigation of dynamical and controlled systems. In this approach as morphisms of system A to the system B smooth maps of the phase space of system A to the phase space of system B are considered, which transform solutions (phase trajectories) of A to the solutions of B. By contrast, in the approach presented here, for the class of all solutions of reduced system B there is a corresponding class of such solutions of original system A, whose graphs could be projected onto the space of dependent and independent variables of B; when we pass to the reduced system, the number of dependent variables remains the same, and the number of independent variables does not increase. Thus the approach proposed is an analog to the sub-object notion (in terminology of [6]) with respect to information about original system solutions, though it is closed to the factor-object notion with respect to relations between original and reduced systems.

If G is symmetry group of E_A, then natural projection $p : N \to N/G$ is admitted by the pair $(A, A/G)$ in the sense of Definition 2, that is our definition is a generalization of the reduction by the symmetry group. Instead of this the general notion of the group analysis we base on a more wide notion "a map admitted by the task". We need not require from the group preserving solution of an interesting class (if even such a group should exist) to be continuous admitted by original system. So we could obtain more general classes of solutions and than classes of invariant solutions of Lie group analysis (though our approach is more laborious owing to non-linearity of a system for admissible map). Besides, when we factorize original system, a factorizing map defined here is a more natural object than the group of transformations operating on space of independent and dependent variables of the original task.

2 Category of parabolic equations

Let us consider subcategory $\mathcal{PE}$ of the general equation category, whose objects are second order parabolic equations:

$$E: \quad u_t = Lu, \quad M = T \times X, \quad K = \mathbb{R},$$

where L is differential operator, depending on the time t, defined on the connected manifold X, which has the following form in any local coordinates $\left(x^i\right)$ on X:

$$Lu = b^{ij}\left(t, x, u\right) u_{ij} + c^{ij}\left(t, x, u\right) u_i u_j + b^i\left(t, x, u\right) u_i + q\left(t, x, u\right).$$

Here a lower index i denotes partial derivative by x^i, form $b^{ij} = b^{ji}$ is positively defined, $c^{ij} = c^{ji}$. Morphisms of $\mathcal{PE}$ are all smooth maps admitted by $\mathcal{PE}$ task pairs. Let us describe this morphisms:

Theorem 1. *Any morphism of the category $\mathcal{PE}$ has the form*

$$\left(t, x, u\right) \to \left(t'\left(t\right), x'\left(t, x\right), u'\left(t, x, u\right)\right). \tag{1}$$

Set of isomorphisms of the category $\mathcal{PE}$ is the set of all one-to-one maps of kind (1).

Let us consider full subcategory $\mathcal{PE}'$ of the category $\mathcal{PE}$, whose objects are equations $u_t = Lu$, where operator L in local coordinates has the following form:

$$Lu = b^{ij}\left(t, x\right)\left(a\left(t, x, u\right) u_{ij} + c\left(t, x, u\right) u_i u_j\right) + b^i\left(t, x, u\right) u_i + q\left(t, x, u\right),$$

and all morphisms are inherited from $\mathcal{PE}$.

Theorem 2. *If set of morphisms* $\mathrm{Mor}_{\mathcal{PE}}\left(A, B\right)$ *is nonempty and $A \in \mathcal{PE}'$, then $B \in \mathcal{PE}'$.*

3 Category of autonomous parabolic equations

Let us call the map (1) *autonomous*, if it has the form

$$\left(t, x, u\right) \to \left(t, x'\left(x\right), u'\left(x, u\right)\right). \tag{2}$$

Let us call a parabolic equation from the category $\mathcal{PE}'$, defined on a Riemann manifold X, *autonomous*, if it has the form:

$$u_t = Lu = a\left(x, u\right)\Delta u + c\left(x, u\right)\left(\nabla u\right)^2 + \xi\left(x, u\right)\nabla u + q\left(x, u\right), \quad \xi\left(\cdot, u\right) \in T^*X.$$

Theorem 3. *Let $F : A \to B$ be a morphism of the category $\mathcal{PE}$, F be an autonomous map, A be an autonomous equation. Then we could endow with Riemann metric the manifold, on which B is posed, in such a way, that B becomes an autonomous equation.*

Let $\mathcal{APE}$ be the subcategory of $\mathcal{PE}'$, objects of which are autonomous parabolic equations, and morphisms are autonomous morphisms of the category $\mathcal{PE}$.

4 Classification of morphisms of nonlinear heat equation

Let us consider a nonlinear heat equation $A \in \mathcal{APE}$, posed on some Riemann manifold X:

$$u_t = a\,(u)\,\Delta u + q(u). \tag{3}$$

(note that any equation $u_t = a(u)\Delta u + c(u)(\nabla u)^2 + q(u)$) is isomorphic to some equation (3) in $\mathcal{APE}$). We will investigate set of morphisms $\mathrm{Mor}\,(A, \mathcal{PE})$ and classes of solutions of equation A, corresponding these morphisms.

Note, that two morphisms $F : A \to B$ and $F' : A \to B'$ are called to be equivalent if there exists such isomorphism $G : B \to B'$ that $F' = G \circ F$. From the point of view of classes of original task solutions obtained from factorization, equivalent morphisms have the same value, that is solution classes are the same for these morphisms. So it is interesting to select from any equivalence class of the simplest (in some sense) morphism, or such morphism for which the factorized equation is the simplest.

When we classify morphisms for the original equation (3), a form of coefficient $a\,(u)$ is important. We will distinguish such option:

$$\begin{tikzcd}
 & \boxed{a(u) \text{ is arbitrary function}} \arrow[dl] \arrow[dr] & \\
\boxed{a(u) = a_0(u-u_0)^\lambda} \arrow[dr] & & \boxed{a(u) = a_0 e^{\lambda u}} \arrow[dl] \\
 & \boxed{a(u) = a_0} &
\end{tikzcd}$$

The lower the option is situated on this scheme, the richer a collection of morphisms is. Note, that similar relation is observed in the group classification of nonlinear heat equation [7].

Theorem 4. *If $a \neq \mathrm{const}$ then for any morphism of equation (3) into the category $\mathcal{PE}$ there exists an equivalent in $\mathcal{PE}$ autonomous morphism (that is morphism of the category $\mathcal{APE}$).*

Let us give a map $p : X \to X'$ from the manifold X to the manifold X' and a differential operator D on X. We will say that D *is projected on* X', if such a differential operator D' on X' exists that the following diagram is commutative:

$$\begin{tikzcd}
C^\infty(X') \arrow[r, "p^*"] \arrow[d, "D'"'] & C^\infty(X) \arrow[d, "D"] \\
C^\infty(X') \arrow[r, "p^*"'] & C^\infty(X)
\end{tikzcd}$$

Theorem 5. *Let $a \neq \mathrm{const}$. For any morphism of the equation A into the category $\mathcal{PE}$ there exists an equivalent in $\mathcal{PE}$ autonomous morphism $(t, x, u) \to (t, y(x), v(x, u))$ A to $B \in \mathcal{APE}$, for which factorized equation B is $v_t = a(v) Lv + Q(v)$, operator L is projection onto Y at map $x \to y(x)$ of the described below operator D (note that this condition is limitation on the projection $y(x)$), where:*

1) if A is arbitrary (not any of the following special form): $D = \Delta$, $v(x, u) = u$;

2) if A is $u_t = a_0 u^\lambda (\Delta u + q_0 u) + q_1 u$ up to shift $u \to u - u_0$, $\lambda \neq 0$, $a_0, q_0, q_1 = \mathrm{const}$: $Df = \beta^{\lambda-1} (\Delta(\beta f) + q_0 \beta f)$ for some function $\beta : X \to \mathbb{R}$, $v(x, u) = \beta^{-1}(x) u$, $Q = q_1 v$;

3) if A is $u_t = a_0 e^{\lambda u} (\Delta u + q_0) + q_1$, $\lambda \neq 0$, $a_0, q_0, q_1 = \mathrm{const}$: $Df = e^{\lambda\beta} (\Delta f + \Delta\beta + q_0)$ for some function $\beta : X \to \mathbb{R}$, $v(x, u) = u - \beta(x)$, $Q = q_1$.

We will call such morphisms "*canonical*". In the category $\mathcal{PE}$ the canonical representative in any class of morphisms is defined uniquely up to diffeomorphism of manifold Y, and in the category $\mathcal{APE}$ it is defined uniquely up to conformal diffeomorphism of Y.

Further we restrict ourselves by the investigation of the canonical maps for the first option, that is will look for such maps p from the given Riemann manifold X onto arbitrary Riemann manifolds Y, for which Laplacian on X is projected to some operator on Y (note that this canonical maps will be canonical for given X in the cases (2), (3) too).

Note that isomorphic autonomous equations B, factorized given A, are distinguished only by arbitrary transformations $v \to v'(y, v)$ and has the same projection $p : x \to y(x)$ up to conformal diffeomorphism of Y. Therefore to find such projection $p : X \to Y$ for canonical morphism is to find all autonomous morphisms from this equivalence class.

5 Factorizing of heat equation in $\mathbb{R}^3$

Let $\mathcal{DAPE}$ be full subcategory of $\mathcal{APE}$, whose objects are autonomous parabolic equations of divergent shape:

$$u_t = c(x, u)^{-1} \operatorname{div}(k(x, u) \nabla u) + q(x, u),$$

and morphisms are autonomous morphisms of the category $\mathcal{APE}$.

Theorem 6. *Let X be a connected region of $\mathbb{R}^3$ with Euclidean metric, Y be a manifold without boundary, A do not have form (2 or 3) from Theorem 5. Then p define canonical morphism of A in $\mathcal{DAPE}$ iff p is restriction on X of factorization $\mathbb{R}^3$ under some (may be discontinuous) group G of isometries.*

6 Factorizing with dimension decrease by 1

Theorem 7. *Let A do not have form (2 or 3) from Theorem 5, and (a) $p : X \to Y$ is a fibering; (b) X and Y are oriented; (c) X is an open domain in complete Riemann space $\tilde{X}$; (d) $\dim Y = \dim X - 1$. Then p define canonical morphism in $\mathcal{DAPE}$ iff the following conditions fulfilled:*

a) p is a superposition of maps $p_1 : X \to Y'$ and $p_0 : Y' \to Y$;

b) $p_1 : X \to Y'$ is a restriction on X of the projection $\tilde{X} \to \tilde{X} \big/ G_1$, where G_1 is some 1-parameter subgroup of group $\mathrm{Isom}\left(\tilde{X}\right)$ of all isometries of $\tilde{X}$;

c) $p_0 : Y' \to \tilde{Y}$ is isomeric covering (for the metric on Y', inherited from X);

d) for the vector field η generating group G_1, the function $\vartheta = \langle \eta, \eta \rangle$, defined on Y', is projectible on Y.

7 Factorizing with dimension decrease by 1: comparison with group analysis

As it was shown in the Section 5, when we were factorizing heat equation in $\mathbb{R}^3$ with Euclidean metric, the class of correspondent (3D) solutions of A coincides with a class of solutions of A, which are invariant under some (maybe discontinuous) group of isometries of $\mathbb{R}^3$.

But these results about coincidence of factorizing maps for the heat equation in $\mathbb{R}^3$ with Euclidean metric with factormaps by symmetry groups (that is isometries groups) are accidental.

At first, projection $p_0 : Y' \to Y$ from previous section is not necessarily generated by some group of transformation of Y'.

At second, let even $Y' = Y/G_0$, where G_0 is some discrete group of the isometries of Y'. The question is: could group G_0 be lifted to some group of the isometries of X, which preserves projection onto Y?

Let the group G_1 be fixed that satisfies conditions of Theorem 7. We consider differential-geometric connection χ on a fibering $p_1 : X \to Y'$ with the structural group G_1, which horizontal planes are orthogonal to G_1 orbits.

Theorem 8. *(necessary condition). If a discrete group G_0, which operates on Y' and satisfies conditions of the Theorem 7, could be lifted to the subgroup of* $\mathrm{Isom}(X)$*, then curvature form $d\chi$, projected on Y', would be invariant respectively G_0.*

Lemma 1. *χ may be decomposed on a sum $\chi = p_{1*}\chi' + dh$, where $\chi' \in T^*Y'$, h is a function from X to $\mathcal{H}$, $\mathcal{H}$ is fiber of p_1 (that is either $\mathbb{R}$, or circle $\mathbb{R} \bmod H$, where $H = \mathrm{const}$ is integral χ on a vertical cycle).*

Theorem 9. *(necessary and sufficient condition). A discrete group G_0, operating on Y' and satisfying conditions of the Theorem 7, could be lifted to the subgroup of* $\mathrm{Isom}\,(X)$*, iff $\forall g \in G_0$ the form $g\chi' - \chi'$ is:*

– exact, if the fiber of p_1 is simply connected;

– closed with periods, multiply H, if the fiber of p_1 is multiply connected.

Particularly, if $X = \mathbb{R}^n$, and G_1 is the rotations group, $\eta = \sum\limits_{i=1}^{m} a_i \partial_{\varphi_i}$, $m \geq 3$, or G_1 is the screw motions group, $\eta = \partial_z + \sum\limits_{i=1}^{m} a_i \partial_{\varphi_i}$, $m \geq 2$, then such groups G_0 exist, which does not lift on X.

8 Factorizing with dimension decrease

Let us equip X with connection generated by planes orthogonal to fibers.

Theorem 10. *Let (a) $p : X \to Y$ be a fibering; (b) $\dim Y < \dim X$. Then p defines canonical morphism to $\mathcal{DAPE}$ iff the following conditions fulfilled:*

1) the fibers of p are parallel;

2) the transformation of a fiber over an initial point to a fiber over a final point changes volumes proportionally when we translate along any curve on Y;

3) the holonomy group preserves volume on a fiber.

Moreover, p define canonical morphism to $\mathcal{APE}$ iff conditions 1)–2) fulfilled.

Example 1. $(\dim X = 4, \dim Y = 2)$. Let $X = \{(x, y, z, w)\}$ with the metric

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+\alpha^2+\beta^2 & \alpha & \beta \\ 0 & \alpha & 1 & 0 \\ 0 & \beta & 0 & 1 \end{pmatrix}, \quad \alpha = xe^w, \quad \beta = xe^z,$$

$Y = \{(x,y)\}$ with the Euclidean metric, $p(x,y,z,w) = (x,y)$. Then map p and equation $v_t = v_{xx} + v_{yy}$ are factorization of the equation

$$u_t = u_{xx} + u_{yy} - 2\alpha u_{yz} - 2\beta u_{yw} + \left(1+\alpha^2\right) u_{zz} \\ + 2\alpha\beta u_{zw} + \left(1+\beta^2\right) u_{ww} + (\alpha\beta)_w u_z + (\alpha\beta)_z u_w,$$

where $\alpha = xe^w$ and $\beta = xe^z$, by the map $p : (x,y,z,w) \to (x,y)$. (The same is true for the equations $v_t = a(v)\Delta v$ on Y and $u_t = a(u)\Delta u$ on X for arbitrary function a, but for simplicity we will write linear equations in examples.) However the only transformations X, under which both the last equation and all it's solutions projected by p are invariant, are $(x,y,z,w) \to (x,y,w,z)$ and identity. Moreover, another transformation with such properties does not exist even locally (i.e. it could not be defined in any small neighborhood on X), even if we replace the requirement "to keep the equation invariant" by the requirement "to be conformal".

Example 2. ($\dim X = 3$, $\dim Y = 2$). Let $\tilde{X} = \mathbb{R}^3 = \{(x,y,z)\}$ with the metric

$$g_{ij} = \begin{pmatrix} 1+z^2 & z & -z \\ z & 2 & -1 \\ -z & -1 & 1 \end{pmatrix},$$

$\tilde{Y} = \{(x,y)\}$ with the Euclidean metric. Let us consider group H of isometries $\tilde{X}$, generated by the screw motion $(x,y,z) \to (x+1,-y,-z)$ (H is projectible on $\tilde{Y}$), $X = \tilde{X}/H$, $Y = \tilde{Y}/H$, $p(x,y,z) = (x,y)$. Y is homeomorphic to the Mobius band without a boundary; X is homeomorphic to the torus without a boundary.

Then map p and equation $e^x v_t = (e^x v_x)_x + (e^x v_y)_y$, or $v_t = v_{xx} + v_{yy} + v_x$ on Y are factorizations of the equation

$$u_t = u_{xx} + u_{yy} + u_x + 2zu_{xz} + 2u_{yz} + \left(\left(2+z^2\right) u_z\right)_z$$

on X. However the only transformation X, under which both the last equation and all projected by p it's solutions are invariant, is identity map. Moreover, there does not exist a non-identity conformal transformation X, under which all projected by p solutions of the last equation are invariant.

Example 3. ($\dim X = 3$, $\dim Y = 1$). Let $X = S^1 \times \mathbb{R}^2 = \{(x,y,z) : x \in \mathbb{R} \bmod 1, y, z \in \mathbb{R}\}$, equipped with the metric

$$g_{ij} = \begin{pmatrix} \alpha^2+\beta^2 & \alpha & \beta \\ \alpha & 1 & 0 \\ \beta & 0 & 1 \end{pmatrix}, \quad \alpha = -e^z, \quad \beta = 2y,$$

$\tilde{Y} = S^1 = \{x \in \mathbb{R} \bmod 1\}$ equipped with the Euclidean metric, $p(x,y,z) = x$. Then map p and equation $a^{-1}(v)v_t = v_{xx}$ on Y are factorizations of the equation

$$u_t = u_{xx} + \left(1+\alpha^2\right) u_{yy} + \left(1+\beta^2\right) u_{zz} + 2\alpha\beta u_{yz} - 2\alpha u_{xy} - 2\beta u_{xz} + (\alpha\beta)_y u_z + (\alpha\beta)_z u_y,$$

on X. However the only transformation X, under which both the last equation and all projected by p it's solutions are invariant, is identity map.

Example 4. ($\dim X = 2$, $\dim Y = 1$). Let $X = \mathbb{R}^2/G$ with the Euclidean metric, when G is the group generated by the sliding symmetry respectively the straight line l. The orthogonal projection of X onto the mean circumference (image of the line l) define equation $v_t = v_{yy}$ on l, factorized the equation $u_t = u_{xx} + u_{yy}$ on X. However the only transformation X, under which both the last equation and all projected by p its solutions are invariant, is reflection with respect to l.

9 Factorization without dimension decrease

If $\dim X = \dim Y$, then $p : X \to \tilde{Y}$ projected Laplacian iff it is isometric projection up to some conformal transformation Y.

Example 5. Let manifold X be a plane without 3 points: $A\,(0,0)$, $B\,(1,0)$ and $C\,(0,2)$. Let's consider heat equation on X with metric $g_{ij} = \lambda^2\,(x)\,\delta_{ij}$:

$$\lambda^2\,(x)\,u_t = u_{11} + u_{22}, \tag{4}$$

where $\lambda\,(x) = \rho\,(x, A)\,\rho\,(x, B)\,\rho\,(x, C)$, ρ is the distance function (in usual plane metric). Let $Y = X$, and map $p : X \to Y$ is given by the formula $y = \frac{1}{4}x^4 - \frac{1+2i}{3}x^3 + ix^2$, where x, y are considered as points at a complex plane.

Because of $|y_x| = |x\,(x-1)\,(x-2i)| = \lambda\,(x)$, heat equation $u_t = u_{11} + u_{22}$ on Y, equipped by Euclidean metric $g_{ij} = \delta_{ij}$, is factorisation of the equation (4) on the manifold $\overset{\circ}{X}$, which is obtained by deleting of pre-images of images of zeroes of λ from X. However, there does not exist a non-identical transformation of $\overset{\circ}{X}$, under which all projected by p solutions of equation (4) are invariant. Moreover, there does not exist a non-identical transformation of any manifold X', under which an equation (4) is invariant, if X' is obtained by deleting an arbitrary discrete set of points from X.

Example 6. Let us consider an equation on $X = \mathbb{R}^2$:

$$u_t = \left(1 + |x|^2\right)^2 (u_{11} + u_{22})\,. \tag{5}$$

Let g be the transformation of $\mathbb{R}^2/\{0\}$ that maps $x \in X$ to the point, obtained from x by inversion under the unit circle with a center in an origin and consequent reflection under this center. Equation (5) is invariant with respect to g, but g is not defined at origin. However the map $p : X \to Y = \mathbb{P}^2$ onto the projective plane, which past together points x and gx at $x \neq 0$, is defined on all X and gives smooth projection. Then inducing on Y heat equation is factorization of original equation on X.

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The Soliton Content of the Camassa–Holm and Hunter–Saxton Equations

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The notion of a scalar equation describing pseudo-spherical surfaces is reviewed. It is shown that if an equation admits this structure, the existence of conservation laws, symmetries, and quadratic pseudo-potentials, can be studied by geometrical means. As an application, it is pointed out that the important Camassa–Holm and Hunter–Saxton equations possess features considered to be characteristic of standard "soliton" equations: an infinite number of local conservation laws, "Miura transformations", a zero curvature formulation, and nonlocal symmetries.

1 Introduction

In this contribution we review some recent developments linking differential geometry of surfaces and integrability of nonlinear partial differential equations. We concentrate on the notion of a scalar equation describing pseudo-spherical surfaces (or "of pseudo-spherical type") introduced by S.S. Chern and Keti Tenenblat [6, 18]: these equations share with the sine–Gordon equation the property that their (suitably generic) solutions determine two-dimensional surfaces equipped with metrics of constant Gaussian curvature -1.

Equations of pseudo-spherical type are introduced in Section 2, and we point out that equations possessing this structure are naturally the integrability condition of an $sl(2,\mathbb{R})$-valued linear problem. We then survey in Section 3 two standard aspects of the geometric theory of differential equations, conservation laws and symmetries: for equations describing pseudo-spherical surfaces, they can be understood by geometrical means. In Section 4 we consider our main application, the Camassa–Holm (Camassa and Holm [5]) and Hunter–Saxton (Hunter and Saxton [8], Hunter and Zheng [9]) equations. We show that for these important examples, the geometric approach reviewed in Sections 2 and 3 allows us to construct explicitly the following: quadratic pseudo-potentials, Miura transformations, "modified" equations, local conservation laws, zero curvature representations, and non-local symmetries.

2 Equations of pseudo-spherical type

This structure was introduced by S.S. Chern and K. Tenenblat in 1986 [6], motivated by the fact that [17] generic solutions of equations integrable by the Ablowitz, Kaup, Newell and Segur (AKNS) inverse scattering scheme determine – whenever their associated linear problems are real – pseudo-spherical surfaces, that is, Riemannian surfaces of constant Gaussian curvature -1.

Definition 1. A scalar differential equation $\Xi(x,t,u,u_x,\ldots,u_{x^nt^m}) = 0$ in two independent variables x, t is of pseudo-spherical type (or, it is said to describe pseudo-spherical surfaces) if there exist one-forms $\omega^\alpha \neq 0$,

$$\omega^\alpha = f_{\alpha 1}(x,t,u,\ldots,u_{x^rt^p})\,dx + f_{\alpha 2}(x,t,u,\ldots,u_{x^st^q})\,dt, \qquad \alpha = 1,2,3 \tag{1}$$

whose coefficients $f_{\alpha\beta}$ are differential functions, such that the one-forms $\overline{\omega}^\alpha = \omega^\alpha(u(x,t))$ satisfy the structure equations

$$d\overline{\omega}^1 = \overline{\omega}^3 \wedge \overline{\omega}^2, \qquad d\overline{\omega}^2 = \overline{\omega}^1 \wedge \overline{\omega}^3, \qquad d\overline{\omega}^3 = \overline{\omega}^1 \wedge \overline{\omega}^2, \tag{2}$$

whenever $u = u(x,t)$ is a solution to $\Xi = 0$.

We recall that a differential function is a smooth function which depends on x, t, and a finite number of derivatives of u [13]. We sometimes use the expression "PSS equation" instead of "equation of pseudo-spherical type". Also, we exclude from our considerations the trivial case when the functions $f_{\alpha\beta}$ all depend only on x, t.

Example 1. Burgers' equation $u_t = u_{xx} + uu_x + h_x(x)$, is a PSS equation with

$$\begin{aligned} \omega^1 &= ((1/2)u - (\beta/\eta))dx + \big((1/2)u_x + (1/4)u^2 + (1/2)h(x)\big)\,dt, \\ \omega^2 &= -\omega^3 = \eta\, dx + ((\eta/2)u + \beta)dt, \end{aligned}$$

in which $\eta \neq 0$ is a parameter, and β is a solution of the equation $\beta^2 - \eta\beta_x + \left(\eta^2/2\right) h(x) = 0$.

The geometric interpretation of Definition 1 is based on the following genericity notions ([15] and references therein):

Definition 2. Let $\Xi = 0$ be a PSS equation with associated one-forms ω^α, $\alpha = 1,2,3$. A solution $u(x,t)$ of $\Xi = 0$ is I-generic if $\left(\omega^3 \wedge \omega^2\right)(u(x,t)) \neq 0$, II-generic if $\left(\omega^1 \wedge \omega^3\right)(u(x,t)) \neq 0$, and III-generic if $\big(\omega^1 \wedge \omega^2)(u(x,t)\big) \neq 0$.

For instance, $u(x,t) = x + t$ is a I- and III-generic solution of the PSS equation $u_t = u_{xx} + u_x$ with associated one-forms $\omega^1 = udx + u_x dt$, $\omega^2 = dx$, and $\omega^3 = udx + u_x dt$.

Proposition 1. *Let $\Xi = 0$ be a PSS equation with associated one–forms ω^α.*

(a) *If $u(x,t)$ is a I–generic solution, $\overline{\omega}^2$ and $\overline{\omega}^3$ determine a Lorentzian metric of Gaussian curvature $K = -1$ on the domain of $u(x,t)$, with connection one–form given by $\overline{\omega}^1$.*

(b) *If $u(x,t)$ is a II–generic solution, $\overline{\omega}^1$ and $-\overline{\omega}^3$ determine a Lorentzian metric of Gaussian curvature $K = -1$ on the domain of $u(x,t)$, with connection one–form given by $\overline{\omega}^2$.*

(c) *If $u(x,t)$ is a III–generic solution, $\overline{\omega}^1$ and $\overline{\omega}^2$ determine a Riemannian metric of Gaussian curvature $K = -1$ on the domain of $u(x,t)$, with connection one–form given by $\overline{\omega}^3$.*

As pointed out above, the main motivation for formulating Definition 1 is its relation with integrable equations. The following notion is implicit in [6]:

Definition 3. An equation is geometrically integrable if it describes a non-trivial one-parameter family of pseudo-spherical surfaces.

Proposition 2. *A geometrically integrable equation $\Xi = 0$ with associated one-forms ω^α, $\alpha = 1,2,3$, is the integrability condition of a one-parameter family of $sl(2,\mathbb{R})$-valued linear problems.*

Proof. The linear problem $d\psi = \Omega\psi$, in which

$$\Omega = U dx + V dt = \frac{1}{2}\begin{pmatrix} \omega^2 & \omega^1 - \omega^3 \\ \omega^1 + \omega^3 & -\omega^2 \end{pmatrix}, \tag{3}$$

is integrable whenever $u(x,t)$ is a solution of $\Xi = 0$. ∎

An important idea in integrable systems is that an equation $\Xi = 0$ is not just the integrability condition of a linear problem $\psi_x = X\psi$, $\psi_t = T\psi$, but that the zero curvature equation $X_t - T_x + [X,T] = 0$ is *equivalent* to $\Xi = 0$. It is a crucial problem to formalize this remark within the context of PSS equations. For evolutionary equations, we proceed thus [10, 15]: if $u_t = F(x,t,u,\dots,u_{x^k})$ is a k^{th} order evolution equation, we consider the differential ideal I_F generated by the two-forms

$$du \wedge dx + F(x,t,u,\dots,u_{x^k})\, dx \wedge dt, \qquad du_{x^l} \wedge dt - u_{x^{l+1}}\, dx \wedge dt, \qquad 1 \le l \le k-1,$$

on a manifold J with coordinates $x,t,u,u_x,\dots,u_{x^k}$.

Definition 4. An evolution equation $u_t = F(x,t,u,\dots,u_{x^k})$ is *strictly pseudo-spherical* if there exist one-forms $\omega^\alpha = f_{\alpha 1}\, dx + f_{\alpha 2}\, dt$, $\alpha = 1,2,3$, whose coefficients $f_{\alpha\beta}$ are smooth functions on J, such that the two-forms

$$\Omega_1 = d\omega^1 - \omega^3 \wedge \omega^2, \qquad \Omega_2 = d\omega^2 - \omega^1 \wedge \omega^3, \qquad \Omega_3 = d\omega^3 - \omega^1 \wedge \omega^2, \tag{4}$$

generate I_F.

Note that local solutions of $u_t = F$ correspond to integral submanifolds of the exterior differential system $\{I_F, dx \wedge dt\}$. It follows that if $u_t = F$ is strictly pseudo-spherical, it is necessary and sufficient for the structure equations $\Omega_\alpha = 0$ to hold. The following lemma [14, 15], used in Section 3 below, allows us to *classify* strictly pseudo-spherical equations [6, 10, 14]:

Lemma 1. *Necessary and sufficient conditions for the k^{th} order equation $u_t = F$ to be strictly pseudo-spherical are the conjunction of* (a) *The functions $f_{\alpha\beta}$ satisfy $f_{\alpha 1, u_{x^a}} = 0$; $f_{\alpha 2, u_{x^k}} = 0$; $f^2_{11,u} + f^2_{21,u} + f^2_{31,u} \neq 0$, in which $a \ge 1$ and $\alpha = 1,2,3$; and* (b) *F and $f_{\alpha\beta}$ satisfy the identities*

$$-f_{\alpha 1,u}F + \sum_{i=0}^{k-1} u_{x^{i+1}} f_{\alpha 2, u_{x^i}} + f_{\delta 1} f_{\gamma 2} - f_{\gamma 1} f_{\delta 2} + f_{\alpha 2,x} - f_{\alpha 1,t} = 0, \tag{5}$$

in which (α,δ,γ) is $(1,2,3)$, $(2,3,1)$, or $(3,2,1)$.

3 Conservation laws and symmetries for PSS equations

By local conservation laws of $\Xi = 0$ we mean one-forms $\theta = f dx + g dt$, f, g differential functions, such that $d_H\theta := (-D_t f + D_x g) dx \wedge dt = 0$ on solutions of $\Xi = 0$, where D_x and D_t denote the total derivatives operators with respect to x and t respectively [13]: cohomology questions [12] are beyond the scope of this paper. Nonlocal conservation laws can be also considered [12], and in fact, it is natural to study both cases simultaneously [18] when treating PSS equations. We begin with a purely geometric result [6, 18]:

Proposition 3. *Given a coframe $\{\overline{\omega}^1, \overline{\omega}^2\}$ and corresponding connection one-form $\overline{\omega}^3$ on a surface M, there exists a new coframe $\left\{\overline{\theta}^1, \overline{\theta}^2\right\}$ and new connection one-form $\overline{\theta}^3$ on M satisfying*

$$d\overline{\theta}^1 = 0, \qquad d\overline{\theta}^2 = \overline{\theta}^2 \wedge \overline{\theta}^1, \qquad \textit{and} \qquad \overline{\theta}^3 + \overline{\theta}^2 = 0, \tag{6}$$

if and only if the surface M is pseudo-spherical.

Proof. Assume that the orthonormal frames dual to the coframes $\{\overline{\omega}^1, \overline{\omega}^2\}$ and $\left\{\overline{\theta}^1, \overline{\theta}^2\right\}$ possess the same orientation. The one-forms $\overline{\omega}^\alpha$ and $\overline{\theta}^\alpha$ are connected by means of

$$\overline{\theta}^1 = \overline{\omega}^1 \cos\rho + \overline{\omega}^2 \sin\rho, \qquad \overline{\theta}^2 = -\overline{\omega}^1 \sin\rho + \overline{\omega}^2 \cos\rho, \qquad \overline{\theta}^3 = \overline{\omega}^3 + d\rho. \tag{7}$$

It follows that $\overline{\theta}^1$, $\overline{\theta}^2$, $\overline{\theta}^3$ satisfying (6) exist if and only if the Pfaffian system

$$\overline{\omega}^3 + d\rho - \overline{\omega}^1 \sin\rho + \overline{\omega}^2 \cos\rho = 0 \tag{8}$$

on the space of coordinates (x, t, ρ) is completely integrable for $\rho(x, t)$, and this happens if and only if M is pseudo-spherical. ■

Equations (6) and (8) determine geodesic coordinates on M. Now, if the equation $\Xi = 0$ describes pseudo-spherical surfaces with associated one-forms $\omega^\alpha = f_{\alpha 1} dx + f_{\alpha 2} dt$, (6) and (8) imply that

$$\omega^3(u(x,t)) + d\rho - \omega^1(u(x,t)) \sin\rho + \omega^2(u(x,t)) \cos\rho = 0 \tag{9}$$

is completely integrable for $\rho(x, t)$ whenever $u(x, t)$ is a local solution of $\Xi = 0$. Equations (6) and (7) then imply that for each solution $u(x, t)$ and a corresponding solution $\rho(x, t)$ of (9), the one-form $\theta^1 = \overline{\omega}^1 \cos\rho + \overline{\omega}^2 \sin\rho$ is closed. If the functions $f_{\alpha\beta}$ can be expanded as power series in a parameter η, so can $\rho(x, t)$ and θ^1. Thus, in principle, geometrically integrable equations possess an infinite number of conservation laws. They may well be nonlocal, however, since they depend on solutions of the Pfaffian system (9), see [18]. The following lemma [14] allows us to construct them explicitly.

Lemma 2. *Let $\Xi = 0$ be a PSS equation with associated one-forms ω^α. Under the changes of variables $\Gamma = \tan(\rho/2)$ and $\hat{\Gamma} = \cot(\rho/2)$, equation (9) and the one-form θ^1 become,*

$$-2d\Gamma = \left(\overline{\omega}^3 + \overline{\omega}^2\right) - 2\Gamma\overline{\omega}^1 + \Gamma^2 \left(\overline{\omega}^3 - \overline{\omega}^2\right), \tag{10}$$

$$\Theta = \overline{\omega}^1 - \Gamma \left(\overline{\omega}^3 - \overline{\omega}^2\right), \qquad (\textit{up to an exact differential form}) \tag{11}$$

and

$$2d\hat{\Gamma} = \left(\overline{\omega}^3 - \overline{\omega}^2\right) - 2\hat{\Gamma}\overline{\omega}^1 + \hat{\Gamma}^2 \left(\overline{\omega}^3 + \overline{\omega}^2\right), \tag{12}$$

$$\hat{\Theta} = -\overline{\omega}^1 + \hat{\Gamma} \left(\overline{\omega}^3 + \overline{\omega}^2\right), \qquad (\textit{up to an exact differential form}). \tag{13}$$

We now turn to (generalized) symmetries. For ease of exposition, we restrict ourselves to strictly pseudo-spherical equations. We recall that a differential function G is a generalized symmetry of $u_t = F$ if and only if $u(x, t) + \tau G(u(x, t))$ is – to first order in τ – a solution of $u_t = F$ whenever $u(x, t)$ is a solution of $u_t = F$.

Let $u_t = F$ be an m^{th} order strictly pseudo-spherical equation with associated one-forms ω^α. Let $u(x, t)$ be a local solution of $u_t = F$, and set $\overline{G} = G(u(x, t))$, in which G is a differential function. We expand $\omega^\alpha(u(x, t) + \tau G(u(x, t))$ about $\tau = 0$, thereby obtaining an infinitesimal deformation $\overline{\omega}^\alpha + \tau\overline{\Lambda}_\alpha$, $\overline{\Lambda}_\alpha = \overline{g}_{\alpha 1} dx + \overline{g}_{\alpha 2} dt$, of the one-forms $\overline{\omega}^\alpha = \omega^\alpha(u(x, t))$. Lemma 1 implies that $\overline{g}_{\alpha 1} = f_{\alpha 1,u}(u(x, t))\,\overline{G}$, and $\overline{g}_{\alpha 2} = \sum_{i=0}^{m-1} f_{\alpha 2, u_{x^i}}(u(x, t)) \left(\partial^i \overline{G}/\partial x^i\right)$.

Theorem 1. *Suppose that $u_t = F(x, t, u, \ldots, u_{x^m})$ is strictly pseudo-spherical with associated one-forms $\omega^\alpha = f_{\alpha 1} dx + f_{\alpha 2} dt$, $\alpha = 1, 2, 3$, and let G be a differential function. The deformed one-forms $\overline{\omega}^\alpha + \tau\overline{\Lambda}_\alpha$ satisfy the structure equations of a pseudo-spherical surface up to terms of order τ^2 if and only if G is a generalized symmetry of $u_t = F$.*

Thus, generalized symmetries of strictly pseudo-spherical equations $u_t = F$ are identified with infinitesimal deformations of the pseudo-spherical structures determined by $u_t = F$ which preserve the Gaussian curvature to first order in the deformation parameter. The proof of Theorem 1 appears in [14, 15]. We note, finally, that nonlocal symmetries (see [4, 12] for a formal definition and applications of this important concept) can be also included in this geometrical framework [15], and that Theorem 1 can be used (see [14, 15]) to show the *existence* of (generalized, nonlocal) symmetries of evolutionary PSS equations.

4 The Camassa–Holm and Hunter–Saxton equations

Several facts about the important Camassa–Holm [5] and Hunter–Saxton equations [8] (the former derivated as a shallow water equation, the later describing weakly nonlinear unidirectional waves) are already known: for example, their inverse scattering solutions have been found (Beals, Sattinger and Szmigielski [2, 3]), their bi-Hamiltonian character has been discussed (Camassa and Holm [5], Hunter and Zheng [9]) and, it has been proven that the Korteweg-de Vries, Camassa–Holm and Hunter–Saxton equations exhaust (in a precise sense) the bi-Hamiltonian equations which can be modeled as geodesic flows on homogeneous spaces related to the Virasoro group (Khesin and Misiołek [11]). It is shown in this section that these three equations are of pseudo-spherical type, and that therefore they can be studied using the results summarized in Sections 2 and 3. We begin with the classical KdV:

Example 2. The KdV equation $u_t = u_{xxx} + 6uu_x$ describes pseudo-spherical surfaces [17, 6] with associated one-forms $\omega^\alpha = f_{\alpha 1}dx + f_{\alpha 2}dt$, in which

$$\omega^1 = (1-u)\,dx + \left(-u_{xx} + \eta u_x - \eta^2 u - 2u^2 + \eta^2 + 2u\right)\,dt, \tag{14}$$
$$\omega^2 = \eta\,dx + \left(\eta^3 + 2\eta u - 2u_x\right)\,dt, \tag{15}$$
$$\omega^3 = (-1-u)\,dx + \left(-u_{xx} + \eta u_x - \eta^2 u - 2u^2 - \eta^2 - 2u\right)\,dt, \tag{16}$$

and η is an arbitrary parameter. After rotating the coframe $\left\{\omega^1, \omega^2\right\}$ and changing Γ for $-\Gamma$, we can write the Pfaffian system (10) as

$$\text{(a) } \Gamma_x = -u - \eta\Gamma - \Gamma^2, \qquad \text{(b) } \Gamma_t = \left(\Gamma_{xx} - 3\Gamma^2\eta - 2\Gamma^3\right)_x.$$

It follows from the fact that KdV is strictly pseudo-spherical that if Γ solves (b), u as given by (a) solves KdV. We thus recover the Miura transformation and the modified KdV equation. Now take a solution $u(x,t)$ of KdV and compute $\Gamma(x,t,\eta)$ from (a). Equation (b) is invariant under the transformation $(\Gamma \mapsto -\Gamma, \eta \mapsto -\eta)$, and therefore (a) implies that $\overline{u}(x,t,\eta) = \Gamma_x(x,t,\eta) - \Gamma(x,t,\eta)\eta - \Gamma(x,t,\eta)^2$ is a one-parameter family of solutions of KdV. It follows that $\overline{u}(x,t,\eta) = u(x,t) + 2\Gamma_x(x,t,\eta)$, and therefore we also recover the classical Darboux transformation!

We now consider the Camassa–Holm (CH)

$$m = u_{xx} - u, \qquad m_t = -m_x\, u - 2\, m\, u_x, \tag{17}$$

and Hunter–Saxton (HS) equations

$$m = u_{xx}, \qquad m_t = -m_x\, u - 2\, m\, u_x. \tag{18}$$

Below and henceforth, we let ϵ be equal to 1 for CH and 0 for HS.

Theorem 2. *The Camassa–Holm and Hunter–Saxton equations, (17) and (18) respectively, describe pseudo-spherical surfaces.*

Proof. We consider one-forms σ^α, $\alpha = 1,2,3$, given by

$$\sigma^1 = \left(m - \beta + \epsilon\,\eta^{-2}(\beta - 1)\right)\,dx$$
$$+ \left(-u_x\beta\,\eta^{-1} - \beta\,\eta^{-2} - u\,m - 1 + u\beta + u_x\,\eta^{-1} + \eta^{-2}\right)\,dt, \tag{19}$$
$$\sigma^2 = \eta\,dx + \left(-\beta\,\eta^{-1} - \eta\,u + \eta^{-1} + u_x\right)\,dt, \tag{20}$$
$$\sigma^3 = (m+1)\;dx + \left(\epsilon\,u\,\eta^{-2}(\beta - 1) - u\,m + \eta^{-2} + \frac{u_x}{\eta} - u - \frac{\beta}{\eta^2} - \frac{u_x\beta}{\eta}\right)\,dt, \tag{21}$$

in which the parameters η and β are constrained by the relation

$$\eta^2 + \beta^2 - 1 = \epsilon \left[\frac{\beta - 1}{\eta} \right]^2 . \tag{22}$$

It is not hard to check that the structure equations (2) are satisfied whenever $u(x,t)$ is a solution of (17) (if $m = u_{xx} - u$) and whenever $u(x,t)$ is a solution of (18) (if $\epsilon = 1$ and $m = u_{xx} - u$) and whenever $u(x,t)$ is a solution of (18) (if $\epsilon = 0$ and $m = u_{xx}$). ■

The fact that the CH equation is of pseudo-spherical type first appeared in [16]. A natural way to dispense with the constraint (22) is by using a parameterization of the curve $\eta^2+\beta^2-1 = \epsilon\,[(\beta-1)/\eta]^2$. We take

$$\eta = \sqrt{\epsilon + 1 - s^2}, \qquad \beta = \frac{\epsilon}{s-1} - s. \tag{23}$$

It follows that the CH and HS equations are geometrically integrable, and it is not difficult to write down $sl(2,\mathbb{R})$-valued linear problems associated to them, simply by applying Proposition 2.

We turn to the quadratic pseudo-potential (12) associated with the CH and HS equations. After parameterizing the one-forms σ^α using (23), rotating the resulting forms via (7), applying the transformation $\hat{\Gamma} \mapsto \gamma + \sqrt{\epsilon+1-s^2}/(1-s)$, and setting $s-1 = 1/\lambda$, we obtain the following result:

Theorem 3. *The CH equation (17) and the HS equation (18) admit quadratic pseudo-potentials γ determined by*

$$m = \gamma_x + \frac{1}{2\lambda}\gamma^2 - \frac{1}{2}\lambda\,\epsilon, \qquad \gamma_t = \frac{\gamma^2}{2}\left[1 + \frac{1}{\lambda}u\right] - u_x\gamma - u\,m + \epsilon\left[\frac{1}{2}u\lambda - \frac{1}{2}\lambda^2\right], \tag{24}$$

in which $\lambda \neq 0$ is a parameter. Moreover, equations (17) and (18) possess the parameter-dependent conservation law

$$\gamma_t = \lambda\left(u_x - \gamma - \frac{1}{\lambda}u\gamma\right)_x . \tag{25}$$

In view of Example 2, it is natural to postulate the first equation of (24) as the analog of the Miura transformation for the CH and HS equations, and (25) as the corresponding "modified" equation. Note that, in contradistinction with the KdV case, the modified CH and HS equations are nonlocal equations for γ. We also remark that equations (24) determine very simple linear problems for the CH and HS equations: setting $\gamma = \psi_1/\psi_2$ and replacing into (24), we find that the compatibility condition of the linear problem $d\psi = (Xdx + Tdt)\psi$, in which $\psi = (\psi_1, \psi_2)^t$, and

$$X = \frac{1}{2}\begin{bmatrix} 0 & \epsilon\,\lambda + 2\,m \\ \lambda^{-1} & 0 \end{bmatrix}, \qquad T = \frac{1}{2}\begin{bmatrix} -u_x & -2u\,m + \epsilon\,\lambda u - \epsilon\,\lambda^2 \\ -1 - u\lambda^{-1} & u_x \end{bmatrix}, \tag{26}$$

is precisely the CH equation (17) (if $m = u_{xx} - u$) and the HS equation (18) (if $m = u_{xx}$). It is not hard to check that this linear problem is related to the one obtained from (3) and (19)–(21) by a $gl(2,\mathbb{R})$-valued gauge transformation.

We now use (24) and (25) to construct conservation laws for the CH and HS equations. Setting $\gamma = \sum_{n=1}^{\infty} \gamma_n \lambda^{n/2}$ yields the conserved densities

$$\gamma_1 = \sqrt{2}\,\sqrt{m}, \qquad \gamma_2 = -\frac{1}{2}\,\ln(m)_x, \qquad \gamma_3 = \frac{1}{2\sqrt{2}\,\sqrt{m}}\left[\epsilon - \frac{m_x^2}{4\,m^2} + \ln(m)_{xx}\right], \tag{27}$$

$$\gamma_{n+1} = -\frac{1}{\gamma_1}\,\gamma_{n,x} - \frac{1}{2\gamma_1}\sum_{j=2}^{n}\gamma_j\,\gamma_{n+2-j}, \qquad n \geq 3, \tag{28}$$

while the expansion $\gamma = \epsilon\,\lambda + \sum_{n=0}^{\infty} \gamma_n \lambda^{-n}$ implies

$$\gamma_{0,x} + \epsilon\,\gamma_0 = m, \qquad \gamma_{n,x} + \epsilon\,\gamma_n = -(1/2)\sum_{j=0}^{n-1} \gamma_j\,\gamma_{n-1-j}, \quad n \geq 1. \tag{29}$$

It is not hard to see [16] that, in the CH case, the local conserved densities γ_n determined by (27) and (28) correspond to the ones found by Fisher and Schiff [7] using an "associated Camassa–Holm equation", while (29) yields the local conserved densities u, $u_x^2 + u^2$, and $uu_x^2 + u^3$, and a sequence of nonlocal conservation laws.

We finish with a theorem on nonlocal symmetries for the Camassa–Holm and Hunter–Saxton equations:

Theorem 4. *Let γ, δ and β be defined by the equations*

$$\gamma_x = m - (1/2\lambda)\,\gamma^2 + \epsilon\,(1/2)\,\lambda, \qquad \gamma_t = \lambda\,(u_x - \gamma - (1/\lambda)u\gamma)_x; \tag{30}$$

$$\delta_x = \gamma, \qquad \delta_t = \lambda\,(u_x - \gamma - (1/\lambda)u\gamma); \tag{31}$$

$$\beta_x = m\,\mathrm{e}^{(1/\lambda)\,\delta}, \qquad \beta_t = \mathrm{e}^{(1/\lambda)\,\delta}\left(-(1/2)\,\gamma^2 + \epsilon\,(1/2)\,\lambda^2 - u\,m\right); \tag{32}$$

which are compatible on solutions of (17) and (18). The systems of equations (17), (30)–(32) and (18), (30)–(32), possess the classical symmetry

$$W = \gamma\,\mathrm{e}^{(1/\lambda)\,\delta}\frac{\partial}{\partial u} + \left(m_x + \frac{2}{\lambda}\gamma\,m\right)\mathrm{e}^{(1/\lambda)\,\delta}\frac{\partial}{\partial m} + m\,\mathrm{e}^{(1/\lambda)\,\delta}\frac{\partial}{\partial \gamma}$$
$$+\,\beta\frac{\partial}{\partial \delta} + \left(m\,\mathrm{e}^{(2/\lambda)\,\delta} + \frac{1}{2\lambda}\,\beta^2\right)\frac{\partial}{\partial \beta}. \tag{33}$$

Thus, in particular, the evolutionary vector field

$$V = \gamma\,\mathrm{e}^{(1/\lambda)\,\delta}\frac{\partial}{\partial u} + \left(m_x + \frac{2}{\lambda}\gamma\,m\right)\mathrm{e}^{(1/\lambda)\,\delta}\frac{\partial}{\partial m}$$

is a one-parameter family of nonlocal symmetries for (17) and (18).

Theorem 4 can be verified using the MAPLE package VESSIOT developed by I. Anderson and his coworkers, see [1]. We remark that it is certainly possible (see [16] for the CH case) to find the flow of W, and therefore Theorem 4 gives us a method to construct solutions to the CH and HS equations!

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The Complete Set of Generalized Symmetries for the Calogero–Degasperis–Ibragimov–Shabat Equation

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We find the complete set of local generalized symmetries (including x,t-dependent ones) for the Calogero–Degasperis–Ibragimov–Shabat (CDIS) equation, and investigate the properties of these symmetries.

1 Introduction

All known today integrable scalar (1+1)-dimensional evolution equations with time-independent coefficients possess infinite-dimensional Abelian algebras of time-independent higher order symmetries (see e.g. [1, 2]). However, the equations of this kind usually do not have *local* time-dependent higher order symmetries. The only known exceptions from this rule seem to occur [3] for linearizable equations like e.g. the Burgers equation, for which the complete set of symmetries was found in [4]. In the present paper we confirm this for a third order linearizable equation (4), which is referred below as Calogero–Degasperis–Ibragimov–Shabat equation, and exhibit the complete set of its time-dependent local generalized symmetries. This equation was discovered by Calogero and Degasperis [5] and studied, among others, by Ibragimov and Shabat [6], Svinolupov and Sokolov [7], Sokolov and Shabat [8], Calogero [9], and by Sanders and Wang [10].

The paper is organized as follows. In Section 2 we recall some well known definitions and results on the symmetries of evolution equations. In Section 3 we present the main result – Theorem 1, giving the complete description of the set of all local generalized symmetries for CDIS equation.

2 Basic definitions and known results

Consider a $(1+1)$-dimensional evolution equation

$$\partial u/\partial t = F(x,u,u_1,\ldots,u_n), \qquad n\geq 2, \qquad \partial F/\partial u_n \neq 0, \tag{1}$$

for a scalar function u, where $u_l = \partial^l u/\partial x^l$, $l=0,1,2,\ldots$, $u_0\equiv u$, and its (local) *generalized symmetries* [1], i.e. the generalized vector fields $\mathcal{G} = G\,\partial/\partial u$, where $G=G(x,t,u,u_1,\ldots,u_k)$, $k\in\mathbb{N}$, is such that the evolution equation $\partial u/\partial\tau = G$ is compatible with (1). Below we shall identify the symmetry $\mathcal{G} = G\,\partial/\partial u$ with its *characteristics* G.

Recall [2, 12] that for any function $H=H(x,t,u,u_1,\ldots,u_q)$ the greatest m such that $\partial H/\partial u_m \neq 0$ is called its *order* and is denoted as $m=\operatorname{ord} H$. We assume that $\operatorname{ord} H = 0$ for any $H=H(x,t)$. A function f of $x,t,u,u_1,\ldots$ is called *local* (cf. [11, 15]), if it has a finite order.

Denote by $S_F^{(k)}$ the space of local generalized symmetries of (1) that are of order not higher than k. Let also

$$S_F = \bigcup_{j=0}^{\infty} S_F^{(j)},\ \Theta_F = \{H(x,t) \mid H(x,t) \in S_F\},\ \mathrm{St}_F = \{G \in S_F \mid \partial G/\partial t = 0\},$$
$$S_{F,k} = S_F^{(k)}/S_F^{(k-1)} \text{ for } k \in \mathbb{N};\ S_{F,0} = S_F^{(0)}/\Theta_F.$$

The set S_F is a Lie algebra with respect to the Lie bracket (see e.g. [1, 15])

$$[H, R] = R_*(H) - H_*(R) = \nabla_H(R) - \nabla_R(H).$$

Here for any local Q we set

$$Q_* = \sum_{i=0}^{\operatorname{ord} Q} \frac{\partial Q}{\partial u_i} D^i, \qquad \nabla_Q = \sum_{i=0}^{\infty} D^i(Q)\, \frac{\partial}{\partial u_i},$$

and $D = \partial/\partial x + \sum\limits_{i=0}^{\infty} u_{i+1}\partial/\partial u_i$ is the total derivative with respect to x.

Note (see e.g. [1]) that a local function G is a symmetry of (1) if and only if

$$\partial G/\partial t = -[F, G]. \tag{2}$$

Equation (2) implies [1, 11]

$$\partial G_*/\partial t \equiv (\partial G/\partial t)_* = \nabla_G(F_*) - \nabla_F(G_*) + [F_*, G_*], \tag{3}$$

where $\nabla_F(G_*) \equiv \sum\limits_{j=0}^{\operatorname{ord} G} \nabla_F\left(\frac{\partial G}{\partial u_j}\right) D^j$ and likewise for $\nabla_G(F_*)$; $[\cdot\,,\cdot]$ stands for the usual commutator of linear differential operators.

Consider also (see e.g. [2, 11, 12] for more information) the set FS of formal series in powers of D, i.e., the expressions of the form $\mathfrak{H} = \sum\limits_{j=-\infty}^{m} h_j D^j$, where h_j are local functions. The greatest k such that $h_k \neq 0$ is called the degree of $\mathfrak{H} \in FS$ and is denoted by $\deg \mathfrak{H}$. Recall that $\mathfrak{R} \in FS$ is called a *formal symmetry* of infinite rank for (1), if it satisfies the relation (see e.g. [2, 12])

$$\partial\mathfrak{R}/\partial t + \nabla_F(\mathfrak{R}) - [F_*, \mathfrak{R}] = 0.$$

3 Symmetries of the CDIS equation

The Calogero–Degasperis–Ibragimov–Shabat (CDIS) equation has the form [5, 6]

$$u_t = u_3 + 3u^2u_2 + 9uu_1^2 + 3u^4u_1. \tag{4}$$

Let us mention that this is the only third order (1+1)-dimensional scalar polynomial λ-homogeneous evolution equation of the form $u_t = u_n + f(u, u_1, \dots, u_{n-1})$ with $\lambda = 1/2$ which possesses infinitely many x, t-independent local generalized symmetries [13]. This equation is linearized into $v_t = v_3$ upon setting $v = \exp(\omega)u$, where $\omega = D^{-1}(u^2)$ [8]. It appears to possess only one local conserved density $\rho = u^2$ (see e.g. [7, 8] and references therein), but it has a Hamiltonian operator and infinitely many conserved densities explicitly dependent on the nonlocal variable ω [7].

In order to refer to the sets of symmetries of the CDIS equation, we shall use the subscript ‘CDIS’ instead of F, i.e., S_{CDIS} will denote the Lie algebra of all generalized symmetries of (4), etc. From now on F will stand for the right-hand side of the CDIS equation, that is, $u_3 + 3u^2u_2 + 9uu_1^2 + 3u^4u_1$.

Let G be a local generalized symmetry of order $k \geq 1$ for (4). Equating to zero the coefficient at D^{k+2} in (3) and solving the arising equation, we obtain (see e.g. [1]) that

$$\partial G/\partial u_k = c_k(t), \tag{5}$$

where $c_k(t)$ is a function of t.

Below we assume without loss of generality that any symmetry $G \in S_{\mathrm{CDIS},k}$, $k \geq 1$, vanishes if the relevant function $c_k(t)$ is identically equal to zero.

Equating to zero the coefficients at D^{k+1} and D^k in (3), we see that for $k \geq 3$ we have $\partial^2 G/\partial x \partial u_{k-1} = 0$ and

$$\partial^2 G/\partial x \partial u_{k-2} = \dot{c}_k(t)/3. \tag{6}$$

Repeatedly using (6) and taking into account that $G \in S_{\mathrm{CDIS}}$ implies $\tilde{G} = \partial^r G/\partial x^r \in S_{\mathrm{CDIS}}$, we find that $\operatorname{ord} \tilde{G} \leq k - 2r$ and

$$\partial \tilde{G}/\partial u_{k-2r} = (1/3)^r d^r c_k(t)/dt^r. \tag{7}$$

For $r = [k/2]$ we have $\operatorname{ord} \tilde{G} \leq 1$. As u_1 is the only generalized symmetry of CDIS equation from $S^{(1)}_{\mathrm{CDIS}}$, and u_1 is time-independent, we see that $c_k(t)$ satisfies the equation $d^m c_k(t)/dt^m = 0$ for $m = [k/2] + 1$. Therefore, $\dim S_{\mathrm{CDIS},k} \leq [k/2] + 1$ for $k \geq 1$.

As all symmetries from $S^{(2)}_{\mathrm{CDIS}}$ are exhausted by u_1, by Theorem 2 of [17] all generalized symmetries of the CDIS equation are polynomial in time t.

Now let us turn to the study of time-independent symmetries of CDIS equation. This equation has infinitely many x,t-independent generalized symmetries, hence [18] a formal symmetry of infinite rank of the form $\mathfrak{L} = D + \sum\limits_{j=0}^{\infty} a_j D^{-j}$, where a_j are some x,t-independent local functions.

Since we have $\deg \nabla_G(F_*) \leq 2$ for any G, by (3) and Lemma 9 from [15] for any $G \in \mathrm{St}_{\mathrm{CDIS}}$, $k = \operatorname{ord} G \geq 2$, we can represent G_* in the form $G_* = \sum\limits_{j=1}^{k} \alpha_j \mathfrak{L}^j + \mathfrak{B}$, where α_j are some constants and $\mathfrak{B}$ is some formal series with time-independent coefficients, $\deg \mathfrak{B} < 1$. We have $\partial \mathfrak{L}/\partial x = 0$, so $\partial G_*/\partial x = \partial \mathfrak{B}/\partial x$ and $\deg \partial G_*/\partial x < 1$.

Thus, any symmetry $G \in \mathrm{St}_{\mathrm{CDIS}}$, $k \equiv \operatorname{ord} G \geq 2$, can be represented in the form

$$G = G_0(u, \ldots, u_k) + Y(x, u). \tag{8}$$

It is obvious that $\partial Y/\partial x = \partial G/\partial x \in \mathrm{St}_{\mathrm{CDIS}}$ and $\operatorname{ord} \partial Y/\partial x = 0$. But the CDIS equation has no generalized symmetries of order zero, so $\partial Y/\partial x = 0$, and thus any time-independent symmetry G of order $k \geq 2$ for CDIS equation is x-independent as well. The straightforward computation shows that the same statement holds true for the symmetries of order lower than 2. Using the symbolic method, it is possible to show [13] that CDIS equation has no even order t,x-independent symmetries. Hence, it has no even order time-independent generalized symmetries at all.

Now let us show that the same is true for time-dependent generalized symmetries as well. Recall that the CDIS equation is invariant under the scaling symmetry $K = 3tF + xu_1 + u/2$. Hence, if a symmetry Q contains the terms of weight γ (with respect to the weighting induced by K, cf. [13, 14]), there exists a homogeneous symmetry $\tilde{Q}$ of the same weight γ. We shall write this as $\mathrm{wt}(\tilde{Q}) = \gamma$. Note that we have $[K, \tilde{Q}] = (\gamma - 1/2)\tilde{Q}$.

If $G \in S_{\mathrm{CDIS},k}$, $k \geq 1$, is a polynomial in t of degree m, then its leading coefficient $\partial G/\partial u_k = c_k(t)$ also is a polynomial in t of degree $m' \leq m$, i.e., $c_k(t) = \sum\limits_{j=0}^{m'} t^j c_{k,j}$, where $c_{k,m'} \neq 0$.

Consider $\tilde{G} = \partial^{m'} G/\partial t^{m'} \in S^{(k)}_{\mathrm{CDIS}}$. We have $\partial \tilde{G}/\partial u_k = \mathrm{const} \neq 0$, hence $\tilde{G}$ contains the terms of the weight $k + 1/2$. Let P be the sum of all terms of weight $k + 1/2$ in $\tilde{G}$. Clearly, P is a homogeneous symmetry of weight $k+1/2$ by construction, $\operatorname{ord} P = k$ and $\partial P/\partial u_k$ is a nonzero constant. Next, $\partial P/\partial t = -[F, P] \in S_{\mathrm{CDIS}}$, and the symmetry $\partial P/\partial t$ is homogeneous of weight $k+7/2$. Obviously, $\operatorname{ord} \partial P/\partial t \le k-1$. By the above, all symmetries in S_{CDIS} are polynomial in t, and thus for any homogeneous $B \in S_{\mathrm{CDIS}}$, $b \equiv \operatorname{ord} B \ge 1$, we have $\partial B/\partial u_b = t^r c_b$, $c_b = \mathrm{const}$ for some $r \ge 0$. Hence, $\mathrm{wt}(B) = b - 3r + 1/2 \le b + 1/2$, and thus for $k \ge 1$ the set S_{CDIS} does not contain homogeneous symmetries B such that $\mathrm{wt}(B) = k + 7/2$ and $\operatorname{ord} B \le k - 1$, so $\partial P/\partial t = 0$.

Taking into account the absence of generalized symmeries of order zero for CDIS equation, we conclude that existence of a time-independent generalized symmetry of order $k \ge 1$ is a necessary condition for the existence of a polynomial-in-time symmetry $G \in S_{\mathrm{CDIS}}$ of the same order. Moreover, by the above all symmetries from S_{CDIS} are polynomial in t. Hence, the absence of time-independent local generalized symmetries of even order for the CDIS equation immediately implies the absence of any *time-dependent* local generalized symmetries of even order.

Thus, we have shown that the CDIS equation has no (local) generalized symmetries of even order and that for any $k \ge 1$ $\dim S_{\mathrm{CDIS},k} \le [k/2] + 1$. Therefore, if for all odd $k = 2l + 1$ we exhibit $l+1$ symmetries of order k, then these symmetries will span the whole Lie algebra S_{CDIS} of (local) generalized symmetries for the CDIS equation.

The symmetries in question can be constructed in the following way.

Let $\tau_{m,0} = x^m u_1 + m x^{m-1} u/2$, $m = 0, 1, 2 \ldots$, and $\tau_{1,1} = x(u_3 + 3u^2 u_2 + 9 u u_1^2 + 3u^4 u_1) + 3u_2/2 + 5u_1 u^2 + u^5/2$. Note that $\tau_{1,1}$ is the first nontrivial master symmetry for the CDIS equation [10, 13]. It is easy to check that in accordance with Theorem 3.18 from [16] we have

$$[\tau_{m,j}, \tau_{m',j'}] = ((2j'+1)m - (2j+1)m')\tau_{m+m'-1,j+j'}, \tag{9}$$

where $\tau_{m,j}$ with $j > 0$ are defined inductively by means of (9), i.e. [16] $\tau_{0,j+1} = \frac{1}{2j+1}[\tau_{1,1}, \tau_{0,j}]$, $\tau_{m+1,j} = \frac{1}{2+4j-m}[\tau_{2,0}, \tau_{m,j}]$.

Thus, the CDIS equation, as well as the Burgers equation, represents a nontrivial example of a (1+1)-dimensional evolution equation possessing a hereditary algebra (9).

Using (9), it can be shown (cf. [16]) that $\mathrm{ad}^{m+1}_{\tau_{0,j}}(\tau_{m,j'}) = 0$, i.e. $\tau_{m,j'}$ are master symmetries of degree m for all equations $u_{t_j} = \tau_{0,j}$, $j = 0, 1, 2, \ldots$. Here $\mathrm{ad}_B(G) \equiv [B, G]$ for any (smooth) local functions B and G.

Let $\exp(\mathrm{ad}_B) \equiv \sum_{j=0}^{\infty} \mathrm{ad}^j_B/j!$. As $\mathrm{ad}^{m+1}_{\tau_{0,j}}(\tau_{m,j'}) = 0$, it is easy to see (cf. [16]) that

$$G^{(k)}_{m,j}(t_k) = \exp(-t_k\,\mathrm{ad}_{\tau_{0,k}})\tau_{m,j} = \sum_{i=0}^{m} \frac{(-t_k)^i}{i!}\,\mathrm{ad}^i_{\tau_{0,k}}(\tau_{m,j}) = \sum_{i=0}^{m} \frac{((2k+1)t_k)^i m!}{i!(m-i)!}\tau_{m-i,j+ik}$$

are time-dependent symmetries for the equation $u_{t_k} = \tau_{0,k}$ and $\operatorname{ord} G^{(k)}_{m,j} = 2(j + mk) + 1$. Note that $G^{(k)}_{m,j}$ obey the same commutation relations as $\tau_{m,j}$, that is

$$[G^{(k)}_{m,j}, G^{(k)}_{m',j'}] = ((2j'+1)m - (2j+1)m')G^{(k)}_{m+m'-1,j+j'}. \tag{10}$$

It is straightforward to verify that $\tau_{0,1} = F = u_3 + 3u^2 u_2 + 9 u u_1^2 + 3u^4 u_1$ and thus $G_{m,j} \equiv G^{(1)}_{m,j}(t) = \exp(-t\,\mathrm{ad}_F)\tau_{m,j}$ are time-dependent symmetries for the CDIS equation.

It is easy to see that the number of symmetries $G_{m,j}$ of given odd order $k = 2l + 1$ equals $[k/2]+1 = l+1$. As $\dim S_{\mathrm{CDIS},k} \le [k/2]+1$, these symmetries exhaust the space $S_{\mathrm{CDIS},k}$. Thus, we have proved the following theorem.

Theorem 1. *Any local generalized symmetry of the CDIS equation is a linear combination of the symmetries $G_{m,j}$ for $m = 0, 1, \cdots$ and $j = 0, 1, 2 \ldots$.*

Note that the technique of [10], based on the representation theory for the algebra $sl(2)$ generated by $\tau_{0,0} = u_1$, $2\tau_{1,0} = 2xu_1 + u$ and $\tau_{2,0} = x^2 u_1 + xu$, enables one to obtain only a part of the symmetries, described in the above theorem. The reason for this is that $\langle \tau_{0,0}, \tau_{1,0}, \tau_{2,0} \rangle$ is a subalgebra of the algebra generated by $\tau_{m,0}, m = 0, 1, \ldots$. This is exactly the same phenomenon as in the case of Lie algebra of vector fields of the form $x^{m+1}\frac{d}{dx}$.

As a final remark, let us mention that, in complete analogy with the above, we can readily obtain the complete description of the set of local generalized symmetries for any of the equations $u_{t_k} = \tau_{0,k}$, $k = 2, 3, \ldots$. In this way we arrive at the following generalization of Theorem 1.

Theorem 2. *Any local generalized symmetry of the equation $u_{t_k} = \tau_{0,k}$, $k \in \mathbb{N}$, is a linear combination of the symmetries $G^{(k)}_{m,j}(t_k)$ for $m = 0, 1, \ldots$ and $j = 0, 1, 2 \ldots$.*

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Method of Group Foliation and Non-Invariant Solutions of Invariant Equations

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Using the heavenly equation as an example, we propose the method of group foliation as a tool for obtaining non-invariant solutions of PDEs with infinite-dimensional symmetry groups. The method involves the study of compatibility of the given equations with a differential constraint, which is automorphic under a specific symmetry subgroup and therefore selects exactly one orbit of solutions. By studying the integrability conditions of this automorphic system, *i.e.* the resolving equations, one can provide an explicit foliation of the entire solution manifold into separate orbits. The new important feature of the method is extensive use of the operators of invariant differentiation for the derivation of the resolving equations and for obtaining their particular solutions. Applying this method we obtain exact analytical solutions of the heavenly equation, non-invariant under any subgroup of the symmetry group of the equation.

1 Introduction

The general standard method for obtaining exact solutions of partial differential equations (PDEs) by symmetry analysis is symmetry reduction which gives only *invariant solutions*, *i.e.* solutions which are invariant with respect to some subgroup of the symmetry group of the PDE.

To be explicit, we consider as an example the heavenly equation

$$u_{z\bar{z}} = \kappa(e^u)_{tt} \quad \Longleftrightarrow \quad u_{xx} + u_{yy} = \kappa(e^u)_{tt}, \qquad \kappa = \pm 1, \tag{1}$$

where $u = u(t, z, \bar{z})$. This equation is a continuous version of the Toda lattice or $SU(\infty)$ Toda field [1, 2]. It appears in the theory of gravitational instantons [3] where it describes self-dual Einstein spaces with Euclidean signature having one rotational Killing vector.

For the point symmetry transformations the symmetry algebra of the equation (1) is realized by vector fields of the form

$$X = \tau\partial_t + \xi\partial_z + \bar{\xi}\partial_{\bar{z}} + \phi\partial_u, \tag{2}$$

where τ, ξ, $\bar{\xi}$ and ϕ are functions of t, z, $\bar{z}$ and u. The condition, which selects invariant solutions with respect to the generator X, is the first order linear equation

$$\tau u_t + \xi u_z + \bar{\xi} u_{\bar{z}} - \phi = 0 \tag{3}$$

which one adds to the studied equation. Solution of the equation (3) depends only on invariants of the corresponding symmetry subgroup, *i.e.* only on 2 variables instead of 3 original variables. Therefore, when this solution is substituted into the equation (1) we obtain the *symmetry reduction*, the *reduced equation* depending only on 2 independent variables since it determines only invariant solutions of the original equation.

We are proposing the method of group foliation as a tool for obtaining non-invariant solutions of non-linear PDEs with infinite dimensional symmetry groups. The idea of the method, belonging to Lie and Vessiot [4, 5], is more than a hundred years old being resurrected in a more modern form by Ovsiannikov 30 years ago (see [6] and references therein).

We have added to this method three important new ideas [7, 8]: the use of *invariant cross-differentiation*, involving the operators of invariant differentiation and their commutator algebra, for the derivation of the resolving equations and for obtaining their particular solutions; the *commutator representation of the resolving system* in terms of the operators of invariant differentiation; the concept of *invariant integration* applied for solving the automorphic system.

In this paper, on the example of the heavenly equation we clarify the main concepts of the method including these three ideas and consider in detail 10 main steps which should be performed for obtaining non-invariant solutions.

2 Symmetry algebra

We start with finding the *symmetry algebra* of generators of the point transformations for the heavenly equation (1) [9]

$$\begin{aligned}
&T = \partial_t, \qquad G = t\partial_t + 2\partial_u,\\
&X_a = a(z)\partial_z + \bar a(\bar z)\partial_{\bar z} - (a'(z) + \bar a'(\bar z))\partial_u,
\end{aligned} \tag{4}$$

where T is the generator of translations in t, G is the generator of a dilation of time accompanied by a shift of u: $t = \tilde t e^{\tau}$, $u = \tilde u + 2\tau$ (τ is a group parameter) and X_a is a generator of the *conformal transformations*

$$z = \phi(\tilde z), \qquad \bar z = \bar\phi(\tilde{\bar z}), \qquad u(z,\bar z,t) = \tilde u(\tilde z,\tilde{\bar z},t) - \ln\bigl(\phi'(\tilde z)\bar\phi'(\tilde{\bar z})\bigr), \tag{5}$$

where $a(z)$ and $\phi(z)$ are arbitrary holomorphic functions of z and prime denotes derivative with respect to argument (see also [10]).

The Lie algebra of symmetry generators (4) is determined by the commutation relations

$$[T, G] = T, \qquad [T, X_a] = 0, \qquad [G, X_a] = 0, \qquad [X_a, X_b] = X_{ab'-ba'}. \tag{6}$$

They show that the generators X_a of conformal transformations form a subalgebra of Lie algebra (6). This subalgebra is infinite dimensional since the generators X_a depend on arbitrary holomorphic function $a(z)$. The corresponding finite transformations (5) form an infinite dimensional symmetry subgroup of the equation (1) since instead of a group parameter they involve an arbitrary holomorphic function $\phi(\tilde z)$.

We choose this infinite dimensional *conformal group* for the group foliation.

3 Differential invariants

Next we find differential invariants of the symmetry subgroup (5) of conformal transformations. *Differential invariants* are the invariants of all the generators X_a in the *prolongation spaces*. This means that they can depend on independent variables, the unknowns and also on the partial derivatives of the unknowns allowed by the order of the prolongation. The *order* of the differential invariant is defined as the order of the highest derivative which this invariant depends on. The number N for the highest order invariant must be larger or equal to the order of the equation ($N \geq 2$) and must satisfy the requirement that there should be n functionally independent invariants with $n > p + q$ where p and q are the number of independent and dependent variables, respectively. In our case we have $p = 3$, $q = 1$ and $n > 4$, $N \geq 2$. We try first $N = 2$ to see if it is sufficient for group foliation.

The determining equation for differential invariants Φ of the order ≤ 2 has the form $\overset{2}{X}_a(\Phi) = 0$ where $\overset{2}{X}_a$ is the second prolongation of the generator X_a of the conformal group defined by standard prolongation formulas. The integration of this equation gives 5 functionally independent differential invariants up to the second order inclusively

$$t, \qquad u_t, \qquad u_{tt}, \qquad \rho = e^{-u} u_{z\bar{z}}, \qquad \eta = e^{-u} u_{zt} u_{\bar{z}t} \tag{7}$$

and all of them are real. This allows us to express the heavenly equation (1) solely in terms of the differential invariants

$$u_{tt} = \kappa\rho - u_t^2. \tag{8}$$

Thus, in our case we have $N = 2$ and $n = 5$. This is enough for the group foliation, and we do not need the set of all 3rd-order invariants.

4 Automorphic system

Next we choose the general form of the automorphic system. We choose $p = 3$ invariants t, u_t, ρ as new *invariant independent variables*, the same number as in the original equation (1), and require that the *remaining invariants be functions of the chosen ones*. This provides us with the general form of the *automorphic system* that also contains the studied equation (8) expressed in terms of invariants (7)

$$\begin{aligned} &u_{tt} = \kappa\rho - u_t^2, \\ &\eta = F(t, u_t, \rho). \end{aligned} \tag{9}$$

The real function F in the right-hand side should be determined from the *resolving equations* which are compatibility conditions of the system (9). Then the system (9) will have the *automorphic property*, *i.e.* any of its solutions can be obtained from any other solution by an appropriate transformation of the conformal group.

5 Operators of invariant differentiation

Our next task is to find *operators of invariant differentiation*. They are linear combinations of the operators of total derivatives D_t, D_z, $D_{\bar{z}}$ with respect to independent variables t, z, $\bar{z}$

$$\delta = \lambda_1 D_t + \lambda_2 D_z + \lambda_3 D_{\bar{z}} = \sum_{i=1}^{3} \lambda_i D_i$$

with the coefficients λ_i which depend on local coordinates of the prolongation space. They are defined by the special property that, acting on any (differential) invariant, they map it again into a differential invariant. Being first order differential operators, they raise the order of a differential invariant by a unit. As a consequence, these differential operators commute with any infinitely prolonged generator $\overset{\infty}{X}_a$ of the conformal symmetry group. This implies the *determining equation* for the coefficients λ_i [6]

$$\overset{\infty}{X}_a(\lambda_i) = \sum_{j=1}^{3} \lambda_j D_j \left[\xi^i\right], \tag{10}$$

where according to the formula (2) we have $\xi^1 = \tau = 0$, $\xi^2 = \xi = a(z)$, $\xi^3 = \bar{\xi} = \bar{a}(\bar{z})$. It is obvious that the total number of independent operators of invariant differentiation is equal to the number of independent variables, *i.e.* 3 in our case.

Solving the equation (10) we obtain a *basis for the operators of invariant differentiation*

$$\delta = D_t, \qquad \Delta = e^{-u} u_{\bar{z}t} D_z, \qquad \bar{\Delta} = e^{-u} u_{zt} D_{\bar{z}}. \tag{11}$$

6 Basis of differential invariants

The next step is to find the basis of differential invariants. The *basis of differential invariants* is defined as a minimal finite set of (differential) invariants of a symmetry group from which any other differential invariant of this group can be obtained by a finite number of invariant differentiations and operations of taking composite functions. The proof of the existence and finiteness of the basis was given by Tresse [11] and in a more modern form by Ovsiannikov [6].

In our example the basis of differential invariants is formed by the set of three invariants t, u_t, ρ, while two other invariants u_{tt} and η of equation (7) are given by the relations

$$u_{tt} = \delta(u_t), \qquad \eta \equiv e^{-u} u_{zt} u_{\bar{z}t} = \Delta(u_t) = \bar{\Delta}(u_t). \tag{12}$$

All other functionally independent higher order invariants can be obtained by acting with operators of invariant differentiation on the *basis* $\{t, u_t, \rho\}$. In particular, the following third order invariants generated from the 2nd-order invariant ρ by invariant differentiations will be involved in our construction

$$\sigma = \Delta(\rho), \qquad \bar{\sigma} = \bar{\Delta}(\rho), \qquad \tau = \delta(\rho) \equiv \rho_t. \tag{13}$$

7 Commutator algebra of operators of invariant differentiation

The operators δ, Δ and $\bar{\Delta}$ defined by the formulas (11) form the *commutator algebra* which is a Lie algebra over the field of invariants of the conformal group [6].

This algebra is simplified by introducing two new operators of invariant differentiation Y and $\bar{Y}$ instead of Δ and $\bar{\Delta}$ and two new variables λ and $\bar{\lambda}$ instead of σ and $\bar{\sigma}$, defined by

$$\Delta = \eta Y, \qquad \bar{\Delta} = \eta \bar{Y}, \qquad \sigma = \eta\lambda, \qquad \bar{\sigma} = \eta\bar{\lambda}. \tag{14}$$

The resulting algebra becomes

$$\begin{aligned}
&\big[\delta, Y\big] = \left(\kappa\bar{\lambda} - 3u_t - \frac{\delta(\eta)}{\eta}\right) Y, \qquad \big[\delta, \bar{Y}\big] = \left(\kappa\lambda - 3u_t - \frac{\delta(\eta)}{\eta}\right) \bar{Y},\\
&\big[Y, \bar{Y}\big] = \frac{(\tau + u_t\rho)}{\eta}\left(Y - \bar{Y}\right).
\end{aligned} \tag{15}$$

With the use of operators δ, Y and $\bar{Y}$ the general form (9) of the automorphic system becomes

$$\begin{aligned}
&\delta(u_t) = \kappa\rho - u_t^2,\\
&Y(u_t) = 1 \qquad (\bar{Y}(u_t) = 1),
\end{aligned} \tag{16}$$

where the first equation is the heavenly equation and the second equation follows from the second relation (12). Here we put $\eta = F$ in the equations (14) and in the commutation relations (15) according to the 2nd equation in (9). Then we obtain $Y = (1/F)\Delta$ and $\bar{Y} = (1/F)\bar{\Delta}$. From the equation (13) we have

$$Y(\rho) = \lambda, \qquad \bar{Y}(\rho) = \bar{\lambda}. \tag{17}$$

8 Derivation of resolving equations

The following step is to derive the *resolving equations*. This is a set of compatibility conditions between the studied equation and those that we have added to obtain the automorphic system. In our case we require compatibility between the two equations (16) which gives restrictions on the function $F(t, u_t, \rho)$ in the right-hand side of the second equation in (9). A new feature in our modification of the method is that we do this in an explicitly invariant manner by using the *invariant cross-differentiation* [7, 8], *i.e.* cross-differentiation with operators of invariant differentiation δ, Y and $\bar{Y}$.

We start with the integrability condition for the system (16) which we obtain by the invariant cross-differentiation with δ and Y using their commutation relation from equation (15)

$$\delta(F) = \big[\kappa(\lambda + \bar{\lambda}) - 5u_t\big]F. \tag{18}$$

The definitions of λ, $\bar{\lambda}$ which appear here are given by two equations (17). The compatibility condition for the equations (17) is obtained by the invariant cross-differentiation with $\bar{Y}$ and Y using their commutation relation from equation (15)

$$F\big(Y(\bar{\lambda}) - \bar{Y}(\lambda)\big) = (\tau + u_t\rho)(\lambda - \bar{\lambda}). \tag{19}$$

The definition of τ which appears here is given in the equation (13)

$$\delta(\rho) = \tau. \tag{20}$$

Using invariant cross-differentiations with δ and Y or $\bar{Y}$, we obtain two compatibility conditions of equation (20) with each of the two equations (17)

$$\delta(\lambda) = Y(\tau) + 2u_t\lambda - \kappa\lambda^2, \tag{21}$$
$$\delta(\bar{\lambda}) = \bar{Y}(\tau) + 2u_t\bar{\lambda} - \kappa\bar{\lambda}^2 \tag{22}$$

which are complex conjugate to each other. One more differential consequence of the obtained resolving equations is the compatibility condition of the equation (19) algebraically solved with respect to $Y(\bar{\lambda})$ together with the equation (22). It is obtained by the invariant cross-differentiation of these equations with δ and Y. Using the other resolving equations it can be brought to the form

$$F\big(Y(\bar{\lambda}) + \bar{Y}(\lambda)\big) = -(\tau + u_t\rho)(\lambda + \bar{\lambda}) + 2\kappa\big[\delta(\tau) + 4u_t\tau + 2F + \kappa\rho^2 + 2u_t^2\rho\big], \tag{23}$$

where no new differential invariants appear.

The resolving equations (18), (19), (21), (22) and (23) form a closed *resolving system* where the 2nd-order differential invariant $\eta = F$ and the 3rd-order differential invariants λ, $\bar{\lambda}$ and τ are functions of t, u_t, ρ. They should be regarded as additional unknowns in these equations, so the resolving system consists of 5 partial differential equations with 4 unknowns F, λ, $\bar{\lambda}$ and τ and 3 independent variables t, u_t, ρ.

Next we project the operators of invariant differentiation on the solution manifold of the heavenly equation and on the space of differential invariants treated as new independent variables. We use the properties of these operators

$$\delta(t) = 1, \qquad \delta(u_t) = \kappa\rho - u_t^2, \qquad \delta(\rho) = \tau$$
$$Y(t) = \bar{Y}(t) = 0, \qquad Y(u_t) = \bar{Y}(u_t) = 1, \qquad Y(\rho) = \lambda, \qquad \bar{Y}(\rho) = \bar{\lambda}$$

following from their definitions and the heavenly equation in (16) to obtain the resulting *projected operators*

$$\delta = \partial_t + \left(\kappa\rho - u_t^2\right)\partial_{u_t} + \tau\partial_\rho, \qquad Y = \partial_{u_t} + \lambda\partial_\rho, \qquad \bar{Y} = \partial_{u_t} + \bar{\lambda}\partial_\rho. \tag{24}$$

When we use these expressions in the resolving equations (18), (19), (21), (22) and (23), we obtain an explicit form of the resolving system. This system is passive, *i.e.* it has no further algebraically independent first order integrability conditions.

The commutator relations (15) were satisfied identically by the operators of invariant differentiation. On the contrary, for the projected operators (24) these commutation relations and even the Jacobi identity

$$\big[\delta, [Y, \bar{Y}]\big] + \big[Y, [\bar{Y}, \delta]\big] + \big[\bar{Y}, [\delta, Y]\big] = 0 \tag{25}$$

are not identically satisfied, but only on account of the resolving equations. It is easy to check that even a stronger statement is valid.

Theorem 1. *The commutator algebra (15) of the operators of invariant differentiation δ, Y, $\bar{Y}$, together with the Jacobi identity (25), is equivalent to the resolving system for the heavenly equation and hence provides a commutator representation for this system.*

This theorem means that the complete set of the resolving equations is encoded in the commutator algebra of the operators of invariant differentiation and provides the easiest way to derive the resolving system [7, 8]. Later we shall see how the commutator representation of the resolving system can lead to a useful Ansatz for finding a particular solution of this system.

9 Criteria for invariant and non-invariant solutions

Since our main goal is to obtain non-invariant solutions, we derive here criteria to distinguish between invariant and non-invariant solutions.

A general form of the generator of a one-parameter symmetry subgroup of the heavenly equation is a linear combination of symmetry generators (4)

$$X = \alpha\partial_t + \beta\,(t\partial_t + 2\partial_u) + a(z)\partial_z + \bar{a}(\bar{z})\partial_{\bar{z}} - \big(a'(z) + \bar{a}'(\bar{z})\big)\,\partial_u, \tag{26}$$

where α and β are arbitrary real constants and $a(z)$ is an arbitrary holomorphic function.

The *infinitesimal criterion for the invariance of the solution* $u = f(t, z, \bar{z})$ with respect to the generator X is $X(f - u)|_{u=f} = 0$ which for X defined by equation (26) becomes

$$(\alpha + \beta t)f_t + a(z)f_z + \bar{a}(\bar{z})f_{\bar{z}} = 2\beta - a'(z) - \bar{a}'(\bar{z}). \tag{27}$$

The invariance criterion can be summed up as follows.

Proposition 1. *If there exists a holomorphic function $a(z)$ and constants α and β, not all equal to zero, such that the equation (27) is satisfied, then the solution $u = f(t, z, \bar{z})$ is invariant. Otherwise this solution is non-invariant.*

In particular, if $\alpha = \beta = 0$, then the equation (27) is a *criterion of the conformal invariance* and the general solution of equation (27) becomes

$$u = \ln f(\xi, t) - \ln a(z) - \ln \bar{a}(\bar{z}),$$

where $\xi = i\left(\int dz/a(z) - \int d\bar{z}/\bar{a}(\bar{z})\right)$. The invariant ρ defined by equation (7) becomes $\rho = \left(f f_{\xi\xi} - f_\xi^2\right)/f^3$ and the formulas (13) give

$$\bar{\sigma} = \sigma \qquad \Longleftrightarrow \qquad \bar{\lambda} = \lambda.$$

This is the *necessary condition for the conformal invariance of a solution.* The converse statement gives the *criterion of conformal non-invariance of a solution* [8].

Corollary 1. *The sufficient condition for a solution of the heavenly equation to be conformally non-invariant is that the following inequality should be satisfied*

$$\bar{\sigma} \neq \sigma \quad \Longleftrightarrow \quad \bar{\lambda} \neq \lambda. \tag{28}$$

This condition must be satisfied by particular solutions of the resolving equations to guarantee that we shall not end up with conformally invariant solutions of the heavenly equation.

10 Particular solutions of resolving system

To find particular solutions of the resolving system, we make various simplifying assumptions. The most obvious ones, like $\bar{Y} = Y$ or $F = 0$, lead to invariant solutions. These we already know, or can obtain by much simpler standard methods. For instance, $\bar{Y} = Y$ implies $\bar{\lambda} = \lambda$, so that the condition (28) of Corollary 1 is not satisfied and we have a good chance to end up with a conformally invariant solution.

The weaker assumption that leads to non-invariant solutions is that the operators Y and $\bar{Y}$ commute

$$[Y, \bar{Y}] = 0 \quad \Longleftrightarrow \quad \tau = -u_t\rho \tag{29}$$

but $\bar{Y} \neq Y$, *i.e.* $\bar{\lambda} \neq \lambda$ and also $F \neq 0$. With this Ansatz we find the *particular solution of the resolving system* [8]

$$F = \rho^3\varphi(\xi,\theta), \quad \tau = -u_t\rho, \quad \lambda = \kappa u_t + i\sqrt{2\kappa\rho - u_t^2}, \quad \bar{\lambda} = \kappa u_t - i\sqrt{2\kappa\rho - u_t^2}, \tag{30}$$

where the condition $2\kappa\rho - u_t^2 \geq 0$ is imposed, φ is an arbitrary real smooth function and

$$\xi = \frac{2\kappa\rho - u_t^2}{\rho^2}, \qquad \theta = t - \frac{\kappa}{\rho}\left(u_t + \sqrt{2\kappa\rho - u_t^2}\right).$$

11 Reconstruction of non-invariant solutions of heavenly equation

To reconstruct solutions of the heavenly equation starting from the particular solution (30) of the resolving system we use the procedure of *invariant integration* which amounts to the transformation of equations to the form of *exact invariant derivative* [8]. Then we drop the operator of invariant differentiation in such an equation adding the term that is an *arbitrary element of the kernel* of this operator. This term plays the role of the integration constant.

To be explicit, we start from our Ansatz (29) in the form $D_t(\ln\rho) = D_t(-u)$. We integrate this equation: $\ln\rho = -u + \ln\gamma_{z\bar{z}}(z,\bar{z})$ where the last term is a function to be determined. This gives $\rho = e^{-u}u_{z\bar{z}} = e^{-u}\gamma_{z\bar{z}}(z,\bar{z})$ and hence $u_{z\bar{z}} = \gamma_{z\bar{z}}(z,\bar{z})$. This implies the following form of solutions

$$u(t,z,\bar{z}) = \gamma(z,\bar{z}) + \alpha(t,z) + \bar{\alpha}(t,\bar{z}), \tag{31}$$

where γ, α and $\bar{\alpha}$ are arbitrary smooth functions of two variables. Then we substitute the expression (31) for u into the heavenly equation (1) with the result

$$e^{\alpha(z,t)+\bar{\alpha}(\bar{z},t)}\left[\alpha_{tt}(z,t) + \bar{\alpha}_{tt}(\bar{z},t) + \left(\alpha_t(z,t) + \bar{\alpha}_t(\bar{z},t)\right)^2\right] = \kappa e^{-\gamma(z,\bar{z})}\gamma_{z\bar{z}}(z,\bar{z}).$$

Next we rewrite the formulas (30) for λ and $\bar\lambda$ in the form of *exact invariant derivatives*

$$Y\left(\sqrt{2\kappa\rho - u_t^2} - i\kappa u_t\right) = 0, \qquad \bar Y\left(\sqrt{2\kappa\rho - u_t^2} + i\kappa u_t\right) = 0.$$

These equations are integrated in the form

$$\sqrt{2\kappa\rho - u_t^2} + i\kappa u_t = \psi(t,z), \qquad \sqrt{2\kappa\rho - u_t^2} - i\kappa u_t = \bar\psi(t,\bar z),$$

where ψ is an arbitrary smooth function and $\bar\psi$ is complex conjugate to ψ.

We skip further details and present only the *final result* [8, 12].

1. *Solution of the heavenly equation* $u_{z\bar z} = (e^u)_{tt}$ *with* $\kappa = 1$:

$$u(t,z,\bar z) = \ln\big(t + b(z)\big) + \ln\big(t + \bar b(\bar z)\big) + \ln c'(z) + \ln \bar c'(\bar z) - 2\ln\big(c(z) + \bar c(\bar z)\big). \tag{32}$$

2. *Solution of the heavenly equation* $u_{z\bar z} = -(e^u)_{tt}$ *with* $\kappa = -1$:

$$u(t,z,\bar z) = \ln\big(t + b(z)\big) + \ln\big(t + \bar b(\bar z)\big) + \ln c'(z) + \ln \bar c'(\bar z) - 2\ln\big(c(z)\bar c(\bar z) + 1\big). \tag{33}$$

Here $b(z)$ and $c(z)$ are arbitrary holomorphic functions. One of them is fundamental and the choice of it corresponds to a particular *orbit of solutions*. The other one is induced by a conformal symmetry transformation and can be transformed away. For example, we can put either $c(z) = z$, or $b(z) = z$.

We have checked that that these solutions are, in general, *not invariant* under any subgroup of the symmetry group. They reduce to invariant solutions only for very special choices of the function $b(z)$ assuming that $c(z) = z$. The full 'black list' of those bad choices of $b(z)$ is obtained for $\kappa = 1$ and $\kappa = -1$ [8]. For all other functions $b(z)$ the formulas (32) and (33) give *non-invariant solutions* of the heavenly equation.

12 Heavenly metrics with Euclidean signature

Solutions of the heavenly equation (1) with $\kappa = -1$ and $\kappa = 1$ generate the metrics which are exact solutions of the Einstein field equations with the Euclidean and ultra-hyperbolic signature, respectively. For non-invariant solutions of the heavenly equation these metrics admit only *one Killing vector*, *i.e.* only one symmetry. The reason for this symmetry is that the heavenly equation is obtained by *symmetry reduction* from the *elliptic complex Monge–Ampère equation* (CMA)

$$u_{1\bar 1}u_{2\bar 2} - u_{1\bar 2}u_{2\bar 1} = 1, \tag{34}$$

where u is a potential of the *Kähler metric*

$$ds^2 = u_{i\bar k}\, dz^i d\bar z^k \tag{35}$$

for a two dimensional complex manifold. The metric (35) is Ricci-flat with the self-dual Riemann curvature [13]. For the solutions of CMA invariant under rotations in the complex z^1-plane the symmetry reduction of the equation (34) together with a Legendre transformation results in the heavenly equation with $\kappa = -1$

$$w_{z\bar z} + (e^w)_{pp} = 0, \tag{36}$$

where $z = z^2$ and w and p come from the Legendre transformation. The Kähler metric now becomes

$$ds^2 = w_p \left(4e^w dz\, d\bar{z} + dp^2\right) + \frac{1}{w_p}\left[d\tau + i(w_z\, dz - w_{\bar{z}}\, d\bar{z})\right]^2. \tag{37}$$

If we use non-invariant solutions of equation (36), then the corresponding metrics will still have only the one Killing vector coming from our choice of rotationally invariant solutions of CMA. Non-invariant solutions will not acquire any new symmetries, hence the metric (37) will not acquire any new Killing vectors.

We use for $w(p, z, \bar{z})$ the non-invariant solutions (33) of the heavenly equation (36) with the corresponding change of notation

$$w = \ln\left|\frac{[p + b(z)]\, c'(z)}{1 + |\, c(z)|^2}\right|^2, \tag{38}$$

where b and c are arbitrary holomorphic functions, one of which can be removed by a conformal symmetry transformation and the prime denotes derivative with respect to argument. The other function is fundamental since a particular choice of it specifies the corresponding orbit of solutions of equation (36). Substituting the solution (38) into the metric (37) we obtain the resulting metric

$$\begin{aligned} ds^2 = {} & (2p + b + \bar{b})\left\{\frac{4\,|c'|^2}{(1 + |c|^2)^2}\, dz\, d\bar{z} + \frac{1}{|p + b|^2}\, dp^2\right\} \\ & + \frac{|p + b|^2}{(2p + b + \bar{b})}\left\{d\tau + 2A_M + i\left(\frac{b' dz}{p + b} - \frac{\bar{b}' d\bar{z}}{p + \bar{b}}\right)\right\}^2, \end{aligned}$$

where

$$A_M = -i\left[\left(\frac{\bar{c}\, c'}{1 + |c|^2} - \frac{c''}{2c'}\right) dz - \left(\frac{c\, \bar{c}'}{1 + |c|^2} - \frac{\bar{c}''}{2\bar{c}'}\right) d\bar{z}\right].$$

13 Heavenly metrics with ultra-hyperbolic signature

By analytic continuation of the metric (37) we obtain metrics with ultra-hyperbolic signature. There are 3 inequivalent choices of such analytic continuation which lead to 3 different ultra-hyperbolic metrics.

One such metric is

$$ds^2 = w_p \left(4e^w dz\, d\bar{z} - dp^2\right) - \frac{1}{w_p}\left[dt + i(w_z\, dz - w_{\bar{z}}\, d\bar{z})\right]^2, \tag{39}$$

where the only Killing vector is a null boost instead of rotation. In this case the Einstein field equations are reduced to the hyperbolic version of the heavenly equation corresponding to $\kappa = 1$: $w_{z\bar{z}} - (e^w)_{pp} = 0$. Its non-invariant solutions are given by

$$w = \ln\left|\frac{[p + b(z)]\, c'(z)}{c(z) + \bar{c}(\bar{z})}\right|^2$$

and the substitution of these into the formula (39) results in the metric with ultra-hyperbolic signature

$$\begin{aligned} ds_1^2 = {} & (2p + b + \bar{b})\left[\frac{4|c'|^2}{(c + \bar{c})^2}\, dz\, d\bar{z} - \frac{1}{|p + b|^2}\, dp^2\right] \\ & - \frac{|p + b|^2}{(2p + b + \bar{b})}\left\{dt + i\left[\left(\frac{2\,\bar{c}'}{c + \bar{c}} - \frac{\bar{c}''}{\bar{c}'} - \frac{\bar{b}'}{p + \bar{b}}\right) d\bar{z} - \left(\frac{2\,c'}{c + \bar{c}} - \frac{c''}{c'} - \frac{b'}{p + b}\right) dz\right]\right\}^2, \end{aligned}$$

where once again one of the arbitrary holomorphic functions $b(z)$ or $c(z)$ can be removed by a conformal transformation.

14 Conclusions and outlook

We conclude that, unlike the method of symmetry reduction, group foliation can be applied for constructing non-invariant solutions of PDEs. A regular approach for solving the resolving equations in terms of invariant derivatives is now in progress. In [7] we constructed the group foliation of the complex Monge–Ampère equation. We hope to obtain its non-invariant solutions generating the metric with no Killing vectors for the gravitational instanton $K3$.

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The Most Symmetric Drift Waves

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Comparative symmetry analysis is done for Korteweg-de Vries and Hasegawa–Mima models, both continuous Lie symmetries and discrete ones are taken into account. The form of the most symmetrical smooth solutions is determined for the Hasegawa–Mima model.

1 Introduction

Low frequency drift oscillations play an important role in the transport processes in magnetized plasmas, so they are intensely studied in recent decades [1]. The main problem in the drift waves investigations is the presence of nonlinear effects even at relatively small amplitudes. Nonlinear generation of the high space harmonics and their accumulation in the initial disturbance zone complicate numerical simulations of the drift waves evolution [2]. Thus some analytical approach based on symmetry analysis of the model is needed.

In the present work, comparative symmetry analysis is carried out for Hasegava–Mima model for the drift waves in a plasma and for the well known Korteweg–de Vries (KdV) model. In the Section 2, symmetries and the most symmetric solutions of the KdV model are reviewed as an illustrative example. This model has sufficiently large symmetry for the existence of a family of the most symmetric stable solutions called solitons. In the Section 3, Hasegawa–Mima model symmetries and solutions are considered, both continuous and discrete symmetries are taken into account. General form of the most symmetrical solutions of the Hasegawa–Mima model is determined.

2 Korteweg-de Vries model (an illustrative example)

Korteweg-de Vries model equation for nonlinear waves with potential nonlinearity (see, e.g. [3])

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

admits the following Lie group of transformations [4]:

a) time and space shifts: $t' = t + C_1$, $x' = x + C_2$;

b) similarity transform: $t' = t\exp(3C_3)$, $x' = x\exp(C_3)$, $u' = u\exp(-2C_3)$;

c) Galilean transform: $x' = x + C_4 t$, $u' = u + C_4$,

where $C_1, \ldots, C_4$ are arbitrary constants.

In addition, KdV equation admits the reflection symmetry transform $t' = -t$, $x' = -x$.

This symmetry group is large enough for the existence of a family of stable solitary wave solutions called solitons. These solutions are the most symmetric localized smooth solutions of the KdV equation. Let us review how they can be obtained by the symmetry approach.

When we introduce the homogeneous boundary condition

$$u \to 0 \quad \text{as} \quad |x| \to \infty$$

the Galilean symmetry is lost, so $C_4 = 0$. Similarity transform invariant solutions are unbounded, so we must put also $C_3 = 0$. Thus, the initial symmetry is reduced to the time and space shifts combined with the reflection transform. As a result, the most symmetric solution must be an even function of the argument $x - vt$:

$$u = u(x - vt), \qquad v = \text{const}, \qquad u(-x) = u(x).$$

Inserting this into the KdV equation and solving the corresponding ordinary differential equation, we obtain a family of solutions

$$u = 12a^2 \text{sech}^2 \left(ax - 4a^3 t \right),$$

where the similarity group orbit is labelled by an arbitrary constant a. These solutions are the well known KdV solitons. Similar considerations allow us to obtain space periodic solutions of the KdV equation known as cnoidal waves.

As for very small amplitude solutions $u = \varepsilon \exp(ikx - i\omega t)$, $|\varepsilon| \ll 1$, they obey the dispersion relation $\omega = -k^3$ and their phase and group velocities are $\frac{\omega}{k} = -k^2$ and $\frac{\partial \omega}{\partial k} = -3k^2$ respectively. So shorter waves have higher velocities and leave the initial disturbance domain faster than longer ones. For the waves with finite but small amplitude this effect compensates the nonlinear breaking of the waves. This is the physical reason for the existence of the stable soliton solutions of the KdV model.

3 Hasegawa–Mima model symmetry and solutions

Let us consider an inhomogeneous plasma slab in the external homogeneous magnetic field. Electrons, unlike ions, are magnetized, smoothing an electrostatic potential Φ along the magnetic field lines. In this case, Hasegawa–Mima model equations hold [1]:

$$\frac{\partial \Psi}{\partial t} + J(\Phi, \Psi) = \frac{\partial \Phi}{\partial y}, \qquad \Psi = \Phi - \Delta_\perp \Phi, \tag{1}$$

where $\Psi \equiv \Psi_z$ is the generalized vorticity, $J(F, G) \equiv \frac{\partial F}{\partial x}\frac{\partial G}{\partial y} - \frac{\partial G}{\partial x}\frac{\partial F}{\partial y}$ the Jacobian nonlinear operator and $\Delta_\perp \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Note that vortex nonlinearity term $J(\Phi, \Psi)$ in (1) is degenerate: zero value of this term means that there exists some functional dependence between the vorticity Ψ and the electrostatic potential Φ. As a consequence, monochromatic wave solutions exist

$$\Phi = \alpha \exp(ik_1 x + ik_2 y - i\omega t), \qquad \Psi = \left(1 + k_1^2 + k_2^2\right) \Phi \tag{2}$$

satisfying the dispersion relation

$$\omega = -\frac{k_2}{1 + k_1^2 + k_2^2}. \tag{3}$$

The amplitude α can be arbitrary, not necessarily small, because of vanishing of the nonlinear term in this case. On the other hand, it is clear from (3) that short waves are slower than the long ones, so dispersion effects cannot balance the nonlinear wave breaking. All these properties are in a sharp contrast with those of the KdV model.

Now let us try to proceed in the way used for the KdV model and obtain the most symmetric solutions of the Hasegawa–Mima equations.

First, it is useful to perform the simplifying transformation

$$\Phi = \Phi(t, x, y + t) - x, \qquad \Psi = \Psi(t, x, y + t) - x. \tag{4}$$

The new RHS functions Φ and Ψ of arguments t, x, $y+t$ satisfy the equations (1) without the dispersion term $\frac{\partial \Phi}{\partial y}$, but the boundary conditions become more complicated:

$$\Phi = x, \quad \Psi = x \quad \text{as} \quad |r| \to \infty, \quad r = \left(x^2 + y^2\right)^{1/2}. \tag{5}$$

The symmetry properties of the Hasegawa–Mima model look simpler in these new notations. The related symmetry group contains four translations

$$t' = t + C_1, \quad x' = x + C_2, \quad y' = y + C_3, \quad \Phi' = \Phi + C_4, \quad \Psi' = \Psi + C_4$$

the rotation around the Oz-axis

$$x' = x \cos C_5 - y \sin C_5, \qquad y' = x \sin C_5 + y \cos C_5$$

and the similarity transform

$$t' = t \exp(C_6), \qquad \Phi' = \Phi \exp(-C_6), \qquad \Psi' = \Psi \exp(-C_6),$$

where $C_1, \ldots, C_6$ are arbitrary constants.

There are also three reflection symmetries:

$$\begin{aligned}
&a)\ x' = -x, \quad \Phi' = -\Phi, \quad \Psi' = -\Psi;\\
&b)\ t' = -t, \quad y' = -y;\\
&c)\ t' = -t, \quad \Phi' = -\Phi, \quad \Psi' = -\Psi.
\end{aligned}$$

Now let us return to the most symmetric solution satisfying the homogeneous boundary conditions which in our new variables have the form (5). Similarity and rotation transforms are incompatible with these conditions, so we must put $C_5 = C_6 = 0$. The only remaining translations are time shift (C_1), y-shift (C_3) and the combination of x, Φ, and Ψ (C_2) shifts:

$$t' = t + C_1, \quad x' = x + C_2, \quad \Phi' = \Phi + C_2, \quad \Psi' = \Psi + C_2, \quad y' = y + C_3,$$

and only (a) and (b) reflections remain.

Returning (by the transformation inverse to (4)) to the initial variables and the initial form (1) of the Hasegawa–Mima equations, we obtain the following symmetries compatible with homogeneous boundary conditions:

$$\begin{aligned}
&t' = t + C_1, \quad x' = x + C_2, \quad y' = y + C_3,\\
&a)\ x' = -x, \quad \Phi' = -\Phi, \quad \Psi' = -\Psi,\\
&b)\ t' = -t, \quad y' = -y.
\end{aligned} \tag{6}$$

The corresponding invariant solutions must have the form

$$\Phi = F(x, y + vt), \qquad \Psi = G(x, y + vt). \tag{7}$$

Here the RHS functions F and G are antisymmetric with respect to their first argument and symmetric with respect to their second argument:

$$\begin{aligned}
F(-x, y + vt) &= -F(x, y + vt), \qquad & F(x, -(y + vt)) &= F(x, y + vt),\\
G(-x, y + vt) &= -G(x, y + vt), \qquad & G(x, -(y + vt)) &= G(x, y + vt).
\end{aligned} \tag{8}$$

In contrast with the KdV model, the value of the constant v in (7) is essential, since no similarity transform connecting solutions with different v's exists for the Hasegawa–Mima model.

Now let us try to obtain the single solution (7) with some definite value of v. Inserting the form (7) in (1) and taking into account the homogeneous boundary condition

$$\Phi = 0, \quad \Psi = 0 \quad \text{as} \quad |r| \to \infty \tag{9}$$

we readily find that only trivial zero smooth solution of this form exists.

Thus, the most symmetric non-trivial smooth localized solutions for the Hasegawa–Mima model must contain the finite or infinite sum of terms with different velocities $v_1, v_2, \ldots$:

$$\begin{aligned} \Phi &= F_1(x, y + v_1 t) + F_2(x, y + v_2 t) + \cdots , \\ \Psi &= G_1(x, y + v_1 t) + G_2(x, y + v_2 t) + \cdots , \end{aligned} \tag{10}$$

where F_i, G_i are antisymmetric functions of their first argument and symmetric functions of their second argument.

Now let us consider the most symmetric periodic in x, y solutions of the Hasegawa–Mima equations. The simplest solutions of this kind are as follows:

$$\Phi = \alpha \sin(k_1 x) \quad \text{and} \quad \Phi = \beta \sin(k_1 x) \cos(\omega t + k_2 y), \tag{11}$$

where ω satisfies the dispersion relation (3). The first solution represents the shear flow along the Oy axis, the second one is the standing wave. Both solutions are trivial inasmuch as the nonlinear term vanishes.

The most symmetric non-trivial periodic solutions for the Hasegawa–Mima equations must have the form (summation over some integer values of m, n, and l is assumed):

$$\Phi = \sum \Phi_{mnl} \sin(m k_1 x) cos(n k_2 y + \omega_{mnl} t). \tag{12}$$

Here Φ_{mnl} and ω_{mnl} are the functions of k_1 and k_2 to be determined from the equations (1). In this way periodicity and symmetry property (8) will be guaranteed. Analytical solutions of this form (except the trivial ones, like (11)) are not known.

Choosing the initial conditions compatible with the symmetry (8), we can proceed in two ways: to find numerical solutions of the Hasegawa–Mima equations or to build perturbative solutions treating the amplitudes as small but finite parameters. For example, numerical simulations were performed and perturbative solution was obtained in [2] for the combination of the shear flow and the standing wave (11). These solutions which keep the symmetric form (12) are characterized by higher harmonics generation and frequency shifts.

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Conditional Symmetry Reduction and Invariant Solutions of Nonlinear Wave Equations

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We obtain sufficient conditions for the solution found with the help of conditional symmetry operators to be an invariant one in classical Lie sense. Several examples of nonlinear partial differential equations are considered.

1 Introduction

It is well known that the symmetry reduction method is very efficient for construction of exact solutions for nonlinear partial differential equations of mathematical physics. With the help of symmetry operators one can find ansatze which reduce partial differential equation to the equation with smaller number of independent variables. Application of conditional symmetry operators essentially widens the class of ansatze reducing initial differential equation [1, 2, 3]. It turns out however that some of these ansatze result in the classical invariant solutions. It is obvious that the existence of conditional symmetry operator does not guarantee that the solution obtained with the help of corresponding ansatz is really new that it is not invariant solution in the classical Lie sense. We have proved theorem allowing us to exclude the operators that lead to the classical invariant solutions.

2 Basic theorem

Let us consider some partial differential equation

$$U(x, u, u_1, \dots, u_k) = 0, \tag{1}$$

where $u \in C^k\left(\mathbb{R}^n, \mathbb{R}^1\right)$, $x \in \mathbb{R}^n$, and u_k denotes all partial derivatives of k-th order. Suppose that the following conditions are fulfilled.

1. Equation (1) is conditionally invariant under involutive family of operators $\{Q_i\}$, $i = \overline{1,p}$

$$Q_i = \xi'^l_i(x,u)\frac{\partial}{\partial x_l} + \eta'_i(x,u)\frac{\partial}{\partial u} \tag{2}$$

and corresponding ansatz reduces this equation to ordinary differential equation.

2. There exists the general solution of reduced equation in the following form

$$u = f(x, C_1, \dots, C_t), \tag{3}$$

where f is arbitrary smooth function of its arguments, $C_1, \dots, C_t$ are arbitrary real constants. The following theorem has been proved.

Theorem 1. *Let equation (1) be invariant under the m-dimensional Lie algebra AG_m with basis elements:*

$$X_j = \tilde{\xi}'^l_j(x)\frac{\partial}{\partial x_l} + \tilde{\eta}_j(x,u)\frac{\partial}{\partial u}, \qquad j = 1, \dots, m, \tag{4}$$

and conditionally invariant with respect to involutive family of operators $\{Q_i\}$ satisfying conditions 1, 2.

If the system

$$\xi'^{l}_{i}\frac{\partial u}{\partial x_i} = \eta'_i(x,u) \tag{5}$$

is invariant under s-dimensional subalgebra $AG_s$with basis elements

$$Y_a = \xi^l_a(x)\frac{\partial}{\partial x_l}, \qquad a = \overline{1,s}, \tag{6}$$

of algebra AG_m and $s \geq t+1$, then conditionally invariant solution of equation (1) with respect to involutive family of operators $\{Q_i\}$ is an invariant solution in the classical Lie sense.

Proof. From the theorem conditions it follows that the system of equations (1), (5) is invariant with respect to AGs algebra with basis elements Y_a. Consider one parameter subgroup of transformations of space $X \times U$ (variables x, u) with infinitesimal operator Y_j. These transformations map any solution from (3) into solution of system (1), (5). Thus the following relations

$$u - f\left(x', C_1, C_2, \ldots, C_t\right) = u - f\left(x, C'_1, \ldots, C'_t\right), \tag{7}$$

where $a \in \mathbb{R}^1$, $C'_1, \ldots C'_t$ depend on $C_1, \ldots C_t, a$, are fulfilled in this case. Equality (7) is true for arbitrary group parameter $a \in \mathbb{R}^1$. Considering it in the vicinity of point $a = 0$ we obtain

$$\xi^l_a(x)\frac{\partial f}{\partial x_l} = -\frac{\partial f}{\partial C_1}\beta_{j1} - \cdots - \frac{\partial f}{\partial C_t}\beta_{jt}, \qquad j = \overline{1,s}, \tag{8}$$

where $\beta_{jk} = \frac{\partial C'_k}{\partial a}$ at the point $a = 0$. As far as the mentioned reasoning is valid for arbitrary operator Y_j then condition (8) is equivalent to the following system of s equations

$$Y_j f = -\sum_{k=1}^{t}\frac{\partial f}{\partial C_k}\beta_{jk}, \qquad 1 \leq j \leq s. \tag{9}$$

From system (9) it follows that there exist such real constants γ_p that the the condition

$$\sum_{p=1}^{s}\gamma_p Y_p f = 0,$$

is true since $s \geq t+1$. Therefore the solution $u = f(x, C_1, \ldots, C_t)$ is invariant with respect to one-parameter Lie group with infinitesimal operator $Q = \sum\limits_{p=1}^{s}\gamma_p Y_p$. ■

Note that theorem can be generalized for infinitesimal operators of the form

$$Y_a = \xi^l_a(x)\frac{\partial}{\partial x_l} + \eta_a(x,u), \tag{10}$$

where

$$\eta_a(x,u) = F_a(x)u + \Phi_a(x), \tag{11}$$

and $F_a(x)$ and $\Phi_a(x)$ are arbitrary smooth functions.

3 Examples

Now consider several examples. We first study nonlinear wave equation

$$u_{xt} = \sin u. \tag{12}$$

We prove that equation (12) is conditionally invariant with respect to the operator

$$X = \left(u_{xx} + \frac{1}{2} \tan u u_x^2 \right) \partial_u. \tag{13}$$

We use the definition of conditional symmetry for arbitrary differential equation given in [4]. Therefore we can use the following differential consequences

$$D_x(u_{xt} - \sin u) = 0, \qquad D_x^2(u_{xt} - \sin u) = 0, \qquad D_x(\eta) = 0, \tag{14}$$

and

$$u_t = -\frac{2\cos u}{u_x}. \tag{15}$$

It is easy to verify that the equality

$$\underset{2}{X}(u_{xt} - \sin u) = 0,$$

where $\underset{2}{X}$ is the extended symmetry operator of the second order, is satisfied on the manifold given by relations (12), (14), (15). Thus equation (12) is conditionally invariant with respect to the Lie–Bäcklund vector field (13). That is why we can reduce it by means of the ansatz which is the solution of the following equation

$$u_{xx} + \frac{1}{2} \tan u u_x^2 = 0 \tag{16}$$

and has an implicit form

$$H(u) = C(t)x + \alpha(t), \tag{17}$$

where

$$H(u) = \int \frac{du}{\sqrt{\cos u}}.$$

Substituting (17) into (12) we receive the reduced system in the form

$$C'(t) = 0, \qquad \frac{1}{2} C(t)\alpha'(t) + 1 = 0.$$

Finally by integrating this system we obtain solution of equation (12)

$$H(u) = C_1 x - \frac{2}{C_1} t + C_2. \tag{18}$$

Both of equations (12) and (16) are invariant with respect to three-dimensional Lie algebra with basis elements

$$Q_1 = u_x \partial_u, \qquad Q_2 = u_t \partial_u, \qquad Q_3 = (t u_t - x u_x)\partial_u.$$

And also the solution depends on two constants. So, the theorem conditions are fulfilled. Thus we conclude that solution (18) is an invariant one in the classical Lie sense as a consequence of theorem. It is obvious that there exist the linear combination of operators Q_1, Q_2, Q_3 such that obtained solution is invariant under transformations generated by this operator.

Now consider equation

$$u_t - u_{xx} = \lambda \exp(u) u_x + u_x^2. \tag{19}$$

It has been proved that equation (19) is conditionally invariant with respect to operator

$$Q = \left(u_{xx} + u_x^2\right) \frac{\partial}{\partial u}.$$

The corresponding ansatz has the form

$$u = \ln(f(t)x + \phi(t)), \tag{20}$$

where f, ϕ are unknown functions. Substitution of (20) in (19) yields the system of two ordinary differential equations in the form

$$f' = \lambda f^2, \qquad \phi' = \lambda f \phi.$$

Having integrated this system one can obtain exact solution of equation (19)

$$u = \ln\left(\frac{x + C_1}{C - \lambda t}\right). \tag{21}$$

Note, that equation

$$u_{xx} + u_x^2 = 0$$

is invariant with respect to three-dimensional algebra with basis elements

$$Q_1 = \frac{\partial}{\partial t}, \qquad Q_2 = \frac{\partial}{\partial x}, \qquad Q_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \frac{\partial}{\partial u}.$$

Thus according to theorem the solution (21) is an invariant one.

It can be verified that solution (21) is invariant with respect to one-parameter transformation group with infinitesimal operator

$$Q = \alpha Q_1 + \beta Q_2 + \gamma Q_3$$

when $\alpha = \gamma C_1$, $\beta = -2\gamma C \lambda^{-1}$.

Finally consider equation

$$u_t - a(u) u_{xx} = u(1 - a(u)), \tag{22}$$

where $a(u)$ is arbitrary smooth function. We have proved that equation (22) is conditionally invariant with respect to operator

$$Q = (u_{xx} - u)\frac{\partial}{\partial u}.$$

The invariance surface condition leads to the following ansatz

$$u(t, x) = \phi_1(t) \exp x + \phi_2(t) \exp(-x),$$

which reduces considered equation. It is easy to construct an exact solution of equation (22) using this approach in the form

$$u = A \exp(t + x) + B \exp(t - x), \tag{23}$$

where A, B are real constants.

It should be noted that the maximal invariance Lie algebra of point transformations is two-dimensional algebra with basis operators ∂_t, ∂_x. But solution (23) depends on two constants. Therefore the theorem conditions are not satisfied. Really it is easy to prove, that solution (23) cannot be constructed by means of Lie point group technique.

4 Conclusion

Thus we obtain a sufficient condition for the solution found with the help of conditional symmetry operators to be an invariant solution in the classical sense. The theorem proved by means of infinitesimal invariance method allows us to optimize the algorithm for construction of conditional symmetry operators, a priori excluding the operators that lead to the classical invariant solutions. It is obvious that this theorem can be generalized and applicable to construction of exact solutions for partial differential equations by using the method of differential constraints, Lie–Bäcklund symmetry method and the approach suggested in [5].

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On the Localized Invariant Traveling Wave Solutions in Relaxing Hydrodynamic-Type Model

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There are presented results of the investigations of a modeling system describing long nonlinear waves propagation in structured media with two relaxing components. A set of traveling wave invariant solutions is analyzed. We determine the conditions assuring the existence of quasiperiodic solutions and show that such analysis is helpful in looking for the localized wave patterns, since the destruction of quasiperiodic regime very often is realized via the homoclinic bifurcation of saddle-focus, which corresponds to the many-hump soliton appearance.

1 Introduction

Actually it is well known [1, 2, 3, 4], that continual models of structured and hierarchic media possess more rich set of solutions than those of structureless media. In this work we analyze maybe the simplest nonlinear hydrodynamic–type system describing long waves propagation in structured media with two relaxing processes on microscopic level.

It is rather difficult to make any general statements concerning the whole family of solutions of a non-linear differential equation, yet, using the group theory and qualitative methods one is able to analyze a set invariant solutions, providing that equation under consideration possesses sufficiently large symmetry group. Very often this set contains interesting and physically meaningful solutions, reporting to attract nearby, not necessarily invariant ones [5, 6, 7]. In this work attention is paid to the problem of extracting localized traveling wave solutions associated with the homoclinic loops of corresponding dynamical system, being obtained from the initial PDE system via the group theory reduction. Actually there does not exist any regular analytical method enabling to predict the appearance of a homoclinic-type solution of a multidimensional dynamical system, yet some information about the possibility of homoclinic bifurcation could be obtained through the analysis of Poincaré canonical form (CPF), corresponding to some type of degeneracy of the linear part of the system. Aside of the case when CPF is Hamiltonian, or close to Hamiltonian [8], this information is incomplete, therefore answering the question on whether or not the homoclinic bifurcation does take place one finally should resort to the numerical simulation.

The contents of this work is following. We consider relatively simple modeling system describing high-rate processes in structured media with two relaxing components. Using the trivial symmetry inherent to any evolution system which does not contain independent variables, we pass to the three-dimensional system of ODE, describing a set of invariant traveling wave solutions. We analyze this system with the help of qualitative theory methods and state the conditions assuring the existence of quasiperiodic solutions in proximity of some stationary point. Since one of the possible scenarios of the quasiperiodic regime destruction is associated with the homoclinic bifurcation, we use the above results to localize the domain of parameter space where the homoclinic bifurcation could take place. In order to capture a family of homoclinic-type solutions we finally use the numerical algorithm outlined in [4].

2 Soliton-like invariant solutions of relaxing hydrodynamics model

The modeling system we are going to analyze is as follows:

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \chi\, \rho^{\sigma-1} \frac{\partial \rho}{\partial x} = \alpha_1 (\rho - 1) + \alpha_2 (\rho - 1)^2 - q_1 \kappa\, \eta \\
&\qquad + q_2 \left[\nu\, \eta - (\sigma - 1)\, \varphi\, (\rho - 1) - \kappa\, (\lambda - \lambda_0)\right], \\
&\frac{\partial \lambda}{\partial t} = -\kappa\, \eta, \\
&\frac{\partial \eta}{\partial t} = \nu\, \eta - \varphi\, \left(\rho^{\sigma-1} - 1\right) + \beta\, (\lambda - \lambda_0)^2 - \kappa\, (\lambda - \lambda_0)\,.
\end{aligned} \tag{1}$$

Let us consider ansatz

$$\rho = R(\omega), \qquad \omega = x - D\, t, \qquad \lambda = L(\omega), \qquad \eta = N(\omega), \tag{2}$$

that describes a family of invariant traveling wave solutions. Inserting (1) into the system (1) we obtain dynamical system

$$\begin{aligned}
&D\left(\chi R^{\sigma-1} - D\right) \dot{R} = D\left[\alpha_1 (R - 1) + \alpha_2 (R - 1)^2\right] - q N \kappa \\
&\qquad + p\left[\nu\, N - \theta\, (\rho - 1) - \kappa\, (L - L_0)\right], \\
&D\left(\chi R^{\sigma-1} - D\right) \dot{L} = \kappa\, \left(\chi R^{\sigma-1} - D\right) N, \qquad\qquad (3) \\
&D\left(\chi R^{\sigma-1} - D\right) \dot{N} = \left(\chi R^{\sigma-1} - D\right) \left(-\left(R^{\sigma-1} - 1\right)\, \varphi - \nu\, N \right.\\
&\qquad \left. - \beta\, (L - L_0)^2 + \kappa\, (L - L_0)\right). \qquad\qquad (4)
\end{aligned}$$

where $\theta = (\sigma - 1)\varphi$, $q = q_1\, D$, $p = q_2\, D$. One easily gets convinced that linear part of system (3) in reference frame $X = R - 1$, $Y = L - L_0$, $Z = N$, centered at the critical point $A(1,\, L_0,\, 0)$, is as follows:

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \begin{bmatrix} \alpha_1 D - p\, \theta & -p\, \kappa & p\, \nu - q\, \kappa \\ 0 & 0 & \kappa\, \Delta \\ \theta\, \Delta & \kappa\, \Delta & -\nu\, \Delta \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \tag{5}$$

where $\Delta = \chi - D$, $h = \theta/\nu$. We are going to employ the methods of local nonlinear analysis to the investigation of quasiperiodic and soliton-like solutions appearance in vicinity of critical point $A(1,\, L_0,\, 0)$. In order for such analysis be effective, restrictions on the parameters should be imposed, assuring that the eigenvalues of the linearization matrix $\hat{M}$ standing at the RHS of system (5) have the $(0,\, \pm i\, \Omega)$ degeneracy [9, 4]. This leads to the following conditions:

$$\alpha_1 = 0, \qquad D = \chi + p\, h, \qquad \Omega^2 = -p\, h^2 \kappa\, (p\, \kappa + \nu\, q) > 0. \tag{6}$$

In order to avoid considering the case that is obviously unstable [10], we should chose the parameters in such a way that the nonlinear wave pack velocity D be greater than the acoustic sound velocity χ. From this requirement we immediately obtain inequality

$$p\, h > 0. \tag{7}$$

Assuming that conditions (6)–(7) are satisfied, let us construct canonical Poincaré form corresponding to system (3). To do that we first make a transition to new variables

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{p}{-\Omega^3\, h} \begin{bmatrix} 0 & 0 & \kappa\, h^2\, (q\,\nu + \kappa\, p) \\ -h\,\theta\,\Omega & -\kappa\, h\,\Omega & \theta\,\Omega \\ \Omega\,\kappa\, p\, h^3 & -q\,\kappa\,\Omega h^2 & -h^2 \kappa\, p\,\Omega \end{bmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix},$$

chosen in such a way that linear part of the system is written in standard quasi-diagonal form:

$$\begin{aligned} x_1' &= -\Omega\, x_2 + \sum_{i+j=2 i\le j} A_{ij} x_i x_j + \sum_{i+j+k=2 i\le j\le k} A_{ijk} x_i x_j x_k + \cdots\,, \\ x_2' &= \Omega\, x_1 + \sum_{i+j=2 i\le j} B_{ij} x_i x_j + \sum_{i+j+k=2 i\le j\le k} B_{ijk} x_i x_j x_k + \cdots\,, \\ x_3' &= \sum_{i+j=2 i\le j} C_{ij} x_i x_j + \sum_{i+j+k=2 i\le j\le k} C_{ijk} x_i x_j x_k + \cdots\,, \end{aligned} \tag{8}$$

where $(\cdot)' = D\,\left(\chi\, R^{\sigma-1} - D\right)\, d\,(\cdot)\,/d\,\omega$. Passage from the representation (8) to CPF

$$\begin{aligned} r' &= a\, r\, z + o\left(r^3,\, |z|^3\right), \\ z' &= b_1 r^2 + b_2 z^2 + f\, z^3 + o\left(r^3,\, |z|^3\right). \end{aligned} \tag{9}$$

is based on standard method that can be found e.g. in [9]. Connections between the coefficients of the CPF and those of system (8) prove to be as follows [11, 12]:

$$a = \frac{(A_{13} + B_{23})}{2}, \qquad b_1 = \frac{(C_{11} + C_{22})}{2}, \qquad b_2 = C_{33}. \tag{10}$$

CPF (9) is obtained from the initial dynamical system through the series of asymptotic transformations [9], followed by the passage to the cylindric coordinates $(r,\, \varphi,\, z)$ and averaging over the "fast" angular variable φ. Therefore the limit cycle, appearing in (9) corresponds to the quasiperiodic solution of the initial PDE system, while the homoclinic trajectory corresponds to the non-classical many-hump solitary wave pack.

As it is shown in [12], stability of the periodic solution of system (9) is determined by the sign of the coefficient f. General expression on this coefficient, given in [11, 12], is too cumbersome to be of any use in analytical treatment, but it drastically simplifies when $C_{33} = 0$. This is so when the following expression holds

$$\alpha_2 = \frac{h\, p\, \nu\left(2\, h\, \beta\, \nu - \kappa^2\right)}{2\, \kappa^2\, (h\, p + \chi)}. \tag{11}$$

Under this condition

$$f = C_{333} + \frac{1}{\Omega}\, (A_{33} C_{23} - B_{33} C_{13})\,. \tag{12}$$

Here and henceforth we put calculation with $\sigma = 3$.

We must stress in this place that, although the CPF (9) serves as a basement for the classification of regimes appearing after the removal of degeneracy, its investigation will not be of any use until the coefficients (10), (12) remain unknown. This problem is rather cumbersome unless one uses some tools for symbolic calculus. Below we give the exact expressions on the CPF coefficients, obtained with the help of the program "Mathematica 4.0":

$$\begin{aligned} a &= \frac{2\,(\nu^2 h^2 p\,\beta\, - \nu\,\kappa^2 \chi) - \nu\,\kappa^2 h\, p}{2\,\kappa}, \qquad b_1 = -\frac{h^4 p^2 \beta}{2\,\kappa}\left[p\left(\kappa^2 + \nu^2\right) - q\,\kappa\,\nu\right], \\ f &= \frac{h^4 p^3 \nu^2 \beta}{\Omega^2}\left(\kappa^2 - 2\, h\,\beta\,\nu\right). \end{aligned} \tag{13}$$

In order to remove the degeneracy of the linear part of system (3) two parameter family of small perturbation is introduced:

$$F \to \alpha_1 (R-1) + \alpha_2 (R-1), \qquad D \to \chi + h\,p\, + \epsilon.$$

This induces the following change of the CPF:

$$\begin{aligned} r' &= \mu_1 r + a\,r\,z + o\left(r^3,\, |z|^3,\, |\mu_i|\right), \\ z' &= \mu_2 z + b_1 r^2 + b_2 z^2 + f\,z^3 + o\left(r^3,\, |z|^3,\, |\mu_i|\right), \end{aligned} \tag{14}$$

where

$$\mu_1 = \frac{\nu}{2}\left(\epsilon - \frac{Dh^2 p\,q\kappa}{\Omega^2}\,\alpha_1\right), \qquad \mu_2 = -\frac{Dh^2p^2\kappa^2}{\Omega^2}\alpha_1. \tag{15}$$

As it is shown in our previous work [12], for sufficiently small $\mu_1\,\mu_2$ the limit cycle appears in system (14) along the manifold

$$\mu_2 + 3f\,(\mu_1/a)^2 = 0, \tag{16}$$

providing that $a\,b_1 < 0$. Here we have two possibilities:

- $f > 0$ and $\mu_1\,\mu_2 < 0$; stable periodic regime exists if the following inequality holds

$$\mu_2 < -3f\,(\mu_1/a)^2 < 0; \tag{17}$$

- $f < 0$ and $\mu_1\,\mu_2 < 0$; unstable periodic regime exists if the following inequality holds

$$\mu_2 > -3f\,(\mu_1/a)^2 > 0 \tag{18}$$

(note that in proximity of the critical point $A(1,\, L_0,\, 0)$ the factor $\left(\chi R^{\sigma-1} - D\right)$ is negative and this circumstance has been taken into account when determining the stability type of the limit cycle).

Numerical simulations show that one of the scenarios of destruction of the limit cycle of system (14) is associated with the appearance of homoclinic loop (for $a = 2$ it can be shown analytically [9]). These regimes correspond to the soliton-like solutions of the initial PDE system. In the first case, corresponding to formula (17), homoclinic trajectory comes out from the critical point $A(1,\, L_0,\, 0)$ (which is a saddle-focus) along the one-dimensional unstable invariant manifold W^u and returns into the critical point along the two-dimensional stable invariant manifold W^s. In the second case, corresponding to formula (18), the direction of movement is opposite. Analysis based on the equations (13)–(18) shows that both of these cases could be realized in the system (1). In the first case we obtain a soliton-like solution with oscillating front while in the second one – a localized pack with oscillating "tail".

As it was mentioned above, a homoclinic bifurcation is not the only scenario leading to the destruction of quasiperiodic movement appearing in system (3). We cannot exclude another scenarios, associated e.g. with a pair of tori appearance (i.e. the case when the Lyapunov indices cross the unit circle having non-zero imaginary part). In this situation final answer on whether or not the soliton-like regimes appear in system could give the numerical experiments.

We employed the numerical procedure put forward in [4] (cf. also [13]). The procedure enables to find out a set of points belonging to parameter plane $(\chi,\, \alpha_1)$, for which trajectory going out of the origin along the one-dimensional unstable invariant manifold W^u returns to the origin along the two-dimensional stable invariant manifold W^s. Having fixed the rest of parameters as follows $\varphi = -1$, $\nu = \kappa = q = 1$, $p = -0.25$, we defined numerically a distance between the

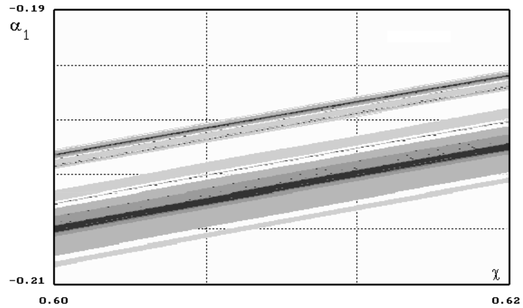

Figure 1.

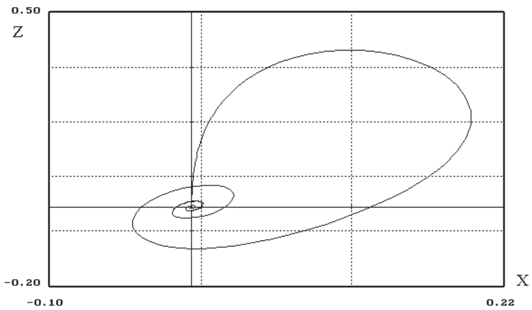

Figure 2.

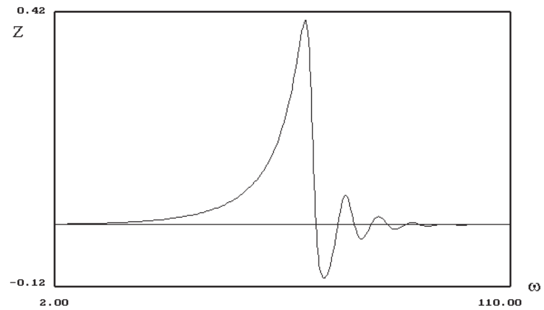

Figure 3.

origin (which coincide with the critical point $A(1,\, L_0,\, 0)$) and the point $X^\Gamma(\omega)$, $Y^\Gamma(\omega)$, $Z^\Gamma(\omega)$ of the phase trajectory $\Gamma(\cdot;\, \chi,\, \alpha_1)$:

$$f^\Gamma(\chi,\, \alpha_1; \omega) = \sqrt{[X^\Gamma(\omega)]^2 + [Y^\Gamma(\omega)]^2 + [Z^\Gamma(\omega)]^2}, \tag{19}$$

starting from the fixed Cauchy data lying on the unstable manifold W^u, close to the origin. Next we determine minimum $f^\Gamma_{\min}(\chi,\, \alpha_1)$ of the function (19) for that part of the trajectory that lies beyond the point at which the distance gains its first local maximum, providing that it remains all the time inside the ball of a fixed (sufficiently large) radius. The results is presented on Fig. 1. Here white color marks values of the parameters χ, α_1 for which $f^\Gamma_{\min} > 1.2$, light grey correspond to the values $0.6 < f^\Gamma_{\min} < 1.2$, grey color - to the values $0.3 < f^\Gamma_{\min} < 0.6$, deep grey – to $0.01 < f^\Gamma_{\min} < 0.3$, black - to $f^\Gamma_{\min} < 0.01$. It is seen on Fig. 1, that the points corresponding to homoclinic loop appearance form a connected set and this is in agreement with the general statements [14]. Let us note that employment of the same procedure for another systems gives a Cantor set instead of the connected curve [4, 13].

Solving numerically system (3) for proper values of the parameters, one is able to obtain a soliton-like solution. Because of the numerical error it is rather impossible to read our form Fig. 1 exact value of the parameters χ, α_1, corresponding to homoclinic loop. Therefore we put $\alpha_1 = -0.2$ and, using the Bolzano–Weierstrass method, varied χ until the homoclinic trajectory was attained at $\chi = 0.619803$. This approach proves to be effective because bifurcation values of the parameters form a smooth curve in the plane $(\chi,\ \alpha_1)$. Projection of the homoclinic trajectory onto the $(X,\ Z)$ plane is shown on Fig. 2; while the corresponding solution $R\,(\omega) = R\,(x - D\,t)$ is shown on Fig. 3.

So we have get convinced that system (1) possesses invariant soliton-like solutions, that look like a many-hump wave pack moving with uniform speed D. Let us note in conclusion that till now we did not touch upon the crucial problem of stability of the soliton-like regimes and their attractive features, but we hope to face this problem it in the nearest future.

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Symmetry Analysis of the Doebner–Goldin Equations

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The paper discusses the application of *MathLie* in connection with Lie group analysis. The examined example is the $(1+1)$-dimensional case of the Doebner–Goldin equation after Madelung transform. The related Lie-algeras are calculated. We present the generators, commutator tables and adjoint representations from the algebras. Furthermore we discuss the reduction of an example to ordinary differential equations and solve it explicitly.

1 Derivation of the Doebner–Goldin equations

During the investigation of Borel quantization for S^1 Dobrev, Doebner and Twarock [1] derived a nonlinear Schrödinger equation of the form (here with $m = 1$, $\hbar = 1$)

$$i\partial_t\psi = -\frac{1}{2}\Delta\psi + V(\vec{x},t)\psi + \frac{i}{2}KR_2(\psi)\psi + \sum_{j=1}^{5} D_j R_j(\psi)\psi. \tag{1}$$

This equation is called Doebner–Goldin equation. Here, the $R_j(\psi)$ with $j \in \{1,2,\ldots,5\}$ are real–valued functionals of the real–valued density $\varrho = \bar{\varrho} = \psi\bar{\psi}$ and the real-valued current $\vec{j} = \bar{\vec{j}} = \frac{i\hbar}{2m}\left(\psi\nabla\bar{\psi} - \bar{\psi}\nabla\psi\right)$. They are given by

$$R_1(\psi) = \frac{\nabla\vec{j}}{\varrho}, \quad R_2(\psi) = \frac{\Delta\varrho}{\varrho}, \quad R_3(\psi) = \frac{\vec{j}^2}{\varrho^2}, \quad R_4(\psi) = \frac{\vec{j}\nabla\varrho}{\varrho^2}, \quad R_5(\psi) = \frac{(\nabla\varrho)^2}{\varrho^2}. \tag{2}$$

To apply Lie theory, we have to write equation (1) as a system of real functions. To do this we use the Madelung transformation [5]

$$\psi = \sqrt{\varrho(\vec{x},t)}\exp(iS(\vec{x},t)), \qquad \bar{\psi} = \sqrt{\varrho(\vec{x},t)}\exp(-iS(\vec{x},t)). \tag{3}$$

Considering the $(1+1)$-dimensional case with the functionals (2) and the Madelung transformation (3) we get the following system of equations:

$$\begin{aligned} &-8\varrho^2V(x,t) + 4i\varrho\varrho_t - 8\varrho^2S_t - \varrho_x^2 - 8D_5\varrho_x^2 \\ &\qquad + 4i\varrho\varrho_xS_x - 8\varrho D_1\varrho_xS_x - 8\varrho D_4\varrho_xS_x - 4\varrho^2S_x^2 - 8\varrho^2D_3S_x^2 \\ &\qquad - 4iK\varrho\varrho_{xx} + 2\varrho\varrho_{xx} - 8D_2\varrho\varrho_{xx} + 4i\varrho^2S_{xx} - 8D_1\varrho^2S_{xx} = 0. \end{aligned} \tag{4}$$

After separating equation (4) into real and imaginary part a system of differential equations in S and ϱ follows:

$$\begin{aligned} &\varrho_t + \varrho_xS_x - K\varrho_{xx} + \varrho S_{xx} = 0, \\ &(1+8D_5)\varrho_x^2 + 2\varrho\left(4D_1\varrho_xS_x + 4D_4\varrho_xS_x + (4D_2-1)\varrho_{xx}\right) \\ &\qquad + 4\varrho^2\left(2V(x,t) + 2S_t + S_x^2 + 2D_3S_x^2 + 2D_1S_{xx}\right) = 0. \end{aligned} \tag{5}$$

By permutating the six parameters $\{K, D_1, D_2, \ldots, D_5\}$ we receive 63 different model equations (see Table 1 below) of a nonlinear Schrödinger type equation called Doebner–Goldin–Madelung equation.

2 Symmetry analysis of the $(1+1)$-dimensional Doebner–Goldin–Madelung equations

In order to find the symmetry group of equations (5) we apply the algorithms described in a lot of textbooks (e.g. [2, 3, 4, 6, 7]). We look for an algebra of vector fields of the form

$$v = \xi[1]\partial_x \cdot + \xi[2]\partial_t \cdot + \phi[1]\partial_\varrho \cdot + \phi[2]\partial_S \cdot,$$

where $\xi[1]$, $\xi[2]$ are functions of x and t and $\phi[1]$ and $\phi[2]$ depend on $\{x, t, \varrho, S\}$.

These coefficients are determined from the requirement that the second prolongation of v should annihilate the equation on the solution set of the equation. This was done using the *Mathematica* program *MathLie* [2].

The application of this theory to the system (5) leads to the following result:

Table 1. Permutation of parameters.

Nr.	Equations	Infinitesimals	Operators
1	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4D_1 S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
2	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2) + \varrho_x^2 +$ $2(4D_2 - 1)\varrho\varrho_{xx} = 0$	$\xi[1] = k_2 + k_5 t + (k_3 + 2k_6 t)x,$ $\xi[2] = k_1 + 2t(k_3 + k_6 t),$ $\phi[1] = (k_7 - 2k_6 t)\varrho,$ $\phi[2] = k_4 + x(k_5 + k_6 x)$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot + x\partial_x\cdot,$ $x\partial_S \cdot + t\partial_x\cdot, \varrho\partial_\varrho\cdot,$ $x^2\partial_S \cdot + 2t^2\partial_t \cdot + 2tx\partial_x \cdot - 2t\varrho\partial_\varrho\cdot$
3	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2) +$ $\varrho_x^2 - 2\varrho\varrho_{xx} = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $\partial_t \cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
4	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2) + \varrho_x^2 +$ $2\varrho(4D_4 S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
5	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2) + (1 + 8D_5)\varrho_x^2 -$ $2\varrho\varrho_{xx} = 0$	$\xi[1] = k_2 + k_5 t + (k_3 + 2k_6 t)x,$ $\xi[2] = k_1 + 2t(k_3 + k_6 t),$ $\phi[1] = (k_7 - 2k_6 t)\varrho,$ $\phi[2] = k_4 + x(k_5 + k_6 x)$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot + x\partial_x\cdot,$ $x\partial_S \cdot + t\partial_x\cdot, \varrho\partial_\varrho\cdot,$ $x^2\partial_S \cdot + 2t^2\partial_t \cdot + 2tx\partial_x \cdot - 2t\varrho\partial_\varrho\cdot$
6	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2) + \varrho_x^2 - 2\varrho\varrho_{xx} = 0$	$\xi[1] = k_2 + k_5 t + (k_3 + 2k_6 t)x,$ $\xi[2] = k_1 + 2t(k_3 + k_6 t),$ $\phi[1] = (k_7 - 2k_6 t)\varrho,$ $\phi[2] = k_4 + x(k_5 + k_6 x)$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2\partial_t \cdot + x\partial_x\cdot,$ $x\partial_S \cdot + t\partial_x\cdot, \varrho\partial_\varrho\cdot,$ $x^2\partial_S \cdot + 2t^2\partial_t \cdot + 2tx\partial_x \cdot - 2t\varrho\partial_\varrho\cdot$
7	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4D_1 S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
8	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2 + 2D_1 S_{xx}) +$ $\varrho_x^2 + 2\varrho(4D_1 S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
9	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4D_1 + D_4)S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
10	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4D_1 S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
11	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4D_1 S_x\varrho_x - \varrho_{xx}) = 0,$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
12	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2) +$ $\varrho_x^2 + 2(4D_2 - 1)\varrho\varrho_{xx} = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
13	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2) + \varrho_x^2 +$ $2\varrho(4D_4 S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
14	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2) + (1 + 8D_5)\varrho_x^2 +$ $2(4D_2 - 1)\varrho\varrho_{xx} = 0$	$\xi[1] = k_2 + k_5 t + (k_3 + 2k_6 t)x,$ $\xi[2] = k_1 + 2t(k_3 + k_6 t),$ $\phi[1] = (k_7 - 2k_6 t)\varrho,$ $\phi[2] = k_4 + x(k_5 + k_6 x)$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t \cdot + x\partial_x\cdot,$ $x\partial_S \cdot + t\partial_x\cdot, \varrho\partial_\varrho\cdot,$ $x^2\partial_S \cdot + 2t^2\partial_t \cdot + 2tx\partial_x \cdot - 2t\varrho\partial_\varrho\cdot$

Nr.	Equations	Infinitesimals	Operators
15	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2) + \varrho_x^2 +$ $2(4D_2 - 1)\varrho\varrho_{xx} = 0$	$\xi[1] = k_2 + k_5 t + (k_3 + 2k_6 t)x,$ $\xi[2] = k_1 + 2t(k_3 + k_6 t),$ $\phi[1] = (k_7 - 2k_6 t)\varrho,$ $\phi[2] = k_4 + x(k_5 + k_6 x)$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $x\partial_S\cdot + t\partial_x\cdot, \varrho\partial_\varrho\cdot,$ $x^2\partial_S\cdot + 2t^2\partial_t\cdot + 2tx\partial_x\cdot - 2t\varrho\partial_\varrho\cdot$
16	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2) + \varrho_x^2 +$ $\varrho(4D_4 S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
17	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2) +$ $(1 + 8D_5)\varrho_x^2 - 2\varrho\varrho_{xx} = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
18	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2) +$ $\varrho_x^2 - 2\varrho\varrho_{xx} = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
19	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2) + (1 + 8D_5)\varrho_x^2 +$ $2\varrho(4D_4 S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
20	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2) + \varrho_x^2 +$ $2\varrho(4D_4 S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
21	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2) +$ $(1 + 8D_5)\varrho_x^2 - 2\varrho\varrho_{xx} = 0$	$\xi[1] = k_2 + k_5 t + (k_3 + 2k_6 t)x,$ $\xi[2] = k_1 + 2t(k_3 + k_6 t),$ $\phi[1] = (k_7 - 2k_6 t)\varrho,$ $\phi[2] = k_4 + x(k_5 + k_6 x)$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\partial_S\cdot + t\partial_x\cdot, \varrho\partial_\varrho\cdot,$ $x^2\partial_S\cdot + 2t^2\partial_t\cdot + 2tx\partial_x\cdot - 2t\varrho\partial_\varrho\cdot$
22	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4D_1 S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
23	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4(D_1 + D_4)S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
24	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4D_1 S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
25	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4D_1 S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
26	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4(D_1 + D_4)S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
27	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2 + 2D_1 S_{xx}) +$ $(1 + 8D_5)\varrho_x^2 + 2\varrho(4D_1 S_x\varrho_x - \varrho_{xx}) = 0,$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
28	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4D_1 S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
29	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4(D_1 + D_4)S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
30	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4(D_1 + D_4)S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
31	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1 S_{xx}) +$ $2\varrho(4D_1 S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
32	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2) +$ $2\varrho(4D_4 S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
33	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2) +$ $(1 + 8D_5)\varrho_x^2 + 2(4D_2 - 1)\varrho\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
34	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2) +$ $2(4D_2 - 1)\varrho\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
35	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2) +$ $2\varrho(4D_4 S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
36	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2) +$ $2\varrho(4D_4 S_x\varrho_x + (4D_2 - 1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
37	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2) + (1 + 8D_5)\varrho_x^2$ $2(4D_2 - 1)\varrho\varrho_{xx} = 0$	$\xi[1] = k_2 + k_5 t + (k_3 + 2k_6 t)x,$ $\xi[2] = k_1 + 2t(k_3 + k_6 t),$ $\phi[1] = (k_7 - 2k_6 t)\varrho,$ $\phi[2] = k_4 + x(k_5 + k_6 x)$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\partial_S\cdot + t\partial_x\cdot, \varrho\partial_\varrho\cdot,$ $x^2\partial_S\cdot + 2t^2\partial_t\cdot + 2tx\partial_x\cdot - 2t\varrho\partial_\varrho\cdot$
38	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $(1 + 8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2) +$ $2\varrho(4D_4 S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
39	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3 S_x^2) +$ $2\varrho(4D_4 S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4 x,$ $\xi[2] = k_2 + 2k_4 t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$

Nr.	Equations	Infinitesimals	Operators
40	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2)+$ $(1+8D_5)\varrho_x^2 - 2\varrho\varrho_{xx} = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
41	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1+8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2)+$ $2\varrho(4D_4S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
42	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2 + 2D_1S_{xx})+$ $2\varrho(4(D_1+D_4)S_x\varrho_x + (4D_2-1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
43	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2 + 2D_1S_{xx})+$ $8D_5\varrho_x^2 + 2\varrho(4D_1S_x\varrho_x + (4D_2-1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
44	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2 + 2D_1S_{xx})+$ $2\varrho(4D_1S_x\varrho_x + (4D_2-1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
45	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $(1+8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1S_{xx})+$ $2\varrho(4(D_1+D_4)S_x\varrho_x + (4D_2-1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
46	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1S_{xx})+$ $2\varrho(4(D_1+D_4)S_x\varrho_x + (4D_2-1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
47	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1+8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1S_{xx})+$ $2\varrho(4D_1S_x\varrho_x + (4D_2-1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
48	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2 + 2D_1S_{xx})+$ $(1+8D_5)\varrho_x^2 + 2\varrho(4(D_1+D_4)S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
49	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2 + 2D_1S_{xx})+$ $2\varrho(4(D_1+D_4)S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
50	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1+8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2 + 2D_1S_{xx})+$ $2\varrho(4D_1S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
51	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1+D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1S_{xx})+$ $2\varrho(4(D_1+D_4)S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
52	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $(1+8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2)+$ $2\varrho(4D_4S_x\varrho_x + (4D_2-1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
53	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2)+$ $2\varrho(4D_4S_x\varrho_x + (4D_2-1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
54	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1+8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2) + 2(4D_2-1)\varrho\varrho_{xx} = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
55	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1+8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2)+$ $2\varrho(4D_4S_x\varrho_x + (4D_2-1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
56	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1+8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2)+$ $2\varrho(4D_4S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
57	$\varrho_t + S_x\varrho_x + \varrho S_{xx} = 0,$ $(1+8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2 + 2D_1S_{xx})+$ $2\varrho(4(D_1+D_4)S_x\varrho_x + (4D_2-1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
58	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2 + 2D_1S_{xx})+$ $2\varrho(4(D_1+D_4)S_x\varrho_x + (4D_2-1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
59	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1+8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2 + 2D_1S_{xx})+$ $2\varrho(4D_1S_x\varrho_x + (4D_2-1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
60	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1+8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_1S_{xx})+$ $2\varrho(4(D_1+D_4)S_x\varrho_x + (4D_2-1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
61	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1+8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2 + 2D_1S_{xx})+$ $2\varrho(4(D_1+D_4)S_x\varrho_x - \varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
62	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1+8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2)+$ $2\varrho(4D_4S_x\varrho_x + (4D_2-1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$
63	$\varrho_t + S_x\varrho_x + \varrho S_{xx} - K\varrho_{xx} = 0,$ $(1+8D_5)\varrho_x^2 + 4\varrho^2(2S_t + S_x^2 + 2D_3S_x^2 + 2D_1S_{xx})+$ $2\varrho(4(D_1+D_4)S_x\varrho_x + (4D_2-1)\varrho_{xx}) = 0$	$\xi[1] = k_3 + k_4x,$ $\xi[2] = k_2 + 2k_4t,$ $\phi[1] = k_5\varrho, \phi[2] = k_1$	$\partial_S\cdot, \partial_t\cdot, \partial_x\cdot,$ $2t\partial_t\cdot + x\partial_x\cdot,$ $\varrho\partial_\varrho\cdot$

We derive from Table 1 the existence of two different algebras A_1, A_2:

$$A_1 = \{\partial_S\cdot, \partial_t\cdot, \partial_x\cdot, 2t\partial_t\cdot + x\partial_x\cdot, \varrho\partial_\varrho\cdot\}.$$

The dimension of the first algebra A_1 is 5.

The following equations are related to A_1: 1, 3, 4, 7, 8, 9, 10, 11, 12, 13, 16, 17, 18, 19, 20, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63.

The second algebra A_2 has dimension 7 and is represented by

$$A_2 = \{\partial_t\cdot, \partial_x\cdot, 2t\partial_t \cdot + x\partial_x\cdot, \partial_S\cdot, \partial_S \cdot + t\partial_x\cdot, x^2\partial_S \cdot + 2t^2\partial_t \cdot + 2tx\partial_x \cdot - 2t\varrho\partial_\varrho\cdot, \varrho\partial_\varrho\cdot\}.$$

The related equations from Table 1 are 2, 5, 6, 14, 15, 21, 37. To calculate the operation of the group we have to solve the following system of ordinary differential equations:

$$\frac{\partial \tilde{x}^n}{\partial \varepsilon} = \xi^n \left[\tilde{x}^i(\varepsilon), \tilde{u}^\beta(\varepsilon)\right], \qquad \frac{\partial \tilde{u}^\alpha}{\partial \varepsilon} = \phi^\alpha \left[\tilde{x}^i(\varepsilon), \tilde{u}^\beta(\varepsilon)\right], \qquad \tilde{x}^n = x^n, \quad \tilde{u}^\alpha = u^\alpha \quad \text{for} \quad \varepsilon = 0.$$

If we do this for the algebra A_1, we find:

$k_1 \neq 0,\ k_2 = k_3 = k_4 = 0$

$$\frac{\partial \tilde{t}}{\partial \varepsilon} = 0 \implies \tilde{t} = t, \qquad \frac{\partial \tilde{x}}{\partial \varepsilon} = 0 \implies \tilde{x} = x, \qquad \frac{\partial \tilde{\varrho}}{\partial \varepsilon} = 0 \implies \tilde{\varrho} = \varrho, \qquad \frac{\partial \tilde{S}}{\partial \varepsilon} = k_1 \implies \tilde{S} = k_1\varepsilon + S;$$

$k_1 = 0,\ k_2 \neq 0,\ k_3 = k_4 = 0$

$$\frac{\partial \tilde{t}}{\partial \varepsilon} = 0 \implies \tilde{t} = t, \qquad \frac{\partial \tilde{x}}{\partial \varepsilon} = k_2 \implies \tilde{x} = k_2\varepsilon + x, \qquad \frac{\partial \tilde{\varrho}}{\partial \varepsilon} = 0 \implies \tilde{\varrho} = \varrho, \qquad \frac{\partial \tilde{S}}{\partial \varepsilon} = 0 \implies \tilde{S} = S;$$

$k_1 = k_2 = 0,\ k_3 \neq 0,\ k_4 = 0$

$$\frac{\partial \tilde{t}}{\partial \varepsilon} = k_3 \implies \tilde{t} = k_3\varepsilon + t, \qquad \frac{\partial \tilde{x}}{\partial \varepsilon} = 0 \implies \tilde{x} = x, \qquad \frac{\partial \tilde{\varrho}}{\partial \varepsilon} = 0 \implies \tilde{\varrho} = \varrho, \qquad \frac{\partial \tilde{S}}{\partial \varepsilon} = 0 \implies \tilde{S} = S;$$

$k_1 = k_2 = k_3 = 0,\ k_4 \neq 0$

$$\frac{\partial \tilde{t}}{\partial \varepsilon} = k_4\tilde{x} \implies \tilde{t} = \frac{\sqrt{2}t + x}{2\sqrt{2}} e^{\sqrt{2}k_4\varepsilon} - \frac{x - \sqrt{2}t}{2\sqrt{2}} e^{-\sqrt{2}k_4\varepsilon},$$

$$\frac{\partial \tilde{x}}{\partial \varepsilon} = 2k_4\tilde{t} \implies \tilde{x} = \frac{\sqrt{2}t + x}{2} e^{\sqrt{2}k_4\varepsilon} - \frac{x - \sqrt{2}t}{2} e^{-\sqrt{2}k_4\varepsilon},$$

$$\frac{\partial \tilde{\varrho}}{\partial \varepsilon} = 0 \implies \tilde{\varrho} = \varrho, \qquad \frac{\partial \tilde{S}}{\partial \varepsilon} = 0 \implies \tilde{S} = S.$$

The investigation of this algebra using *MathLie* [3] gives the following results. The commutator table is given by:

$[v_i, v_j]$	v_1	v_2	v_3	v_4	v_5
v_1	0	0	0	0	0
v_2	0	0	0	$2v_2$	0
v_3	0	0	0	v_3	0
v_4	0	$-2v_2$	$-v_3$	0	0
v_5	0	0	0	0	0

The non–trivial algebra elements are $\{v_1, v_2, v_3, v_4, v_5\}$ and the algebra generating elements reads $\{v_1, v_2, v_3, v_4, v_5\}$. Also the Cartan matrix is given by:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Furthermore, we see that the algebra is not semisimple and not nilpotent, but it is solvable. We find the following subalgebras: A subalgebra with zero elements:$\{\ \}$; subalgebras with one element: $\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{v_4\}$, $\{v_5\}$; subalgebras with two elements: $\{v_1,v_2\}$, $\{v_1,v_3\}$, $\{v_1,v_4\}$, $\{v_1,v_5\}$, $\{v_2,v_3\}$, $\{v_2,v_4\}$, $\{v_2,v_5\}$, $\{v_3,v_4\}$, $\{v_3,v_5\}$, $\{v_4,v_5\}$; subalgebras with three elements: $\{v_1,v_2,v_3\}$, $\{v_1,v_2,v_4\}$, $\{v_1,v_2,v_5\}$, $\{v_1,v_3,v_4\}$, $\{v_1,v_3,v_5\}$, $\{v_1,v_4,v_5\}$, $\{v_2,v_3,v_4\}$, $\{v_2,v_3,v_5\}$, $\{v_2,v_4,v_5\}$, $\{v_3,v_4,v_5\}$; subalgebras with four elements: $\{v_1,v_2,v_3,v_4\}$, $\{v_1,v_2,v_3,v_5\}$, $\{v_1,v_2,v_4,v_5\}$, $\{v_1,v_3,v_4,v_5\}$, $\{v_2,v_3,v_4,v_5\}$; subalgebra with five elements: $\{v_1,v_2,v_3,v_4,v_5\}$. Ideals of the algebra A_1 are: $\{\ \}$, $\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{v_5\}$, $\{v_1,v_2\}$, $\{v_1,v_3\}$, $\{v_1,v_5\}$, $\{v_2,v_3\}$, $\{v_2,v_5\}$, $\{v_3,v_5\}$, $\{v_1,v_2,v_3\}$, $\{v_1,v_2,v_5\}$, $\{v_1,v_3,v_5\}$, $\{v_2,v_3,v_4\}$, $\{v_2,v_3,v_5\}$, $\{v_1,v_2,v_3,v_4\}$, $\{v_1,v_2,v_3,v_5\}$, $\{v_2,v_3,v_4,v_5\}$, $\{v_1,v_2,v_3,v_4,v_5\}$. The radical is $\{v_1,v_2,v_3,v_4,v_5\}$. The center of the algebra is $\{v_1,v_5\}$. The adjoint representation in matrix–form is given by

$$\mathrm{Ad}(\varepsilon_1 v_1)=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\qquad \mathrm{Ad}(\varepsilon_2 v_2)=\begin{pmatrix}1&0&0&0&0\\0&1&0&-2\varepsilon_2&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},$$

$$\mathrm{Ad}(\varepsilon_3 v_3)=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&-\varepsilon_3&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},\qquad \mathrm{Ad}(\varepsilon_4 v_4)=\begin{pmatrix}1&0&0&0&0\\0&e^{2\varepsilon_4}&0&0&0\\0&0&e^{\varepsilon_4}&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix},$$

$$\mathrm{Ad}(\varepsilon_5 v_5)=\begin{pmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{pmatrix}.$$

The second algebra A_2 is

$$v_1=\partial_t\cdot,\quad v_2=\partial_x\cdot,\quad v_3=2t\partial_t\cdot+x\partial_x\cdot,\quad v_4=\partial_S\cdot,\quad v_5=x\partial_S\cdot+t\partial_x\cdot,$$
$$v_6=x^2\partial_S\cdot+2t^2\partial_t\cdot+2tx\partial_x\cdot-2t\varrho\partial_\varrho\cdot,\quad v_7=\varrho\partial_\varrho\cdot$$

with the commutator table

$[v_i,v_j]$	v_1	v_2	v_3	v_4	v_5	v_6	v_7
v_1	0	0	$2v_1$	0	v_2	$2v_3-2v_7$	0
v_2	0	0	v_2	0	v_4	$2v_5$	0
v_3	$-2v_1$	$-v_2$	0	0	v_5	$2v_6$	0
v_4	0	0	0	0	0	0	0
v_5	$-v_2$	$-v_4$	$-v_5$	0	0	0	0
v_6	$-2v_3+2v_7$	$-2v_5$	$-2v_6$	0	0	0	0
v_7	0	0	0	0	0	0	0

The non–trivial algebra elements are $\{v_1,v_6\}$ and the algebra generating elements read $\{v_1,v_2,v_3,v_6\}$, $\{v_1,v_2,v_6,v_7\}$, $\{v_1,v_3,v_5,v_6\}$, $\{v_1,v_5,v_6,v_7\}$, $\{v_1,v_2,v_3,v_4,v_6\}$, $\{v_1,v_2,v_3,v_5,v_6\}$, $\{v_1,v_2,v_3,v_6,v_7\}$, $\{v_1,v_2,v_4,v_6,v_7\}$, $\{v_1,v_2,v_5,v_6,v_7\}$, $\{v_1,v_3,v_4,v_5,v_6\}$, $\{v_1,v_3,v_5,v_6,v_7\}$,

$\{v_1, v_4, v_5, v_6, v_7\}$, $\{v_1, v_2, v_3, v_4, v_5, v_6\}$, $\{v_1, v_2, v_3, v_4, v_6, v_7\}$, $\{v_1, v_2, v_3, v_5, v_6, v_7\}$, $\{v_1, v_2, v_4, v_5, v_6, v_7\}$, $\{v_1, v_3, v_4, v_5, v_6, v_7\}$, $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. The Cartan matrix can be calculated as

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -10 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Furthermore, the algebra A_2 is not semisimple, not solvable and not nilpotent. The following subalgebras can be calculated: A subalgebra with zero element $\{\ \}$; subalgebras with one element: $\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{v_4\}$, $\{v_5\}$, $\{v_6\}$, $\{v_7\}$; subalgebras with two elements: $\{v_1, v_2\}$, $\{v_1, v_3\}$, $\{v_1, v_4\}$, $\{v_1, v_7\}$, $\{v_2, v_3\}$, $\{v_2, v_4\}$, $\{v_2, v_7\}$, $\{v_3, v_4\}$, $\{v_3, v_5\}$, $\{v_3, v_6\}$, $\{v_3, v_7\}$, $\{v_4, v_5\}$, $\{v_4, v_6\}$, $\{v_4, v_7\}$, $\{v_5, v_6\}$, $\{v_5, v_7\}$, $\{v_6, v_7\}$; subalgebras with three elements: $\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_4\}$, $\{v_1, v_2, v_7\}$, $\{v_1, v_3, v_4\}$, $\{v_1, v_3, v_7\}$, $\{v_1, v_4, v_7\}$, $\{v_2, v_3, v_4\}$, $\{v_2, v_3, v_7\}$, $\{v_2, v_4, v_5\}$, $\{v_2, v_4, v_7\}$, $\{v_3, v_4, v_5\}$, $\{v_3, v_4, v_6\}$, $\{v_3, v_4, v_7\}$, $\{v_3, v_5, v_6\}$, $\{v_3, v_5, v_7\}$, $\{v_3, v_6, v_7\}$, $\{v_4, v_5, v_6\}$, $\{v_4, v_5, v_7\}$, $\{v_4, v_6, v_7\}$, $\{v_5, v_6, v_7\}$; subalgebras with four elements: $\{v_1, v_2, v_3, v_4\}$, $\{v_1, v_2, v_3, v_7\}$, $\{v_1, v_2, v_4, v_5\}$, $\{v_1, v_2, v_4, v_7\}$, $\{v_1, v_3, v_4, v_7\}$, $\{v_1, v_3, v_6, v_7\}$, $\{v_2, v_3, v_4, v_5\}$, $\{v_2, v_3, v_4, v_7\}$, $\{v_2, v_4, v_5, v_6\}$, $\{v_2, v_4, v_5, v_7\}$, $\{v_3, v_4, v_5, v_6\}$, $\{v_3, v_4, v_5, v_7\}$, $\{v_3, v_4, v_6, v_7\}$, $\{v_3, v_5, v_6, v_7\}$, $\{v_4, v_5, v_6, v_7\}$; subalgebras with five elements: $\{v_1, v_2, v_3, v_4, v_5\}$, $\{v_1, v_2, v_3, v_4, v_7\}$, $\{v_1, v_2, v_4, v_5, v_7\}$, $\{v_1, v_3, v_4, v_6, v_7\}$, $\{v_2, v_3, v_4, v_5, v_6\}$, $\{v_2, v_3, v_4, v_5, v_7\}$, $\{v_2, v_4, v_5, v_6, v_7\}$, $\{v_3, v_4, v_5, v_6, v_7\}$; subalgebras with six elements: $\{v_1, v_2, v_3, v_4, v_5, v_7\}$, $\{v_2, v_3, v_4, v_5, v_6, v_7\}$; subalgebra with seven elements: $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$.

The algebra–ideals are $\{\ \}$, $\{v_4\}$, $\{v_7\}$, $\{v_4, v_7\}$, $\{v_2, v_4, v_5\}$, $\{v_2, v_4, v_5, v_7\}$, $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and the algebra–radical is $\{v_2, v_4, v_5, v_7\}$. The algebra–center reads $\{v_4, v_7\}$. The adjoint representation of the algebra is

$$\mathrm{Ad}(\varepsilon_1 v_1) = \begin{pmatrix} 1 & 0 & -2\varepsilon_1 & 0 & 0 & 2\varepsilon_1^2 & 0 \\ 0 & 1 & 0 & 0 & -\varepsilon_1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -2\varepsilon_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\varepsilon_1 & 1 \end{pmatrix},$$

$$\mathrm{Ad}(\varepsilon_2 v_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\varepsilon_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\varepsilon_2 & \varepsilon_2^2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2\varepsilon_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathrm{Ad}(\varepsilon_3 v_3) = \begin{pmatrix} e^{2\varepsilon_3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\varepsilon_3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-\varepsilon_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-2\varepsilon_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathrm{Ad}(\varepsilon_4 v_4) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathrm{Ad}(\varepsilon_5 v_5) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\varepsilon_5^2 & \varepsilon_5 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathrm{Ad}(\varepsilon_6 v_6) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2\varepsilon_6 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2\varepsilon_6 & 0 & 0 & 1 & 0 & 0 \\ 2\varepsilon_6^2 & 0 & 2\varepsilon_6 & 0 & 0 & 1 & 0 \\ 2\varepsilon_6 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathrm{Ad}(\varepsilon_7 v_7) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

3 Optimal systems

We consider a system of differential equations with a r-parametric symmetry group

$$F\left(\vec{x}, \vec{u}, \vec{u}^{(r)}\right) = \left(F_1\left(\vec{x}, \vec{u}, \vec{u}^{(r)}\right), \ldots, F_m\left(\vec{x}, \vec{u}, \vec{u}^{(r)}\right)\right).$$

For every s-paramteric subgroup H we can find under the assumption that $s < \min(r, n') -$ n' = number of the independent variables $-$ a family of similarity solutions. It is impossible to calculate all kinds of similarity solutions because there are infinitely many such subgroups.

In this set of similarity solutions there are such solutions, which can be calculated by transformation of the symmetry group from other similarity solutions. The aim is to calculate a minimal set of similarity solutions from which one can gain all the other similarity solutions by transformation. Such a list is called optimal system and the elements are essentially different similarity solutions [7, 8, 9]. With group theoretical and algebraical considerations we can transform this problem to that of classifying the Lie subalgebras [9, 10]. The tools to do this are the Campbell–Baker–Hausdorff formula and the adjoint representation of the Lie algebra. These tools are now applied to the algebra A_1. The general adjoint representation which can be calculated by the matrix product of all adjoint representations of A_1 is

$$\mathrm{Ad}_g = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & e^{2\varepsilon_4} & 0 & -2\varepsilon_2 e^{2\varepsilon_4} & 0 \\ 0 & 0 & e^{\varepsilon_4} & -3\varepsilon_3 e^{\varepsilon_4} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have to simplify the following system of equations:

$$\frac{1}{a}\mathrm{Ad}_g \cdot \vec{\alpha} = \vec{\beta} \qquad \text{with} \quad \vec{\alpha} = (\alpha_1, \ldots, \alpha_5)^T \quad \text{and} \quad \vec{\beta} = (\beta_1, \ldots, \beta_5)^T.$$

This is the system

$$\frac{1}{a}\alpha_1 = \beta_1, \tag{6}$$

$$\frac{1}{a}\left(e^{2\varepsilon_4}\alpha_2 - 2\varepsilon_2 e^{2\varepsilon_4}\alpha_4\right) = \beta_2, \tag{7}$$

$$\frac{1}{a}\left(e^{\varepsilon_4}\alpha_3 - 3\varepsilon_3 e^{\varepsilon_4}\alpha_4\right) = \beta_3, \tag{8}$$

$$\frac{1}{a}\alpha_4 = \beta_4, \tag{9}$$

$$\frac{1}{a}\alpha_5 = \beta_5. \tag{10}$$

Special cases:

- $\alpha_4 \neq 0$. From equation (7), (8) follows that $\varepsilon_2 = \frac{\alpha_2}{2\alpha_4}$, $\varepsilon_3 = \frac{\alpha_3}{3\alpha_4}$. Therefore we have $\vec{\beta} = (1, 0, 0, 1, 1)$;
- $\alpha_4 = 0$, $\alpha_2 \neq 0$. From equation (7) and (8) we have $\frac{1}{a}\left(e^{2\varepsilon_4}\alpha_2\right) = \pm 1$. It follows that $e^{\varepsilon_4} = \sqrt{\frac{1}{|\alpha_2|}}$. Furthermore it is $\frac{1}{a}\left(e^{\varepsilon_4}\alpha_3\right) = \pm 1$ from which we get $\pm\frac{1}{a}\sqrt{\frac{1}{|\alpha_2|}} = \pm 1$. Therefore $\vec{\beta}$ is $\vec{\beta} = (1, \pm 1, \pm 1, 0, 1)$. This gives four linearly independent vectors: $\vec{\beta}_1 = (1, 1, 1, 0, 1)$, $\vec{\beta}_2 = (1, 1, -1, 0, 1)$, $\vec{\beta}_3 = (1, -1, 1, 0, 1)$, $\vec{\beta}_4 = (1, -1, -1, 0, 1)$;
- $\alpha_4 = 0$, $\alpha_2 = 0$, $\alpha_3 \neq 0$. From equation (8) we have $\frac{1}{a}\left(e^{\varepsilon_4}\alpha_3\right) = \pm 1$ from which follows $e^{\varepsilon_4} = \pm\frac{1}{\alpha_3}$. In this case $\vec{\beta} = (1, 0, \pm 1, 0, 1)$. This gives two linearly independent vectors: $\vec{\beta}_1 = (1, 0, 1, 0, 1)$, $\vec{\beta}_2 = (1, 0, -1, 0, 1)$;
- $\alpha_4 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$. Therefore $\vec{\beta}$ is $\vec{\beta} = (1, 0, 0, 0, 1)$;
- $\alpha_4 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, $\alpha_1 = 0$. Therefore $\vec{\beta}$ is $\vec{\beta} = (0, 0, 0, 0, 1)$.

4 Solutions of the Doebner–Goldin–Madelung equations

During the last part of this paper we want to show how to construct solutions for special subalgebras. To do this we choose for example the algebra A_2. Here we consider the generators v_5 and v_6 which generates an Abelian group. The algorithm for calculating solutions is represented in the literature [3, 11]. Because of the fact that the choosen subalgebra is Abelian we have to solve the following equations:

$$v_5 I = 0, \qquad v_6 I = 0,$$

where I are the invariants. We start with the second equation containing v_6. The connected system of characteristics results:

$$\frac{\partial x}{\partial s} = 2t(s)x(s), \qquad \frac{\partial t}{\partial s} = 2t(s)^2, \qquad \frac{\partial \varrho}{\partial s} = -2t(s)\varrho(s), \qquad \frac{\partial S}{\partial s} = x(s)^2. \tag{11}$$

This system can be solved by repeated isolation of s from the other variables of the system (11). The solution is

$$t = \frac{1}{-2s - C_1}, \qquad x = \frac{C_2}{2s + C_1}, \qquad \varrho = (2s + C_1)C_3, \qquad S = \frac{-C_2^2}{2(2s + C_1)} + C_4. \tag{12}$$

Here C_1, C_2, C_3, C_4 are constants of integration. They play the role of the invariants. To eliminate the parameter s we solve the first equation of (12) with respect to the parameter s and put the result into the other equations. We then get

$$s = \frac{1 + tC_1}{-2t}, \qquad x = -tC_2, \qquad \varrho = -\frac{C_3}{t}, \qquad S = \frac{1}{2}\frac{x^2}{t} + C_4.$$

Therefore the invariants are received by isolating the constants:

$$I_1 = \frac{x}{t} = -C_2, \qquad I_2 = t\varrho = -C_3, \qquad I_3 = S - \frac{1}{2}\frac{x^2}{t} = C_4.$$

The next step is to transform the generator v_5 into the basis of the invariants. This means that it is of the form

$$v_5 = (v_5 I_1)\frac{\partial\cdot}{\partial I_1} + (v_5 I_2)\frac{\partial\cdot}{\partial I_2} + (v_5 I_3)\frac{\partial\cdot}{\partial I_3}.$$

Here, the result simply is $v_5 = \frac{\partial\cdot}{\partial I_1}$. Now we have to calculate the invariants from this generator. The system of characteristic equations is

$$\frac{\partial I_1}{\partial s} = 1, \qquad \frac{\partial I_2}{\partial s} = 0, \qquad \frac{\partial I_3}{\partial s} = 0,$$

with the solution

$$I_1 = s + C_1, \qquad J_2 = I_2 = t\varrho, \qquad J_3 = I_3 = S - \frac{1}{2}\frac{x^2}{t},$$

where $C_1 = J_1$, J_2 and J_3 are the invariants of this generator. Now it is

$$J_2 = \phi(J_1), \qquad J_3 = \psi(J_1).$$

With $s = 0$ this leads to

$$\varrho = \frac{\phi(\alpha)}{t}, \qquad S = \psi(\alpha) + \frac{x^2}{2t}, \qquad \alpha = \frac{x}{t}. \tag{13}$$

In this case we can put $s = 0$ because of the fact that we freely can choose the origin of the coordinate system. For calculating the solution we choose equation 2 from Table 1. By introducing the expressions for ϱ and S from the equations (13) we find the following ordinary differential equations:

$$\phi_\alpha(\alpha)\psi_\alpha(\alpha) + \phi(\alpha)\psi_{\alpha\alpha}(\alpha) = 0,$$
$$\phi_\alpha(\alpha)^2 + 2\phi(\alpha)\big(2\phi(\alpha)\psi_\alpha(\alpha)^2 + (4D_2 - 1)\phi_{\alpha\alpha}(\alpha)\big) = 0.$$

It is easy to integrate the first equation to $\psi_\alpha(\alpha) = \frac{A}{\phi(\alpha)}$. By introducing this into the second equation we find

$$4A^2 + \phi_\alpha(\alpha)^2 + 2(4D_2 - 1)\phi(\alpha)\phi_{\alpha\alpha}(\alpha) = 0.$$

To solve this equation, we first divide by the coefficient in front of the second derivative. This provides

$$\phi_{\alpha\alpha}(\alpha) + \frac{\phi_\alpha(\alpha)^2}{2(4D_2 - 1)\phi(\alpha)} + \frac{4A^2}{2(4D_2 - 1)\phi(\alpha)} = 0.$$

Now we substitute $p(\phi) = \phi_\alpha(\alpha)$. The result is

$$pp_\phi + \frac{p^2}{2(4D_2 - 1)\phi} + \frac{4A^2}{2(4D_2 - 1)\phi} = 0$$

with the solution:

$$p = \pm\sqrt{C[1]^2\phi(\alpha)^{\frac{1}{1-4D_2}} - 4A^2}.$$

In the next step we have to calculate $p(\phi) = \phi_\alpha(\alpha)$. With $D_2 = \frac{1}{8}$ this leads to the solution:

$$\phi(\alpha) = \frac{4A^2 e^{\pm\alpha C[1] - C[2]}}{C[1]} + \frac{e^{\mp\alpha C[1] + C[2]}}{4C[1]}.$$

For the other function we find :

$$\psi(\alpha) = B + \arctan\left((4A)^{\pm 1} e^{\alpha C[1] \mp C[2]}\right).$$

This leads to

$$\begin{aligned} \varrho(x,t) &= \frac{4A^2 e^{\pm \frac{x}{t} C[1] - C[2]}}{C[1]t} + \frac{e^{\mp \frac{x}{t} C[1] + C[2]}}{4C[1]t}, \\ S(x,t) &= B + \frac{x^2}{2t} + \arctan\left((4A)^{\pm 1} e^{\frac{x}{t} C[1] \mp C[2]}\right). \end{aligned} \tag{14}$$

To prove that these are solutions of the original equations we put these expressions into system 2 of Table 1. The result is valid with $D_2 = \frac{1}{8}$. The original form of the Doebner–Goldin equation then can be solved by a $\psi(x,t)$ according to the Madelung transformation (3) in $1+1$ dimensions. Especially when discussing a time–independent potential $V(x,t) = \delta(x-a)$ with natural boundary conditions, the results (14) can be useful.

Another special case is given with $D_2 = \frac{1}{4}$ leading to the following differential equation:

$$4A^2 + \phi_\alpha(\alpha)^2 = 0.$$

To get a real–valued density $\varrho(x,t)$ out of this system, $A = i\aleph$ is an imaginary integration constant leading to the following result which is valid formally for any D_2, but for $A = \aleph = 0$ and no imaginary rest in the original Schrödinger context only:

$$\varrho(x,t) = \frac{C[1]}{t} - \frac{2\aleph x}{t^2}, \qquad S(x,t) = B + \frac{x^2}{2t} - \frac{i}{2} \ln\left(\frac{2\aleph x}{t} - C[1]\right).$$

The next case is $A = 0$. The connected differential equation simply is

$$\phi_\alpha^2 + 2(4D_2 - 1)\phi\phi_{\alpha\alpha} = 0$$

with the obvious solution

$$\phi(\alpha) = \left(\frac{(\pm\alpha C[1] - C[2])\,(8D_2 - 1)}{8D_2 - 2}\right)^{\frac{8D_2-2}{8D_2-1}}.$$

Therefore the solution of the original system is

$$\varrho(x,t) = \frac{\left(\frac{(\pm\frac{x}{t} C[1] - C[2])(8D_2-1)}{8D_2-2}\right)^{\frac{8D_2-2}{8D_2-1}}}{t}, \qquad S(x,t) = B + \frac{x^2}{2t},$$

where B, $C[1]$, $C[2]$ are constants. We note that the special case $D_2 = \frac{1}{8}$ leads to a different solution of type (14).

5 Conclusion

This paper demonstrates the solution of the system of the Doebner–Goldin–Madelung equations. The main tool for solving this system was the computer algebra package *MathLie* written in *Mathematica*. By applying this tool to equation (5) we have derived analytical solutions. We demonstrated that *MathLie* is also able to examine the Lie group and the related Lie algebra.

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Transformation of Scientific System of Knowledge in Educational: Symmetry Analysis of Equations of Mathematical Physics

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The paper presents an outline for introduction of symmetry ideas and techniques for solution of partial differential equations into the curriculum for training of teachers of mathematics. The main goal of such special course would be integration of the previously gained knowledge in calculus, algebra and ordinary differential equations together with showing students some developments in modern mathematical science.

1 Introduction

Physical or mathematical theory as the most developed form of scientific knowledge cannot be transferred into a learning process completely, as the system of knowledge used in learning and the scope of scientific knowledge are not identical. The scientific system of knowledge is based on research process directly, and learning system is based on research only indirectly.

In this respect the following questions arise: 1) Whether the system of scientific knowledge that is being studied, is a theory in a strictly logical sense? 2) Whether this system of scientific knowledge would enable formation of theoretical thinking of students and give them comprehensive image of the learning object?

The same learning material can be structured in different ways depending on aims and techniques of learning. In the process of advanced studying physics and mathematics the main attention is concentrated on presentation of the core of fundamental physical theories and enabling students to master their deep essence. In this way the teaching system that employs scientific system of knowledge develops productive forms of reasoning, provides theoretical knowledge for understanding of the physical picture of the world by means of a mathematical model.

Real physical processes are mostly described by nonlinear partial differential equations. The search for exact solutions of such equations is one of the most important stages for mathematical description of nature. At present many efficient methods for solving PDEs: separation of variables, Poisson method, decomposition into Fourier series, inverse problem and others. All these methods are based on the ideas of symmetry and are efficiently employed for solving those problems that have implicit or explicit symmetry.

Mathematical foundation for the theory of symmetry of differential equations was created by prominent Norwegian mathematician Sophus Lie as far as in 1881–1885. Modern development for this theory was provided, in particular, by Ovsiannikov [1], Olver [2], Bluman and Kumei [3], Fushchych and his collaborators [4]. These books together with numerous manuals targeted towards post-graduate students may provide a basis for development of courses on application of symmetry methods to partial differential equations. There is some advancement in Ukraine as to introduction of symmetry methods into university courses, mostly as special courses for graduate students. However, basic calculus and algebra courses, especially for mathematical education students, underwent little changes through the recent half a century.

All that stimulated the author into development of a course “Group Theoretical Techniques for Investigation and Solution of Partial Differential Equations”. The main goal of the course is formation of modern perception of symmetry and ensuring fundamental mathematical background for future teachers of mathematics. The curriculum for this course determines the scope of knowledge needed for professional formation at Specialist’s and Master’s degrees in Mathematics and Mathematical Education. Added value of the course is integration of knowledge and skills gained in other mathematical courses – calculus, algebra, ordinary and partial differential equations, together with being a good foundation for preparation of degree theses.

2 Plan for the course on group theoretical techniques for investigation and solution of partial differential equations

The course is targeted at mathematical education students that are interested in gaining advanced knowledge in mathematics. It is assumed that students’ background includes calculus, algebra and ordinary differential equations. As a preparation to studying symmetry techniques, the course includes also basics of partial differential equations. The curriculum for the course includes 16 hours of lectures and 16 hours of practical seminars. Evaluation of the course results is planned on the basis of the individual written paper. Layout of the course is suggested as follows:

	Topic of the lecture	No. of hours
1.	Partial differential equations. Main Definitions. Examples. Cauchy problem and boundary problem. First order uniform linear partial differential equations. First order quasilinear equations. Examples.	2
2.	Classification of second order PDEs. Reduction of an equation to the canonical form. Survey of methods for solution of differential equations: d’Alembert method and Fourier method for separation of variables.	2
3.	Introduction to the theory of Lie groups. Historical survey of development of the symmetry theory. One-parameter transformation groups. Definitions, examples. The problem of construction of a group. Lie theorem. Operators of translation, rotation, scale transformation.	2
4.	Infinitesimal operator. Prolongation of an operator. Invariance criterion. Lie algorithm for investigation of symmetry of PDEs.	2
5.	Symmetry of the d’Alembert equation. The conformal group $C(1,3)$. Basis operators and invariance transformations.	2
6.	Group analysis of a nonlinear wave equation. Euler–Lagrange system. Construction of a maximal invariance algebra.	2
7.	Reduction and exact solutions of nonlinear wave equation. Derivation of new solutions.	2
8.	Symmetry of a first-order nonlinear equation. Symmetry properties and exact solutions for the eikonal equation.	2

The course accounts for the fact that partial differential equations are not included into basis courses for graduate students in Mathematical Education. First two lectures of the course present to the students main concepts of partial differential equations. Core part or the course is example-based explanation of the main concepts and techniques of the group analysis of PDEs with special consideration given to application and strengthening of the skills obtained in the previous basic courses.

3 A practical problem with solution

Here we show an example that may be presented to students with the reference to the needed basic skills. The task is to show that the the heat equation

$$u_t - u_{xx} = 0 \tag{1}$$

is invariant under the Galilei transformations

$$t' = t, \qquad x' = x + 2at, \qquad u' = ue^{-(ax+a^2t)}. \tag{2}$$

Invariance of the equation (1) can be checked by the formulae of substitution of variables in the partial derivatives:

$$u'_{t'} = \left(u_t + a^2u - 2au_x\right) e^{-(ax+a^2t)}, \qquad u'_{x'} = (u_x - au)e^{-(ax+a^2t)},$$
$$u'_{x'x'} = \left(u_{xx} + a^2u - 2au_x\right) e^{-(ax+a^2t)}.$$

From here we derive an equality

$$u'_{t'} - u'_{x'x'} = (u_t - u_{xx})e^{-(ax+a^2t)}$$

showing that under the action of the transformations (2) the equation (1) is transformed into the same equation in the transformed variables, so the invariance in proved.

The next task is to find an infinitesimal operator corresponding to the transformations (2). This task requires differentiation of the expressions for the transformations (2) and substitution of the condition $a = 0$:

$$\begin{aligned} X &= \left(\frac{dt'}{da}\frac{\partial}{\partial t} + \frac{dx'}{da}\frac{\partial}{\partial x} + \frac{du'}{da}\frac{\partial}{\partial u}\right)\Big|_{a=0} \\ &= \left(0 \cdot \frac{\partial}{\partial t} + 2t\frac{\partial}{\partial x} + ue^{-(ax+a^2t)}(-(x+2at))\frac{\partial}{\partial u}\right)\Big|_{a=0} = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}. \end{aligned}$$

We see that appropriate breaking of the group analysis techniques into steps makes them accessible for mathematical education students with reasonable basic background in mathematics, providing them with practice in their calculation skills together with understanding of the value of this skills for practical problem solving and research.

4 Conclusions

Such special course contributes to widening of scientific horizons of students who are mainly future teachers of mathematics, deeper understanding of the role of modern mathematics and gives insight to development of modern Ukrainian mathematical school of thought.

The author also used simplified version of the course with more links to school geometry and physics in the optional secondary school course, with student research papers prepared as a result. One of these papers by Julia Lashkevych received a 2nd award at the 1998 Ukrainian Competition of Secondary School Student Research Papers.

Another aspect that provides particular benefits of introduction of the course into advanced mathematics teachers' training curriculum is wide use of symmetry in the secondary school courses, mainly in geometry and physics. School courses deal mostly with discrete symmetries in the Euclidean space (translation, rotation, mirror and central symmetries), and widening of the notion of symmetry seems appropriate for teachers' training. Further development of the course, if more hours could be used, may be towards introduction of the notions of conservation laws and the relevant links with the course of physics.

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Differential Invariants and Construction of Conditionally Invariant Equations

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New concept of conditional differential invariant is discussed that would allow description of equations invariant with respect to an operator under a certain condition. Example of conditional invariants of the projective operator is presented.

1 Introduction

Importance of investigation of symmetry properties of differential equations is well-established in mathematical physics. Classical methods for studying symmetry properties and their utilisation for finding solutions of partial differential equations were originated in the papers by S. Lie, and developed by modern authors (see e.g. [1, 2, 3, 4]).

We start our consideration from some symmetry properties and solutions of the nonlinear wave equation

$$\Box u = F(u, u^*) \tag{1}$$

for the complex-valued function $u = u(x_0, x_1, \ldots, x_n)$, $x_0 = t$ is the time variable, $x_1, \ldots, x_n$ are n space variables. F is some function. $\Box u$ is the d'Alembert operator

$$\Box u = -\frac{\partial^2 u}{\partial x_0^2} + \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}. \tag{2}$$

It is well-known that the equation (1) may be reduced to a nonlinear Schrödinger equation with the number of space dimensions smaller by 1, when the nonlinearity F has a special form $F = uf(|u|)$, where $|u| = (uu^*)^{1/2}$, an asterisk designates complex conjugation.

Further we are trying to generalise this relation between the nonlinear wave equation and the nonlinear Schrödinger equation into a relation between differential invariants of the respective invariance algebrae, and introduce new concepts of the reduction of fundamental sets of differential invariants and of conditional differential invariants. Conditional differential invariants may be utilised to describe conditionally invariant equations under certain operators and with the certain conditions, in the same manner as absolute differential invariants of a Lie algebra may be utilised for description of all equations invariant under this algebra.

The concept of non-classical, or conditional symmetry, originated in its various facets in the papers [5, 6, 7, 8, 9, 10] and later by numerous authors was developed into the theory and a number of algorithms for studying symmetry properties of equations of mathematical physics and for construction of their exact solutions. Here we will use the following definition of the conditional symmetry:

Definition 1. The equation $F(x, u, \underset{1}{u}, \ldots, \underset{l}{u}) = 0$ where $\underset{k}{u}$ is the set of all kth-order partial derivatives of the function $u = (u^1, u^2, \ldots, u^m)$, is called conditionally invariant under the operator

$$Q = \xi^i(x, u)\partial_{x_i} + \eta^r(x, u)\partial_{u^r} \tag{3}$$

if there is an additional condition

$$G(x, u, \underset{1}{u}, \dots, \underset{l_1}{u}) = 0, \tag{4}$$

such that the system of two equations $F = 0$, $G = O$ is invariant under the operator Q.

If (4) has the form $G = Qu$, then the equation $F = 0$ is called Q-conditionally invariant under the operator Q.

2 Differential invariants and description of invariant equations

Differential invariants of Lie algebrae present a powerful tool for studying partial differential equations and construction of their solutions [21, 22, 23].

Now we will present some basic definitions that we will further generalise. For the purpose of these definitions we deal with Lie algebrae consisting of the infinitesimal operators

$$X = \xi^i(x, u)\partial_{x_i} + \eta^r(x, u)\partial_{u^r}. \tag{5}$$

Here $x = (x_1, x_2, \dots, x_n)$, $u = (u^1, u^2, \dots, u^m)$.

Definition 2. The function $F = F(x, u, \underset{1}{u}, \dots, \underset{l}{u})$, is called a differential invariant for the Lie algebra L with basis elements X_i of the form (5) ($L = \langle X_i \rangle$) if it is an invariant of the l th prolongation of this algebra:

$$\underset{l}{X_s} F(x, u, \underset{1}{u}, \dots, \underset{l}{u}) = \lambda_s(x, u, \underset{1}{u}, \dots, \underset{l}{u})F, \tag{6}$$

where the λ_s are some functions; when $\lambda_i = 0$, F is called an absolute invariant; when $\lambda_i \neq 0$, it is a relative invariant.

Further when writing “differential invariant” we would imply “absolute differential invariant”.

Definition 3. A maximal set of functionally independent invariants of order $r \leq l$ of the Lie algebra L is called a functional basis of the l th-order differential invariants for the algebra L.

While writing out lists of invariants we shall use the following designations

$$\begin{gathered} u_a \equiv \frac{\partial u}{\partial x_a}, \qquad u_{ab} \equiv \frac{\partial^2 u}{\partial x_a \partial x_b}, \qquad S_k(u_{ab}) \equiv u_{a_1 a_2} u_{a_2 a_3} \cdots u_{a_{k-1} a_k} u_{a_k a_1}, \\ S_{jk}(u_{ab}, v_{ab}) \equiv u_{a_1 a_2} \cdots u_{a_{j-1} a_j} v_{a_j a_{j+1}} \cdots v_{a_k a_1}, \\ R_k(u_a, u_{ab}) \equiv u_{a_1} u_{a_k} u_{a_1 a_2} u_{a_2 a_3} \cdots u_{a_{k-1} a_k}. \end{gathered} \tag{7}$$

In all the lists of invariants j takes the values from 0 to k. We shall not discern the upper and lower indices with respect to summation: for all Latin indices $x_a x_a \equiv x_a x^a \equiv x^a x_a = x_1^2 + x_2^2 + \cdots + x_n^2$.

Fundamental bases of differential invariants for the standard scalar representations of the Poincaré and Galilei algebra of the types (17), (12) were found in [24]. Fundamental bases of differential invariants allow describing all equations invariant under the respective Lie algebrae.

Construction of conditional differential invariants would allow describing all equations, conditionally invariant with respect to certain operators under certain conditions.

Definition 4. $F = F(x, u, \underset{1}{u}, \dots, \underset{l}{u})$ is called a conditional differential invariant for the operator with X of the form (5) if under the condition

$$G(x, u, \underset{1}{u}, \dots, \underset{l_1}{u}) = 0, \tag{8}$$

$$\underset{l_{\max}}{X} F(x, u, \underset{1}{u}, \dots, \underset{l}{u}) = 0, \qquad \underset{l_{\max}}{X} G(x, u, \underset{1}{u}, \dots, \underset{l_1}{u}) = 0, \tag{9}$$

$\underset{l_{\max}}{X}$ being the $l_{\max}$th prolongation of the operator X. The order of the prolongation $l_{\max} = \max(l, l_1)$.

3 Nonlinear wave equation, nonlinear Schrödinger equation and relation between their symmetries

The Galilei algebra for $n-1$ space dimensions is a subalgebra of the Poincaré algebra for n space dimensions (see e.g. [11]) and references therein), and this fact allows reduction of the nonlinear wave equation (1) to the Schrödinger equation. We will consider the nonlinear wave equations for three space variables, and its symmetry properties in relation to the symmetry properties of the nonlinear Schrödinger equation for two space variables. However, all the results can be easily generalised for arbitrary number of space dimensions.

Reduction of the nonlinear wave equation (1) to the Schrödinger equation can be performed by means of the ansatz

$$u = \exp((-im/2)(x_0 + x_3))\Phi(x_0 - x_3, x_1, x_2). \tag{10}$$

Substitution of the expression (10) into (1) gives the equation $\exp((-im/2)(x_0+x_3))(2im\Phi_\tau + \Phi_{11} + \Phi_{22}) = F(u, u^*)$. Here we adopted the following notations: $\tau = x_0 + x_3$ is the new time variable, $\Phi_\tau = \frac{\partial\Phi}{\partial\tau}$, $\Phi_a = \frac{\partial\Phi}{\partial x_a}$, $\Phi_{ab} = \frac{\partial^2\Phi}{\partial x_a \partial x_b}$.

Further on we adopt the convention that summation is implied over the repeated indices. If not stated otherwise, small Latin indices run from 1 to 2.

If the nonlinearity in the equation (1) has the form $F = uf(|u|)$, then it reduces to the Schrödinger equation

$$2im\Phi_\tau + \Phi_{11} + \Phi_{22} = \Phi f(|\Phi|). \tag{11}$$

Such reduction allowed construction of numerous new solutions for the nonlinear wave equation by means of the solutions of a nonlinear Schrödinger equation [12, 13]. We show that this reduction allowed also to describe additional symmetry properties for the equation (1), related to the symmetry properties of the equation (11).

Lie symmetry of the equation (11) was described in [14, 16]. With an arbitrary function f it is invariant under the Galilei algebra with basis operators

$$\begin{aligned} &\partial_\tau = \frac{\partial}{\partial\tau}, \qquad \partial_a = \frac{\partial}{\partial x_a}, \qquad J_{12} = x_1\partial_2 - x_2\partial_1, \\ &G_a = t\partial_a + ix_a(\Phi\partial_\Phi - \Phi^*\partial_{\Phi^*}) \quad (a = 1, 2), \qquad J = (\Phi\partial_\Phi - \Phi^*\partial_{\Phi^*}). \end{aligned} \tag{12}$$

When $f = \lambda|u|^2$, where λ is an arbitrary constant, the equation (11) is invariant under the extended Galilei algebra that contains besides the operators (12) also the dilation operator

$$D = 2\tau\partial_\tau + x_a\partial_a - I, \tag{13}$$

where $I = \Phi\partial_\Phi + \Phi^*\partial_{\Phi^*}$, and the projective operator

$$A = \tau^2\partial_\tau + \tau x_a\partial_a + \frac{im}{2}x_a x_a J - \tau I. \tag{14}$$

Lie reductions and families of exact solutions for multidimensional nonlinear Schrödinger equations were found at [15, 16, 17, 18, 19, 20]. Note that the ansatz (10) is the general solution of the equation

$$u_0 + u_3 + imu = 0. \tag{15}$$

We can regard the equation (15) as the additional condition imposed on the nonlinear wave equation with the nonlinearity $F = \lambda u|u|^2$. Solution of the resulting system

$$\Box u = \lambda u|u|^2, \tag{16}$$

with the equation (15) would allow to reduce number of independent variables by one, and obtain the same reduced equation, invariant under the extended Galilei algebra with the projective operator. This allows establishing conditional invariance of the nonlinear wave equation (16) under the projective operator. It is well-known that it is not invariant under this operator in the Lie sense.

The maximal invariance algebra of the equation (1) that may be found according to the Lie algorithm (see e.g. [1, 2, 3, 4]) is defined by the following basis operators:

$$p_\mu = ig_{\mu\nu}\frac{\partial}{\partial x_\nu}, \qquad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \tag{17}$$

where μ, ν take the values $0, 1, \ldots, 3$; the summation is implied over the repeated indices (if they are small Greek letters) in the following way: $x_\nu x_\nu \equiv x_\nu x^\nu \equiv x^\nu x_\nu = x_0^2 - x_1^2 - \cdots - x_n^2$, $g_{\mu\nu} = \mathrm{diag}\,(1, -1, \ldots, -1)$.

However, summation for all derivatives of the function u is assumed as follows: $u_\nu u_\nu \equiv u_\nu u^\nu \equiv u^\nu u_\nu = -u_0^2 + u_1^2 + \cdots + u_n^2$.

Unlike the standard convention on summation of the repeated upper and lower indices we consider x_ν and x^ν equal with respect to summation not to mix signs of derivatives and numbers of functions.

Theorem 1. *The nonlinear wave equation (16) is conditionally invariant with the condition (15) under the projective operator*

$$\begin{aligned} A_1 = {}& \frac{1}{2}(x_0 - x_3)^2(\partial_0 - \partial_3) + (x_0 - x_3)(x_1\partial_1 + x_2\partial_2) \\ & + \frac{imx^2}{2}(u\partial_u - u^*\partial_{u^*}) + \frac{n-1}{2}(x_0 - x_3)(u\partial_u + u^*\partial_{u^*}). \end{aligned} \tag{18}$$

To prove Theorem 1 it is sufficient to show that the system (16), (15) is invariant under the operator (18) by means of the classical Lie algorithm.

Our further study aims at construction of other Poincaré-invariant equations possessing the same conditional invariance property.

4 Example: construction of conditional differential invariants

Now we adduce fundamental bases of differential invariants that will be utilised for construction of our example of conditional differential invariants.

First we present a functional basis of differential invariants for the Poincaré algebra (17) of the second order for the complex-valued scalar function $u = u(x_0, x_1, \ldots, x_3)$. It consists of 24 invariants

$$u^r, \qquad R_k\left(u_\mu^r, u_{\mu\nu}^1\right), \qquad S_{jk}\left(u_{\mu\nu}^r, u_{\mu\nu}^1\right). \tag{19}$$

In (19) everywhere $k = 1, \ldots, 4$; $j = 0, \ldots, k$. A functional basis of differential invariants for the Galilei algebra (12), mass $m \neq 0$, of the second order for the complex-valued scalar function $\Phi = \Phi(\tau, x_1, \ldots, x_2)$ consists of 16 invariants.

For simplification of the expressions for differential invariants we introduced the following notations:

$$\Phi = \exp\phi, \qquad \operatorname{Im}\Phi = \arctan\frac{\operatorname{Re}\phi}{\operatorname{Im}\phi}.$$

The elements of the functional basis may be chosen as follows:

$$\begin{gathered}\phi+\phi^*, \quad M_1 = 2im\phi_t + \phi_a\phi_a, \quad M_1^*, \quad M_2 = -m^2\phi_{tt} + 2im\phi_a\phi_{at} + \phi_a\phi_b\phi_{ab}, \quad M_2^*, \\ S_{jk}(\phi_{ab}, \phi_{ab}^*), \quad R_j^1 = R_j(\theta_a, \phi_{ab}), \quad R_j^2 = R_j(\theta_a^*, \phi_{ab}), \quad R_j^3 = R_j(\phi_a + \phi_a^*, \phi_{ab}) \end{gathered} \tag{20}$$

Here $\theta_a = im\phi_{at} + \phi_a\phi_{ab}, \phi_{ab}$ are covariant tensors for the Galilei algebra.

A functional basis of differential invariants for the Galilei algebra (12) extended by the dilation operator (13) and the projective operator (14) may be chosen as follows:

$$\begin{gathered}N_1 e^{-2(\phi+\phi^*)}, \quad \frac{N_1}{N_1^*}, \quad \frac{N_2}{N_1^2}, \quad \frac{N_2^*}{(N_1^*)^2}, \quad S_{jk}(\rho_{ab}, \rho_{ab}^*), \quad R_j(\rho_a, \rho_{ab}), \\ R_j(\rho_a^*, \rho_{ab}), \quad R_j(\phi_a + \phi_a^*, \rho_{ab})N_1^{-1}, \quad (\phi_{aa} + \phi_{aa}^*)N_1^{-1}, \end{gathered} \tag{21}$$

where

$$N_1 = M_1 + \phi_{aa} = 2im\phi_t + \phi_{aa} + \phi_a\phi_a, \qquad N_2 = \frac{1}{n}\phi_{aa}N_1 + \frac{\phi_{aa}^2}{2n} + M_2 \tag{22}$$

and the covariant tensors have the form

$$\rho_a = \theta_a N_1^{-3/2}, \qquad \rho_{ab} = \left(\phi_{ab} - \frac{\delta_{ab}}{n}\phi_{cc}\right)N_1^{-1}.$$

An algorithm for construction of conditional differential invariants may be derived directly from the Definition 4. Such invariants may be found by means of the solution of the system (9), (8).

We can construct conditional differential invariants of the Poincaré algebra (17) and the projective operator (18) solving the system

$$\underset{2}{A_1} F(\operatorname{Inv}_P) = 0, \qquad u_0 + u_3 + imu = 0,$$

where Inv_P are all differential invariants (19) of the Poincaré algebra (17). Using the fact that the ansatz (10) is the general solution of the additional condition (15), we can directly substitute this ansatz into differential invariants (19). To avoid cumbersome formulae here we did not list expressions for all differential invariants from (19).

The expression $\Box u$ transforms into the following:

$$\Box u = u(2im\phi_\tau + \phi_{aa} + \phi_a\phi_a),$$

where N_1 is an expression entering into expression for differential invariants (20). Further we get

$$\begin{aligned} u_\mu u_\mu &= u^2(2im\phi_t + \phi_a\phi_a), \\ u_\mu u_\nu u_{\mu\nu} &= u^3(\phi_a\phi_b\phi_{ab} + (\phi_a\phi_a)^2 - m^2(\phi_{tt} + 4\phi_t^2) + \phi_a\phi_b\phi_{ab} + (\phi_a\phi_a)^2 \\ &\qquad - m^2(\phi_{tt} + 4\phi_t^2) + 2im\phi_a\phi_{at} + 4im\phi_t\phi_a\phi_a), \end{aligned} \tag{23}$$

Substituting the ansatz (10) to all elements of the fundamental basis (19) of second-order differential invariants of the Poincaré algebra similarly to (23), we can obtain reduced basis of differential invariants, that may be used for construction of all equations reducible by means of this ansatz. We can give the following representation of the Poincaré invariants using expressions M_k (20) and N_k (21), where in the expressions for M_k, N_k $(k = 1, 2)$ time variable is $\tau = x_0 - x_3$:

$$\begin{aligned}
&\Box u = uN_1, \quad u_\mu u_\mu = u^2 M_1, \quad u_\mu u_\nu u_{\mu\nu} = u^3\left(M_2 + M_1^2\right), \\
&u_{\mu\nu} u_{\mu\nu} = u^2\left(2M_2 + M_1^2 + \phi_{ab}\phi_{ab}\right), \\
&u_\mu u_\mu^* = \frac{uu^*}{2}\left(M_1 + M_1^* - (\phi_a + \phi_a^*)(\phi_a + \phi_a^*)\right).
\end{aligned} \tag{24}$$

Here a, b take values from 1 to 2.

Whence

$$\begin{aligned}
&M_1 = u_\mu u_\mu u^{-2}, \quad \phi_{aa} = N_1 - M_1 = \frac{u\Box u - u_\mu u_\mu}{u^2}, \\
&M_2 = u_\mu u_\nu u_{\mu\nu} u^{-3} - (u_\mu u_\mu)^2 u^{-4}, \quad N_1 = \frac{\Box u}{u}, \\
&N_2 = \frac{1}{n}\phi_{aa} N_1 + \frac{\phi_{aa}^2}{2n} + M_2 = u_\mu u_\nu u_{\mu\nu} u^{-3} - (u_\mu u_\mu)^2 u^{-4} \\
&\qquad + \frac{1}{n}\frac{\Box u}{u}\frac{u\Box u - u_\mu u_\mu}{u^2} + \frac{1}{2n}\frac{(u\Box u - u_\mu u_\mu)^2}{u^4}, \\
&R_1(\phi_a + \phi_a^*, \rho_{ab})N_1^{-1} = (\phi_a + \phi_a^*)(\phi_a + \phi_a^*)N_1^{-1} \\
&\qquad = \left(N_1 + N_1^* - \frac{2}{uu^*}u_\mu u_\mu^*\right)N_1^{-1} = \frac{u^*\Box u + u\Box u^* - 2u_\mu u_\mu^*}{u^*\Box u}.
\end{aligned} \tag{25}$$

We construct Poincaré-invariant conditional differential invariants of the projective operator (18) under the condition (15) using differential invariants (20)

$$\begin{aligned}
&I_1 = N_1 e^{-2(\phi+\phi^*)} = \frac{\Box u}{u(uu^*)^2}, \quad I_2 = \frac{N_1}{N_1^*} = \frac{u^*\Box u}{u\Box u^*}, \\
&I_3 = \frac{N_2}{N_1^2} = \left(uu_\mu u_\nu u_{\mu\nu} + \frac{3}{2n}u^2(\Box u)^2 + \left(\frac{1}{2n} - 1\right)(u_\mu u_\mu)^2 - \frac{2}{n}u\Box u(u_\mu u_\mu)\right)\left(u^2(\Box u)^2\right)^{-1}, \\
&I_4 = R_1(\phi_a + \phi_a^*, \rho_{ab})N_1^{-1} = \frac{u^*\Box u + u\Box u^* - 2u_\mu u_\mu^*}{u^*\Box u}.
\end{aligned} \tag{26}$$

Whence, we may state that all equations of the form $F(I_1, I_2, I_3, I_4) = 0$ are conditionally invariant with respect to the operator A_1 (18) with the additional condition (15).

Finding similar representations for all elements of the functional basis (20) of the second-order differential invariants of the Galilei algebra (12) extended by the dilation operator (13) and the projective operator (14), we can construct functional basis of conditional differential operators. Such basis would allow to describe all Poincaré-invariant equations for the scalar complex-valued functions that are conditionally invariant under the operator A_1 (18).

5 Conclusion

The procedure for finding conditional differential invariants outlined above may be used for other cases when the additional condition (8) has the general solution that may be used as ansatz, and when a functional basis of the operator (9) in the variables involved in such reduction is already known.

Besides finding new conditionally invariant equations, further developments of the ideas presented in this paper may be description of all equations reducible by means of a certain ansatz, and search of methods for restoration of original equations from the reduced equations.

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Symmetries of Integro-Differential Equations

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The Ovsiannikov method of finding Lie symmetries is generalized to the case of point transformations of integro-differential equations. The new method is direct and applicable to practical cases, for instance to Vlasov–Maxwell equations of plasmas.

1 Introduction

We present a general and direct method of determination of symmetry groups of point transformations for integro-differential equations. The method is a natural generalization of the Ovsiannikov method for differential equations [1, 2, 3, 4, 5, 6].

We consider a system of integro-differential equations (IDE's) of the form

$$W\left(F(x, y, \underset{1}{y}, \dots, \underset{m}{y}), \int_X dx^1 \cdots dx^l f(x, y, \underset{1}{y}, \dots, \underset{k}{y})\right) = 0, \tag{1}$$

where n, m, k, l are arbitrary natural numbers ($l \leq n$), $x = (x^1, \dots, x^n)$, functions W, F and f are arbitrary but sufficiently regular to secure the existence of solutions to (1), limits of integrations (region X) are also arbitrary. The symbol $\underset{m}{y}$ denotes the set of all partial derivatives of m-order:

$$\underset{m}{y} = \left\{ \frac{\partial^m y}{\partial x^{i_1} \cdots \partial x^{i_m}} \equiv \partial_{x^{i_1}} \cdots \partial_{x^{i_m}} y \equiv y_{i_1 \cdots i_m} \right\}.$$

The equation (1) reduces to a differential equation for $f = 0$, thus our method contains the Ovsiannikov method as a particular case.

Earlier approaches to investigations of symmetries of IDE's can be found in [7], in CRC Handbook [8], and in references therein. The lack of a general and universal method has led to many attempts using various methods which often constitute *ad hoc* means adapted for each case. For example, specific kinds of IDE's were chosen so that certain methods could be used effectively. In [9] the integral term of the IDE has the form of a square root of a differential operator. The method used there consists in finding a partial differential equation (PDE) with the space of solutions containing the solutions of the considered IDE. After finding symmetries of the auxiliary PDE by standard method the symmetries of IDE are found by inspection. In [10] the IDE with the integral in the form of a Fourier transform is considered. In this case the Lie derivative is found effectively and used for the determination of symmetries.

Methods called *indirect* methods form a separate class. They are based on a transformation of a given set of IDE's to an equivalent set of auxiliary equations for which symmetries are known or can be found by known methods. Then symmetries of the initial system of IDE's can be reconstructed. Usually, this auxiliary set of equations consists of PDE's as, for example, in Taranov's method [11]. He transformed the Vlasov–Maxwell equations for one-component plasma into an infinite chain of differential equations for the moments of a distribution function. Another indirect approach is based on an extension of the Harrison and Estabrook method [12] to the case of IDE's. A given set of equations is transformed to an equivalent set of differential forms. This method was used in [13] for IDE's invariant with respect to Galilei, Poincaré, Schrödinger and conformal groups, in [14] for the Boltzmann equation and in [15, 16] for IDE's of Hartree type. These methods are encumbered with the usual burden of indirect methods. They involve the necessary movement there and back with the crucial problem of equivalence and an interpretation of results. Moreover, quite often an auxiliary problem is more complicated than the initial one when our direct method is applied.

A direct method is presented in [5] and in Chapter 5 of Vol.3 of CRC Handbook [8]. It consists in assuming equal to zero the derivative with respect to the group parameter of a transformed IDE (depending on the parameter) at zero value of this parameter. When this condition is properly evaluated, that is when the dependence of limits of an integral on the group parameter is taken into account, then it leads to our criterion of symmetry of IDE's (8). However, this evaluation must be done every time when this condition is used. This may be suitable for a computer (see [17]) but not for a man. The dependence of limits of an integral (even constant limits!) on the group parameter is sometimes overlooked in certain papers. Moreover, it is more appropriate to consider a region of integration since the expression of the n-dimensional integral by the n-fold integral is not invariant with respect to point transformations. The problem disappears for Bäcklund symmetries in the form of vertical transformations because there is no transformation of independent variables in this case. The method was used in [18, 19] for finding symmetries of the Boltzmann equation of a special kind.

The general and sophisticated method of Vinogradov and Krasilshchik [20, 21] has arisen from a simple idea of elimination of integrals from IDE's by virtue of the fundamental theorem of calculus by further prolongation to *nonlocal* variables: the primitive functions of dependent variables. This is natural in the case of IDE's with variable limits of integrals, for example for the Volterra type of IDE's. However, the most important IDE's in physics, such as equations of kinetic theory, contain integrals with constant limits. Then, this construction is somewhat artificial and complicated. The method becomes indirect since it leads to the so called *boundary-differential* equations [21]. The method requires advanced and sophisticated mathematics, for example the theory of coverings of a system of differential equations and the prolongation procedure for boundary–differential equations. The method was used in [22, 23] for finding symmetries of the coagulation kinetic equation.

Since, in general, an integral structure of equations cannot be transformed into an algebraic one by admitting nonlocal variables, we stay in a jet space to deal with derivatives, as in the Ovsiannikov method, and find a new infinitesimal criterion of symmetry in our case of IDE's. This criterion is the essence of our direct method.

2 Extension of a group

We look for a Lie symmetry group of *point transformations*

$$\widetilde{x}^i = e^{\epsilon G} x^i = x^i + \epsilon \xi^i(x,y) + \mathcal{O}\left(\epsilon^2\right), \qquad \widetilde{y} = e^{\epsilon G} y = y + \epsilon \eta(x,y) + \mathcal{O}\left(\epsilon^2\right), \tag{2}$$

with the *infinitesimal generator* (summation over repeated indices is assumed)

$$G = \xi^i(x,y)\partial_{x^i} + \eta(x,y)\partial_y, \tag{3}$$

admitted by the system of IDE's (1). As in the Ovsiannikov method we extend the group of point transformations (2) to a jet space of independent and dependent variables and derivatives of dependent variables in the usual way [1, 2, 3, 4, 5, 6]

$$
\begin{aligned}
&\widetilde{x}^i = e^{\epsilon G^{(m)}} x^i = x^i + \epsilon \xi^i(x, y) + \mathcal{O}\left(\epsilon^2\right), \\
&\widetilde{y} = e^{\epsilon G^{(m)}} y = y + \epsilon \eta(x, y) + \mathcal{O}\left(\epsilon^2\right), \\
&\widetilde{y}_i = e^{\epsilon G^{(m)}} y_i = y_i + \epsilon \eta_i(x, y, \underset{1}{y}) + \mathcal{O}\left(\epsilon^2\right), \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\widetilde{y}_{i_1\cdots i_m} = e^{\epsilon G^{(m)}} y_{i_1\cdots i_m} = y_{i_1\cdots i_m} + \epsilon \eta_{i_1\cdots i_m}(x, y, \underset{1}{y}, \ldots, \underset{m}{y}) + \mathcal{O}\left(\epsilon^2\right),
\end{aligned}
\tag{4}
$$

where the *extended generator* is of the form

$$
G^{(m)} = G + \eta_i \partial_{y_i} + \cdots + \eta_{i_1\cdots i_m} \partial_{y_{i_1\cdots i_m}}. \tag{5}
$$

The coefficients $\eta_i, \ldots, \eta_{i_1\cdots i_m}$, defining the extended group, are given by the recursion relations:

$$
\begin{aligned}
&\eta_i = D_i \eta - y_j D_i \xi^j, \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
&\eta_{i_1\cdots i_m} = D_{i_m} \eta_{i_1\cdots i_{m-1}} - y_{i_1\cdots i_{m-1} j} D_{i_m} \xi^j
\end{aligned}
\tag{6}
$$

and the total derivative D_i is defined as follows

$$
D_i = \partial_i + y_i \partial_y + y_{ij} \partial_{(y_j)} + \cdots + y_{i i_1\cdots i_n} \partial_{(y_{i_1\cdots i_n})} + \cdots .
$$

3 Criterion of invariance of integro-differential equations

Invariance of an equation means invariance of the space of its solutions. Thus, point transformation (2) maps any solution $y(x)$ of the equation (1) into another solution $\widetilde{y}(\widetilde{x})$ of the equation. In our geometric language, where solutions $y(x)$ are represented by their graphs in a jet space, it means that the following implication holds

$$
W(F, I) = 0 \quad \Longrightarrow \quad W(\widetilde{F}, \widetilde{I}) = 0, \tag{7}
$$

where I means integral term in (1), $\widetilde{F} \equiv F(\widetilde{\cdot})$ and $\widetilde{I}$ are obtained by extended transformations (4).

According to the definition (7), we act on the integro-differential equation (1) by extended transformations (4) writing down explicitly only terms that are linear with respect to the parameter ϵ. Next, by expanding functions W, F and f in their Taylor series and changing variables in the integral, we express the change of (1) in terms of the extended generator (5). From the definition of symmetry (7), this change must be equal to zero for all values of ϵ. Thus, we obtain an infinitesimal criterion of invariance of the equation (1).

We restrict our considerations to the one scalar equation of the type (1) for the sake of simplicity of notation. For a system of equations with p dependent variables $y = (y^1, \ldots, y^p)$ some minor changes are evident and the resulting criterion is to be applied to each equation of the system. Expanding the function W in a Taylor series, we obtain

$$
W(\widetilde{F}, \widetilde{I}) = W(F, I) + \frac{\partial W}{\partial F} \Delta F + \frac{\partial W}{\partial I} \Delta I + \cdots .
$$

The change ΔF of the differential term of (1) is calculated by expanding the function F in a Taylor series

$$\begin{aligned}
\Delta F &= F(\widetilde{x}, \widetilde{y}, \underset{1}{\widetilde{y}}, \ldots, \underset{m}{\widetilde{y}}) - F(x, y, \underset{1}{y}, \ldots, \underset{m}{y}) \\
&= F\Big(x^1 + \epsilon\xi^1 + \mathcal{O}\left(\epsilon^2\right), \ldots, x^n + \epsilon\xi^n + \mathcal{O}\left(\epsilon^2\right), y + \epsilon\eta + \mathcal{O}\left(\epsilon^2\right), y_1 + \epsilon\eta_1 + \mathcal{O}\left(\epsilon^2\right), \\
&\quad \ldots, y_n + \epsilon\eta_n + \mathcal{O}\left(\epsilon^2\right), y_{i_1\ldots i_m} + \epsilon\eta_{i_1\cdots i_m} + \mathcal{O}\left(\epsilon^2\right)\Big) - F(x, y, \underset{1}{y}, \ldots, \underset{m}{y}) \\
&= \epsilon\left[\xi^i\partial_{x^i}F + \eta\partial_y F + \eta_i\partial_{y_i}F + \cdots + \eta_{i_1\cdots i_m}\partial_{y_{i_1}}\cdots\partial_{y_{i_m}}F\right] + \mathcal{O}\left(\epsilon^2\right).
\end{aligned}$$

Due to the definition of the extended generator (5) we can rewrite the above result in the form

$$\Delta F = \epsilon G^{(m)} F(x, y, \underset{1}{y}, \ldots, \underset{m}{y}) + \mathcal{O}\left(\epsilon^2\right).$$

Thus, the condition $\Delta F = 0$ leads to the Ovsiannikov infinitesimal criterion of invariance of differential equation $G^{(m)}F(x, y, \underset{1}{y}, \ldots, \underset{m}{y}) = 0$.

Let us consider the change of an integral term in the equation (1)

$$\Delta I = \int_{\widetilde{X}} d\widetilde{x}^1 \cdots d\widetilde{x}^l f(\widetilde{x}, \widetilde{y}, \underset{1}{\widetilde{y}}, \ldots, \underset{k}{\widetilde{y}}) - \int_X dx^1 \cdots dx^l f(x, y, \underset{1}{y}, \ldots, \underset{k}{y})$$

under the extended transformations (4). To this end, we change variables in the first integral according to the transformations (4):

$$\left\{\widetilde{x}^1, \ldots, \widetilde{x}^l\right\} \mapsto \left\{x^1, \ldots, x^l\right\}.$$

By virtue of (4) the elements of Jacobi's matrix are equal

$$\frac{\partial \widetilde{x}^i}{\partial x^j} = \delta_{ij} + \epsilon\frac{\partial \xi^i}{\partial x^j} + \mathcal{O}\left(\epsilon^2\right), \qquad i, j = 1, \ldots, l.$$

Because the off-diagonal elements of the matrix are of the order $\mathcal{O}\left(\epsilon^2\right)$, thus the linear contribution to the Jacobian comes only from the product of the diagonal elements:

$$\frac{\partial\left(\widetilde{x}^1 \cdots \widetilde{x}^l\right)}{\partial\left(x^1 \cdots x^l\right)} = \left(1 + \epsilon\frac{\partial \xi^1}{\partial x^1}\right) \cdots \left(1 + \epsilon\frac{\partial \xi^l}{\partial x^l}\right) + \mathcal{O}\left(\epsilon^2\right) = 1 + \epsilon\sum_{i=1}^{l}\frac{\partial \xi^i}{\partial x^i} + \mathcal{O}\left(\epsilon^2\right).$$

We do not use the summation convention when summation goes over the range $1, \ldots, l \leq n$ only.

Consequently, the change ΔI of the integral term is equal

$$\begin{aligned}
&\int_X dx^1 \cdots dx^l \Bigg[\left(1 + \epsilon\sum_{i=1}^{l}\frac{\partial \xi^i}{\partial x^i}\right) f\Big(x^1 + \epsilon\xi^1 + \mathcal{O}\left(\epsilon^2\right), \ldots, x^n + \epsilon\xi^n + \mathcal{O}\left(\epsilon^2\right), \\
&\qquad y + \epsilon\eta + \mathcal{O}(\epsilon^2), y_1 + \epsilon\eta_1 + \mathcal{O}\left(\epsilon^2\right), \ldots, y_n + \epsilon\eta_n + \mathcal{O}\left(\epsilon^2\right), \ldots, \\
&\qquad y_{i_1\cdots i_k} + \epsilon\eta_{i_1\cdots i_k} + \mathcal{O}\left(\epsilon^2\right)\Big) - f(x, y, \underset{1}{y}, \ldots, \underset{k}{y})\Bigg] + \mathcal{O}\left(\epsilon^2\right).
\end{aligned}$$

Expanding the function f into a Taylor series we obtain

$$\Delta I = \epsilon\int_X dx^1 \cdots dx^l \left[\xi^i\partial_{x_i}f + \eta\partial_y f + \eta_i\partial_{y_i}f + \cdots + \eta_{i_1\cdots i_k}\partial_{y_{i_1}}\cdots\partial_{y_{i_k}}f + f\sum_{i=1}^{l}\frac{\partial \xi^i}{\partial x^i}\right] + \mathcal{O}\left(\epsilon^2\right).$$

In view of the definition of the extended generator (5) we can rewrite the above result as follows

$$\Delta I = \epsilon \int_X dx^1 \cdots dx^l \left[G^{(k)} f(x, y, \underset{1}{y}, \ldots, \underset{k}{y}) + f(x, y, \underset{1}{y}, \ldots, \underset{k}{y}) \sum_{i=1}^{l} \frac{\partial \xi^i}{\partial x^i} \right] + \mathcal{O}\left(\epsilon^2\right).$$

From the calculations performed above we see that the implication (7) leads to the following infinitesimal *criterion of invariance* of integro-differential equations of the type (1) under the point transformations (2):

$$\frac{\partial W}{\partial F} G^{(m)} F + \frac{\partial W}{\partial I} \int_X dx^1 \cdots dx^l \left[G^{(k)} f + f \sum_{i=1}^{l} \frac{\partial \xi^i}{\partial x^i} \right] = 0 \qquad \text{on solutions of (1).} \tag{8}$$

For a system of equations of the type (1) we apply the criterion (8) to each equation of the system as was mentioned earlier. Generalization to the case of more than one integral term

$$I_1 = \int_{X_1} dx^1 \cdots dx^l f(\,\cdot\,), \qquad I_2 = \int_{X_2} dx^1 \cdots dx^p g(\,\cdot\,), \qquad \ldots$$

in the equation (1) is simple. Then, in the resulting criterion we get the following summation

$$\frac{\partial W}{\partial I_1} \int_{X_1} dx^1 \cdots dx^l \left[G^{(k)} f + f \sum_{i=1}^{l} \frac{\partial \xi^i}{\partial x^i} \right] + \frac{\partial W}{\partial I_2} \int_{X_2} dx^1 \cdots dx^p \left[G^{(k)} g + g \sum_{i=1}^{p} \frac{\partial \xi^i}{\partial x^i} \right] + \cdots.$$

In the case of $W = F + I$, which corresponds to our example of the Vlasov–Maxwell equations, the criterion (8) takes the form

$$G^{(m)} F + \int_X dx^1 \cdots dx^l \left[G^{(k)} f + f \sum_{i=1}^{l} \frac{\partial \xi^i}{\partial x^i} \right] = 0 \qquad \text{on solutions of (1).}$$

According to the criterion (8) we have to take into account the equation (1), which is now a constraint on extended variables. Using this equation we can eliminate some of them. Remaining variables are essentially independent, thus the equation (8) must be satisfied identically with respect to them. It means that the coefficients at independent expressions, involving these variables, must be equal to zero. This leads to the system of the so called *determining equations* for the integro-differential equation (1). They are homogeneous and *linear* integro-differential equations for coefficients ξ^i, η determining the generator (3) and the point transformations (2). In applications, we have additional information in each particular case. Often, this information enables us to go to the integrands in integral determining equations by using the Lagrange lemma of variational calculus [24]. This leads to differential determining equations.

The criterion (8) is a *necessary* condition for symmetry of the equation (1), so it allows us to find all *possible* symmetry transformations of (1). The difficult task to obtain is to find a sufficient condition of symmetry. To this end we need a theorem on *global* existence and uniqueness of the solutions of the equation (1). The latter problem is far from being solved, see [25]. From a practical point of view the necessary condition is more important and useful than the sufficient one as the main task is to find symmetry transformations. A possible symmetry transformation of the equation (1) can be easily verified by inspection and this should be done anyway.

4 Symmetries of Vlasov–Maxwell equations

Let us consider the Vlasov–Maxwell system of equations for collisionless, multicomponent, one-dimensional plasmas with no magnetic field:

$$\begin{aligned}
&\partial_t f_\alpha + u\partial_x f_\alpha + \frac{q_\alpha}{m_\alpha} E\partial_u f_\alpha = 0,\\
&\partial_t E + \sum_\alpha \frac{q_\alpha}{\epsilon_0}\int_{-\infty}^{\infty} du\, u f_\alpha = 0, \qquad \partial_x E - \sum_\alpha \frac{q_\alpha}{\epsilon_0}\int_{-\infty}^{\infty} du f_\alpha = 0,
\end{aligned} \tag{9}$$

where $E = E(t,x)$ is the x-component of electric vector field $\boldsymbol{E} = (E,0,0)$, u is the x-component of vector velocity $\boldsymbol{v} = (u,0,0)$, $f_\alpha = f_\alpha(t,x,u)$ is the distribution function of α-plasma component, q_α, m_α are charge and mass of α-particles, respectively and ϵ_0 is electric permittivity of free space.

In this case, the generators (3) of point transformations (2) take the form

$$G = \tau\partial_t + \xi\partial_x + \rho\partial_u + \sum_\alpha \eta_\alpha \partial_{f_\alpha} + \zeta\partial_E. \tag{10}$$

Using the criterion (8) we obtain

$$0 = \partial_{f_\alpha}\tau = \partial_{f_\alpha}\xi = \partial_{f_\alpha}\rho = \partial_E\tau = \partial_E\xi = \partial_E\rho,$$

and the following determining equations (limits $\pm\infty$ of integrals are dropped):

$$\begin{aligned}
&0 = \partial_t\zeta = \partial_x\zeta = \partial_{f_\alpha}\zeta, \qquad 0 = u\partial_u\tau - \partial_u\xi,\\
&0 = \partial_t\eta_\alpha + u\partial_x\eta_\alpha + \frac{q_\alpha}{m_\alpha}E\partial_u\eta_\alpha, \qquad 0 = u\partial_t\tau - \partial_t\xi + \rho + u^2\partial_x\tau - u\partial_x\xi,\\
&0 = \sum_\beta E\left(\frac{q_\alpha}{m_\alpha} - \frac{q_\beta}{m_\beta}\right)(\partial_u f_\beta)\partial_{f_\beta}\eta_\alpha, \qquad 0 = \sum_\beta \frac{q_\beta}{\epsilon_0}\left(u\int du f_\beta - \int du\, u f_\beta\right)\partial_E\eta_\alpha,\\
&0 = \frac{q_\alpha}{m_\alpha}\left(\partial_t\tau + u\partial_x\tau + \frac{q_\alpha}{m_\alpha}E\partial_u\tau - \partial_u\rho\right)E + \frac{q_\alpha}{m_\alpha}\zeta - \partial_t\rho - u\partial_x\rho,\\
&0 = (\partial_t\tau - \partial_E\zeta)\sum_\beta \frac{q_\beta}{\epsilon_0}\int du\, u f_\beta - (\partial_t\xi)\sum_\beta \frac{q_\beta}{\epsilon_0}\int du f_\beta + \sum_\beta \frac{q_\beta}{\epsilon_0}\int du(\rho f_\beta + u\eta_\beta + u f_\beta\partial_u\rho),\\
&0 = (\partial_E\zeta - \partial_x\xi)\sum_\beta \frac{q_\beta}{\epsilon_0}\int du f_\beta + (\partial_x\tau)\sum_\beta \frac{q_\beta}{\epsilon_0}\int du\, u f_\beta - \sum_\beta \frac{q_\beta}{\epsilon_0}\int du(\eta_\beta + f_\beta\partial_u\rho).
\end{aligned}$$

Except for the nonphysical case of a constant charge to mass ratio $q_\alpha/m_\alpha = \mathrm{const}$ we easily find from the differential determining equations that

$$0 = \partial_u\tau = \partial_u\xi = \partial_t\eta_\alpha = \partial_x\eta_\alpha = \partial_u\eta_\alpha = \partial_E\eta_\alpha = \partial_{f_\beta}\eta_\alpha \quad \text{for} \quad \alpha \neq \beta, \qquad \zeta = \lambda_1 E.$$

Then, the last two integro-differential lead to

$$0 = \int du\,[f_\alpha(u\partial_u\tau - \lambda_1 u + \rho + u\partial_u\rho) + u\eta_\alpha], \qquad 0 = \int du[f_\alpha(\lambda_1 - \partial_x\xi + u\partial_x\tau - \partial_u\rho) + \eta_\alpha].$$

for every α. We assume that the point transformations (2) are analytic functions of the point (t,x,u,f_α). In general, analyticity with respect to the parameter ϵ and infinite differentiability with respect to the point is assumed for Lie groups. However, the latter dependence is in fact also analytic due to a physical interpretation. Expanding $\eta_\alpha(f_\alpha)$ in the Taylor series, using the generalized mean value theorem and well known special solutions of the Vlasov–Maxwell equations (9), that is the stationary solutions depending only on velocity and BGK solutions, we find that the coefficients η_α can depend on f_α only linearly $\eta_\alpha = \lambda_2 f_\alpha$. Thus, we can apply the Lagrange lemma calculus of variations [24] and obtain *differential* equations for integrands.

Solutions of the determining equations are given by

$$\tau = -\frac{1}{3}(\lambda_1 + \lambda_2)t + \lambda_3, \qquad \xi = \frac{1}{3}(\lambda_1 - 2\lambda_2)x + \lambda_4 t + \lambda_5, \qquad \rho = \frac{1}{3}(2\lambda_1 - \lambda_2)u + \lambda_4,$$
$$\eta_\alpha = \lambda_2 f_\alpha, \qquad \zeta = \lambda_1 E,$$

where $\lambda_1, \ldots, \lambda_5$ are arbitrary parameters. Substituting the solutions into (10) and choosing all parameters equal to zero except one, which is assumed to be equal to 1, in each case, we derive the following five generators

$$G_1 = \partial_t, \qquad G_2 = \partial_x, \qquad G_3 = t\partial_x + \partial_u,$$
$$G_4 = -t\partial_t + x\partial_x + 2u\partial_u + 3E\partial_E, \qquad G_5 = -t\partial_t - 2x\partial_x - u\partial_u + 3\sum_\alpha f_\alpha \partial_{f_\alpha}, \tag{11}$$

which span the Lie algebra of the group of point symmetry transformations of the Vlasov–Maxwell equations (9). Non-vanishing commutators between these generators are given by

$$[G_1, G_3] = G_2, \qquad [G_1, G_4] = -G_1, \qquad [G_1, G_5] = -G_1, \qquad [G_2, G_4] = G_2,$$
$$[G_2, G_5] = -2G_2, \qquad [G_3, G_4] = 2G_3, \qquad [G_3, G_5] = -G_3.$$

The algebra is solvable.

Summing up the Lie series we obtain one-parameter subgroups of the symmetry group of transformations corresponding to the generators (11). For G_1 and G_2 we have translations in time and translations in space respectively. These symmetries follow from the fact, that coefficients of equation (9) do not depend on time and space variables, and lead to the conservation laws of energy and momentum respectively. For G_3 we have Galilean transformations. The above three kinetic symmetries are obvious as they express the geometric properties of space-time in nonrelativistic theory. The dynamical symmetries, which depend on details of interaction, are more interesting. In the case of the Vlasov–Maxwell equations they are generated by G_4 and G_5 and have the form of scaling transformations. We can construct a general symmetry transformation of the Vlasov–Maxwell equation (9) from the above one-parameter transformations.

Other approaches to the problem of finding symmetries of Vlasov–Maxwell equations can be found in papers [26, 27, 28] and in Chapter 16 of Vol.2 of CRC Handbook [8].

5 Conclusions

It has been shown that there is no need for a nonlocal extension of a symmetry group in the case of integro-differential equations. It is sufficient to stay in a jet space as in the case of differential equations. The generalization of the Ovsiannikov method consists in the change of the infinitesimal criterion of symmetry. The method has been successfully applied to significant integro-differential equations. In addition to the Vlasov–Maxwell equations we have also determined the symmetry group of the nonlocal NLS equation for modulated Langmuir waves in plasmas. In this last case a further generalization of the Ovsiannikov method to equations with delayed arguments is needed.

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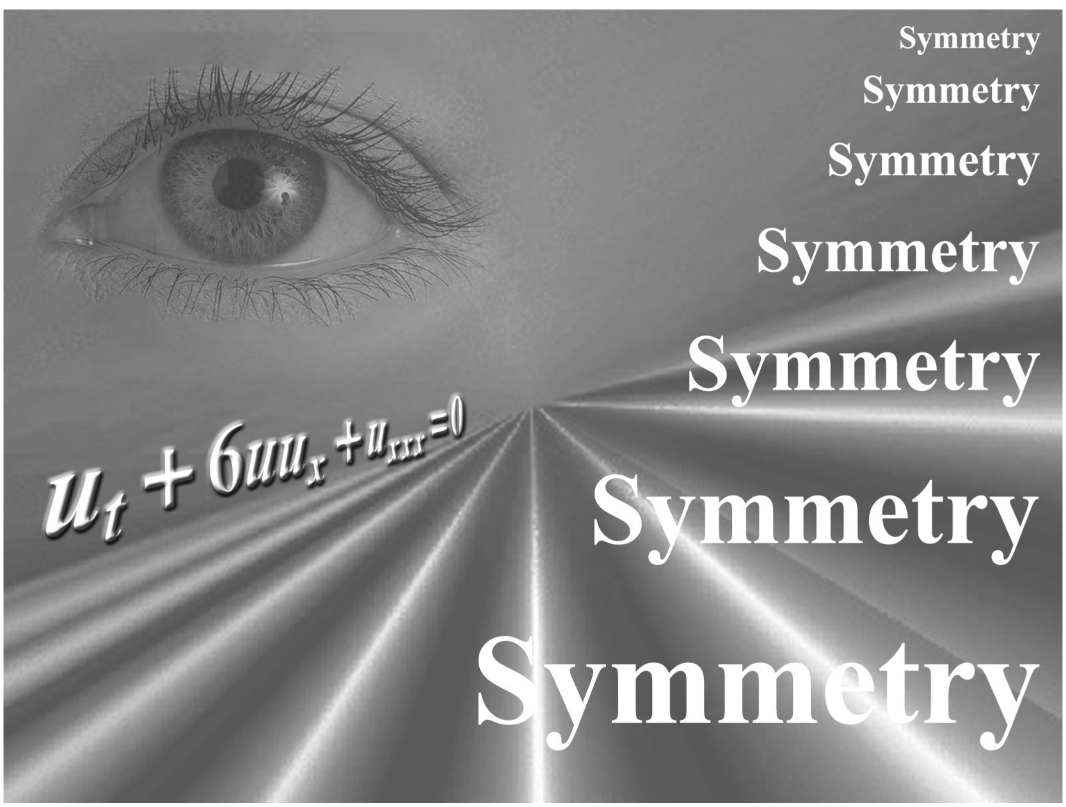
Solitons and Integrability



Symmetry in Nonlinear Mathematical Physics

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Spectra of the Schrödinger Operators with Finite-Gap Potentials and Integrable Systems

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The report presents results of studies of spectral properties of the Schrödinger operators with finite-gap potentials along with their applications to problems of physics of solids. As an example of these applications we consider the Peierls problem.

1 Introduction

Recently a deep connection of spectral properties of the Schrödinger operators with periodic finite- and infinite-gap potentials was discovered, and also their relations with the theory of integrable systems were established. Last property allows to describe effectively the spectrum, eigenvalues, eigenfunctions and also matrix elements of any observable analytically. Application of these results to physics of solids appears to be very successful for explanation and quantitative description of many phenomena and properties of solids. Construction of the separable many-dimensional generalization of the one-dimensional finite-gap potentials is very interesting since it which allows to solve a number of 3-dimensional practical problems of solid state physics.

2 The Schrödinger operator with finite-gap potential

2.1 The one-gap potential

Let us consider the Schrödinger equation with the Weierstrass function $\wp(z)$ as the potential,

$$-\partial_x^2\psi(x) + u(x)\psi(x) = \epsilon\psi(x), \qquad u(x) = -2\wp(\imath x + \omega), \quad x \in \mathbb{R}.$$

We use here and further the traditional notations of the theory of elliptic functions. An average value of the potential is

$$\langle u(x)\rangle = -2\langle\wp(\imath x + \omega)\rangle = 2\eta'/\omega'.$$

Here and further

$$\langle g\rangle = \lim_{l\to\infty} L^{-1}\int_0^L g(x)\mathrm{d}x.$$

The following statements are simple consequences of the formulae from the elliptic functions theory.

The potential $u(x) = -2\wp(\imath x + \omega)$ has a real period $T = -\imath 2\omega'$ and an imaginary period $T' = \imath 2\omega$. The potential $u(x)$ is a linear superposition of solitons,

$$u(x) = 2\frac{\eta}{\omega} - 2\left(\frac{\pi}{2\omega}\right)^2 \sum_{n=-\infty}^{+\infty} \cosh^{-2}\left[\frac{\imath\pi}{2\omega}(\imath x - n2\omega')\right].$$

We can present the eigenfunction of the Schrödinger operator with the one-gap potential in terms of the σ-functions

$$\psi_{\pm}(x) = \frac{\sigma(\imath x + \omega \pm z)}{\sigma(\imath x + \omega)\sigma(z)} e^{\mp \imath x \zeta(z)},$$

or in terms of the potential $u(x)$, or $\wp$-function,

$$\psi(x,\varepsilon) = [\langle \chi^{-1}(x,\varepsilon)\rangle \chi(x,\varepsilon)]^{-1/2} \exp\left(\imath \int^x \mathrm{d}x\, \chi(x,\varepsilon)\right),$$
$$\chi(x,\varepsilon) = [P(\varepsilon)]^{1/2}\big(\varepsilon - \gamma(x)\big)^{-1}, \qquad P(\varepsilon) = (\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)(\varepsilon-\varepsilon_3),$$
$$\gamma(x) = (1/2)\big(\varepsilon_1+\varepsilon_2+\varepsilon_3 - u(x)\big).$$

In the last form the wave function is normalized by a condition $\langle |\psi|^2\rangle = 1$.

The wave function is defined on the two sheets of a Riemann surface given by the equation $\mu^2 = P(\varepsilon) = (\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)(\varepsilon-\varepsilon_3)$.

The spectrum Σ of the Schrödinger operator with the one-gap potential is

$$\epsilon = \wp(z),$$
$$z = \omega' + t\omega, \quad 0 \le t \le 1, \quad \text{or}$$
$$z = \omega + t\omega', \quad 0 \le t \le 1.$$

Therefore the spectrum Σ has one finite energy band $e_3 \le \epsilon \le e_2$ when $z = \omega' + t\omega$, $0 \le t \le 1$, the energy gap $e_2 \le \epsilon \le e_1$ when $z = \omega + t\omega'$, $0 \le t \le 1$, and one infinite energy band $e_1 \le \epsilon < +\infty$ when $z = \omega + t\omega$, $0 \le t \le 1$.

The eigenfunctions $\psi_{\pm}(x)$ satisfy the Floquet equality $\psi(x+T) = e^{\imath kT}\psi(x)$ and have the Bloch form $\psi(x) = A(x)e^{\imath kx}$ where $A(x)$ is the periodic amplitude, $A(x+T) = A(x)$, and k is the wave vector

$$k = \zeta(z) - z(\eta'/\omega'), \quad z = \omega' + t\omega, \quad 0 \le t \le 1;$$
$$k = -[\zeta(z) - z(\eta'/\omega')], \quad z = \omega + t\omega, \quad 0 \le t \le 1.$$

Due to the oscillation theorem the density of states $n = k/\pi$.

A number of states over a unit of length with the energy less than ε is

$$n(u(\cdot),\varepsilon) = \frac{1}{\pi}\langle \chi(x,\varepsilon)\rangle = \frac{1}{\pi}\sqrt{(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)(\varepsilon-\varepsilon_3)}\langle(\varepsilon - \gamma(x))^{-1}\rangle.$$

Here ε_1, ε_2, ε_3, $\varepsilon_1 \le \varepsilon_2 \le \varepsilon_3$ are the boundaries of the spectrum.

This quantity satisfies important relations:

$$\frac{\mathrm{d}n}{\mathrm{d}\varepsilon}(u(\cdot),E) = \frac{1}{2\pi}\langle \chi^{-1}(x,\varepsilon)\rangle, \qquad \frac{\delta n}{\delta u(x)}(u(\cdot),\varepsilon) = -\frac{1}{2\pi}\chi^{-1}(x,\varepsilon).$$

Here $(\delta n/\delta u(x))(u(\cdot),\varepsilon)$ is a variational derivative of the number of states n with respect to such variations of the potential which do not change its periods.

3 The finite-gap potentials

3.1 One-dimensional finite-gap potentials

A one-gap potential $u(z) = 2\wp(z)$ is a special case of the Lame potentials

$$u(z) = n(n+1)\wp(z), \qquad n \in \mathbb{N}.$$

The Schrödinger operator with the Lamé potential has the n-gap spectrum and the following two linearly independent eigenfunctions:

$$\Psi_{\pm}(x;z) = \prod_{r=1}^{n} \left\{ \frac{\sigma(a_r + x)}{\sigma(a_r)\sigma(x)} \right\} \exp\left\{ \pm x \sum_{r=1}^{n} \zeta(a_r) \right\}.$$

The Lame potentials present a particular case of the Darboux potentials which are linear superpositions of the Weierstrass functions with shifts of the following form

$$u(x) = n_0(n_0+1)\wp(x) + \sum_{i=1}^{3} n_i(n_i+1)\wp(x+\omega_i),$$

where ω_i, $i = 1,2,3$ are half-periods. If $n_i = 0$, $i = 1,2,3$ we have the Lame potential.

The Darboux potential is characterized completely by the four integers $n_0, n_1, n_2, n_3 \in \mathbb{Z}_+$. If the Darboux potential is associated with the hyperelliptic curve $w^2 = P_{2g+1}(E)$ then its genus is equal to g. This curve covers N-sheetedly an elliptic curve and

$$N = \frac{1}{2}\left\{ n_0(n_0+1) + \sum_{i=1}^{3} n_i(n_i+1) \right\}.$$

A list of some Darboux potentials [15].

- $\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}$, $g = 0$, $N = 0$.
  $u(x) = 0$; $w^2 = \mu$.
- $\mathbf{1},\mathbf{0},\mathbf{0},\mathbf{0}$, $g = 1$, $N = 1$.
  $u(x) = 2\wp(x)$; $w^2 = (\mu - e_1)(\mu - e_2)(\mu - e_3)$.
- $\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{0}$, $g = 1$, $N = 2$.
  $u(x) = -2e_1 + 2\wp(x) + 2\wp(x+\omega_1)$;
  $w^2 = (\mu + 2e_1)(\mu^2 - 2e_1\mu - 11e_1^2 + g_2)$.
- $\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{0}$, $g = 2$, $N = 3$.
  $u(x) = 2\wp(x) + 2\wp(x+\omega_1) + 2\wp(x+\omega_2)$;
  $w^2 = (\mu + 3e_1)(\mu + 3e_2)(\mu + 3e_3)(\mu^2 - 3g_2)$.
- $\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1}$, $g = 1$, $N = 4$.
  $u(x) = 2\wp(x) + 2\wp(x+\omega_1) + 2\wp(x+\omega_2) + 2\wp(x+\omega_3)$;
  $w^2 = (\mu - 4e_1)(\mu - 4e_2)(\mu - 4e_3)$.

The periodic elliptic finite-gap potentials are a very special case of the general finite-gap potentials which are described in terms of the hyperelliptic functions and in generic situation are the quasi-periodic functions.

Definition 1. The almost-periodic function $u(x)$ is called a finite-band potential if the spectrum of the Schrödinger operator $L(u) = -\partial_x^2 + u(x)$ is a union of the finite set of segments of a Lebesgue (double absolutely continuous) spectrum.

Starting directly from this definition we can derive an explicit expression of a finite-band potential.

Theorem 1 ([13]). *The potential $u(x)$ of the Schrödinger operator $L(u) = -\partial_x^2 + u(x)$ with the g-gap Lebesgue spectrum*

$$\Sigma = [E_1, E_2] \cup \cdots \cup [E_{2g+1}, \infty)$$

has the form

$$u(x) = -2\partial_x^2 \ln \theta \left(\imath \vec{U} x - \vec{A}(D) + \vec{K}, B \right).$$

Here B is the matrix of the periods of normalized holomorphic differentials $\vec{\omega}$ on the hyperelliptic Riemann surface X, defined by the equation

$$\mu^2 = \prod_{i=1}^{2g+1} (\lambda - E_i),$$

$\vec{K}$ is the vector of Riemann constants, $\vec{U}$ is the vector of the periods of the normalized Abelian differential Ω of the second kind, which at infinity has a second-order pole with zero residue, D is a non-special divisor, $\vec{A}(D) = \int_\infty^P \vec{\omega}$ is an Abelian mapping.

By means of the Weierstrass–Poincaré theory of the Abelian function reduction we can point out conditions when these general finite-gap potentials become elliptic and fulfill this reduction effectively [7, 8].

3.2 Separable many-dimensional finite-gap potential

In the framework of the complex analysis there are no many-dimensional generalizations of the finite-gap potentials besides only trivial (i.e. separable) many-dimensional finite-gap potentials [11].

Let us consider a d-dimensional Schrödinger operator

$$H = -\Delta + U(x_1, x_2, \ldots, x_d) \tag{1}$$

with a periodic separable finite-gap potential of the form

$$U(x_1, x_2, \ldots, x_d) = \sum_{i=1}^{d} u(x_i), \tag{2}$$

where $u(x_i)$, $i = 1, 2, \ldots, d$ are some 1-dimensional finite-gap potentials (e.g. the Lamé potentials, the Darboux potentials etc.). In spite of triviality of this Hamiltonian it appears to be successful in applications. As an example we apply it to the quantum theory of solids with $d = 3$.

A limit case $U(x_1, x_2, x_3) \to 0$ of the operator (1) has been used with success in theory of solids in order to describe the electron energy spectra and classify all possible Fermi surfaces of metals [12]. Harrison's method is in fact a theory of almost free electrons when the potential is considered as a perturbation. Usage of the separable finite-gap potentials (2) is an essential step ahead with respect to Harrison's method since it allows consider limit cases of the free and strongly bound electrons in frame of the same model.

We have used the separable finite-gap potentials to describe many phenomena and properties of solids: the electron energy spectra of solids and Fermi surfaces of metals, the scattering and absorption of electromagnetic and other waves by finite-gap solids, the electron-phonon interaction, the Peierls transition and Fröhlich conductivity, the classification of the quasi-one-dimensional conductors, the oscillations in solids due to isospectral deformations of finite-gap potentials etc. Results of these studies are presented in recent publications [1, 2, 6].

4 The Peierls problem

The Peierls problem is to find eigenstates and eigenvalues of the electron-phonon Hamiltonian, defined on a segment $[0, L]$,

$$H = \sum_{k,\sigma} \epsilon_k a^+_{k,\sigma} a_{k,\sigma} + \sum_q \omega_q b^+_q b_q + L^{-1/2} \sum_q (\lambda_q L_q b^+_q + \lambda^*_q L^+_q b_q),$$
$$L_q = \sum_{k,\sigma} a^+_{k,\sigma} a_{k+q,\sigma}.$$

Here $a^+_{k,\sigma}$, $a_{k,\sigma}$ are electron operators; b^+_q, b_q are phonon operators; L^+_q, L_q are operators for charge density waves; ϵ_k, ω_q are the energies of electrons and phonons appropriately, λ_q is the electron-phonon interaction constant.

The exact solution of the Peierls problem was found recently [3, 4, 5].

When the mean field approximation is valid then we have in thermodynamic limit the condensation of phonons and charge density waves,

$$L^{-1/2} b^\#_q = \xi^\#_q + (L^{-1/2} b^\#_q - \xi^\#_q), \qquad \xi^\#_q = L^{-1/2} \langle b^\#_q \rangle,$$
$$L^{-1} L^\#_q = \eta^\#_q + (L^{-1} L^\#_q - \eta^\#_q), \qquad \eta^\#_q = L^{-1} \langle L^\#_q \rangle.$$

Here $\langle F \rangle$ stands for the statistical average of the operator F and therefore $\xi^\#_q$, $\eta^\#_q$ are statistical averages of the operators $b^\#_q$, $L^\#_q$. The operators $L^{-1/2} b^\#_q - \xi^\#_q$, $L^{-1} L^\#_q - \eta^\#_q$ describe the quantum fluctuations with respect to mean values.

Thus we can represent the Fröhlich Hamiltonian in the following form

$$H = H_A + V = (H_0 + H_e + H_{ph}) + V,$$

where

$$H_0 = -L \sum_q \left(\frac{|\lambda_q|^2 |\eta_q|^2}{\omega_q} + \lambda_q \eta_q \xi^*_q + \lambda^*_q \eta^*_q \xi_q \right),$$
$$H_e = \sum_{k,\sigma} \epsilon_k a^+_{k,\sigma} a_{k,\sigma} + \sum_q (L_q \lambda_q \xi^*_q + L^+_q \lambda^*_q \xi_q),$$
$$H_{ph} = L \sum_q \omega_q \left(\frac{b^+_q}{L^{1/2}} + \frac{\lambda^*_q \eta^*_q}{\omega_q} \right) \left(\frac{b_q}{L^{1/2}} + \frac{\lambda_q \eta_q}{\omega_q} \right),$$
$$V = \sum_q \left[\lambda_q \left(\frac{L_q}{L} - \eta_q \right) \left(\frac{b^+_q}{L^{1/2}} - \xi^*_q \right) + \lambda^*_q \left(\frac{L^+_q}{L} - \eta^*_q \right) \left(\frac{b_q}{L^{1/2}} - \xi_q \right) \right].$$

The electron Hamiltonian H_e describes a motion of the electrons in the classical field, which appears as a result of the condensation of phonons, the phonon Hamiltonian H_{ph} describes quantum fluctuations of phonons with respect to their classical average values and the interaction part of the Hamiltonian V describes the interaction of the quantum fluctuation of phonons and the charge density waves.

Let us assume that the self-consistency condition is satisfied,

$$L^{-1/2} \langle b_q \rangle = \xi_q = -(\lambda_q/\omega_q)\eta_q = -(\lambda_q/\omega_q) L^{-1} \langle L_q \rangle.$$

Under this condition in the thermodynamic limit we can neglect the interaction V and as a result of that the Fröhlich Hamiltonian H appears to be thermodynamically equivalent to the quadratic approximating Hamiltonian H_A. It means that the densities of the thermodynamic potentials for both Hamiltonians are equal.

We assume further that

$$\epsilon_k = k^2, \qquad \omega_q \sim q, \qquad \lambda_q \sim q^{1/2}.$$

The last two equalities are valid for acoustic phonons. With this assumption we have

$$\omega_q/|\lambda_q|^2 = \kappa,$$

and therefore the self-consistency condition attains the form

$$\eta_q = -\kappa\lambda_q^*\xi_q.$$

Multiplying the self-consistency condition by $\exp(\imath qx)$ and summing up over q we present this condition in the form

$$\sum_\sigma \langle\psi^+(x,\sigma)\psi(x,\sigma)\rangle = -\kappa u(x).$$

Here we have introduced notations

$$\psi(x,\sigma) = L^{-1/2}\sum_{k,\sigma} a_{k,\sigma}\exp(\imath kx), \qquad u(x) = \sum_q \lambda_q\xi_q\exp(\imath qx),$$

where $\psi(x,\sigma)$ is the electron field operator and $u(x)$ is the classical potential, in which the non-interacting electrons move, according to the electron Hamiltonian H_e.

Diagonalizing the Hamiltonian H_e we can present the electron field operator in the form

$$\psi(x,\sigma) = \sum_{E,\sigma} a_{E,\sigma}\phi(x,E).$$

Here eigenfunctions $\phi(x,E)$ are solutions of the Schrödinger equation

$$\partial_x^2\phi(x,E) + (E - u(x))\phi(x,E) = 0,$$

and the operators $a_{E,\sigma}^{\#}$ satisfy the relations

$$[a_{E,\sigma}^+, a_{E',\sigma'}] = \delta_{E,E'}\delta_{\sigma,\sigma'}, \qquad \langle a_{E,\sigma}^+ a_{E',\sigma'}\rangle = \delta_{E,E'}\delta_{\sigma,\sigma'}f(E),$$

where $f(E)$ is the Fermi distribution function. Using this form of the electron field operator we can present the self-consistency condition as follows,

$$\sum_{E,\sigma} f(E)|\phi(x,E)|^2 = -\kappa u(x).$$

The function $|\phi(x,E)|^2$ as a product of two solutions of the Schrödinger equation satisfy the equation

$$\left(\partial_x^3 - 4u(x)\partial_x - 2\partial_x u(x)\right)|\phi(x,E)|^2 = -4E|\phi(x,E)|^2.$$

Let us multiply this equation by the Fermi–Dirac function $f(E)$, sum it up over E and take in account the self-consistency condition. Then we get

$$u_{xxx} - 6uu_x = 4\kappa^{-1}\partial_x\sum_{E,\sigma} Ef(E)|\phi(x,E)|^2.$$

The sum in the r.h.s. of this equation is the energy of electrons and in equilibrium must not depend on a point x (otherwise the electrons will move from one point to another). Therefore

$$u_{xxx} - 6uu_x = 0.$$

Integrating this equation twice we get

$$(u_x)^2 = 2u^3 - 2g_2u - g_3.$$

Since the Weierstrass elliptic function satisfy the equation

$$\left[\wp(z)'\right]^2 = 4\wp^3(z) - g_2\wp(z) - g_3,$$

we can express the potential $u(x)$ in terms of this function [9, 10]

$$u(x) = -2\wp(\imath x + \omega).$$

If we put in the self-consistency condition

$$\sum_{\varepsilon,\sigma} f(\varepsilon)|\phi(x,\varepsilon)|^2 = \kappa u(x)$$

the expression

$$|\phi(x,\varepsilon)|^2 = \left[\langle\chi^{-1}(x,\varepsilon)\rangle\chi(x,\varepsilon)\right]^{-1} = \frac{\varepsilon - \gamma(x)}{\varepsilon - \langle\gamma(x)\rangle},$$

we can present this condition in the form

$$\frac{1}{2\pi}\int_{\varepsilon\in\Sigma} d\varepsilon f(\varepsilon)\frac{\varepsilon - \gamma(x)}{\sqrt{(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)(\varepsilon-\varepsilon_3)}} = -\kappa u(x).$$

Equating coefficients at the constant term and $u(x)$ at both sides of the last equality we get two equations

$$\frac{1}{2\pi}\int_{\varepsilon\in\Sigma} d\varepsilon f(\varepsilon)\frac{\varepsilon - (1/2)(\varepsilon_1+\varepsilon_2+\varepsilon_3)}{\sqrt{(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)(\varepsilon-\varepsilon_3)}} = 0,$$
$$\frac{1}{4\pi}\int_{\varepsilon\in\Sigma} d\varepsilon f(\varepsilon)\frac{1}{\sqrt{(\varepsilon-\varepsilon_1)(\varepsilon-\varepsilon_2)(\varepsilon-\varepsilon_3)}} = -\kappa.$$

Solving these equations we can define two parameters that characterize the potential $u(x)$, i.e. g_2, g_3, or ω, ω'.

We can solve these equations easily for the absolute zero of temperature when

$$f(\varepsilon) = \theta(\mu - \varepsilon).$$

In such a way we obtain the Peierls equation

$$\omega' = \imath/2\mathcal{N},$$

and the Fröhlich equation

$$\omega = 2\pi\kappa.$$

Here $\mathcal{N}$ is the number of states in the finite energy band.

5 Conclusion

Algebraic finite-gap potentials have deep connections to many areas of mathematics and physics, e.g. to complex analysis, integrable systems etc. They have proved also to be useful in different applications. For example, since we can approximate any smooth periodic potential by the finite-gap one with any desired accuracy [14] we obtain possibility to describe spectral properties of the Schrödinger operator with periodic potential of general form. Although the separable many-dimensional generalizations of the one-dimensional finite-gap are simple they appears also to be effective for solution of a number problems of 3-dimensional physics of solids due to their tight relation to the well known Harrison method. We believe that these potentials will have a lot of interesting applications in future.

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Integrable Equations and Motions of Plane Curves

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Motions of plane curves in Klein geometry is studied. It is shown that the KdV, Harry–Dym, Sawada–Kotera, Burgers, the defocusing mKdV hierarchies and the Kaup–Kupershmidt equation naturally arise from motions of plane curves in affine, centro-affine and similarity geometries. These local and nonlocal dynamics conserve global geometric quantities of curves such as perimeter and enclosed area.

1 Introduction

The connection between motion of space or plane curves and integrable equations has drawn wide interest in the past and many results have been obtained. The pioneering work is due to Hasimoto where he showed in [1] that the nonlinear Schrödinger equation describes the motion of an isolated non-stretching thin vortex filament. Lamb [2] used the Hasimoto transformation to connect other motions of curves to the mKdV and sine-Gordon equations. Lakshmanan [3] related the Heisenberg spin model to the motion of space curves in the Euclidean space. Langer and Perline [4] obtained the Schrödinger heirarchy from motions of the non-stretching thin vortex filament. Motions of curves in S^2 and S^3 were considered by Doliwa and Santini [5]. Nakayama [6, 7] investigated motions of curves in Minkowski space and obtained the Regge–Lund equation, a couple of systems of the KdV equations and their hyperbolic type. In contrast to the motions of curves in space, only two types of integrable equations have been shown to be associated to motions of plane curves. In fact, Goldstein and Petrich [8] discovered that the dynamics of a non-stretching string on the plane produces the recursion operator of the mKdV hierarchy. Nakayama, Segur and Wadati [9] obtained the sine-Gordon equation by considering a nonlocal motion. They also pointed out that the Serret–Frenet equations for curves in $\mathbf{E}^2$ and $\mathbf{E}^3$ are equivalent to the AKNS-ZS spectral problem without spectral parameter [10, 11]. It is commonly believed that the KdV equation does not occur in the motion of plane curves.

The purpose of this paper is to study motions of plane curves in Klein geometries. These geometries are characterized by their associated Lie algebras of vector fields in $\mathbf{E}^2$. We shall see that the KdV, Harry–Dym and Sawada–Kotera hierarchies and the Kaup–Kupershmidt equation naturally arise from the motions of plane curves in affine, centro-affine and similarity geometries. The outline of this paper is as follows. In Section 2, we give a brief discussion on the Klein geometry. In Sections 3, 4, and 5, we discuss motion laws of plane curves respectively in affine, centro-affine and similarity geometries. Section 7 is concluding remarks about this work.

2 Klein geometry

In this section, we give an extremely brief account of the Klein geometry. Our basic reference is [12].

Let $\mathcal{G}$ be a Lie transformation group acts locally and effectively on the plane. Its Lie algebra $\mathfrak{g}$ can be identified with a subalgebra of the Lie algebra of all smooth vector fields in $\mathbf{E}^2$ under the usual Poisson bracket. According to the Erlanger Programme, every $\mathcal{G}$ or $\mathfrak{g}$ determines a Klein geometry for plane curves via its invariants. To describe the invariants, let us assume a curve γ and its image γ' under a typical element g in $\mathcal{G}$ are represented locally as graphs $(x, u(x))$ and $(y, v(y))$ over some intervals I and J respectively. A differential invariant of $\mathfrak{g}$ is a n-th smooth function Φ defined on the n-jet space $X \times U^{(n)}$ for some $n \geq 1$ satisfying $\Phi\left(x, u(x), \ldots, u^{(n)}(x)\right) = \Phi\left(y, v(y), \ldots, v^{(n)}(y)\right)$ for all $g \in \mathcal{G}$. An invariant one-form, or, more precisely, a horizontal contact-invariant form, is a one-form defined in the n-jet space $X \times U^{(n)}$, locally in the form $d\sigma = P\left(x, u(x), \ldots, u^{(n)}(x)\right) dx$, satisfying

$$\int_I P\left(x, u(x), \ldots, u^{(n)}(x)\right) dx = \int_J P\left(y, v(y), \ldots, v^{(n)}(y)\right) dy,$$

for all g in $\mathcal{G}$. Let

$$\mathbf{v} = \xi(x,u)\frac{\partial}{\partial x} + \phi(x,u)\frac{\partial}{\partial u}$$

be an arbitrary vector field in $\mathfrak{g}$. We denote its n-th prolongation vector field on $X \times U^{(n)}$ by $\mathbf{pr}^{(n)}\mathbf{v}$. The infinitesimal criterion for the invariance of Φ and $d\sigma$ are given respectively by

$$\mathbf{pr}^{(n)}\mathbf{v}(\Phi) = 0,$$

and

$$\mathbf{pr}^{(n)}\mathbf{v}(P) + P\mathrm{div}\xi = 0,$$

where $\mathrm{div}\,\xi = \xi_x + \xi_u u_x$. A basic result is

Theorem 1 ([12]). *For any Lie transformation group acting locally and effectively on the plane, there exist an invariant one-form $d\sigma = Pdx$ and a differential invariant Φ, both of lowest order such that every differential invariant can be written as a function of Φ and its derivatives $D\Phi$, $D^2\Phi, \ldots$, where*

$$D = \frac{1}{P}\frac{d}{dx}.$$

Moreover, every invariant one-form is of the form $Id\sigma$ where I is a differential invariant.

Here "order" refers to the highest number of derivatives involved in the local expression for P and Φ.

Definition 1. Invariant one-forms and differential invariant of lowest order of the Lie group are respectively called the *group arclength* and the *group curvature.*

Example 1. We look at the Euclidean geometry which is the Klein geometry associated to the Lie algebra by $\{\partial_x, \partial_u, x\partial_u - u\partial_x\}$. It is readily verified that one can choose its group arclength to be

$$ds = \sqrt{1 + u_x^2}dx$$

and group curvature to be

$$\kappa = \left(1 + u_x^2\right)^{-\frac{3}{2}} u_{xx}.$$

Example 2. Consider the affine geometry which is associated to $SA(2)$ by $\{\partial_x, \partial_u, x\partial_x - u\partial_u, x\partial_u, u\partial_x\}$. Its group arclength and group curvature are

$$d\rho = \kappa^{\frac{1}{3}} ds, \qquad \mu = \kappa^{\frac{4}{3}} + \frac{1}{3}\left(\kappa^{-\frac{5}{3}}\kappa_s\right)_s. \tag{1}$$

One may consult [13] for a discussion on affine geometry.

In the following we shall consider motions of plane curves in affine, centro-affine and similarity geometries. For any parametrized curve γ, we define its *group tangent* and *group normal* to be $\mathbf{T} = \gamma_\sigma$ and $\mathbf{N} = \gamma_{\sigma\sigma}$ respectively, where σ is the group arc-length. A group invariant motion is of the form

$$\frac{\partial \gamma}{\partial t} = f\mathbf{N} + g\mathbf{T}, \tag{2}$$

where f and g are functions of the group curvature. With a given motion law, the equation for its curvature can be obtained in the following four steps. First, we determine the Serret–Frenet formulas for each geometry. It is of the form

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}_\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}. \tag{3}$$

In some occasions this system is the AKNS system without spectral parameter. Second, we compute the first variation for the group perimeter

$$L = \oint_\gamma d\sigma,$$

for a closed curve driven under (2) to obtain

$$\frac{dL}{dt} = \oint_\gamma F d\sigma, \tag{4}$$

where F depends on f and g in (2). By choosing f and g such that F vanishes pointwisely we ensure that $[\partial/\partial t, \partial/\partial\sigma] = 0$, i.e. $\partial/\partial t$ and $\partial/\partial\sigma$ commute. Third, we compute the time evolution of $\mathbf{T}$ and $\mathbf{N}$ to get

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}_t = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}. \tag{5}$$

Finally, the compatibility condition between (3) and (5)

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}_{t\sigma} = \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}_{\sigma t}$$

gives the general equation for the curvature. By choosing f and g suitably we obtain integrable equations. This procedure has been used in [8, 9] to obtain the mKdV and sine-Gordon equations in the Euclidean geometry. Similarly, some other mKdV equations are obtained by Doliwa–Santini [5] in the “restricted conformal” $SO(3)$-geometry.

3 Motion of curves in affine geometry

This is the classical geometry invariant under the unimodular transformations

$$\begin{pmatrix} x' \\ u' \end{pmatrix} = A \begin{pmatrix} x \\ u \end{pmatrix} + B,$$

where $A \in SL(2,\mathbb{R})$, $B \in \mathbb{R}^2$. The affine arc-length $d\rho$ and curvature μ are given in terms of the Euclidean arc-length and curvature by (1), where and hereafter κ and ds always denote the Euclidean curvature and arclength.

The affine Serret–Frenet formulas are given by

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}_\rho = \begin{pmatrix} 0 & 1 \\ -\mu & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}. \tag{6}$$

The affine tangent and normal are related to the Euclidean tangent $\mathbf{t}$ and normal $\mathbf{n}$ via

$$\mathbf{t} = k^{\frac{1}{3}}\mathbf{T}, \qquad \mathbf{n} = \frac{1}{3}k^{-\frac{5}{3}}k_s\mathbf{T} + k^{-\frac{1}{3}}\mathbf{N}.$$

We relate the motion (2) with the motion in Euclidean geometry

$$\gamma_t = \tilde{f}\mathbf{n} + \tilde{g}\mathbf{t}, \tag{7}$$

where

$$\tilde{f} = k^{\frac{1}{3}}f, \qquad \tilde{g} = k^{-\frac{1}{3}}g - \frac{1}{3}k^{-\frac{5}{3}}k_s f.$$

By a direct computation

$$\begin{aligned} \tilde{f}_{ss} &= \frac{1}{3}\left(k^{-\frac{2}{3}}k_s\right)_s f + k^{-\frac{1}{3}}k_s f_\rho + k f_{\rho\rho}, \\ \tilde{g}_s - k\tilde{f} &= g_\rho - \frac{1}{3}k^{-\frac{4}{3}}k_s(g + f_\rho) - \mu f. \end{aligned}$$

Substituting these equations into the evolution equations for s and k [8, 9], we have

$$\begin{aligned} s_t &= s\left[g_\rho - \frac{1}{3}k^{-\frac{4}{3}}k_s(g + f_\rho) - \mu f\right], \\ k_t &= k\left[f_{\rho\rho} + k^{-\frac{4}{3}}k_s(f_\rho + g) + \mu f\right]. \end{aligned}$$

Hence, the first variation of the affine perimeter satisfies

$$\begin{aligned} \frac{dL}{dt} &= \oint_\gamma \left(\frac{k_t}{3k} + \frac{s_t}{s}\right) d\rho, \\ &= \oint \left(\frac{1}{3}f_{\rho\rho} - \frac{2}{3}\mu f + g_\rho\right) d\rho. \end{aligned}$$

We impose

$$\oint \mu f d\rho = 0, \tag{8}$$

and

$$g = -\frac{1}{3}f_\rho + \frac{2}{3}\partial_\rho^{-1}(\mu f). \tag{9}$$

On the other hand, we have

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}_t = \begin{pmatrix} g_\rho - \mu f & f_\rho + g \\ H_1 & H_2 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}, \tag{10}$$

where $H_1 = g_{\rho\rho} - 2\mu f_\rho - \mu_\rho f - \mu g$ and $H_2 = f_{\rho\rho} + 2g_\rho - \mu f$. Under (8) and (9), $[\partial/\partial\rho, \partial/\partial t] = 0$, and so the compatibility condition between (6) and (10) implies

$$\mu_t = \frac{1}{3}\left(D_\rho^4 + 5\mu D_\rho^2 + 4\mu_\rho D_\rho + \mu_{\rho\rho} + 4\mu^2 + 2\mu_\rho \partial_\rho^{-1}\mu\right) f, \tag{11}$$

after using (9).

If we take $f = -3\mu_\rho$ in (11), we get the Sawada–Kotera equation [14, 15]

$$\mu_t + \mu_5 + 5\mu\mu_3 + 5\mu_1\mu_2 + 5\mu^2\mu_1 = 0. \tag{12}$$

If we take $f = -3(\mu_3 + 2\mu\mu_1)$, we obtain a seventh-order Sawada–Kotera equation

$$\mu_t + \mu_7 + 7\mu\mu_5 + 14\mu_1\mu_4 + 21\mu_2\mu_3 + 14\mu^2\mu_3 + 42\mu\mu_1\mu_2 + 7\mu_1^3 + \frac{28}{3}\mu^3\mu_1 + a\mu_1 = 0.$$

In general, we take $f = -3\left(D_\rho^2 + \mu + \mu_\rho\partial_\rho^{-1}\right)u$, $u = \Omega^{n-1}(\mu)\mu_\rho$, where

$$\Omega(\mu) = \left(D_\rho^3 + 2\mu D_\rho + 2D_\rho\mu\right)\left(D_\rho^3 + D_\rho^2\mu\partial_\rho^{-1} + \partial_\rho^{-1}\mu D_\rho^2 + \frac{1}{2}\left(\mu^2\partial_\rho^{-1} + \partial_\rho^{-1}\mu^2\right)\right),$$

is the recursion operator of the Sawada–Kotera equation [16]. By a direct computation, the following identity holds

$$\begin{aligned}&\left(D_\rho^4 + 5\mu D_\rho^2 + 4\mu_\rho D_\rho + \mu_{\rho\rho} + 4\mu^2 + 2\mu_\rho\partial_\rho^{-1}\mu\right)\left(D_\rho^2 + \mu + \mu_\rho\partial_\rho^{-1}\right)\\ &\quad = \left(D_\rho^3 + 2\mu D_\rho + 2D_\rho\mu\right)\left(D_\rho^3 + D_\rho^2\mu\partial_\rho^{-1} + \partial_\rho^{-1}\mu D_\rho^2 + \frac{1}{2}\left(\mu^2\partial_\rho^{-1} + \partial_\rho^{-1}\mu^2\right)\right).\end{aligned}$$

Using this identity we see that μ satisfies the Sawada–Kotera hierarchy

$$\mu_t = -\Omega^n(\mu)\mu_\rho. \tag{13}$$

4 Motion of plane curves in centro-affine geometry

The geometrical quantities in centro-affine geometry are invariant under the transformations

$$\begin{pmatrix} x' \\ u' \end{pmatrix} = A\begin{pmatrix} x \\ u \end{pmatrix},$$

where $A \in SL(2,\mathbb{R})$. Let $\gamma(p) = (\gamma_1(p), \gamma_2(p))$ be a parametrized curve in $\mathbf{E}^2$. We define its centro-affine arclength $d\tilde{s}$ as

$$d\tilde{s} = (\gamma_1\gamma_2' - \gamma_1'\gamma_2)dp = h ds,$$

where $h = -\gamma \cdot \mathbf{n}$ is the support function of γ [17]. The centro-affine curvature ϕ is given by

$$\phi = \kappa h^{-3}.$$

The centro-affine tangent and normal vectors are given by $\mathbf{T} = \gamma_{\tilde{s}}$ and $\mathbf{N} = \gamma_{\tilde{s}\tilde{s}}$ respectively. They are related to the Euclidean tangent and normal by

$$\mathbf{T} = h^{-1}\mathbf{t}, \qquad \mathbf{N} = \kappa h^{-2}\mathbf{n} - h^{-3}h_s\mathbf{t}.$$

Notice that this frame is centro-affine invariant in the sense that $\mathbf{T}' = A\mathbf{T}$ and $\mathbf{N}' = A\mathbf{N}$. Using the Serret–Frenet formulas in $\mathbf{E}^2$

$$\mathbf{t}_s = \kappa\mathbf{n}, \qquad \mathbf{n}_s = -\kappa\mathbf{t},$$

and the following identities

$$\kappa^{-1}h^2\left(h^{-3}\kappa_s - 3h^{-4}\kappa h_s\right) = \frac{\phi_{\tilde{s}}}{\phi},$$
$$h_{ss} = \kappa^{-1}\kappa_s h_s + \kappa - \kappa^2 h,$$

we obtain the centro-affine Serret–Frenet formulas

$$\mathbf{T}_{\tilde{s}} = \mathbf{N}, \qquad \mathbf{N}_{\tilde{s}} = \frac{\phi_{\tilde{s}}}{\phi}\mathbf{N} - \phi\mathbf{T}. \tag{14}$$

Now we first compute the first variation of the centro-affine perimeter $L = \oint d\tilde{s}$. To this purpose, we express (2) in the form (7), where now

$$\tilde{f} = \kappa h^{-2} f, \qquad \tilde{g} = h^{-1}g - h^{-3}h_s f.$$

By the formulas in $\mathbf{E}^2$ [8, 9]

$$s_t = s\left(\tilde{g}_s - \kappa\tilde{f}\right), \qquad \kappa_t = \tilde{f}_{ss} + \kappa_s\tilde{g} + \kappa^2\tilde{f},$$

and

$$h_t = -\tilde{f} + \left(\tilde{f}_s + \kappa\tilde{g}\right)\gamma\cdot\mathbf{t},$$

we have

$$L_t = \oint\left(h^{-1}h_t + s^{-1}s_t\right)d\tilde{s},$$
$$= \oint\left(g_{\tilde{s}} - 2\phi f\right)d\tilde{s}.$$

As parallel to the affine case, we require f to satisfy

$$\oint \phi f d\tilde{s} = 0, \tag{15}$$

and choose

$$g = 2\partial_{\tilde{s}}^{-1}(\phi f), \tag{16}$$

so that $[\frac{\partial}{\partial t}, \frac{\partial}{\partial\tilde{s}}] = 0$. By (14) and (7) we obtain the time evolution for tangent and normal vectors:

$$\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\end{pmatrix}_t = \begin{pmatrix}A & C\\ B & D\end{pmatrix}\begin{pmatrix}\mathbf{T}\\ \mathbf{N}\end{pmatrix}, \tag{17}$$

where

$$A = g_{\tilde{s}} - \phi f, \qquad B = f_{\tilde{s}} + g + \frac{\phi_{\tilde{s}}}{\phi}f, \qquad C = A_{\tilde{s}} - \phi B, \qquad D = B_{\tilde{s}} + A + \frac{\phi_{\tilde{s}}}{\phi}B.$$

The compatibility condition between (14) and (17) gives the equation for the curvature

$$\phi_t = \phi f_{\tilde{s}\tilde{s}} + 2\phi_{\tilde{s}}f_{\tilde{s}} + (\phi_{\tilde{s}\tilde{s}} + 4\phi^2)f + 2\phi_{\tilde{s}}\partial_{\tilde{s}}^{-1}(\phi f), \tag{18}$$

after using (16), where we always assume (15) holds. We now consider several cases:

Case 1. $f = u/\phi$. In this case, (18) becomes

$$\phi_t = \left(D_{\tilde{s}}^2 + 4\phi + 2\phi_{\tilde{s}}\partial_{\tilde{s}}^{-1}\right) u.$$

Setting $u = -\Omega_1^{n-1}\phi_{\tilde{s}}$. We get the KdV hierarchy

$$\phi_t = -\Omega_1^n \phi_{\tilde{s}}, \qquad n \geq 1,$$

where $\Omega_1 = D_{\tilde{s}}^2 + 4\phi + 2\phi_{\tilde{s}}\partial_{\tilde{s}}^{-1}$ is the recursion operator of the KdV equation

$$\phi_t + \phi_{\tilde{s}\tilde{s}\tilde{s}} + 6\phi\phi_{\tilde{s}} = 0. \tag{19}$$

Setting $u = -\phi^{-3/2}\phi_{\tilde{s}}\partial_{\tilde{s}}^{-1}(\phi q) + 2\phi^{1/2}q$, $\psi = \phi^{-1/2}$, ψ satisfies

$$\psi_t = -\left[\psi\left(\psi D_{\tilde{s}}^2 - \psi_{\tilde{s}} D_{\tilde{s}} + \psi_{\tilde{s}\tilde{s}} + \psi^2\psi_{\tilde{s}\tilde{s}\tilde{s}}\partial_{\tilde{s}}^{-1}\psi^{-2}\right) + 4\right] q. \tag{20}$$

Taking $q = 0$ in (20), we get the Harry Dym equation

$$\psi_t + \psi^3\psi_{\tilde{s}\tilde{s}\tilde{s}} = 0.$$

Setting $q = \Omega_2^{n-1}\left(\psi^3\psi_{\tilde{s}\tilde{s}\tilde{s}}\right)$, we get the Harry Dym hierarchy

$$\psi_t = -\Omega_2^n\left(\psi^3\psi_{\tilde{s}\tilde{s}\tilde{s}}\right),$$

where

$$\Omega_2 = \psi^2 D_{\tilde{s}}^2 - \psi\psi_{\tilde{s}} D_{\tilde{s}} + \psi\psi_{\tilde{s}\tilde{s}} + \psi^3\psi_{\tilde{s}\tilde{s}\tilde{s}}\partial_{\tilde{s}}^{-1}\psi^{-2} + 4,$$

is a recursion operator of the Harry Dym equation [18].

Case 2. $f = u_{\tilde{s}\tilde{s}\tilde{s}}/\phi + u_{\tilde{s}}$. In this case, (18) becomes

$$\phi_t = \left[D_{\tilde{s}}^5 + 5\phi D_{\tilde{s}}^3 + 4\phi_{\tilde{s}} D_{\tilde{s}}^2 + \left(\phi_{\tilde{s}\tilde{s}} + 4\phi^2\right) D_{\tilde{s}} + 2\phi_{\tilde{s}}\partial_{\tilde{s}}^{-1}\phi D_{\tilde{s}}\right] u.$$

Taking $u = -\phi$, we get the Sawada–Kotera equation (12). Next, we take $u = -\partial_{\tilde{s}}^{-1}\left(D_{\tilde{s}}^2 + \phi + \phi_{\tilde{s}}\partial_{\tilde{s}}^{-1}\right) q$, $q = \Omega^{n-1}(\phi)\phi_{\tilde{s}}$, where $\Omega(\phi)$ is the recursion operator of the Sawada–Kotera equation, we obtain the Sawada–Kotera hierarchy (13).

Case 3. $f = -(\phi_{\tilde{s}\tilde{s}\tilde{s}}/\phi + 16\phi_{\tilde{s}})$. The resulting equation is the Kaup–Kupershmidt equation [19, 20]

$$\phi_t + \phi_5 + 20\phi\phi_3 + 50\phi_1\phi_2 + 80\phi^2\phi_1 = 0.$$

Case 4. $f = \phi^{-4}\phi_{\tilde{s}}$. We have

$$\phi_t = \frac{1}{2}\left(\phi^{-2}\right)_{\tilde{s}\tilde{s}\tilde{s}} + 3\left(\phi^{-1}\right)_{\tilde{s}},$$

which is an integrable equation [21].

It is easy to see that in these four cases the motions also conserve the enclosed area of the curves. Notice that the area does not change under centro-affine action and so it makes sense in the centro-affine geometry. A fuller discussion on the integrable equations can be found in Chou–Qu [22].

5 Motions of curves in similarity geometry

The similarity algebra is obtained by adding the dilatation to $\mathbf{E}^2$. The $Sim(2)$ arc-length is given by $d\theta$, where θ is the angle between the tangent and the x-axis. The curvature in similarity geometry is related to the Euclidean curvature by the Cole–Hopf transformation

$$\chi = (\ln k)_\theta = k^{-2}k_s.$$

Using $\mathbf{T} = k^{-1}\mathbf{t}$ and $\mathbf{N} = k^{-1}\mathbf{n} - k^{-3}k_s\mathbf{t}$, the Serret–Frenet formulas in similarity geometry are given by

$$\mathbf{T}_\theta = \mathbf{N}, \qquad \mathbf{N}_\theta = -2\chi\mathbf{N} - \left(\chi_\theta + \chi^2 + 1\right)\mathbf{T}. \tag{21}$$

We express the motion (2) in the form (7), where now $\tilde{f} = k^{-1}f$ and $\tilde{g} = k^{-1}(g - \chi f)$. The first variation of the similarity perimeter is given by

$$\frac{dL}{dt} = \oint (k_t s + k s_t)dp = \oint \left[\tilde{f}_{ss} + (\chi g)_s\right] ds.$$

Hence dL/dt always vanishes for any closed curve. However, it still makes sense to set

$$g = -f_\theta + 2\chi f + a, \qquad a = \text{const}, \tag{22}$$

so that $[\partial/\partial\theta, \partial/\partial t] = 0$ for any f and g related by (22). The evolution of the similarity tangent and normal are given respectively by

$$\begin{aligned}
\mathbf{T}_t &= -k^{-2}k_t\mathbf{t} + k^{-1}\mathbf{t}_t = -k^{-1}(\mathcal{L}f + a\chi)\mathbf{t} + ak^{-1}\mathbf{n} = -\mathcal{L}f\mathbf{T} + a\mathbf{N},\\
\mathbf{N}_t &= k^{-1}(\mathbf{n} - \chi\mathbf{t})_t - k^{-2}k_t(\mathbf{n} - \chi\mathbf{t})\\
&= -(\mathcal{L}f + 2a\chi)(\mathbf{N} + \chi\mathbf{T}) - [(\mathcal{L}f + a\chi)_\theta - \chi(\mathcal{L}F + a\chi) + a]\mathbf{T}\\
&= -(\mathcal{L}f + 2a\chi)\mathbf{N} - [(\mathcal{L}f + a\chi)_\theta + a(\chi^2 + 1)]\mathbf{T}.
\end{aligned}$$

Hence

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}_t = \begin{pmatrix} -\mathcal{L}f & a \\ Q & P \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \end{pmatrix}, \tag{23}$$

where $Q = -(\mathcal{L}f + a\chi)_\theta - a(\chi^2+1)$, $P = -(\mathcal{L}f + 2a\chi)$ and $\mathcal{L} = (\partial_\theta - \chi)^2 + 1$ is a linear operator.

The compatibility condition between (21) and (23) yields the following equation for the $Sim(2)$-curvature χ after using (22),

$$\chi_t = \left[D_\theta^3 - 2\chi D_\theta^2 - \left(3\chi_\theta - \chi^2 - 1\right) D_\theta - (\chi_{\theta\theta} - 2\chi\chi_\theta)\right] f + a\chi_\theta, \tag{24}$$

where f is an arbitrary function.

The simplest choice is $f = -1$. Then (24) becomes the Burgers equation

$$\chi_t = \chi_{\theta\theta} - 2\chi\chi_\theta + a\chi_\theta.$$

The next choice is $f = \chi$, which yields the third order Burgers equation

$$\chi_t = \chi_{\theta\theta\theta} - 3\chi\chi_{\theta\theta} - 3\chi_\theta^2 + 3\chi^2\chi_\theta + (a+1)\chi_\theta.$$

In general, setting $f = \partial_\theta^{-1}u$, the equation becomes

$$\chi_t = \left[\left(D_\theta - \chi - \chi_\theta\partial_\theta^{-1}\right)^2 + 1\right] u + a\chi_\theta.$$

Setting $u = \Omega_3^{n-2}\chi_\theta$, we obtain the Burgers hierarchy

$$\chi_t = \left(\Omega_3^n + \Omega_3^{n-2} + a\right)\chi_\theta,$$

where $\Omega_3 = D_\theta - \chi - \chi_\theta \partial_\theta^{-1}$ is the recursion operator of the Burgers equation. These equations can be linearized by the Cole–Hopf transformation $\chi = (\ln \eta)_\theta$, where η is the reciprocal of the Euclidean curvature $\eta = 1/k$. Indeed, the hierarchy is transformed to

$$\eta_t = D_\theta^n \eta + D_\theta^{n-2}\eta + a\eta.$$

It is noted that the motion conserves enclosed area of the curve only when n is odd.

6 Concluding remarks

We have shown that many well-known integrable equations including KdV, Sawada–Kotera, Harry Dym, Burgers hierarchies and Kaup–Kupershmidt equation naturally arise from motions of plane curves in affine, centro-affine and similarity geometries. The mKdV equation in the Euclidean space $\mathbf{E}^2$, the KdV equation in the centro-affine geometry and the Sawada–Kotera equation in the affine geometry, are all obtained by choosing the normal velocity to be the derivative of the curvature with respect to the arclength. A further analysis shows that the N-soliton of the mKdV and the Sawada–Kotera equation gives N-loop curves respectively in Euclidean and affine geometries and the N-soliton of the KdV equation gives $N-1$-loop curve in centro-affine geometry [22]. Similar properties also hold for space curves [23]. These analogies suggest that the KdV equation and Sawada–Kotera equation are respectively the centro-affine version and affine version of the mKdV equation.

The vector fields of Lie algebras acting on the plane have been completely classified [12]. Recently we have investigated motions of curves in these geometries and found many associated integrable hierarchies. The reader is referred to as [24] for all details.

Finally we point out that the equivalence between integrable equations for the curvature and invariant motion leads to some new integrable equations. For example, in the Euclidean case, suppose the mKdV flow can be expressed as the graph of $(x, u(x,t))$ of some function u over x-axix, one finds that u satisfies the well-known WKI equation [25]

$$u_t = \left[\frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}}\right]_x, \tag{25}$$

which can be solved by the inverse scattering method. Similarly in the affine geometry, Sawada–Kotera flow can be expressed by the following integrable equation

$$u_t = -\left[u_{xx}^{-\frac{5}{3}}u_{xxxx} - \frac{5}{3}u_{xx}^{-\frac{8}{3}}u_{xxx}^2\right]_x. \tag{26}$$

The WKI equation (25) and equation (26) have many similarities, such as they are derived in the same manner, can be solved by the inverse scattering method and have N-loop solitons. A detail analysis to (26) is presented in [24].

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Asymptotics of the Coupled Solutions of the Modified Kadomtsev–Petviashvili Equation

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We determine a subset of $\mathbb{R}^2$ and a measure on this set which allow to construct coupled non-localized solutions $u^+(x,y,t)$ and $u^-(x,y,t)$ of the modified KP-I equation, which are connected by the relation $u^-(x,y,t) = -u^+(-x,y,-t)$, and split into asymptotic solitons as $t \to \infty$ in the neighbourhood of the leading edge of the solutions. The solitons corresponding to each of the solutions have different amplitudes and lines of constant phase, and are not connected by the above relation.

1 Introduction

In 1974 V.E. Zakharov and A.B. Shabat [1] proposed a very effective scheme of the inverse scattering method for the integration of nonlinear evolution equations with one and two spatial variables, so called *dressing method*. It turns out very convenient to obtain wide classes of solutions avoiding the difficult stage of the solving of the inverse scattering problem for the corresponding differential operator.

We consider the modified Kadomtsev–Petviashvili (mKP) equation [2]

$$u_t + \frac{1}{4}u_{xxx} - \frac{3}{2}\alpha^2 \left(u^2 u_x - \frac{1}{2\alpha}\hat{D}^{\pm}[u_{yy}] + u_x \hat{D}^{\pm}[u_y] \right) = 0, \tag{1}$$

with $u = u(x,y,t)$, $\alpha = i$ for the mKP-I equation, $\alpha = 1$ for the mKP-II equation, and operators $\hat{D}^{\pm}$ choosen as $\hat{D}^-[u(x,y,t)] = \int_{-\infty}^{x} u(s,y,t)ds$ or $\hat{D}^+[u(x,y,t)] = -\int_{x}^{\infty} u(s,y,t)ds$ simultaneously in both of the summands. These equations plays an important role in the understanding of the various properties of the Kadomtsev–Petviashvili equation and generalized Miura transformation. In both cases ($\alpha^2 = \pm 1$) (1) has exact solutions in the form of plane solitons

$$u(x,y,t) = \pm \frac{2q^2}{p + \sqrt{p^2+q^2}\cosh\left[2q(x - 2py - (q^2 - 3p^2)t - \beta)\right]},$$

($\beta = \mathrm{const}$, $(p,q) \in \mathbb{R}^+$ – upper half-plane of $\mathbb{R}$) the lines of constant phase of which $x = 2py + \left(q^2 - 3p^2\right)t + \beta$ are straight lines in (x,y)-plane.

In 1986, V.E. Zakharov constructed some solutions of the KP-II equation that were interpreted as curved solitons [3]. In 1994, using the dressing method, there were constructed KP-II non-localized solutions vanishing as $x \to \infty$, which split in the neighbourhood of the leading edge into infinite series of curved solitons for $t \to \infty$ [4]. These solitons are represented in the form of plane solitons, but their lines of constant phase are curves in the (x,y)-plane, and the depend on the parameter $Y = y/t$. Analogous solutions of the KP-I, mKP-I and Johnson equation (also called cylindrical KP) were constructed in [5, 6, 7], and their long-time asymptotic behaviour was investigated.

In this note we use the scheme of the dressing method for the mKP equation introduced in [5], and the fact that each constructed solution $u^+(x,y,t)$ of the mKP-I with operator $\hat{D}^+$ generates another solution $u^-(x,y,t) = -u^+(-x,y,-t)$ of the mKP-I with operator $\hat{D}^-$. So each solution $u^+(x,y,t)$ vanishing as $x \to +\infty$ [5, 7] generates $u^-(x,y,t)$, which vanishes as $x \to -\infty$. Such pair of solutions we call *coupled solutions*. However, direct application of the change of variables to the asymptotic formulae is not correct, and the investigation of the asymptotics of the new constructed solution requires special consideration.

We determine some subset in $\mathbb{R}^2$ and a measure on this set which allow to construct real coupled solutions of the mKP-1 equation, which split into asymptotic solitons as $t \to \infty$ in the neighbourhood of the leading edge of the solutions. The asymptotic solitons corresponding to each of both solutions have a different form and are not connected by the above transformation.

2 Construction of the mKP-I equation solution

According to the scheme of the dressing method [5] the mKP solution can be represented as follows:

$$u(x,y,t) = \frac{1}{\alpha}\frac{d}{dx}\left(1 \mp \hat{D}^{\pm}[K(x,s,y,t)]\right), \tag{2}$$

where $K(x,z,y,t)$ is a solution of Marchenko integral equation

$$K(x,z,y,t) + F(x,z,y,t) \mp \hat{D}^{\pm}[K(x,s,y,t)F(s,z,y,t)] = 0, \tag{3}$$

and $\hat{D}^{\pm}$ are operators with respect to the argument s. The kernel $F(x,z,y,t)$ of (3) satisfies the system of linear differential equations

$$\begin{aligned} &F_t + F_{xxx} + F_{zzz} = 0, \\ &\alpha F_y + F_{xx} - F_{zz} = 0. \end{aligned} \tag{4}$$

We start from the solution $u^+(x,y,t)$ corresponding to the operator $\hat{D}^+$ in (1)–(3). A wide class of solutions of (4) for $\alpha = i$ in this case can be found by the Fourier method in the form:

$$F(x,z,y,t) = \iint_{\Omega} \exp\left[ip(x-z) - q(x+z) + 4pqy + 2q\left(q^2 - 3p^2\right)t\right] d\mu(p,q), \tag{5}$$

where $\Omega \subset \mathbb{R}^2$ and $d\mu(p,q)$ is some measure on Ω.

To construct a mKP-I solution by the scheme (2)–(5) we must define the set Ω in (5) and the measure $d\mu(p,q)$ over this set. For this goal we introduce the functions $C^{\pm}(s)$ and $g(s)$ which play an important role in the construction of the solution and in the investigation of its asymptotic behaviour. For the sake of simplicity we restrict ourselves by a special choice of these functions in this note.

Let $b = \text{const} > 0$. The functions $C^+(s)\colon \mathbb{R} \to \mathbb{R}^+$ and $C^-(s)\colon \mathbb{R} \to \mathbb{R}^+$ are defined by

$$C^+(s) = s^2 + b^2, \qquad C^-(s) = (|s| + b)^2.$$

We denote

$$f^{\pm}(p,q,s) = 2ps \pm \left(q^2 - 3p^2\right).$$

The curve $q = h^+(p) = \sqrt{2p^2 + b^2}$ is envelope of the family of hyperbolas

$$E^+(p,q;s) = \left\{(p,q) \in \mathbb{R}^2 \mid f^+(p,q,s) = C^+(s)\right\}_{s\in\mathbb{R}}$$

with a contact at $(p_0^+(s), q_0^+(s))$ defined by

$$p_0^+(s) = \frac{C_s^+(s)}{2} = s,$$
$$q_0^+(s) = \sqrt{C^+(s) + (3/4)(C_s^+(s))^2 - sC^+(s)} = \sqrt{2s^2 + b^2}. \tag{6}$$

The curve $q = h^-(p) = \sqrt{4p^2 - 2|p|\,b}$ $(|p| \geq b/2)$ is envelope of the family of hyperbolas

$$E^-(p, q; s) = \left\{(p, q) \in \mathbb{R}^2 \mid f^-(p, q, s) = C^-(s)\right\}_{s\in\mathbb{R}},$$

with a contact at $(p_0^-(s), q_0^-(s))$ defined by

$$p_0^-(s) = \frac{C_s^+(s)}{2} = \operatorname{sign} s(|s| + b),$$
$$q_0^-(s) = \sqrt{\frac{3}{4}(C_s^-(s))^2 + sC^-(s) - C^-(s)} = \sqrt{2(|s| + b)(2|s| + b)}. \tag{7}$$

We consider a subset $\Omega \subset \mathbb{R}^2$ of the form

$$\Omega = \left\{(p, q) \in \mathbb{R}^2 \mid -\infty < p < \infty,\ q \in Q\right\}, \tag{8}$$

where

$$Q = \left\{q \mid q \geq \delta > 0\right\} \cap \left\{q \mid q \geq h^-(p)\right\} \cap \left\{q \mid q \leq h^+(p)\right\}$$

and $\delta < b$. It is easy to show that this choice of Ω implies

$$C^+(s) = \max_{(p,q)\in\Omega} f^+(p, q, s), \qquad C^-(s) = \max_{(p,q)\in\Omega} f^-(p, q, s). \tag{9}$$

Moreover, the maximum value of $f^\pm(p, q, s)$ is attained at the unique point $(p_0^\pm(s), q_0^\pm(s))$ respectively.

About the function $g(s)$ and the measure $d\mu$ we assume that

$$g(s)\colon \mathbb{R} \to \mathbb{R}^+ \quad \text{is} \quad \mathcal{C}^\infty, \qquad \tilde{g}\colon \Omega \to \mathbb{R}^+ \quad \text{is} \quad \mathcal{C}^\infty,$$
$$\tilde{g}(p_0(\kappa), q_0(\kappa)) = g(\kappa), \qquad d\mu(p, q) = \sqrt{\frac{p - iq}{p + iq}}\,\tilde{g}(p, q)dpdq. \tag{10}$$

Lemma 1. *Assume that Ω has the form (8) and $d\mu$ satisfies (9). Then the scheme (2)–(5) determines smooth real coupled solutions of the mKP-I equation vanishing as $x \to \pm\infty$ respectively, and bounded for all fixed x, y, t.*

Proof. The proof is based on the fact that the function $F(x, z, y, t)$ (5) generates a self-adjoint positive compact operator $\hat{F}[z]\colon L^2([x, \infty)) \to L^2([x, \infty))$. Then, by Fredholm theory [8] (4) has a unique solution $K(x, z, y, t)$, which is $\mathcal{C}^\infty$ with respect to all variables. Moreover, the self-adjointness of F and the special choice of the measure $d\mu$ (10) leads to the reality of $u(x, y, t)$. Thus the solution of the mKP-I equation constructed by (2) has all properties of the Lemma. We denote this solution $u^+(x, y, t)$.

This solution $u^+(x, y, t)$ of the mKP-I equation generates a new solution $u^-(x, y, t)$ by the change $u^-(x, y, t) = -u^+(-x, y, -t)$. This solution has the same properties as $u^+(x, y, t)$, boundedness, smoothness and reality, but it vanishes as $x \to -\infty$. ■

In Section 3 we describe the long time asymptotic behaviour of these solutions.

3 Theorem about soliton asymptotics of $u^{\pm}(x,y,t)$

The asymptotic behaviour of $u^{\pm}(x,y,t)$ for $t\to\infty$ is investigated in the following domains of the leading edge of the solutions ($M>2$):

$$G^{\pm}(M)=\left\{(x,y,t)\in\mathbb{R}^3\,\middle|\, t>t_0(M),\, Y=\frac{y}{t}\in I^{\pm},\, x\gtrless C^{\pm}(Y)t\mp\frac{M+1}{2q_0^{\pm}(Y)}\ln t\right\}, \tag{11}$$

where $I^{+}=\left[-\frac{1+\sqrt{3}}{2}b+\varepsilon,\ \frac{1+\sqrt{3}}{2}b-\varepsilon\right]$, $I^{-}=\left[-\frac{\sqrt{3}-1}{2}b+\varepsilon,\ -\varepsilon\right]\bigcup\left[\varepsilon,\ \frac{\sqrt{3}-1}{2}b-\varepsilon\right]$, $\varepsilon>0$, and $t_0(M)$ is large enough.

It is described by the following theorem.

Theorem 1. *The solution $u^{\pm}(x,y,t)$ of the mKP-I equation constructed in Lemma 1 have the following asymptotics in the domains $G^{\pm}_M(t)$ as $t\to\infty$:*

$$u^{\pm}(x,y,t)=\mp\sum_{n=1}^{[M-1]}u_n^{\pm}(x,y,t)+O\left(\frac{1}{t^{1/2-\varepsilon_1}}\right),\qquad (0<\varepsilon_1<1/2), \tag{12}$$

$$u_n^{\pm}(x,y,t)=\frac{2q_0^{\pm}(Y)^2}{p_0^{\pm}(Y)+\sqrt{(p_0^{\pm}(Y))^2+(q_0^{\pm}(Y))^2}\cosh\left[2q_0^{\pm}(Y)\kappa_n^{\pm}(x,y,t)\right]}, \tag{13}$$

where

$$\kappa_n(x,y,t)=x\mp C^{\pm}(Y)t\pm\frac{1}{2q_0^{\pm}(Y)}\left(\ln t^{n+1/2}-\ln g(Y)\phi_n^{\pm}(Y)\right),$$

$$\phi_n^{\pm}(Y)=\frac{(C_{YY}^{\pm}(Y))^{n-1/2}((q_0^{\pm}(Y))^2+(3p_0^{\pm}(Y)-Y)^2)^{n-1}\Omega^{(n)}I^{(n)}}{2^{4n+1}(q_0^{\pm}(Y))^{5n-3/2}[(n-1)!]^2\Omega^{(n-1)}I^{(n-1)}},$$

$p_0^{\pm}(Y)$, $q_0^{\pm}(Y)$ are defined in (6), (7), $C_{YY}^{\pm}=\frac{d^2C^{\pm}(Y)}{dY^2}$, and $\Gamma^{(n)}$, $Q^{(n)}>0$ are determinants of $n\times n$ matrices with entries $(0\le i,k\le n-1)$

$$\Gamma^{(n)}_{i+1,k+1}=\Gamma\left(\frac{i+k+1}{2}\right)\left(1+(-1)^{i+k}\right),\qquad Q^{(n)}_{i+1,k+1}=\Gamma(i+k+1).$$

Here the asymptotic representation (12) is uniform with respect to x and y in $G^{\pm}_M(t)$ for any fixed $M>2$.

Proof. The statement of the theorem concerning the solution $u^{+}(x,y,t)$ was proved in [5, 7]. We present here a scheme of the proof for the case of $u^{-}(x,y,t)$. First of all we apply the transformation $(x,t)\mapsto(-x,-t)$ to (2)–(5) with Ω (8) and $d\mu$ (10), and obtain the corresponding formulas for the solution $u^{-}(x,y,t)$. (3) is transformed into the equation containing integration from $-\infty$ to x with the kernel

$$F^{-}(x,z,y,t)=\iint_{\Omega}\exp\left[-ip(x-z)+q(x+z)+4pqy+2q\left(3p^2-q^2\right)t\right]d\mu(p,q). \tag{14}$$

On the next stage we study the asymptotics of (14) as $t\to\infty$. After change of variables $x=-C^{-}(Y)t-\xi$, $z=-C^{-}(Y)t-\zeta$ (14) acquires the form:

$$\begin{aligned}\tilde F^{-}(\xi,\zeta,y,t)&=F^{-}(C^{-}(Y)t+\xi,C^{-}(Y)t+\zeta,y,t)\\&=\iint_{\Omega}\exp\left[-ip(\xi-\zeta)+q(\xi+\zeta)-2q(C^{-}(Y)-f^{-}(p,q,Y))t\right]d\mu(p,q).\end{aligned} \tag{15}$$

Using property (9), we apply Laplace method to (15) and prove that

$$\tilde{F}^{-}(\xi,\zeta,y,t) = F_N(\xi,\zeta,y,t) + \tilde{G}(\xi,\zeta,y,t),$$

in the domains

$$\zeta > \xi < -\frac{1}{2q_0^-(Y)} \ln t^M, \qquad Y = \frac{y}{t}, \qquad t \to \infty, \tag{16}$$

where $F_N(\xi,\zeta,y,t)$ is a degenerate kernel ($N = [2M-3]$, $M > 2$ is an arbitrary integer), and $\|\tilde{G}(\xi,\zeta,y,t)\|_{L_2([x,\infty))} = O\left(1/t^{\varepsilon_1}\right)$, $0 < \varepsilon_1 < 1/2$.

On the third stage we prove that the degenerate kernel $F_N(\xi,\zeta,y,t)$ brings the main contribution into the asymptotics of Marchenko's equation in (16). After the solving of the equation and analysis of the corresponding determinant formulae we obtain the statement of the theorem. ■

Thus we have constructed non-localized coupled solutions $u^{\pm}(x,y,t)$ of the mKP-I equation, which split into infinite series of the curved asymptotic solitons in the domains of the leading edge of the solutions as $t \to \infty$. These asymptotic solitons are generated by the neighbourhoods of the curves $q = h^{\pm}(p)$ respectively. Both of them have the varying amplitude, width, and are diverged as t increases, but they are not connected by the transformation $u^+(x,y,t) \mapsto u^-(x,y,t)$.

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Exact Solutions of Matrix Generalizations of Some Integrable Systems

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Exact solutions of a matrix generalization of nonlinear Yajima–Oikawa model are built in an explicit form. The Melnikov-like system was also integrated.

1 Introduction

The hierarchy of Kadomtsev–Petviashvili equations can be given as an infinite sequence of the Sato–Wilson operator equations [1, 2]

$$\alpha_n W_{t_n} = -\left(W\mathcal{D}^n W^{-1}\right)_- W, \qquad n \in \mathbb{N}, \quad \alpha_n \in \mathbb{C}, \tag{1}$$

where $W = 1 + w_1\mathcal{D}^{-1} + w_2\mathcal{D}^{-2} + \cdots$ is a microdifferential operator (MDO) with coefficients w_i, $i \in \mathbb{N}$, depending on the variables $\boldsymbol{t} = (t_1, t_2, \ldots)$, $t_1 := x$ and $\mathcal{D} := \frac{\partial}{\partial x}$, $\mathcal{D}\mathcal{D}^{-1} = 1$. Differential and integral parts of the microdifferential operator $W\mathcal{D}^n W^{-1}$ are denoted by $\left(W\mathcal{D}^n W^{-1}\right)_+$ and $\left(W\mathcal{D}^n W^{-1}\right)_-$ respectively. In the algebra MDO ζ:

$$\zeta = \left\{ \sum_{i=-\infty}^{n(L)} a_i \mathcal{D}^i : a_i = a_i(\boldsymbol{t}) \in \mathcal{A};\ i, n(L) \in \mathbb{Z} \right\},$$

the operation of multiplication is induced by the generalized Leibnitz rule

$$\mathcal{D}^n f := \sum_{j=0}^{\infty} \binom{n}{j} f^{(j)} \mathcal{D}^{n-j}, \quad n \in \mathbb{Z}, \qquad \mathcal{D}^m(f) := \frac{\partial^m f}{\partial x^m} = f^{(m)}, \quad m \in \mathbb{Z}_+,$$

where $\mathcal{D}^n\mathcal{D}^m := \mathcal{D}^m\mathcal{D}^n := \mathcal{D}^{n+m}$, $n, m \in \mathbb{Z}$, and f is the operator of multiplication by a function $f(\boldsymbol{t})$, which belongs to the same functional space $\mathcal{A}$ that the coefficients of microdifferential operators $L \in \zeta$.

With the aid of the MDO L is defined by formula $L := W\mathcal{D}W^{-1} = \mathcal{D} + U\mathcal{D}^{-1} + U_2\mathcal{D}^{-2} + \cdots$ system (1) can be rewritten in the form of the Lax representation

$$\alpha_n L_{t_n} = [B_n, L] := B_n L - L B_n, \tag{2}$$

where $B_n = (L^n)_+ = (W\mathcal{D}^n W^{-1})_+$, $n \in \mathbb{N}$.

Nonlocal reduced hierarchy of Kadomtsev–Petviashvili is the system of operator equations (2) with the additional restriction so-called *k-constraint* of the form [3, 4, 5, 6, 7] (see also [8]) $L^k := (L^r)^k = B_k + \sum_{i=1}^{l} q_i \mathcal{D}^{-1} r_i^\top$, where "$\top$" denotes transposition which is in accordance with dynamics of system (2), if field-variables q_i, r_i satisfy the system of the following equations:

$$\alpha_n q_{it_n} = B_n(q_i), \qquad \alpha_n r_{it_n} = -B_n^\tau(r_i),$$

the symbol "τ" denotes the transposition of differential operator.

Equations from k–reduced hierarchy of Kadomtsev–Petviashvili allow the Lax representation

$$\Big[B_k + \boldsymbol{q}\mathcal{D}^{-1}\boldsymbol{r}^\top, \alpha_n\partial_{t_n} - B_n\Big] = 0, \qquad n \in \mathbb{N}. \tag{3}$$

2 Exact solutions of a matrix generalization of Yajima–Oikawa model

In the present paper we consider the matrix case of (3): $k = 2$, $n = 2$ and $U_1 := U, U_2, U_3, \ldots \in \mathrm{Mat}_{N\times N}(\mathbb{C})$, $\boldsymbol{q}, \boldsymbol{r} \in \mathrm{Mat}_{N\times N'}(\mathbb{C})$ and obtain the system:

$$\alpha_2\boldsymbol{q}_{t_2} = \boldsymbol{q}_{xx} + 2U\boldsymbol{q}, \qquad \alpha_2 U_{t_2} = (\boldsymbol{q}\mathbf{r}^\top)_x, \qquad \alpha_2\boldsymbol{r}_{t_2} = -\boldsymbol{r}_{xx} - 2\boldsymbol{r}U. \tag{4}$$

Introduce the additional reductions of complex conjugation $\alpha_2 = i$, $t_2 = t$, $U = U^* := \bar{U}^\top$, $\boldsymbol{r} = i\bar{\boldsymbol{q}}M^\top$, where $M \in \mathrm{Mat}_{N'\times N'}(\mathbb{C}), M = M^*$. System (4) can be represented as:

$$i\boldsymbol{q}_t = \boldsymbol{q}_{xx} + 2U\boldsymbol{q}, \qquad U_t = (\boldsymbol{q}M\boldsymbol{q}^*)_x. \tag{5}$$

System (5) is a matrix generalization of Yajima–Oikawa model [9]. Operators of this system in the Lax representation ($[L, A] = 0$) have the form:

$$L = \mathcal{D}^2 + 2U + i\boldsymbol{q}M\mathcal{D}^{-1}\boldsymbol{q}^*, \qquad A = i\partial_t - \mathcal{D}^2 - 2U.$$

Proposition 1 ([2, 10]). *Let $B = B_+$ be a differential operator; $\boldsymbol{f}\mathcal{D}^{-1}\boldsymbol{g}$, $\tilde{\boldsymbol{f}}\mathcal{D}^{-1}\tilde{\boldsymbol{g}} \in \zeta$. Then the following relations hold:*

$$B\boldsymbol{f}\mathcal{D}^{-1}\boldsymbol{g}^\top = \Big(B\boldsymbol{f}\mathcal{D}^{-1}\boldsymbol{g}^\top\Big)_+ + B(\boldsymbol{f})\mathcal{D}^{-1}\boldsymbol{g}^\top,$$

$$\boldsymbol{f}\mathcal{D}^{-1}\boldsymbol{g}^\top B = \Big(\boldsymbol{f}\mathcal{D}^{-1}\boldsymbol{g}^\top B\Big)_+ + \boldsymbol{f}\mathcal{D}^{-1}\Big(B^\tau\boldsymbol{g}\Big)^\top,$$

$$\boldsymbol{f}\mathcal{D}^{-1}\boldsymbol{g}^\top\tilde{\boldsymbol{f}}\mathcal{D}^{-1}\tilde{\boldsymbol{g}}^\top = \boldsymbol{f}\left(\int \boldsymbol{g}^\top\tilde{\boldsymbol{f}}\right)\mathcal{D}^{-1}\tilde{\boldsymbol{g}}^\top - \boldsymbol{f}\mathcal{D}^{-1}\left(\int \boldsymbol{g}^\top\tilde{\boldsymbol{f}}\right)\tilde{\boldsymbol{g}}^\top. \tag{6}$$

In formulas (6) the symbol $\int \boldsymbol{g}^\top\tilde{\boldsymbol{f}}$ stands for an arbitrary fixed primitive of $\Big(\boldsymbol{g}^\top\tilde{\boldsymbol{f}}\Big)(x, t_2)$ as a function of x.

Let φ, ψ be smooth complex matrix ($N \times K$) functions of real variables $x, t_2 \in \mathbb{R}$, $C = (C_{mn}) = \mathrm{const} \in \mathrm{Mat}_{K\times K}(\mathbb{C})$, and also:

1) the improper integral $\int_{-\infty}^x \psi^\top\varphi\, ds := \int_{-\infty}^x \psi^\top(s, t_2)\varphi(s, t_2)\, ds$ converges absolutely $\forall\ (x, t_2) \in \mathbb{R}\times\mathbb{R}_+$ and admits differentiation by the parameter $t_2 \in \mathbb{R}_+$;

2) the matrix-function $\Omega(x, t_2) := C + \int_{-\infty}^x \psi^\top\varphi\, ds$ is nondegenerate in $(x, t_2) \in \sigma \subset \mathbb{R}\times\mathbb{R}_+$.

Define the functions $\Phi = \Phi(x, t_2)$, $\Psi = \Psi(x, t_2)$ and MDO W by the following way:

$$\Phi = \varphi\Omega^{-1}, \qquad \Psi^\top = \Omega^{-1}\psi^\top, \qquad W = 1 - \Phi\mathcal{D}^{-1}\psi^\top. \tag{7}$$

Lemma 1. *The components Φ_{ij}, Ψ_{ij}, $i = \overline{1, N}$, $j = \overline{1, K}$, of matrix functions Φ, Ψ (7) can be given as:*

$$\Phi_{ij} = \left(\varphi\Omega^{-1}\right)_{ij} = (-1)^{K+j}\frac{\begin{vmatrix}\Omega_{(j)}\\ \varphi_i\end{vmatrix}}{|\Omega|}, \tag{8}$$

$$\Psi_{ij} = \left(\psi\Omega^{\top-1}\right)_{ij} = (-1)^{K+j}\frac{\begin{vmatrix}\Omega^\top_{(j)}\\ \psi_i\end{vmatrix}}{|\Omega|}. \tag{9}$$

Here $\Omega_{(j)}$ is obtained from Ω by deletion of j-line; φ_i, ψ_i are i-lines of matrixes φ, ψ.

Proof. In order to prove (8), (9) we use a well-known algebraic equality for framed determinant:

$$\det\begin{pmatrix}\Omega & \psi_j^\top\\ \varphi_i & \alpha\end{pmatrix} := \begin{vmatrix}\Omega & \psi_j^\top\\ \varphi_i & \alpha\end{vmatrix} = \alpha\det\Omega - \varphi_i\Omega^C\psi_j^\top,$$

where Ω^C is the matrix of cofactors.

$$\Phi_{ij} = \left(\varphi\Omega^{-1}\right)_{ij} = \varphi_i\Omega^{-1}e_j^\top = (-1)^{K+j}\frac{\begin{vmatrix}\Omega_{(j)}\\ \varphi_i\end{vmatrix}}{|\Omega|}.$$

Here $e_i = (e_{i_1},\ldots,e_{i_K})$, $e_{i_i} = 1$, $e_{i_j} = 0$ for $i,j = \overline{1,K}$, $i \neq j$.

By the similar reasoning, formula (9) can be proved. ■

Theorem 1 ([10]). *MDO W has an inverse operator W^{-1} and:*

$$W^{-1} = 1 + \varphi\mathcal{D}^{-1}\Psi^\top.$$

Proposition 2. *For MDO W (7) the equalities are true:*

$$\begin{aligned}W\mathcal{D}^2W^{-1} &= \left(I - \Phi\mathcal{D}^{-1}\psi^\top\right)\mathcal{D}^2\left(I + \varphi\mathcal{D}^{-1}\Psi^\top\right) = \mathcal{D}^2 + 2\left(\varphi\Omega^{-1}\psi^\top\right)_x\\ &- \Phi\mathcal{D}^{-1}\left(\psi_{xx}^\top - \int_{-\infty}^x \psi_{ss}^\top\varphi\, ds\,\Psi^\top\right) + \left(\varphi_{xx} - \Phi\int_{-\infty}^x \psi^\top\varphi_{ss}\, ds\right)\mathcal{D}^{-1}\Psi^\top,\\ W\left(i\partial_t - \mathcal{D}^2\right)W^{-1} &= i\partial_t - \mathcal{D}^2 - 2\left(\varphi\Omega^{-1}\psi^\top\right)_x\\ &+ \Phi\mathcal{D}^{-1}\left\{\left(i\psi_t^\top + \psi_{xx}^\top\right) - \int_{-\infty}^x\left(i\psi_t^\top + \psi_{ss}^\top\right)\varphi\, ds\,\Psi^\top\right\}\\ &+ \left\{(i\varphi_t - \varphi_{xx}) - \Phi\int_{-\infty}^x \psi^\top(i\varphi_t - \varphi_{ss})\, ds\right\}\mathcal{D}^{-1}\Psi^\top.\end{aligned}$$

The proof of the Proposition 2 is based on the using of formulas (6) and the generalized Leibnitz rule.

Consider operators $L_0 = \mathcal{D}^2$, $A_0 = i\partial_t - \mathcal{D}^2$, $\hat{L} = WL_0W^{-1}$, $\hat{A} = WA_0W^{-1}$.

Theorem 2. *Let:*

a) φ be a solution of the equation $i\varphi_t = \varphi_{xx}$;

b) $\varphi_{xx} = \varphi\Lambda$, where $\Lambda = \operatorname{diag}\left(\lambda_1^2, \lambda_2^2, \ldots, \lambda_K^2\right) = \mathrm{const} \in \mathrm{Mat}_{K\times K}(\mathbb{C})$;

c) $\psi = \bar{\varphi}$;

d) $C = C^*$.

Then

1) $\Psi = \bar{\Phi}$;

2) $\hat{L} = \mathcal{D}^2 + 2\left(\varphi\Omega^{-1}\varphi^*\right)_x + \Phi J\mathcal{D}^{-1}\bar{\Phi}^\top$, where $J = C\Lambda - \Lambda^*C$;

3) $\hat{A} = i\partial_t - \mathcal{D}^2 - 2\left(\varphi\Omega^{-1}\varphi^*\right)_x$.

Proof. 1) From definitions (7) and condition d we have:

$$\bar{\Phi} = \overline{\varphi\left(C + \int_{-\infty}^x \varphi^*\varphi\, ds\right)^{-1}} = \bar{\varphi}\left(\bar{C} + \int_{-\infty}^x \varphi^\top\bar{\varphi}\, ds\right)^{-1} = \psi\left(C^\top + \int_{-\infty}^x \varphi^\top\psi\, ds\right)^{-1} = \Psi.$$

2) From Proposition 2, condition b and properties:

$$\Phi\int_{-\infty}^x \psi^\top\varphi\, ds = \varphi - \Phi C \qquad \text{and} \qquad \int_{-\infty}^x \psi^\top\varphi\, ds\Psi^\top = \psi^\top - C\Psi^\top,$$

it follows that:

$$\hat{L} = \mathcal{D}^2 + 2\left(\varphi\Omega^{-1}\varphi^*\right)_x - \Phi\mathcal{D}^{-1}\Lambda^*\psi^\top + \Phi\mathcal{D}^{-1}\Lambda^*\int_{-\infty}^x \psi^\top\varphi\, ds\Psi^\top$$
$$+\varphi\Lambda\mathcal{D}^{-1}\Psi^\top - \Phi\int_{-\infty}^x \psi^\top\varphi\, ds\Lambda\mathcal{D}^{-1}\Psi^\top = \mathcal{D}^2 + 2\left(\varphi\Omega^{-1}\varphi^*\right)_x + \Phi\left(C\Lambda - \Lambda^*C\right)\mathcal{D}^{-1}\Psi^\top.$$

3) The validity of this item follows from Proposition 2 and conditions a), c). ■

Proposition 3. *The matrix $J = (J_{mn})$, $m, n = \overline{1, K}$ has the following properties:*
1) $J = -J^*$;
2) $J_{mn} = C_{mn}\left(\lambda_m^2 - \overline{\lambda_n^2}\right)$;
3) if the matrix C is diagonal, then: $J = \operatorname{diag}\left(2ic_1\operatorname{Im}\lambda_1^2, 2ic_2\operatorname{Im}\lambda_2^2, \ldots, 2ic_K\operatorname{Im}\lambda_K^2\right)$.

Proof. 1) $J^* = (C\Lambda - \Lambda^*C)^* = \Lambda^*C^* - C^*\Lambda = \Lambda^*C - C\Lambda = -J$.
The proof of the 2), 3) is based on the using of formulas of operations with matrices. ■

Corollary 1. *Let the matrix J be defined by the following condition: $\Phi J\Phi^* = i\boldsymbol{q}M\boldsymbol{q}^*$, then the functions $\boldsymbol{q} = (q_{ij})$ and $U = (u_{kl})$, $i, k, l = \overline{1, N}$, $j = \overline{1, K}$, where*

$$q_{ij} = (-1)^{j+K}\frac{\begin{vmatrix}\Omega_{(j)}\\ \varphi_i\end{vmatrix}}{|\Omega|}, \qquad u_{kl} = \left(\begin{vmatrix}\Omega & \bar{\varphi}_l^\top\\ \varphi_k & 0\end{vmatrix}|\Omega|^{-1}\right)_x$$

are solutions of system (5).

The proof of corollary is based on the using of formula (8), equality for framed determinant and Theorem 2.

Consider the simplest case of matrix equation (5): $N = 2$, $K = 1$. Then $\varphi_1 = \hat{c}e^{\lambda x - i\lambda^2 t}$, $\varphi_2 = \hat{c}e^{-\lambda x - i\lambda^2 t}$, $M = \mu \in \mathbb{R}$ and under conditions $\operatorname{Re}\lambda > 0$, $\mu = 4\operatorname{Re}\lambda\operatorname{Im}\lambda\cdot C$ solutions have the form:

$$q_1 := q_{11} = \frac{4\hat{c}\operatorname{Re}\lambda\operatorname{Im}\lambda\, e^{\lambda x - i\lambda^2 t}}{\mu + 4|\hat{c}|^2\operatorname{Im}\lambda\,\sinh\left(2\operatorname{Re}\lambda\cdot x\right)e^{4\operatorname{Re}\lambda\operatorname{Im}\lambda\, t}},$$
$$q_2 := q_{21} = \frac{4\hat{c}\operatorname{Re}\lambda\operatorname{Im}\lambda\, e^{-\lambda x - i\lambda^2 t}}{\mu + 4|\hat{c}|^2\operatorname{Im}\lambda\,\sinh(2\operatorname{Re}\lambda\cdot x)e^{4\operatorname{Re}\lambda\operatorname{Im}\lambda\, t}},$$
$$u_{11} = \frac{4|\hat{c}|^2\operatorname{Re}\lambda\operatorname{Im}\lambda\, e^{2\operatorname{Re}\lambda\, x + 4\operatorname{Re}\lambda\operatorname{Im}\lambda\, t}}{\mu + 4|\hat{c}|^2\operatorname{Im}\lambda\,\sinh(2\operatorname{Re}\lambda\cdot x)e^{4\operatorname{Re}\lambda\operatorname{Im}\lambda\, t}},$$
$$u_{12} = \bar{u}_{21} = \frac{4|\hat{c}|^2\operatorname{Re}\lambda\operatorname{Im}\lambda\, e^{4\operatorname{Re}\lambda\operatorname{Im}\lambda\, t + 2i\operatorname{Re}\lambda\, x}}{\mu + 4|\hat{c}|^2\operatorname{Im}\lambda\,\sinh(2\operatorname{Re}\lambda\cdot x)e^{4\operatorname{Re}\lambda\operatorname{Im}\lambda\, t}},$$
$$u_{22} = \frac{4|\hat{c}|^2\operatorname{Re}\lambda\operatorname{Im}\lambda\, e^{-2\operatorname{Re}\lambda x + 4\operatorname{Re}\lambda\operatorname{Im}\lambda\, t}}{\mu + 4|\hat{c}|^2\operatorname{Im}\lambda\,\sinh(2\operatorname{Re}\lambda\cdot x)e^{4\operatorname{Re}\lambda\operatorname{Im}\lambda\, t}}.$$

3 Exact solutions of higher equation from matrix hierarchy of Yajima–Oikawa

Let us consider the following operators:

$$L = \mathcal{D}^2 + 2U + i\boldsymbol{q}M\mathcal{D}^{-1}\boldsymbol{q}^*, \qquad A = \partial_t - \mathcal{D}^3 - 3U\mathcal{D} - \frac{3}{2}U_x - \frac{3}{2}i\boldsymbol{q}M\boldsymbol{q}^*.$$

The result of equation $[L, A] = 0$ will be the system:

$$\boldsymbol{q}_t = \boldsymbol{q}_{xxx} + 3U\boldsymbol{q}_x + \frac{3}{2}U_x\boldsymbol{q} + \frac{3}{2}i\mu\boldsymbol{q}\boldsymbol{q}^*\boldsymbol{q}, \tag{10}$$

$$U_t = \frac{1}{4}U_{xxx} + 3UU_x + \frac{3}{4}i\mu(\boldsymbol{q}_{xx}\boldsymbol{q}^* - \boldsymbol{q}\boldsymbol{q}^*_{xx}). \tag{11}$$

This system is a matrix generalization of Melnikov model [7, 11].

Proposition 4. *For MDO W the equality is true:*

$$\begin{aligned} W\left(\partial_t - \mathcal{D}^3\right)W^{-1} &= \partial_t - \mathcal{D}^3 - 3\left(\varphi\Omega^{-1}\psi^\top\right)_x\mathcal{D} \\ &- \frac{3}{2}\left(\varphi_{xx}\Omega^{-1}\psi^\top - \varphi\Omega^{-1}\psi^\top_{xx} + \varphi\Omega^{-1}\psi^\top_x - \varphi\Omega^{-1}\psi^\top\varphi_x\Omega^{-1}\psi^\top\right) \\ &+ \Phi\mathcal{D}^{-1}\left\{\left(\psi^\top_t - \psi^\top_{xxx}\right) - \int_{-\infty}^{x}\left(\psi^\top_t - \psi^\top_{sss}\right)\varphi\,ds\,\Psi^\top\right\} \\ &+ \left\{(\varphi_t - \varphi_{xxx}) - \Phi\int_{-\infty}^{x}\psi^\top(\varphi_t - \varphi_{sss})\,ds\right\}\mathcal{D}^{-1}\Psi^\top. \end{aligned}$$

The proof of the proposition is based on the formulas (6).

Consider operators $L_0 = \mathcal{D}^2$, $A_0 = \partial_t - \mathcal{D}^3$, $\hat{L} = WL_0W^{-1}$, $\hat{A} = WA_0W^{-1}$.

Theorem 3. *Let:*
a) φ be a solution of the equation $\varphi_t = \varphi_{xxx}$;
b) $\varphi_{xx} = \varphi\Lambda$, where $\Lambda = \operatorname{diag}\left(\lambda_1^2, \lambda_2^2, \ldots, \lambda_K^2\right) = \mathrm{const} \in \mathrm{Mat}_{K\times K}(\mathbb{C})$;
c) $\psi = \bar{\varphi}$;
d) $C = C^*$.
Then
1) $\hat{L} = \mathcal{D}^2 + 2\left(\varphi\Omega^{-1}\varphi^*\right)_x + \Phi J\mathcal{D}^{-1}\bar{\Phi}^\top$, where $J = C\Lambda - \Lambda^*C$;
2) $\hat{A} = \partial_t - \mathcal{D}^3 - 3\left(\varphi\Omega^{-1}\varphi^*\right)_x\mathcal{D} - \frac{3}{2}\left(\varphi_{xx}\Omega^{-1}\varphi^* - \varphi\Omega^{-1}\varphi^*_{xx} + \varphi\Omega^{-1}\varphi^*_x - \varphi\Omega^{-1}\varphi^*\varphi_x\Omega^{-1}\varphi^*\right)$.

Proof. 2) The validity of this item follows from Proposition 4 and conditions a), c). ■

Remark 1. For system (10) the corollary of the previous part is true (see above).

Consider the case $N = 2$, $K = 1$. Then $\varphi_1 = \hat{c}e^{\lambda x+\lambda^3 t}$, $\varphi_2 = \hat{c}e^{-\lambda x-\lambda^3 t}$, $M = \mu \in \mathbb{R}$ and under conditions $\operatorname{Re}\lambda > 0$, $\mu = 4\operatorname{Re}\lambda\operatorname{Im}\lambda\cdot C$ solutions will be of the form:

$$\begin{aligned} q_1 := q_{11} &= \frac{4\hat{c}\operatorname{Re}\lambda\operatorname{Im}\lambda\, e^{\lambda x+\lambda^3 t}}{\mu + 4|\hat{c}|^2\operatorname{Im}\lambda\cdot\sinh\left(2\operatorname{Re}\lambda\cdot x + 2\left(\operatorname{Re}^3\lambda - 3\operatorname{Re}\lambda\operatorname{Im}^2\lambda\right)t\right)}, \\ q_2 := q_{21} &= \frac{4\hat{c}\operatorname{Re}\lambda\operatorname{Im}\lambda\, e^{-\lambda x-\lambda^3 t}}{\mu + 4|\hat{c}|^2\operatorname{Im}\lambda\cdot\sinh\left(2\operatorname{Re}\lambda\cdot x + 2\left(\operatorname{Re}^3\lambda - 3\operatorname{Re}\lambda\operatorname{Im}^2\lambda\right)t\right)}, \\ u_{11} &= \frac{4|\hat{c}|^2\operatorname{Re}\lambda\operatorname{Im}\lambda\, e^{2\left(\operatorname{Re}\lambda\cdot x+\left(\operatorname{Re}^3\lambda-3\operatorname{Re}\lambda\operatorname{Im}^2\lambda\right)t\right)}}{\mu + 4|\hat{c}|^2\operatorname{Im}\lambda\cdot\sinh\left(2\operatorname{Re}\lambda\cdot x + 2\left(\operatorname{Re}^3\lambda\cdot x - 3\operatorname{Re}\lambda\operatorname{Im}^2\lambda\right)t\right)}, \\ u_{12} = \bar{u}_{21} &= \frac{4|\hat{c}|^2\operatorname{Re}\lambda\operatorname{Im}\lambda\, e^{2i\left(\operatorname{Im}\lambda\cdot x+\left(3\operatorname{Re}^2\lambda\operatorname{Im}\lambda-\operatorname{Im}^3\lambda\right)t\right)}}{\mu + 4|\hat{c}|^2\operatorname{Im}\lambda\cdot\sinh\left(2\operatorname{Re}\lambda\cdot x + 2\left(\operatorname{Re}^3\lambda\cdot x - 3\operatorname{Re}\lambda\operatorname{Im}^2\lambda\right)t\right)}, \\ u_{22} &= \frac{4|\hat{c}|^2\operatorname{Re}\lambda\operatorname{Im}\lambda\, e^{-2\left(\operatorname{Re}\lambda\cdot x+\left(\operatorname{Re}^3\lambda-3\operatorname{Re}\lambda\operatorname{Im}^2\lambda\right)t\right)}}{\mu + 4|\hat{c}|^2\operatorname{Im}\lambda\cdot\sinh\left(2\operatorname{Re}\lambda\cdot x + 2\left(\operatorname{Re}^3\lambda\cdot x - 3\operatorname{Re}\lambda\operatorname{Im}^2\lambda\right)t\right)}. \end{aligned}$$

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New Exact Solutions of Some Two-Dimensional Integrable Nonlinear Equations via $\overline{\partial}$-Dressing Method

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Recently obtained via $\overline{\partial}$-dressing method new exact solutions of some $(2+1)$-dimensional integrable nonlinear evolution equations such as Nizhnik–Veselov–Novikov (NVN), generalized Kaup–Kuperschmidt (2DKK) and generalized Savada–Kotera (2DSK) equations are discussed.

1 Introduction

In the last two decades the Inverse Spectral Transform (IST) method has been generalized and successfully applied to various $(2+1)$-dimensional nonlinear evolution equations such as Kadomtsev–Petviashvili, Davey–Stewardson, Nizhnik–Veselov–Novikov, Zakharov–Manakov system, Ishimory, two dimensional integrable sine-Gordon and others (see books [1, 2, 3, 4] and references therein). The nonlocal Riemann–Hilbert [5], $\overline{\partial}$-problem [6] and more general $\overline{\partial}$-dressing method of Zakharov and Manakov [7, 8] are now basic tools for solving $(2+1)$-dimensional integrable nonlinear equations (see also the reviews [10, 11, 12] and books [1, 2, 3, 4]).

In the present short paper new exact solutions calculated via $\overline{\partial}$-dressing method of some two-dimensional integrable nonlinear equations such as Nizhnik–Veselov–Novikov (NVN) [13, 14], generalized Kaup–Kuperschmidt (2DKK) [16, 17] and generalized Savada–Kotera (2DSK) [16, 17] equations are reviewed.

It is well known that $\overline{\partial}$-dressing method is very powerful method for the solution of integrable nonlinear evolution equations. This method has been discovered by Zakharov and Manakov [7, 8] (see also the books [3, 4]) and applies now successfully as to $(1+1)$-dimensional and also to $(2+1)$-dimensional integrable nonlinear evolution equations. The $\overline{\partial}$-dressing method allows to construct Lax pairs (auxiliary linear problems); to solve initial and boundary value problems, to calculate the broad classes of exact solutions of integrable nonlinear equations. By the use of $\overline{\partial}$-dressing method one can construct simultaneously broad classes of exactly solvable potentials (variable coefficients of linear PDE's) and corresponding wave functions of auxiliary linear problems.

Let us remind following to [7, 8] basic ingredients of $\overline{\partial}$-dressing method for $(2+1)$-dimensional case. At first one postulates nonlocal $\overline{\partial}$-problem:

$$\frac{\partial\chi(\lambda,\overline{\lambda})}{\partial\overline{\lambda}} = (\chi * R)(\lambda,\overline{\lambda}) = \iint_C d\lambda'\wedge d\overline{\lambda'}\chi(\lambda'.\overline{\lambda'})R(\lambda',\overline{\lambda'};\lambda,\overline{\lambda}). \tag{1}$$

For the sake of definiteness we restrict the attention to the case of the scalar complex-valued functions χ and R with the canonical normalization ($\chi \to \chi_0 = 1$, as $\lambda \to \infty$). We assume

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also that the problem (1) is uniquely solvable. The equation (1) defines behavior of the wave function χ in the spectral or momentum space.

Then one introduces dependence of kernel R and consequently the function χ on space and time variables ξ, η, t:

$$\begin{aligned}
\frac{\partial R}{\partial \xi} &= I_1(\lambda')R(\lambda',\overline{\lambda'};\lambda,\overline{\lambda};\xi,\eta,t) - R(\lambda',\overline{\lambda'};\lambda,\overline{\lambda};\xi,\eta,t)I_1(\lambda),\\
\frac{\partial R}{\partial \eta} &= I_2(\lambda')R(\lambda',\overline{\lambda'};\lambda,\overline{\lambda};\xi,\eta,t) - R(\lambda',\overline{\lambda'};\lambda,\overline{\lambda};\xi,\eta,t)I_2(\lambda),\\
\frac{\partial R}{\partial t} &= I_3(\lambda')R(\lambda',\overline{\lambda'};\lambda,\overline{\lambda};\xi,\eta,t) - R(\lambda',\overline{\lambda'};\lambda,\overline{\lambda};\xi,\eta,t)I_3(\lambda),
\end{aligned} \tag{2}$$

i.e.

$$R(\lambda',\overline{\lambda'};\lambda,\overline{\lambda};\xi,\eta,t) = R_0(\lambda',\overline{\lambda'};\lambda,\overline{\lambda})\exp(F(\lambda')-F(\lambda)), \tag{3}$$

where

$$F(\lambda) := I_1(\lambda)\xi + I_2(\lambda)\eta + I_3(\lambda)t. \tag{4}$$

Here $I_i(\lambda)$ $(i = 1, 2, 3)$ are some polynomial or rational functions of λ, the choice of these functions depends on concrete integrable equation. The role of the variables ξ, η, t will be played by the usual space and time variables x, y, t or their combinations $\xi = x+\sigma y$, $\eta = x-\sigma y$ with $\sigma^2 = \pm 1$. By introducing the "long" derivatives

$$D_\xi = \partial_\xi + I_1(\lambda), \qquad D_\eta = \partial_\eta + I_2(\lambda), \qquad D_t = \partial_t + I_3(\lambda) \tag{5}$$

dependence of R on ξ, η, t can be expressed in the form

$$[D_\xi, R] = 0, \qquad [D_\eta, R] = 0, \qquad [D_t, R] = 0. \tag{6}$$

By use of derivatives (5) one constructs then linear operators

$$L = \sum u_{lmn}(\xi,\eta,t)D_\xi^l D_\eta^m D_t^n \tag{7}$$

which satisfy to the condition $\left[\frac{\partial}{\partial\overline{\lambda}}, L\right] = 0$ of absence of singularities on λ. For such operators L the function $L\chi$ obeys the same $\overline{\partial}$-equation as the function χ. If there are several operators L_i of this type then by virtue of the unique solvability of (1) one has: $L_i\chi = 0$.

The solution of $\overline{\partial}$-problem (1) with the canonical normalization $\chi_0 = 1$ is equivalent to the solution of the following singular integral equation:

$$\chi(\lambda) = 1 + \iint_C \frac{d\lambda'\wedge d\overline{\lambda'}}{2\pi i(\lambda'-\lambda)} \iint_C d\mu\wedge d\overline{\mu}\chi(\mu,\overline{\mu})R_0(\mu,\overline{\mu};\lambda',\overline{\lambda'})e^{F(\mu)-F(\lambda')}. \tag{8}$$

From (8) one obtains for the coefficients $\tilde{\chi}_0$, χ_{-1} and χ_{-2} of series expansion of χ near the points $\lambda = 0$ and $\lambda = \infty$ ($\chi = \tilde{\chi}_0 + \chi_1\lambda + \cdots$ and $\chi = \chi_0 + \frac{\chi_{-1}}{\lambda} + \cdots$):

$$\tilde{\chi}_0 = 1 + \iint_C \frac{d\lambda\wedge d\overline{\lambda}}{2\pi i\lambda} \iint_C d\mu\wedge d\overline{\mu}\chi(\mu,\overline{\mu})R_0(\mu,\overline{\mu};\lambda,\overline{\lambda})e^{F(\mu)-F(\lambda)}, \tag{9}$$

$$\begin{aligned}
\chi_{-1} &= -\iint_C \frac{d\lambda\wedge d\overline{\lambda}}{2\pi i} \iint_C d\mu\wedge d\overline{\mu}\chi(\mu,\overline{\mu})R_0(\mu,\overline{\mu};\lambda,\overline{\lambda})e^{F(\mu)-F(\lambda)},\\
\chi_{-2} &= -\iint_C \frac{d\lambda\wedge d\overline{\lambda}}{2\pi i}\lambda \iint_C d\mu\wedge d\overline{\mu}\chi(\mu,\overline{\mu})R_0(\mu,\overline{\mu};\lambda,\overline{\lambda})e^{F(\mu)-F(\lambda)},
\end{aligned} \tag{10}$$

where $F(\lambda)$ is given by the formula (4). Through the coefficients $\tilde{\chi}_0$ and χ_{-1} usually the reconstructions formulas for the potentials are defined. In order to calculate via $\overline{\partial}$-dressing method exact solutions of integrable nonlinear equations and auxiliary linear problems one must to solve for given kernel R of $\overline{\partial}$-problem (3) (usually one chooses the degenerate kernels) singular integral equation (8) for wave function χ. Then by some reconstruction formulas one calculates exact solutions. Going by this way one must to satisfy important reality, potentiality or another conditions for the solutions.

In conclusion of this section let us obtain some useful general formulas for calculations of soliton and rational solutions of integrable nonlinear equations. Soliton solutions can be generated by the following delta-kernel $R_0(\mu,\overline{\mu};\lambda,\overline{\lambda})$ (3) of $\overline{\partial}$-problem (1):

$$R_0(\mu,\overline{\mu};\lambda,\overline{\lambda}) = \frac{\pi}{2}\sum_{p=1}^{N} A_p\delta(\mu-\Lambda_p)\delta(\lambda-\Sigma_p) \tag{11}$$

which has nonzero values at the set of points

$$\Lambda := (\Lambda_1,\dots,\Lambda_N), \qquad \Sigma := (\Sigma_1,\dots,\Sigma_N) \tag{12}$$

of complex plane, where A_p are arbitrary complex constants; here and below $\delta(\mu-\Lambda_p)$ and $\delta(\lambda-\Sigma_p)$ are complex δ-functions. Using (11) one obtains from (8) the following linear algebraic system of equations for calculating the quantities $\chi(\Lambda_p)e^{F(\Lambda_p)}$:

$$\sum_{q=1}^{N} A_{pq}\chi(\Lambda_q)e^{F(\Lambda_q)} = e^{F(\Lambda_p)}, \qquad A_{pq} := \delta_{pq} + \frac{iA_q e^{F(\Lambda_p)-F(\Sigma_q)}}{\Lambda_p-\Sigma_q}, \quad (p,q=1,\dots,N). \tag{13}$$

Coefficients χ_{-1} and χ_{-2} due to (10) are given by expressions:

$$\chi_{-1} = -i\sum_{p=1}^{N} A_p\chi(\Lambda_p)e^{F(\Lambda_p)-F(\Sigma_p)}, \qquad \chi_{-2} = -i\sum_{p=1}^{N} A_p\chi(\Lambda_p)\Sigma_p e^{F(\Lambda_p)-F(\Sigma_p)}. \tag{14}$$

Here as supposed all denominators in the formula (13) have nonzero values.

Rational solutions of integrable nonlinear equations can be generated by another delta-kernel $R_0(\mu,\overline{\mu};\lambda,\overline{\lambda})$ (3) of $\overline{\partial}$-problem (1):

$$R_0(\mu,\overline{\mu};\lambda,\overline{\lambda}) = \frac{\pi}{2}\sum_{p=1}^{N} A_p\,\delta(\mu-\Lambda_p)\,\delta(\lambda-\Lambda_p) \tag{15}$$

which has nonzero values at the set of isolated points

$$\Lambda := (\Lambda_1,\Lambda_2,\dots,\Lambda_N) \tag{16}$$

of complex plane, where for simplicity we choose A_p as some complex constants. Using (15) in (9) and (10) one obtains for $\widetilde{\chi}_0$ and χ_{-1},χ_{-2} the expressions:

$$\widetilde{\chi}_0 = 1 + \sum_{p=1}^{N}\frac{A_p}{\Lambda_p}\chi(\Lambda_p), \qquad \chi_{-1} = -i\sum_{p=1}^{N} A_p\,\chi(\Lambda_p), \qquad \chi_{-2} = -i\sum_{p=1}^{N} A_p\Lambda_p\chi(\Lambda_p). \tag{17}$$

For the quantities $\chi(\Lambda_p)$ from integral equation (8) a simple algebraic system of equations follows:

$$\sum_{q=1}^{N} A_{pq}\,\chi(\Lambda_q) = 1, \quad A_{pq} = \delta_{pq}(1 + i\,A_pF'(\Lambda_p)) + \frac{i\,A_q(1-\delta_{pq})}{\Lambda_p-\Lambda_q}, \quad (p,q=1,\dots,N). \tag{18}$$

The main problem in constructing soliton and rational solutions is the problem of choice of the sets of points Λ and Σ (12), (16) and constants A_p in (11), (15) in order to satisfy the conditions of reality, potentiality and so on.

2 Exact rational solutions of NVN equations

In this section we present some new rational solutions with constant asymptotic values at infinity of the famous (2+1)-dimensional Nizhnik–Veselov–Novikov (NVN) integrable equations [13, 14]:

$$u_t + \kappa_1 u_{\xi\xi\xi} + \kappa_2 u_{\eta\eta\eta} + 3\kappa_1 \left(u\partial_\xi^{-1} u_\eta\right)_\eta + 3\kappa_2 \left(u\partial_\eta^{-1} u_\xi\right)_\xi = 0 \tag{19}$$

where $u(\xi, \eta, t)$ is a scalar function; κ_1, κ_2 are arbitrary constants, $\partial_\xi = \partial_x + \sigma\partial_y$, $\partial_\eta = \partial_x - \sigma\partial_y$ and $\sigma^2 = \pm 1$. Equation (19) was first introduced by Nizhnik [13] for $\sigma = 1$ and independently by Veselov and Novikov [14] for $\sigma = i$, $\kappa_1 = \kappa_2 = 1$. Here and below ∂_ξ^{-1}, ∂_η^{-1} denote operators inverse to ∂_ξ, ∂_η: $\partial_\xi^{-1}\partial_\xi = \partial_\eta^{-1}\partial_\eta = 1$.

The integrability of (19) by IST and by another means is based on the representation of this equation as the compatibility condition for two linear auxiliary problems

$$\begin{aligned} &L_1\psi = (\partial_{\xi\eta}^2 + U)\psi = 0, \\ &L_2\psi = (\partial_t + \kappa_1\partial_\xi^3 + \kappa_2\partial_\eta^3 + 3\kappa_1(\partial_\xi^{-1} u_\eta)\partial_\xi + 3\kappa_2(\partial_\eta^{-1} u_\xi)\partial_\eta)\psi = 0 \end{aligned} \tag{20}$$

in the form of Manakov's triad

$$[L_1, L_2] = BL_1, \qquad B := 3(\kappa_1\partial_\xi^{-1} U_{\eta\eta} + \kappa_2\partial_\eta^{-1} U_{\xi\xi}). \tag{21}$$

Integration of NVN equation (19) has remarkable history. In the work of Nizhnik [13] the equation (19) with $\sigma = 1$ has been integrated by the technique of inverse problems for hyperbolic systems on the plane. In the paper of Veselov and Novikov [14] for the construction of the periodic finite-zone exact solutions of (19) with $\sigma = i$ algebraic geometric methods have been used. There exist several other beautiful works of Grinevich and Manakov, Grinevich and S. Novikov, Grinevich and R. Novikov, Grinevich in which the problem of construction of exact solutions of Veselov–Novikov (VN) equation [14] and transparent potentials for 2D stationary Schrödinger equations via $\overline{\partial}$-problem combined with nonlocal Riemann–Hilbert problem and so on have been discussed (see [18, 19] and references therein).

Here we present some rational solutions of NVN equations (19) obtained recently in the paper [20]. In the paper [20] the $\overline{\partial}$-dressing method is applied to bare operators of linear auxiliary problems (20) with constant asymptotic value of U at infinity

$$U(\xi, \eta, t) \underset{x^2+y^2\longrightarrow\infty}{\longrightarrow} -\epsilon \neq 0. \tag{22}$$

In this case the first linear auxiliary problem (20) has the form:

$$(\partial_{\xi\eta}^2 + \widetilde{U})\psi = \epsilon\psi. \tag{23}$$

For $\sigma = 1$ (23) can be interpreted ($\xi \Rightarrow t - x$, $\eta \Rightarrow t + y$) as one-dimensional Klein–Gordon or perturbed telegraph equation; for $\sigma = i$ (23) is nothing but the two-dimensional 2D stationary Schrödinger equation. Construction of exact solutions of (19) with constant asymptotic values at infinity means simultaneously calculation of exact wave function ψ and exactly solvable corresponding potentials for above mentioned classical linear equations; here we present also new exact rational potentials for two-dimensional stationary Schrödinger equation which correspond to two-pole wave functions. Our results partially interplay in the case $\sigma = i$ with that obtained by different methods in the papers of Grinevich and his co-authors (see [18, 19] and references therein). The use of the celebrated $\overline{\partial}$-method of Zakharov and Manakov for the construction of new exact solutions for NVN equations (19) by our opinion is very instructive and useful.

The long derivatives (5) in the case of NVN equations (19) have the form:

$$D_1 = \partial_\xi + i\lambda, \qquad D_2 = \partial_\eta - i\epsilon/\lambda, \qquad D_3 = \partial_t + i\left(\kappa_1\lambda^3 - \kappa_2\epsilon^3/\left(\lambda^3\right)\right). \tag{24}$$

One can construct in this case two linear auxiliary problems of the type (7):

$$\begin{aligned} &L_1\chi = (D_1D_2 + V_1D_1 + V_2D_2 + U)\chi = 0,\\ &L_2\chi = \left(D_3 + \kappa_1D_1^3 + \kappa_2D_2^3 + W_1D_1^2 + W_2D_2^2 + W_3D_1 + W_4D_2 + W\right)\chi = 0 \end{aligned} \tag{25}$$

satisfying to the condition of absence of singularities in λ. Reconstruction formulae for V_1, V_2, U in the considered case have the form [20]:

$$V_1 = -\chi_{0\eta}/\chi_0, \qquad V_2 = -\widetilde{\chi}_{0\xi}/\widetilde{\chi}_0, \qquad U = -\epsilon - i\chi_{-1\eta} = -\epsilon + i\chi_{1\xi}. \tag{26}$$

Due to canonical normalization of χ, $\chi_0 = 1$ and $V_1 = 0$. Potentiality condition for the operator L_1 in (25) means $V_2 = 0$ or due to (26) $\widetilde{\chi}_0 = \text{const}$, say $\widetilde{\chi}_0 = 1$, and according to (9) has the form:

$$\iint_C \frac{d\lambda\wedge d\overline{\lambda}}{\lambda}\iint_C d\mu\wedge d\overline{\mu}\,\chi(\mu,\overline{\mu})\,R_0(\mu,\overline{\mu};\lambda,\overline{\lambda})e^{F(\mu)-F(\lambda)} = 0, \tag{27}$$

where due to (4) and (24)

$$F(\lambda) := i\left(\lambda\xi - \frac{\epsilon}{\lambda}\eta\right) - i\left(\kappa_1\lambda^3 - \kappa_2\frac{\epsilon^3}{\lambda^3}\right)t. \tag{28}$$

The conditions of reality U and of potentiality of the operator L_1 give some restrictions on the kernel R_0 of $\overline{\partial}$-problem (1). In Nizhnik case ($\sigma = 1$) of NVN equations (19) with real $\xi = x + y$, $\eta = x - y$ space variables and $\overline{\kappa_1} = \kappa_1$, $\overline{\kappa_2} = \kappa_2$ in the limit of weak fields from (10) and (26) one can easily obtain the following restriction on the kernel R_0 (3) of $\overline{\partial}$-problem:

$$R_0(\mu,\overline{\mu};\lambda,\overline{\lambda}) = \overline{R_0(-\overline{\mu},-\mu;\,-\overline{\lambda},-\lambda)}. \tag{29}$$

To the Veselov–Novikov case ($\sigma = i$, $\kappa_1 = \overline{\kappa_2} = \kappa$) of NVN equations (19) with $z = \xi = x + iy$, $\overline{z} = \eta = x - iy$ the condition of reality of U leads from (10) and (26) in the limit of weak fields to another restriction on the kernel R_0 of $\overline{\partial}$-problem:

$$R_0(\mu,\overline{\mu};\lambda,\overline{\lambda}) = \frac{\epsilon}{|\mu|^2|\lambda|^2\overline{\mu}\lambda}\,\overline{R_0\left(-\frac{\epsilon}{\overline{\lambda}} - \frac{\epsilon}{\lambda}; -\frac{\epsilon}{\overline{\mu}}, -\frac{\epsilon}{\mu}\right)}. \tag{30}$$

Various choices for the kernel R of $\overline{\partial}$-problem (1) satisfying to restrictions (27), (29) and (30) lead to various classes of exact solutions of integrable nonlinear NVN equations (19).

In conclusion of this section let us cite several simplest exact rational solutions of NVN equations (19) calculated in the paper [20].

Nizhnik equation, $\boldsymbol{\sigma = 1}$. 1. The kernel R_0 has the form (15) with $N = 2$, $A_2 = \overline{A_1}$, with the set (16) $\Lambda = (\lambda_1, -\lambda_1)$, $\overline{\lambda_1} = \lambda_1$; the potentiality condition (27) is satisfied for $\frac{1}{A_1} - \frac{1}{\overline{A_1}} = \frac{i}{\lambda_1}$. The solution U of Nizhnik version of equations (19) has the form:

$$U = -\epsilon - \frac{2\epsilon}{\left(\xi\lambda_1 + \frac{\epsilon}{\lambda_1}\eta + 3\left(\kappa_1\lambda_1^3 + \kappa_2\frac{\epsilon^3}{\lambda_1^3}\right)t - a_1\lambda_1\right)^2} \tag{31}$$

with the wave function of equation (23) of the following form [20]:

$$\psi = \frac{\exp\left[\pm i\left(\lambda_1\xi - \left(\epsilon/\lambda_1^2\right)\eta + \left(\kappa_1\lambda_1^3 - \kappa_2\epsilon^3/\lambda_1^3\right)t\right)\right]}{\xi + \left(\epsilon/\lambda_1^2\right)\eta + 3\left(\kappa_1\lambda_1^2 + \kappa_2\epsilon^3/\lambda_1^4\right)t - a_1}. \tag{32}$$

2. The kernel R_0 has the form (15) with $N = 2$ and the set (16) $\Lambda = (i\alpha_1, -i\alpha_1)$, $\overline{\alpha_1} = \alpha_1$; the potentiality condition (27) is satisfied for $\frac{1}{A_2} - \frac{1}{A_1} = \frac{i}{\lambda_1}$. The solution U of Nizhnik version of equations (19) has the form:

$$U = -\epsilon - \frac{2\epsilon}{\left(\xi\alpha_1 - \frac{\epsilon}{\alpha_1}\eta - 3\left(\kappa_1\alpha^3 - \kappa_2\frac{\epsilon^3}{\alpha_1^3}\right)t - a_1\alpha_1\right)^2} \tag{33}$$

with the wave function of equation (23) of the following form [20]:

$$\psi = \frac{\exp\left[\pm\left(\alpha_1\xi + \left(\epsilon/\alpha_1^2\right)\eta - \left(\kappa_1\alpha_1^3 - \kappa_2\epsilon^3/\alpha_1^3\right)t\right)\right]}{\xi - \left(\epsilon/\alpha_1^2\right)\eta - 3\left(\kappa_1\alpha_1^2 - \kappa_2\epsilon^3/\alpha_1^4\right)t - a_1}. \tag{34}$$

The solutions (31), (33) and wave functions (32), (34) evidently are singular.

3. The kernel R_0 has the form (15) with $N = 4$, $A_2 = \overline{A_1}, A_4 = \overline{A_3}$, with the set (16) $\Lambda = (\lambda_1, -\overline{\lambda_1}, -\lambda_1, \overline{\lambda_1})$, the potentiality condition (27) is satisfied for $\frac{1}{\overline{A_3}} - \frac{1}{A_1} = \frac{i}{\lambda_1}$. The solution U of Nizhnik version of equations (19) has the form:

$$U(\xi,\eta,t) = -\epsilon - 2\epsilon\frac{(\lambda_1 X(\lambda_1))^2 + \left(\overline{\lambda_1 X}(\lambda_1)\right)^2 - 1/2\left(\lambda_{1I}^2 - \lambda_{1R}^2\right)^2/\left(\lambda_{1I}^2\lambda_{1R}^2\right)}{\left(|\lambda_1 X(\lambda_1)|^2 + \frac{|\lambda_1|^2}{4}\left(\frac{1}{\lambda_{1I}^2} - \frac{1}{\lambda_{1R}^2}\right)\right)^2} \tag{35}$$

with $X(\lambda_1) = \xi + \frac{\epsilon}{\lambda_1^2}\eta + 3\left(\kappa_1\lambda_1^2 + \kappa_2\frac{\epsilon^3}{\lambda_1^4}\right)t - a_1$ and $\lambda_1 = \lambda_{1R} + i\lambda_{2I}$. The solution (35) evidently nonsingular for $|\lambda_{1I}| < |\lambda_{1R}|$.

Quite analogously one calculates rational solutions of Veselov–Novikov version of equations (19) [20].

Veselov–Novikov equation, $\boldsymbol{\sigma = i}$. 1. The kernel R_0 has the form (15) with $N = 2$, $A_2 = \overline{A_1}\lambda_1/\overline{\lambda_1}$, with the set (16) $\Lambda = (\lambda_1, -\lambda_1)$, $|\lambda_1|^2 = \epsilon$; the potentiality condition (27) is satisfied for $\frac{\overline{\lambda_1}}{\overline{A_1}\lambda_1} - \frac{1}{A_1} = \frac{i}{\lambda_1}$. The solution U of Veselov–Novikov version of equations (19) has the form:

$$U(z,\overline{z},t) = -\epsilon - \frac{2\epsilon}{\left(\lambda_1 z + \overline{\lambda_1}\overline{z} + 3\left(\kappa\lambda_1^3 + \overline{\kappa}\overline{\lambda_1}^3\right)t - \widetilde{a}_1\lambda_1\right)^2} \tag{36}$$

with the wave function of equation (23) (in this case of 2D stationary Schrödinger equation) of the following form [20]:

$$\psi = \frac{\exp\left[\pm i\left(\lambda_1 z - \overline{\lambda_1}\overline{z} + 3\left(\kappa\lambda_1^3 - \overline{\kappa}\overline{\lambda_1}^3\right)t\right)\right]}{\lambda_1 z + \overline{\lambda_1}\overline{z} + 3\left(\kappa\lambda_1^3 + \overline{\kappa}\overline{\lambda_1}^3\right)t - \widetilde{a}_1\lambda_1}. \tag{37}$$

2. The kernel R_0 has the form (15) with $N = 2$, $\overline{A_k}\lambda_k/\overline{\lambda_k} = -A_k$, $k = 1, 2$; with the set (16) $\Lambda = (\lambda_1, -\lambda_1)$, $|\lambda_1|^2 = -\epsilon$; the potentiality condition (27) is satisfied for $\frac{1}{A_2} - \frac{1}{A_1} = \frac{i}{\lambda_1}$. The solution U of Veselov–Novikov version of equations (19) has the form:

$$U(z,\overline{z},t) = -\epsilon - \frac{2\epsilon}{\left(\lambda_1 z + \overline{\lambda_1}\overline{z} + 3\left(\kappa\lambda_1^3 + \overline{\kappa}\overline{\lambda_1}^3\right)t - \widetilde{a}_1\lambda_1\right)^2} \tag{38}$$

with the wave function of equation (23) (in this case of 2D stationary Schrödinger equation) of the following form [20]:

$$\psi = \frac{\exp\left[\pm i\left(\lambda_1 z - \overline{\lambda_1}\overline{z} + 3\left(\kappa\lambda_1^3 - \overline{\kappa}\overline{\lambda_1}^3\right)t\right)\right]}{\lambda_1 z + \overline{\lambda_1}\overline{z} + 3\left(\kappa\lambda_1^3 + \overline{\kappa}\overline{\lambda_1}^3\right)t - \widetilde{a}_1\lambda_1}. \tag{39}$$

The solutions (36), (38) and wave functions (37), (39) evidently are singular.

3. The kernel R_0 has the form (15) with $N = 4$, $A_{2k} = \epsilon\overline{A_{2k-1}}/\overline{\lambda_k}^2$, $k = 1,2$; with the set (16) $\Lambda = (\lambda_1, -\epsilon/\overline{\lambda_1}, -\lambda_1, \epsilon/\overline{\lambda_1})$, $|\lambda_1|^2 = -\epsilon$; the potentiality condition (27) is satisfied for $\frac{1}{A_2} - \frac{1}{A_1} = \frac{i}{\lambda_1}$. The solution U of Veselov–Novikov version of equations (19) has the form:

$$U(z,\overline{z},t) = -\epsilon - 2\epsilon\frac{\lambda_1^2 X(\lambda_1)^2 + \overline{\lambda_1}^2\overline{X}(\lambda_1)^2 + 2\left[\left(\epsilon^2 + |\lambda_1|^4\right)^2/\left(\epsilon^2 - |\lambda_1|^4\right)^2\right]}{\left(|\lambda_1 X(\lambda_1)|^2 - \left[2\epsilon|\lambda_1|^2\left(\epsilon^2 + |\lambda_1|^4\right)/\left(\epsilon^2 - |\lambda_1|^4\right)^2\right]\right)^2}. \tag{40}$$

The solution (40) evidently nonsingular for $\epsilon < 0$ and have been obtained earlier (see [18, 19] and references therein) by another method.

Let us mention that one can consider also the kernels R_0 (3) of $\overline{\partial}$-problem (1) with products of delta-functions with derivatives, for example the kernel R_0

$$\begin{aligned} R_0(\mu,\overline{\mu};\lambda,\overline{\lambda}) = \frac{\pi}{2}\sum_{k=1}^{N}\Bigg[& A_k\,\delta_\mu(\mu-\lambda_k)\delta_\lambda(\lambda-\lambda_k) + \frac{\epsilon^3\overline{A}_k}{|\mu|^2|\lambda|^2\overline{\mu}\overline{\lambda}}\delta_{\epsilon/\lambda}\left(\frac{\epsilon}{\lambda}+\overline{\lambda_k}\right)\delta_{\epsilon/\mu}\left(\frac{\epsilon}{\mu}+\overline{\lambda_k}\right) \\ & + B_k\,\delta_\mu(\mu+\lambda_k)\,\delta_\lambda(\lambda+\lambda_k) + \frac{\epsilon^3\overline{B}_k}{|\mu|^2|\lambda|^2\overline{\mu}\overline{\lambda}}\,\delta_{\epsilon/\lambda}\left(\frac{\epsilon}{\lambda}-\overline{\lambda_k}\right)\delta_{\epsilon/\mu}\left(\frac{\epsilon}{\mu}-\overline{\lambda_k}\right)\Bigg] \end{aligned} \tag{41}$$

in the form of products of derivatives of the first order of complex delta functions which have non-zero values on the set Λ of complex plane consisting N quartets of complex isolated points $\Lambda := \bigcup_{k=1}^{N}(\lambda_k, -\epsilon/\overline{\lambda_k}, -\lambda_k, \epsilon/\overline{\lambda_k})$ arranged symmetrically near the origin and going to each other by inversion relative to the origin and/or to the circle of radius $\sqrt{|\epsilon|}$; A_k, B_k $(k = 1,\ldots,N)$ are some complex constants. Such kernels correspond in the case $\sigma = i$ to so called multiple-pole (to pole of order two in considered case) wave functions of 2D stationary Schrödinger equation (23). Recently following to the paper [21] new exact rational potentials of equation (23) by Dubrovsky and Formusatik have been calculated. For the case $|\lambda_1|^2 = \epsilon > 0$ and one quartet of the points the potentiality condition (27) satisfies for the following choice of parameters $1/B_1 - 1/A_1 = i/\left(2\lambda_1^3\right)$, $\overline{\lambda_1^3/A_1} = -\lambda_1^3/A_1$, $\overline{\lambda_1^3/B_1} = -\lambda_1^3/B_1$ and the corresponding exact potential has the form:

$$U = -2\epsilon\frac{4(\tilde{x}-\tilde{y})^6 - 9\left(\tilde{x}^2-\tilde{y}^2\right)^2}{\left[2(\tilde{x}-\tilde{y})^4 + 3\left(\tilde{x}^2+\tilde{y}^2\right)\right]^2}, \qquad \tilde{x} := \lambda_R(x-\tilde{x}_0), \qquad \tilde{y} := \lambda_I(y-\tilde{y}_0), \tag{42}$$

where

$$\tilde{x}_0 := -\frac{\alpha_1\lambda_I}{|\lambda|^2} + x_0, \qquad \tilde{y}_0 := -\frac{\alpha_1\lambda_R}{|\lambda|^2} + y_0, \qquad \lambda_1 := \lambda_R + i\lambda_I. \tag{43}$$

For another case $|\lambda_1|^2 = -\epsilon > 0$ and one quartet of the points the potentiality condition (27) satisfies for the choice of parameters $1/B_1 - 1/A_1 = i/\left(2\lambda_1^3\right)$, $\overline{\lambda_1^3/A_1} = \lambda_1^3/B_1$ and the corresponding exact potential has the form:

$$U = 2\epsilon\frac{4(\tilde{x}+\tilde{y})^6 + 9\left(\tilde{x}^2-\tilde{y}^2\right)^2}{\left[2(\tilde{x}+\tilde{y})^4 - 3\left(\tilde{x}^2+\tilde{y}^2\right)\right]^2}, \qquad \tilde{x} := \lambda_I(x-\tilde{x}_0), \qquad \tilde{y} := \lambda_R(y-\tilde{y}_0), \tag{44}$$

where

$$\tilde{x}_0 := -\frac{r_1\lambda_R}{|\lambda|^2} + x_0, \qquad \tilde{y}_0 := \frac{r_1\lambda_I}{|\lambda|^2} + y_0, \qquad \lambda_1 := \lambda_R + i\lambda_I. \tag{45}$$

In the formulas (42)–(45) x_0, y_0, α_1, r_1 are some real parameters. It occurs that the main problem in calculating rational solutions corresponding to multiple pole wave functions of 2D stationary Schrödinger equation (23) as in the case of wave functions with simple poles is the fulfillment to the potentiality condition (27), in order to achieve this goal one must to choose in (41) appropriately the constants A_k, B_k and the set Λ.

3 Exact solutions of 2DKK and 2DSK equations

In this section following to the work [22] exact solutions of two-dimensional generalizations of Sawada–Kotera and Kaup–Kuppershmidt equations [16, 17] are considered. The $\overline{\partial}$-dressing method can be applied also to the study of $(2+1)$-dimensional integrable generalizations of Kaup–Kuperschmidt (2DKK)

$$U_t = U_{xxxxx} + \frac{25}{2}U_xU_{xx} + 5UU_{xxx} + 5U_x^2 + 5U_{xxy} - 5\partial_x^{-1}U_{yy} + 5UU_y + 5U_x\partial_x^{-1}U_y \tag{46}$$

and Sawada–Kotera (2DSK)

$$U_t = U_{xxxxx} + 2U_xU_{xx} + 5UU_{xxx} + 5U_x^2 + 5U_{xxy} - 5\partial_x^{-1}U_{yy} + 5UU_y + 5U_x\partial_x^{-1}U_y \tag{47}$$

equations. These equations were discovered in papers [16, 17], now they are known also as a members of the so called CKP hierarchy [16] and can be represented as the compatibility conditions in the Lax form $[L_1, L_2] = 0$; for the 2DKK equation – of the following two linear auxiliary problems [17]:

$$\begin{aligned}
L_1\Psi &= \left(\partial_x^3 + U\partial_x + \frac{1}{2}U_x + \partial_y\right)\Psi = 0,\\
L_2\Psi &= \Bigg[\partial_t - 9\partial_x^5 - 15U^2\partial_x^3 - \frac{45}{2}U_x\partial_x^2\\
&\quad - \left(\frac{35}{2}U_{xx} + 5U^2 - 5\partial_x^{-1}U_y\right)\partial_x - \left(5UU_x - \frac{5}{2}U_y + 5U_{xxx}\right)\Bigg]\Psi = 0
\end{aligned} \tag{48}$$

and for 2DSK equation – of another two linear auxiliary problems [17]:

$$\begin{aligned}
L_1\Psi &= \left(\partial_x^3 + U\partial_x + \partial_y\right)\Psi = 0,\\
L_2\Psi &= \left[\partial_t - 9\partial_x^5 - 15U^2\partial_x^3 - 15U_x\partial_x^2 - \left(10U_{xx} + 5U^2 - 5\partial_x^{-1}U_y\right)\partial_x\right]\Psi = 0.
\end{aligned} \tag{49}$$

Here and bellow $\partial_x \equiv \partial/\partial_x$, $\partial_y \equiv \partial/\partial_y$, $\partial_t \equiv \partial/\partial_t$ and ∂_x^{-1} is an operator inverse to ∂_x. The first linear auxiliary differential problems in (48) and (49) are of the third order on ∂_x, such problems in general position have several fields as the coefficients at the various degrees of ∂_x, the 2DKK equation (46) and 2DSK equation (47) arise as special reductions of some integrable nonlinear systems for these fields. It is well known that study of special reductions requires more attention and may be more difficult than the consideration of nonlinear equations integrable by auxiliary linear problems in general position. By our opinion application of $\overline{\partial}$-dressing method in nonstandard situations of special reductions may be very instructive and useful (in our case some nonlinear constraint on the wave functions of the linear auxiliary problems must be satisfied).

The long derivatives (5) in the considered case have the form:

$$D_1 = \partial_x + i\lambda, \qquad D_2 = \partial_y + i\lambda^3, \qquad D_3 = \partial_t + 9i\lambda^5. \tag{50}$$

By the use of these derivatives one can construct two linear operators (7):

$$L_1\chi = \left(D_2 + D_1^3 + UD_1 + V\right)\chi = 0, \tag{51}$$

$$L_2\chi = \left(D_3 - 9D_1^5 + w_3D_1^3 + w_2D_1^2 + w_1D_1 + w_0\right)\chi = 0 \tag{52}$$

satisfying to the condition of absence of singularities on λ. After simple calculations the following reconstruction formulas for the potentials U and V:

$$U = -3i\chi_{-1x}, \qquad V = -3i\chi_{-1xx} + 3\chi_{-2x} - 3\chi_{-1}\chi_{-1x} \tag{53}$$

and some formulas for potentials w_0, w_1, w_2 can be obtained [22].

It was shown in the paper [17] that to the $(2+1)$-dimensional integrable generalizations of nonlinear Kaup–Kuperschmidt (46) and Sawada–Kotera (47) equations correspond the reductions:

$$(2\mathrm{DKK}): \quad V = \frac{1}{2}U_x, \qquad (2\mathrm{DSK}): \quad V = 0. \tag{54}$$

In terms of the wave function $\chi = \chi_0 + \chi_{-1}/\lambda + \chi_{-2}/\lambda^2 + \cdots$ the reductions (54) can be expressed as the nonlinear constraint on the coefficients χ_{-1} and χ_{-2} (10):

$$(2\mathrm{DKK}): \quad \chi_{-2x} - \frac{i}{2}\chi_{-1xx} - \chi_{-1}\chi_{-1x} = 0, \tag{55}$$

$$(2\mathrm{DSK}): \quad \chi_{-2x} - i\chi_{-1xx} - \chi_{-1}\chi_{-1x} = 0. \tag{56}$$

As usual the solution of the $\overline{\partial}$-problem (1) with canonical normalization $\chi_0 = 1$ is equivalent to the solution of the singular integral equation (8) with $F(\lambda)$ (4) given due to (4), (5) and (50) by the expression:

$$F(\lambda) := i\left(9\lambda x + \lambda^3 y + 9\lambda^5 t\right). \tag{57}$$

The coefficients χ_{-1} and χ_{-2} of Taylor expansion of $\chi(\lambda)$ near the point $\lambda = \infty$ by the formulas (10) are given.

One can easily obtain the restrictions following from reality $\overline{U} = U$ of U on the kernel R_0 of $\overline{\partial}$-problem (1), one has in the limit of weak fields from (10) and (53):

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \overline{R_0(-\overline{\mu}, -\mu; -\overline{\lambda}, -\lambda)}, \qquad R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \overline{R_0(\overline{\lambda}, \lambda; \overline{\mu}, \mu)}. \tag{58}$$

It is evident that the conditions (58) are the same for both 2DKK and 2DSK equations (46) and (47) but the nonlinear constraint (55) and (56) for these equations have different forms. So in order to calculate the exact solutions of 2DKK (46) and 2DSK (47) equations via $\overline{\partial}$-dressing method one must satisfy the conditions of reality (58) and the nonlinear constraint (55) and (56).

Let us consider some new solutions of 2DKK (46) and 2DSK (47) equations obtained recently in the work [22]

Exact solutions of 2DKK equation. 1. In the case of line soliton solutions to the conditions of reality for U (58) the following delta-kernel R_0 of $\overline{\partial}$-problem (1) satisfies:

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \frac{\pi}{2}\sum_{k=1}^{N} A_k \delta(\mu - i\alpha_k)\delta(\lambda + i\alpha_k) \tag{59}$$

with nonzero values at the sets (12) of pure imaginary points $\Lambda := (i\alpha_1, \ldots, i\alpha_N)$, $\Sigma := (-i\alpha_1, \ldots, -i\alpha_N)$ of the complex plane; A_p $(p = 1, \ldots, N)$ are arbitrary real constants; α_p are chosen so that $|\alpha_1| < |\alpha_2| < \cdots < |\alpha_N|$ and consequently $\alpha_p + \alpha_q \neq 0$ for all p, q.

As was shown recently [22] the nonlinear constraint (55) for such kernel (59) in the case of 2DKK (46) equation is satisfied and for the N-soliton solution of 2DKK equation one obtains simple determinant formula:

$$U(x, y, t) = 3\frac{\partial^2}{\partial x^2}\ln\det A \tag{60}$$

with matrix A given by (13). In the simplest case $N = 1$ of the kernel R_0 (59) with one term in the sum using (13) and (60) one obtains one-soliton solution of 2DKK equation (46):

$$U(x, y, t) = \frac{3\alpha_1^2}{\cosh^2[\alpha_1 x - \alpha_1^3 y + 9\alpha_1^5 t - a_1]}, \tag{61}$$

where on the constants A_1 and α_1 additional condition $0 < \frac{A_1}{2\alpha_1} := e^{2a_1}$ is imposed. The general formula (60) represents the superposition of N line soliton solutions of the type (61) interacting with each other elastically. The solutions corresponding to the kernel R_0 of the type (59) have been obtained recently [23] for $N = 1, 2$ via the direct Hirota method by adjusting parameters of solutions using symbolic calculations with well known software package Mathematica. The application of $\overline{\partial}$-dressing method leads immediately to general (N-arbitrary) determinant formula (60).

2. By $\overline{\partial}$-dressing method one can also effectively calculate rational solutions of integrable nonlinear equations. To the rational solutions of 2DKK equation leads for example the following delta-kernel R_0 of $\overline{\partial}$-problem (1):

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \frac{\pi}{2} \sum_{k=1}^{N} [A_k \delta(\mu - i\alpha_k)\delta(\lambda - i\alpha_k) + A_k \delta(\mu + i\alpha_k)\delta(\lambda + i\alpha_k)] \tag{62}$$

which has nonzero values at the following set (16) Λ of pure imaginary points of complex plane $\Lambda := (\Lambda_1, \ldots, \Lambda_{2N}) = (i\alpha_1, -i\alpha_1, \ldots, i\alpha_N, -i\alpha_N)$. Constants A_k in (62) are arbitrary real constants. It is evident that such kernel satisfies the conditions of reality for U (58). One can show [22] that for such kernel the constraint (55) is also satisfied. For the rational solutions of 2DKK equation (46) corresponding to the kernel (62) one obtains again the simple determinant formula (60) with the matrix A given by (18). In the simplest $N = 1$ case of two terms in the sum (62) one has from (60) using (18) the following nonsingular rational solution of 2DKK equation:

$$U(x, y, t) = 6 \frac{\frac{1}{4\alpha_1^2} - \left(X(i\alpha_1) - \frac{1}{A_1}\right)^2}{\left[\frac{1}{4\alpha_1^2} + \left(X(i\alpha_1) - \frac{1}{A_1}\right)^2\right]^2}, \qquad X(i\alpha_1) = x - 3\alpha_1^2 y + 45\alpha_1^4 t. \tag{63}$$

The expression (63) represents nonsingular line lump solution of 2DKK equation (46). The general formula (60) with matrix A (18) gives the superposition of N line nonsingular lumps of the type (63) interacting with each other elastically.

3. Quite analogously to previous case one can show that to reality condition (58) and to the constraint (55) also satisfies the following kernel R_0 of $\overline{\partial}$-problem (1):

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \frac{\pi}{2} \sum_{k=1}^{N} [A_k \delta(\mu - \lambda_k)\delta(\lambda - \lambda_k) + A_k \delta(\mu + \lambda_k)\delta(\lambda + \lambda_k)] \tag{64}$$

which has nonzero values at the set Λ (16) with N pairs $(\lambda_k, -\lambda_k)$, $k = 1, \ldots, N$ of real points of complex plane; here A_k are arbitrary real constants. The calculations at the present case are the same as at the previous one [22]. By the general formulas (15)–(18) and (53) one obtains the solution of 2DKK equation in the simple determinant form (60) with some matrix A of the form (18). In the simplest case $N = 1$ of two terms in the sum (64) due to (18) and (60) the solution $U(x, y, t)$

$$U(x, y, t) = -6 \frac{\left(X(\lambda_1) - \frac{1}{A_1}\right)^2 + \frac{1}{4\lambda_1^2}}{\left[\left(X(\lambda_1) - \frac{1}{\lambda_1}\right)^2 - \frac{1}{4\lambda_1^2}\right]^2}, \qquad X(\lambda_1) = x - 3\lambda_1^2 y + 45\lambda_1^4 t \tag{65}$$

of 2DKK equation (46) represents singular line lump. The general formula (60) gives the superposition of N singular line lumps of the type (65) and also is the singular solution of 2DKK equation (46).

Exact solutions of 2DSK equation. $(2+1)$-dimensional integrable generalization of Sawada–Kotera (2DSK) equation (47) differs from 2DKK equation (46) only by the constant coefficient under the term U_xU_{xx}. These equations are different reductions (54) ($V = 1/2U_x$ and $V = 0$) of some integrable $(2+1)$-dimensional nonlinear system of equations for the fields U and V. Due to this fact these equations have different linear auxiliary problems (48), (49) and as consequence they have different constraints (55) and (56). The main problem in calculations of exact solution of 2DSK equation (47) (as also for 2DKK equation (46)) is the choice of the kernel R_0 of $\overline{\partial}$-problem (1) in the way that the reality conditions (58) and constraint (56) should be satisfied.

1. In order to calculate line soliton solutions of 2DSK equation (47) let us start from the delta-kernel R_0 of the type (11):

$$R_0(\mu,\overline{\mu};\lambda,\overline{\lambda}) = \frac{\pi}{2}\sum_{k=1}^{N}[A_k\delta(\mu - i\alpha_k)\delta(\lambda - i\beta_k) + B_k\delta(\mu + i\beta_k)\delta(\lambda + i\alpha_k)] \tag{66}$$

with nonzero values at the sets (12) $\Lambda = (i\alpha_1, -i\beta_1, \ldots, i\alpha_N, -i\beta_N)$ and $\Sigma = (i\alpha_1, -i\beta_1, \ldots, i\alpha_N, -i\beta_N)$; here A_k, B_k, α_k, β_k ($k = 1, \ldots, N$) are arbitrary real constants. Analogously to the calculations in the case of 2DKK equation (46) one can show [22] that constraint (56) with the kernel R_0 (66) is satisfied if the following relation between constants A_k and B_k is fulfilled: $A_k\alpha_k = B_k\beta_k$. The general formulas (11)–(14) and (60) are valid in the present case and the N-soliton solution $U(x, y, t)$ of 2DSK equation is given by the simple determinant formula (60) with some matrix A of the type (13) [22]. In the simplest case $N = 1$ of two terms in the sum (66) using (11)–(14) and (60) one obtains typical line-soliton solution of 2DSK equation (47):

$$U(x,y,t) = \frac{3(\alpha_1 - \beta_1)^2}{2\cosh^2\frac{1}{2}[(\alpha_1 - \beta_1)x - (\alpha_1^3 - \beta_1^3)y + 9(\alpha_1^5 - \beta_1^5)t - 2a_1]}, \tag{67}$$

where on constants A_1, α_1, β_1 the condition $0 < \frac{A_1(\alpha_1+\beta_1)}{2\beta_1(\alpha_1-\beta_1)} := e^{2a_1}$ is imposed. The general formula (60) with matrix (13) represents the superposition of N line soliton solutions of the type (67) which interact with each other elastically.

2. As the second example let us calculate rational solutions of 2DSK equation (47) which correspond to the delta-kernel R_0 of the type (15):

$$R_0(\mu,\overline{\mu};\lambda,\overline{\lambda}) = \frac{\pi}{2}\sum_{k=1}^{N}[A_k\delta(\mu - i\alpha_k)\delta(\lambda - i\alpha_k) + A_k\delta(\mu + i\alpha_k)\delta(\lambda + i\alpha_k)] \tag{68}$$

with nonzero values at the set (16) $\Lambda = (i\alpha_1, -i\alpha_1, \ldots, i\alpha_N, -i\alpha_N)$; here A_k, B_k, α_k ($k = 1, \ldots, N$) are arbitrary real constants. Analogously to the calculations in the case of 2DKK equation one can show [22] that constraint (56) with the kernel R_0 (68) is satisfied if the following relation between constants A_k and B_k is fulfilled: $\frac{1}{B_k} - \frac{1}{A_k} = \frac{1}{\alpha_k}$; from the last relation follows the parameterizations: $\frac{1}{B_k} = a_k + \frac{1}{2\alpha_k}$, $\frac{1}{A_k} = a_k - \frac{1}{2\alpha_k}$ with a_k ($k = 1, \ldots, N$) – arbitrary real constants. The general formulas (15)–(18) are valid in the present case and the rational solution of 2DSK equation (47) corresponding to the kernel R_0 (68) is given by the simple determinant formula (60) with some matrix A of the type (18). In the simplest case $N = 1$ of two terms in the sum (68) using (18), (60) and (68) one obtains [22] singular rational solution of 2DSK equation (47):

$$U(x,y,t) = \frac{6}{(x - 3\alpha_1^2 y + 45\alpha_1^4 t - a_1)^2}. \tag{69}$$

The general formula (60) represents the superposition of N line lump solutions of the type (69) interacting with each other elastically, these solutions are also singular.

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$\boldsymbol{R}$-Matrix Approach to the Krall–Sheffer Problem

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The complete set of commuting invariants for integrable systems arising in the framework of the Krall–Sheffer problem is derived using the classical R-matrix approach, based on the loop algebra $\widetilde{sl}(2)_R$. The separating coordinates are also deduced from this framework.

1 Introduction

Krall and Sheffer studied the problem of finding all polynomial eigenfunctions of second order linear differential operators in two variables having polynomial coefficients of degree equal to the order of derivative under certain further restrictions relating to its symmetrizability and the orthogonality of its eigenfunctions (for details see [2]). They classified all possible normal forms of the operators satisfying the required properties. It was shown in [3] that all the operators in the Krall–Sheffer list are reducible by gauge transformations to the form of a Laplace–Beltrami operator on a space of constant curvature plus some potential, the magnetic field being absent. Moreover, they all are related to two-dimensional superintegrable systems on spaces of constant curvature [2].

In this paper we show how to construct a complete set of commuting invariants to the integrable systems arising in the Krall–Sheffer framework using the classical R-matrix approach, based on the loop algebra $\widetilde{sl}(2)_R$. We give both the quantum and classical formulations in terms of Lax matrices depending on a loop parameter. The main construction is based on the well-known procedure of symmetry reduction from a free system in a higher dimension space (in particular, quadrics in $\mathbb{R}^6$ or $\mathbb{C}^6$). Classically this corresponds to reduction of geodesic flow, while quantum mechanically it involves reduction of the Laplacian. The reduction process leaves a residue of the original system, providing a complete set of commuting integrals.

2 General construction scheme

We begin with a phase space $\mathbb{M}$ of $\dim\mathbb{M} = 12$, with canonical variables $(x_i, y_i)_{i=1,\dots,6}$ which form the components of a pair (X, Y) of (either real or complex) column vector.

From these we form a Lax matrix $N(\lambda)$, depending on a spectral parameter $\lambda \in \mathbb{C}$ as follows:

$$N(\lambda) := \frac{1}{2}\left(Y^T, -X^T J\right)(\lambda - A)^{-1}(X, JY) = \sum_{i=1}^{n}\sum_{a=1}^{m_i}\frac{N_i^a}{(\lambda-\alpha_i)^a},$$

where A, J are fixed 6×6 matrices with A having either $n = 1, 2$ or 3 distinct eigenvalues $\{\alpha_i\}_{i=1,\dots,n}$ and minimal polynomial

$$\prod_{i=1}^{n} (\lambda - \alpha_i)^{m_i}$$

and J is a symmetric real matrix with antidiagonal blocks of the form

$$\begin{pmatrix} 0 & 0 & \dots & 1 \\ 0 & \dots & 1 & 0 \\ \dots & & & \\ 1 & \dots & & \end{pmatrix}$$

for each Jordan block of A.

The dynamics is generated by Hamiltonians chosen from the algebra of spectral invariants of $N(\lambda)$. Classically, these Poisson commute and hence generate isospectral flows satisfying a Lax equation:

$$\frac{dN}{dt} = [B, N] \, .$$

It is easily verified that $N(\lambda)$ satisfies the standard rational R-matrix Poisson bracket relations:

$$\{N(\lambda) \otimes N(\mu)\} = [r(\lambda), N(\lambda) \otimes \mathbb{I} + \mathbb{I} \otimes N(\mu)],$$

where both sides are viewed, for fixed $\lambda \neq \mu$ as elements of $\mathrm{End}\left(\mathbb{C}^6 \otimes \mathbb{C}^6\right)$ and

$$r(\lambda) = \frac{P_{1,2}}{(\lambda - \mu)}, \qquad P_{1,2}(u \otimes v) = v \otimes u.$$

In the cases considered below, we only study Hamiltonians that are $O(6, J)$ invariant and restrict to the quadric defined by

$$X^T J X = 1.$$

Quotienting by the stabilizer $G_A \subset O(6, J)$ of A we reduce to a 2-dimensional configuration space, however the reduced system is no longer free.

In this case the algebra of spectral invariants is generated by the coefficients of:

$$-\frac{1}{2} \operatorname{Tr} N(\lambda)^2 = \sum_{i=1}^{n} \sum_{d=1}^{2m_i} \frac{H_i}{(\lambda - \alpha_i)^d}$$

with $2m_i \leq n_i$. The numerators H_i of this partial fraction expansion all Poisson commute and generate the algebra of spectral invariants. They are not all independent, however, since:

$$\sum_{i=1}^{n} H_{id} = 0$$

and H_{id} with $m_i < d \leq 2m_i$ are Casimir invariants.

The connection between configuration space coordinates in 6-dimensional space and the separating coordinates λ_1, λ_2 in the reduced 2-dimensional space is given by

$$X^T J (\lambda - A)^{-1} X = \frac{(\lambda - \lambda_1)(\lambda - \lambda_2)}{a(\lambda)},$$

where $a(\lambda)$ is the minimal polynomial of the matrix A.

The quantum version of this approach is simply obtained through canonical quantization with conjugate (momentum) variables y_j replaced by the partial derivatives $i\,\partial/\partial x_j$. The relation between the quantum integrals and the ones in the corresponding Krall–Sheffer cases is obtained applying a suitable gauge transformation.

3 Case 1. Sphere. Neuman–Rosochatius system

In the case of a sphere in $\mathbb{R}^6$, the matrices A and J are just:

$$A = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma \end{pmatrix}, \qquad J = id$$

with $\alpha \neq \beta \neq \gamma$. The symmetry algebra g_A corresponding to the stabilizer $G_A \subset O(6, \mathbb{R})$ is a maximal torus with generators

$$\{x_1 y_2 - x_2 y_1,\ x_3 y_4 - x_4 y_3,\ x_5 y_6 - x_6 y_5\}$$

and the Lax matrix has the form:

$$N(\lambda) = \frac{N_1}{(\lambda - \alpha)} + \frac{N_2}{(\lambda - \beta)} + \frac{N_3}{(\lambda - \gamma)} = \begin{pmatrix} h(\lambda) & f(\lambda) \\ e(\lambda) & -h(\lambda) \end{pmatrix},$$

where the N_i are elements of $sl(2)$

$$N_1 = \frac{1}{2} \begin{pmatrix} x_1 y_1 + x_2 y_2 & y_1^2 + y_2^2 \\ -x_1^2 - x_2^2 & -x_1 y_1 - x_2 y_2 \end{pmatrix},$$
$$N_2 = \frac{1}{2} \begin{pmatrix} x_3 y_3 + x_4 y_4 & y_3^2 + y_4^2 \\ -x_3^2 - x_4^2 & -x_3 y_3 - x_4 y_4 \end{pmatrix},$$
$$N_3 = \frac{1}{2} \begin{pmatrix} x_5 y_5 + x_6 y_6 & y_5^2 + y_6^2 \\ -x_5^2 - x_6^2 & -x_5 y_5 - x_6 y_6 \end{pmatrix}.$$

The invariants are the coefficients of:

$$-\frac{1}{2} \operatorname{Tr} N(\lambda)^2 = \frac{H_1}{(\lambda - \alpha)} + \frac{H_2}{(\lambda - \beta)} + \frac{H_3}{(\lambda - \gamma)} + \frac{\mu_1^2}{(\lambda - \alpha)^2} + \frac{\mu_2^2}{(\lambda - \beta)^2} + \frac{\mu_3^2}{(\lambda - \gamma)^2}.$$

Here μ_1, μ_2 and μ_3 are constants defining the restriction to level sets of invariants of motion under the reduction procedure (the components of the moment map generating the torus action), namely:

$$\mu_1 = x_1 y_2 - x_2 y_1, \qquad \mu_2 = x_3 y_4 - x_4 y_3, \qquad \mu_3 = x_5 y_6 - x_6 y_5.$$

Integrals H_1, H_2 and H_3 are not all independent, since their sum is equal to zero. The Hamiltonian of the problem is given by the linear combination:

$$H = \alpha H_1 + \beta H_2 + \gamma H_3.$$

The constraint to a sphere $\mathbb{S}^5 \subset \mathbb{R}^6$ is given by $X^T J X = 1$:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = 1.$$

The reduced ambient coordinates are given by the radial distance in three planes (X_1, X_2), (X_3, X_4) and (X_5, X_6):

$$s_1^2 = x_1^2 + x_2^2, \qquad s_2^2 = x_3^2 + x_4^2, \qquad s_3^2 = x_5^2 + x_6^2.$$

The reduction of the constraint gives

$$s_1^2 + s_2^2 + s_3^2 = 1.$$

The reduced Hamiltonian is:

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + p_3^2\right) + \frac{\mu_1^2}{s_1^2} + \frac{\mu_2^2}{s_2^2} + \frac{\mu_3^2}{s_3^2}.$$

which is the kinetic energy on the sphere in $\mathbb{R}^3$ plus Rosochatius potential. Here (p_1, p_2, p_3) are canonical conjugate to (s_1, s_2, s_3).

The reduced separating coordinates (λ_1, λ_2) in this case are sphero-conical coordinates related to (s_1, s_2, s_3) by:

$$s_1^2 = \frac{(\alpha - \lambda_1)(\alpha - \lambda_2)}{(\alpha - \beta)(\alpha - \gamma)}, \qquad s_2^2 = \frac{(\beta - \lambda_1)(\beta - \lambda_2)}{(\beta - \alpha)(\beta - \gamma)}, \qquad s_3^2 = \frac{(\gamma - \lambda_1)(\gamma - \lambda_2)}{(\gamma - \alpha)(\gamma - \beta)}.$$

In terms of the reduced ambient space coordinates the integrals H_1, H_2 and H_3 are:

$$\begin{aligned}
H_1 &= -\frac{1}{2}\frac{L_{13}^2 + \mu_3^2 s_1^2/s_3^2 + \mu_1^2 s_3^2/s_1^2}{\alpha - \gamma} - \frac{1}{2}\frac{L_{12}^2 + \mu_1^2 s_2^2/s_1^2 + \mu_2^2 s_1^2/s_2^2}{\alpha - \beta},\\
H_2 &= -\frac{1}{2}\frac{L_{23}^2 + \mu_3^2 s_2^2/s_3^2 + \mu_2^2 s_3^2/s_2^2}{\beta - \gamma} + \frac{1}{2}\frac{L_{12}^2 + \mu_1^2 s_2^2/s_1^2 + \mu_2^2 s_1^2/s_2^2}{\alpha - \beta},\\
H_3 &= \frac{1}{2}\frac{L_{23}^2 + \mu_3^2 s_2^2/s_3^2 + \mu_2^2 s_3^2/s_2^2}{\beta - \gamma} + \frac{1}{2}\frac{L_{13}^2 + \mu_3^2 s_1^2/s_3^2 + \mu_1^2 s_3^2/s_1^2}{\alpha - \gamma},
\end{aligned}$$

where $L_{ij} = s_1 p_2 - s_2 p_1$. The quantum versions of these integrals are denoted by $(\hat{H}_1, \hat{H}_2, \hat{H}_3)$ and are obtained by replacing the matrix elements of $N(\lambda)$ by the corresponding differential operators. This leads to replacing L_{ij} by their quantum version:

$$\hat{L}_{ij} = \sqrt{-1}(s_i \partial/\partial s_j - s_j \partial/\partial s_i).$$

Note that whereas the Hamiltonian H is independent of the parameters (α, β, γ), which only serve to determine the separating coordinate system, the invariants H_1, H_2 individually do depend on those. Therefore, different choices for these parameters give distinct integrals that commute with H, but do not commute with each other. This provides an explanation for the superintegrability of this system.

To relate the invariants to the ones obtained in [2] for the corresponding Krall–Sheffer case we apply the gauge transformation consisting of conjugation by the function:

$$\Phi = x^{d_1} y^{d_2} (1 - x - y)^{d_3},$$

where

$$d_1 = \frac{1}{2}(d_{00} + 1/2), \qquad d_2 = \frac{1}{2}(e_{00} + 1/2), \qquad d_3 = \frac{1}{2}(1/2 - d_{00} - e_{00} - B)$$

and d_{00}, e_{00}, B are the parameters appearing in Krall–Sheffer setting (see [2]).

The following are the relations between the integrals constructed in these two approaches:

$$\tilde{H}_1 = 4\frac{\alpha_1 - \gamma_1}{\beta_1 - \gamma_1}\hat{I}_x + 4\hat{I}_y - 4\hat{L} - c_0, \qquad \tilde{H}_2 = 4\frac{\gamma_1 - \beta_1}{\gamma_1 - \alpha_1}\hat{I}_y + 4\hat{I}_x - 4\hat{L} - c_1,$$

where $\tilde{H}_i = \Phi \hat{H} \Phi^{-1}$ and $\hat{L}$ is the Krall–Sheffer operator corresponding to case I, c_0 and c_1 depend on α, β, γ, d_{00}, e_{00}, B.

4 Case 2. Hyperboloid

For the case of a hyperboloid embedded in $\mathbb{R}^6$, matrices (A, J) may be taken as

$$A = \begin{pmatrix} \alpha & 1 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 1 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 1 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that J has an antidiagonal block corresponding to each Jordan block of A and a diagonal block corresponding to the diagonal part of A.

The symmetry algebra g_A again has three generators

$$\{x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4,\ x_2y_1 - x_4y_3,\ x_5y_6 - x_6y_5\}$$

but the Lax matrix now has a second order pole at $\lambda = \alpha$:

$$N(\lambda) = \frac{N_1}{(\lambda - \alpha)} + \frac{N_2}{(\lambda - \alpha)^2} + \frac{N_3}{(\lambda - \beta)},$$

where

$$N_1 = \frac{1}{2}\begin{pmatrix} x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 & 2y_1y_4 + 2y_2y_3 \\ -2x_1x_4 - 2x_2x_3 & -x_1y_1 - x_2y_21x_3y_3 - x_4y_4 \end{pmatrix},$$
$$N_2 = \frac{1}{2}\begin{pmatrix} -x_4y_3 + x_2y_1 & -2y_3y_1 \\ 2x_2x_4 & -x_2y_1 + x_4y_3 \end{pmatrix},$$
$$N_3 = \frac{1}{2}\begin{pmatrix} x_5y_5 + x_6y_6 & y_5^2 + y_6^2 \\ -x_5^2 - x_6^2 & -x_5y_5 - x_6y_6 \end{pmatrix}.$$

Here (N_1, N_2) should be viewed as an element of the jet extension $sl(2)^{(1)*}$ while $N_3 \in sl(2)$. The invariants again give us only two independent H_1 and H_2

$$-\frac{1}{2}\operatorname{Tr} N(\lambda)^2 = \frac{H_1}{(\lambda - \alpha)} + \frac{H_2}{(\lambda - \alpha)^2} - \frac{\mu_1\mu_2}{(\lambda - \alpha)^3} + \frac{\mu_2^2}{2(\lambda - \alpha)^4} + \frac{H_3}{(\lambda - \beta)} - \frac{\mu_3^2}{2(\lambda - \beta)^2},$$

where

$$H_1 + H_3 = 0.$$

The Hamiltonian is now defined by:

$$H = (\alpha - \beta)H_1 + H_2 - \frac{1}{2}\mu_3^2.$$

The reduced ambient space coordinates (s_1, s_2, s_3) are now defined by:

$$s_1^2 = \frac{(x_1x_4 + x_2x_3)^2}{2x_2x_4}, \qquad s_2^2 = 2x_2x_4, \qquad s_3^2 = x_5^2 + x_6^2.$$

The constraint to the quadric $X^TJX = 1$ reduces to define a hyperboloid in $\mathbb{R}^3$

$$2s_1s_2 + s_3^2 = 1.$$

In these coordinates the integrals H_1 and H_2 are

$$
\begin{aligned}
H_1 &= \frac{(s_1p_3 - s_3p_2)(s_3p_1 - s_2p_3) - \mu_3^2 s_1 s_2/s_3^2 + \mu_1\mu_2 s_3^2/s_2^2 - \mu_2^2 s_1 s_3^2/s_2^2}{\alpha - \beta} \\
&\quad - \frac{(s_3p_1 - s_2p_3)^2 + \mu_3^2 s_2^2/s_3^2 - \mu_2^2 s_3^2/s_2^2}{2(\alpha-\beta)^2}, \\
H_2 &= \frac{1}{2}(s_1p_1 - s_2p_2)^2 - 2\frac{\mu_2^2 s_1^2}{s_2^2} + 2\frac{\mu_1\mu_2 s_1}{s_2} + \frac{(s_3p_1 - s_2p_3)^2 + \mu_3^2 s_2^2/s_3^2 - \mu_2^2 s_3^2/s_2^2}{2(\alpha-\beta)}.
\end{aligned}
$$

The quantized operators $\hat{H}_1$, $\hat{H}_2$, $\hat{H}_3$ are obtained as before by replacing all conjugate variables by corresponding differential operators. And again, whereas Hamiltonian H does depend on the parameters (α, β) the integrals H_1, H_2 do, thereby again providing an explanation for the superintegrability in this case.

5 Case 3. Pseudoeuclidean plane

Matrix A in this case has only one degenerate eigenvalue:

$$
A = \begin{pmatrix}
\alpha & 1 & 0 & 0 & 0 & 0 \\
0 & \alpha & 1 & 0 & 0 & 0 \\
0 & 0 & \alpha & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha & 1 & 0 \\
0 & 0 & 0 & 0 & \alpha & 1 \\
0 & 0 & 0 & 0 & 0 & \alpha
\end{pmatrix}, \qquad
J = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

J is antidiagonal.

The symmetry algebra g_A is generated by

$$
\{-x_1y_4 - x_2y_5 - x_3y_6 + x_4y_1 + x_5y_2 + x_6y_3,\; x_6y_1 - x_3y_4,\; -x_2y_4 - x_3y_5 + x_5y_1 + x_6y_2\}
$$

and the Lax matrix is of the form:

$$
N(\lambda) = \frac{N_1}{(\lambda-\alpha)} + \frac{N_2}{(\lambda-\alpha)^2} + \frac{N_3}{(\lambda-\alpha)^3},
$$

where

$$
\begin{aligned}
N_1 &= \frac{1}{2}\begin{pmatrix}
\begin{aligned} &x_1y_1 + x_2y_2 + x_3y_3 \\ &\quad + x_4y_4 + x_5y_5 + x_6y_6 \end{aligned} & 2y_1y_3 + y_2^2 + 2y_4y_6 + y_5^2 \\
-2x_1x_3 - x_2^2 - 2x_4x_6 - x_5^2 & \begin{aligned} &-x_1y_1 - x_2y_2 - x_3y_3 \\ &\quad - x_4y_4 - x_5y_5 - x_6y_6 \end{aligned}
\end{pmatrix}, \\
N_2 &= \frac{1}{2}\begin{pmatrix}
-x_3y_2 - x_2y_1 - x_6y_5 - x_5y_4 & -2y_2y_1 - 2y_4y_5 \\
2x_2x_3 + 2x_5x_6 & x_3y_2 + x_2y_1 + x_6y_5 + x_5y_4
\end{pmatrix}, \\
N_3 &= \frac{1}{2}\begin{pmatrix}
x_3y_1 + x_6y_4 & y_1^2 + y_4^2 \\
-x_3^2 - x_5^2 & -x_3y_1 - x_6y_4
\end{pmatrix}.
\end{aligned}
$$

The trace formula again gives only two independent integrals H_1 and H_2

$$
-\frac{1}{2}\operatorname{Tr} N(\lambda)^2 = \frac{H_1}{(\lambda-\alpha)^2} + \frac{H_2}{(\lambda-\alpha)^3} - \frac{2\mu_1\mu_2 - \mu_3^2}{2(\lambda-\alpha)^4} + \frac{\mu_2\mu_3}{2(\lambda-\alpha)^5} - \frac{\mu_2^2}{2(\lambda-\alpha)^6}.
$$

The Hamiltonian of the problem is:

$$H = -2p_1p_3 - p_2^2 + 2\gamma_1\gamma_3 + \gamma_2^2,$$
$$\gamma_1 = \frac{\mu_1}{s_1} - \frac{\mu_2 s_2}{s_1^2} - \frac{\mu_3 s_2^2}{s_1^2} - \frac{\mu_3 s_3}{s_1^2}, \qquad \gamma_2 = \frac{\mu_2}{s_1} - \frac{\mu_3 s_2}{s_1^2}, \qquad \gamma_3 = \frac{\mu_3}{s_1}.$$

In this case the parameter α may be absorbed in the definition of λ and therefore no parameter dependence appears in the integrals H_1 and H_2:

$$H_1 = (p_2s_3 - s_2p_1)(s_1p_1 - s_3p_3) - 2s_2s_3\left(p_2^2 + 2p_1p_3\right) - \frac{\mu_1\mu_2}{s_1^2} - \frac{3\mu_3\mu_2 s_3}{s_1^3}$$
$$- \frac{\mu_3\mu_2 s_1}{s_2^2} - \frac{4s_3\mu_3\mu_1(1-2s_1s_2)}{s_1^4} - \frac{\mu_3 s_2^2}{s_1^2} - \frac{(\mu_2^2 + \mu_3\mu_1)s_2}{s_1^3},$$
$$H_2 = \left(p_2^2 + 2p_1p_3\right)\left(s_2^2 + 2s_1s_3\right) + \frac{2\mu_3^2 s_1}{s_2^2} + \frac{4\mu_3^2 s_3^2}{s_1^2}$$
$$+ \frac{4\mu_3\mu_2 s_2}{s_1^3} - \frac{\mu_2^2 - 2\mu_3\mu_1}{s_1^2} + \frac{\mu_3^2\left(1 - 2s_2^2\right)}{s_1^4}.$$

Reduced coordinates in $\mathbb{R}^3$

$$s_1^2 = -\frac{(x_1x_3 + x_4x_6)^2}{x_3^2 + x_6^2}, \qquad s_2^2 = x_2^2 + x_5^2, \qquad s_3^2 = -\left(x_3^2 + x_6^2\right).$$

The constraint to the quadric $X^TJX = 1$ reduces to $2s_1s_3 + s_2^2 = 1$.

6 Conclusions

The approach based on Lax matrices satisfying the rational R-matrix structure gives a systematic way to derive the Hamiltonians and commuting invariants for these three cases corresponding to Krall–Sheffer operators on quadrics. This also provides a prescription for the separating coordinates, both in the classical and quantum cases. The presence of the additional parameters (α, β, γ) in the Case I, and (α, β) in the case II provides an explanation for their superintegrability.

A similar analysis may be made for the cases of Euclidean space arised in the Krall–Sheffer problem, they may be obtained as limiting cases of the above, providing an R-matrix approach to the remaining Krall–Sheffer operators. The details for all these cases will be provided elsewhere.

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On Integrability of Some Nonlinear Model with Variable Separant

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In this paper a new integrable nonlinear Hamiltonian system in $(1+1)$-dimension is introduced. Nontrivial connection with well-known multicomponent nonlinear Schrödinger model is found.

Let us consider a non-linear Hamiltonian system

$$\psi_t = \{\psi, H\} \tag{1}$$

in the Schwarz space of smooth fast decreasing on the $\pm\infty$ complex value l-component vector-functions $\psi = (\psi_1, \dots, \psi_l)(x)$, $l \in \mathbb{N}$ of the variable $x \in \mathbb{R}$ with the Hamiltonian

$$H = \int_{-\infty}^{+\infty} |\psi_x|^2 dx, \tag{2}$$

and local brackets of Poisson for dynamic variables ψ_m, ψ_n, $m, n = \overline{1, l}$:

$$\left\{\psi_n(x), \bar{\psi}_m(y)\right\} := i\delta_n^m \left(c + |\psi|^2(x)\right)^2 \delta(x-y), \tag{3}$$

where δ_n^m is the Kronecker symbol, $\delta(z)$ is the Dirac function, $c \in \mathbb{R}$.

System (1)is non-linear evolutionary system of differential equations with variable separant (coefficient at higher derivative) and has the next form:

$$i\psi_t = -\left(c + |\psi|^2\right)^2 \frac{\delta H}{\delta \psi^*} = \left(c + |\psi|^2\right)^2 \psi_{xx}, \tag{4}$$

where $\frac{\delta}{\delta \psi^*}$ is the Euler operator of variative derivative over the vector-function $\psi^* := \bar{\psi}^\top$.

Proposition 1. *Hamiltonian system (1)–(4) is formally integrable (by Lax) and assumes infinitive hierarchy non-trivial local laws of motion.*

Proof. For simplicity we restrict ourselves with Lax commutative representation discovered by us $[L, M] := LM - ML = 0$ in algebra of integro-differential operators [1, 2] which is equivalent to system (4), where

$$L = \left(c + |\psi|^2\right)\mathcal{D} + \psi_x\psi^* - \psi_x\mathcal{D}^{-1}\psi_x^*, \tag{5}$$

$$M = i\partial_t - \left(c + |\psi|^2\right)^2 \mathcal{D}^2 - 2\left(c + |\psi|^2\right)|\psi|_x^2\mathcal{D} = i\partial_t - \left(L^2\right)_{>0}, \tag{6}$$

and, as consequence of operators commutativity in (5)–(6), known [1] procedure for finding density ρ_k of first integrals $H_k := \int_{-\infty}^{+\infty} \rho_k dx$:

$$\rho_k = \operatorname{Res}\left(L^k\right), \qquad k \in \mathbb{Z}. \tag{7}$$

■

Remark 1. Obviously, $k = 1$ corresponds to Hamiltonian $H(2)$, and one of the simplest first integrals ($k = -1$) in the formula (7) has the form:

$$H_{-1} = \int_{-\infty}^{+\infty} \frac{|\psi|^2}{c + |\psi|^2} dx, \qquad c \in \mathbb{R} \setminus \{0\}.$$

Remark 2. In the formula (5) integral item $\psi_x \mathcal{D}^{-1} \psi_x^*$ is a symbol of skew-Hermitian operator of Volterra $\widehat{V}$ with the degenerated kernel $V(x,s) := \frac{\partial\psi(x)}{\partial x}\frac{\partial\psi^*(s)}{\partial s}$

$$\left(\widehat{V} f\right)(x) = \frac{1}{2}\left\{\int_{-\infty}^{x} \sum_{i=1}^{l} \frac{\partial\psi_i(x)}{\partial x}\frac{\partial\bar{\psi}_i(s)}{\partial s} f(s) ds - \int_{x}^{+\infty} \sum_{i=1}^{l} \frac{\partial\psi_i(x)}{\partial x}\frac{\partial\bar{\psi}_i(s)}{\partial s} f(s) ds\right\}.$$

The symbol $\left(L^k\right)_{>0}$ strands for the differential part without free term (multiplier operator by function) of an integro-differential operator L^k.

Proposition 2. *The followng non-local replacement of variables* $(t, x, \psi) \to (\tau, y, \varphi)$:

$$\tau = t, \qquad y'_x = \frac{1}{c + |\psi|^2}, \qquad \varphi(\tau, y) = \frac{\psi_y}{c + |\psi|^2} \exp \int_{-\infty}^{y} \frac{\psi_y \psi^*}{c + |\psi|^2} dy \tag{8}$$

transforms non-linear system (4) into the multicomponent non-linear equation of Schrödinger [3]

$$i\varphi_\tau = \varphi_{xx} + 2|\varphi|^2\varphi. \tag{9}$$

Proof. The proof is conducted by direct calculation. We restrict ourselves by the Lax operator (5). Making replacement (8) we get

$$L = \left(c + |\psi|^2\right) \mathcal{D}_x + \psi_x \psi^* - \psi_x \mathcal{D}_x^{-1} \psi^* \to \widetilde{L} = \mathcal{D}_y + \frac{\psi_y \psi^*}{c + |\psi|^2} - \frac{\psi_y}{c + |\psi|^2} \mathcal{D}_y^{-1} \psi_y^*,$$

and after gauge trasformation $\widetilde{L} \to \Phi \widetilde{L} \Phi^{-1}$ with the function $\Phi = \exp \int_{-\infty}^{y} \frac{\psi_y \psi^*}{c + |\psi|^2} dy$ the operator L to pass into the Lax operator L_{NS} [2, 4, 5] for the model (9):

$$L_{NS} = \Phi \widetilde{L} \Phi^{-1} = \mathcal{D}_y - \varphi \mathcal{D}^{-1} \varphi^*,$$

where the dynamic variable $\varphi = \varphi(\tau, y)$ is defined by substitution (8). ■

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Aspects of Symmetry in Sine-Gordon Theory

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As a prototype of powerful non-Abelian symmetry in an Integrable System, I will show the appearance of a Witt algebra of vector fields in the SG theory. This symmetry does not share anything with the well-known Virasoro algebra of the conformal $c = 1$ unperturbed limit. Although it is quasi-local in the SG field theory, nevertheless it gives rise to a local action on N-soliton solution variables. I will explicitly write the action on special variables, which possess a beautiful geometrical meaning and enter the Form Factor expressions of quantum theory. At the end, I will also give some preliminary hints about the quantisation.

1 Introduction

Nowadays the very peculiar rôle of symmetries is clearly recognised in all the areas of Mathematical Physics also thanks to the recent developments of Quantum Physics. In fact, it was in the context of Classical Physics that Liuoville defined as integrable a system having a number of local integrals of motion in involution (LIMI's) equal to the degrees of freedom and proposed a theorem (Liouville–Arnold theorem [1]) to *solve the motion up to quadratures* – in the case of finite number of degrees of freedom. Nevertheless, there is no equivalent theorem when the degrees of freedom become infinite as well as the number of Abelian symmetries: the classical field theories represent an important example which attracted more and more interest. The situation is even more complicated when the system is a quantum field theory: in this case we may be interested, for instance, in the energy spectrum [2, 3] or in the spectrum of fields or in the correlation functions of those fields [4], as the usual meaning of motion is definitely lost. In fact, in systems with infinite degrees of freedom non-Abelian symmetries revealed to be more useful: let us think of Classical Inverse Scattering Method [5] and Bethe Ansatz [2, 3] as two illustrative examples among the others. Moreover, the Virasoro algebra in two Dimensional Conformal Field Theories (CFT's) represents perhaps the most successful example of how a non-Abelian symmetry can *solve* a quantum field theory and in this case a theory realising a physical system at the very important *critical point* [6].

Unfortunately, this Virasoro algebra does not exist any longer if the system is pushed out of the critical point, still preserving Liouville integrability [6]. For instance, the Sine-Gordon (SG) theory is one of the simplest massive Integrable Field Theories (IFT's), although it is the first theory in a series of structure richer theories, the Affine Toda Field Theories (ATFT's) [7] and possesses all the features peculiar to the more general IFT's [8]. Actually, non-Abelian infinite-dimensional symmetries were found in all Toda theories and they are called dressing symmetries at classical level [9] and become (level 0) affine quantum algebras after quantisation [10]. Nevetheless, because of their affine and highly non-local characters those symmetries are not of large use.

In this talk I present the appearance of infinitesimal symmetry transformations (vector fields) acting on the boson field of the classical Sine-Gordon theory. These vector fields turn out to close a Witt (centerless Virasoro) algebra. Since the only *ingredient of the recipe* is the Lax pair formulation of SG equation, it is clear how to generalise the construction to more general field theories like, for instance, ATFT's. Nevertheless, I rather would like to focus my attention on the origin and form of the infinitesimal transformations in the particular case of SG theory.

Specifically, I will show how to introduce the SG theory starting from the simpler Korteweg-de Vries (KdV) theory and how to frame this symmetry inside the KdV theory. Actually, I will not give a complete proof of all the statements I will formulate, leaving this part to a more systematic publication [11]. On the contrary, the restriction of these vector fields on the variables of the N-soliton solutions was described and analysed in [12]: in this talk I sketch only how to derive this action on a more intuitive ground. In the soliton phase space the infinitesimal transformations are realised in a much simpler form and in particular they become local contrary to the field theory case (in which these are quasi-local). At the end, I will deliver few comments about how much easier quantisation of the soliton phase space might appear.

2 The action of the Witt symmetry on fields

Let me recall the construction of the Witt symmetry in the context of (m)KdV theory [13, 14]. It was shown in [14], following the so-called matrix approach, that it appears as a generalisation of the ordinary dressing transformations of integrable models. As integrable system the mKdV equation enjoys a zero-curvature representation

$$[\partial_t - A_t, \partial_x - A_x] = 0, \tag{1}$$

where the Lax connections A_x, A_t belong to a finite dimensional representation of some loop algebra and contain the fields and their derivatives. In this particular case the first Lax operator $\mathcal{L}$ is given by

$$A_x = \begin{pmatrix} \phi' & \lambda \\ \lambda & -\phi' \end{pmatrix}, \tag{2}$$

where I have denoted with ϕ' the mKdV field (prime means derivative with respect to the *space variable* x), with λ the $A_1^{(1)}$ loop algebra parameter (spectral parameter) and A_t can be found using the dressing procedure I am going to describe [15]. The KdV variable $u(x)$ is connected to the mKdV field ϕ' by the Miura transformation:

$$u = -(\phi')^2 - \phi''. \tag{3}$$

Key objects in the following construction are solutions $T(x, \lambda)$ of the so-called associated linear problem

$$(\partial_x - A_x(x, \lambda))T(x, \lambda) = 0, \tag{4}$$

which may be called monodromy matrices. A formal (suitably normalised) solution of (4) can be formally expressed by

$$T_{\text{reg}}(x, \lambda) = e^{H\phi(x)} \mathcal{P} \exp \left(\lambda \int_0^x dy \left(e^{-2\phi(y)} E + e^{2\phi(y)} F \right) \right). \tag{5}$$

Of course, this solution is just an infinite series in positive powers of $\lambda \in \mathbb{C}$ with an infinite radius of convergence. I shall often refer to (5) as *regular expansion*. It is also clear from (5) that any solution $T(x, \lambda)$ possesses an essential singularity at $\lambda = \infty$ where it is governed by the corresponding *asymptotic expansion*. In consequence, an asymptotic expansion has been derived in detail in [15]

$$T_{\text{asy}}(x, \lambda) = KG(x, \lambda) e^{-\int_0^x dy D(y)}, \tag{6}$$

where K and G and D are written explicitly in [15]. In particular the matrix

$$D(x,\lambda)=\sum_{i=-1}^{\infty}\lambda^{-i}d_i(x)H^i, \qquad H=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \tag{7}$$

contains the local conserved densities $d_{2n+2}(x)$.

Obviously, a gauge transformation for A_x

$$\delta A_x(x,\lambda)=[\theta(x,\lambda),\mathcal{L}] \tag{8}$$

preserves the zero-curvature form (1) if an analogous one applies to A_t: the vector field δ defines a symmetry of *the equation of motion* (1) in the usual sense, mapping a solution into another solution. Moreover, to build up a consistent gauge connection $\theta(x,\lambda)$ for the previous infinitesimal transformation, I must pay attention to the fact that the r.h.s. needs to be independent of λ since the l.h.s. is, as consequence of (2). Hence, a suitable choice for the gauge connection goes through the construction of the following object

$$Z^X(x,\lambda)=T(x,\lambda)XT(x,\lambda)^{-1}, \tag{9}$$

where X is such that

$$[\partial_x,X]=0. \tag{10}$$

Indeed, it is obvious from the previous definition that it satisfies the *resolvent condition*

$$\left[\mathcal{L},Z^X(x,\lambda)\right]=0, \tag{11}$$

for the first Lax operator $\mathcal{L}=\partial_x-A_x(x,\lambda)$. Now, this property implies

$$\left[\mathcal{L},(Z^X(x,\lambda))_-\right]=-\left[\mathcal{L},\left(Z^X(x,\lambda)\right)_+\right], \tag{12}$$

where the subscript $-$ $(+)$ means that I restrict the series only to negative (non-negative) powers of λ, and hence yields the construction of a consistent gauge connection defined as

$$\theta^X(x,\lambda)=\left(Z^X(x,\lambda)\right)_- \qquad \text{or} \qquad \theta^X(x,\lambda)=\left(Z^X(x,\lambda)\right)_+. \tag{13}$$

Further, I have to impose one more consistency condition implied by the explicit form of A_x (2), namely δA_x must be diagonal

$$\delta^X A_x=H\delta^X\phi'. \tag{14}$$

This implies restrictions about the indices of the transformations [16]. After posing $T=T_{\mathrm{reg}}$ I obtain the so-called dressing symmetries [15] and the indices are even for $X=H$ and odd for $X=E,F$. Instead, after posing $T=T_{\mathrm{asy}}$ I get for $X=H$ the commuting (m)KdV flows (or the (m)KdV hierarchy), which define the different *time* t_{2k+1}, $k=0,1,2,\ldots$ *evolutions* and in particular (1) with $t=t_3$ [16].

At this point I want to make an important observation. Let me consider the KdV variable x as a *space direction* x_- of some more general system (and $\partial_-=\partial_x$ as a space derivative). Let me introduce the *time* variable x_+ through the corresponding evolution flow

$$\partial_+=\left(\delta^E_{-1}+\delta^F_{-1}\right), \tag{15}$$

defined by a zero curvature condition of the form (8)

$$\partial_+A_{x_-}(x_-,x_+;\lambda)=[\theta_+(x_-,x_+;\lambda),\mathcal{L}]. \tag{16}$$

This specific $\theta_+(x_-, x_+; \lambda)$ is derived using (13) with the regular expansion. Then, it can be proved [16] that the equation of motion for ϕ becomes:

$$\partial_+\partial_-\phi = 2\sinh(2\phi), \qquad \text{or} \quad \text{if} \quad \phi \to i\phi, \qquad \partial_+\partial_-\phi = 2\sin(2\phi), \tag{17}$$

i.e. the Sine-Gordon equation. As we will see later, this observation will appear very fruitful for my purpose since it provides an introduction of Sine-Gordon dynamics as a vector field in the powerful algebraic framework of the KdV hierarchy and its symmetries. For instance, I obtain as simple by-product the fact that mKdV hierarchy is a symmetry for SG equation. Of course, the Hamiltonians – given by the part of the dressing charges corresponding to the (16) and the higher flows $\partial_{2k+1} = \delta^E_{-2k-1} + \delta^F_{-2k-1}$ [15] – coincide with the well-known ones [5].

Now, let me explain how the Witt symmetry appears in the KdV system [14]. The main idea is that one may dress not only the generators of the underlying $A_1^{(1)}$ algebra but also an arbitrary differential operator in the spectral parameter. I take for example $\lambda^{m+1}\partial_\lambda$ which are the well known vector fields of the diffeomorphisms of the unit circumference and close a Witt algebra. Then I proceed in the same way as above defining the resolvent associated to the circumference diffeomorphisms

$$Z^V(x,\lambda) = T(x,\lambda)\partial_\lambda T(x,\lambda)^{-1}. \tag{18}$$

When I consider the asymptotic case, i.e. I take $T = T_{\text{asy}}$ in (18), I obtain the non-negative Witt flows. In general they are written in terms of recursive quasi-local expressions $\alpha^V_{2m}(x)$, $m \geq 0$ as

$$\delta^V_{2m}\phi(x) = \alpha^V_{2m}(x), \tag{19}$$

where

$$\alpha^V_0(x) = -x\phi'(x), \qquad \alpha^V_{2m+2}(x) = \left[-\phi'\partial_x^{-1}\phi'\partial_x + \frac{1}{4}\partial_x^2\right]\alpha^V_{2m}(x). \tag{20}$$

Let me highlight the appearance of the pseudodifferential operator ∂_x^{-1}, acting on a function $f(x)$ as

$$\partial_x^{-1}f(x) = \int^x dy f(y), \tag{21}$$

which is responsible (together with the form of the initial condition (20)) for the non complete locality. From these vector fields I can deduce the action on $u(x)$ using (3)

$$\delta^V_{2m}u(x) = 2\partial_x\beta^V_{2m+1}(x) \tag{22}$$

again in terms of recursive quasi-local expressions $\beta^V_{2m-1}(x)$

$$\beta^V_{-1} = -x, \qquad \beta^V_{2m+1}(x) = \left[\frac{1}{2}\left(u + \partial_x^{-1}u\partial_x + \frac{1}{2}\partial_x^2\right)\right]\beta^V_{2m-1}(x). \tag{23}$$

For instance, the first two of (19) can be written as

$$\delta^V_0\phi = -x\phi', \qquad \delta^V_2\phi = \frac{1}{2}\phi'\left(\partial_x^{-1}\phi'^2\right) - \frac{1}{2}\phi'' - \frac{1}{2}x\left[\frac{1}{2}\phi''' - (\phi')^3\right]. \tag{24}$$

The negative Witt transformations can also be built up by taking $T = T_{\text{reg}}$ in (18) in such a way to complete the algebra [14]. Unfortunately, those vector fields do not act as gauge transformations on the SG equation of motion (16), actually they are not true symmetry transformations [16].

Nevertheless, thanks to the way (16) I have introduced SG theory through, I am in the position to extend the (half) Witt symmetry algebra jumping from (m)KdV to the SG theory. For obvious reasons I will rename in the following the KdV variable u with

$$u^-(x_-,x_+) = -(\partial_-\phi(x_-,x_+))^2 - \partial_-^2\phi(x_-,x_+). \tag{25}$$

Hence, after looking at the symmetric rôle that the derivatives ∂_- and ∂_+ play in the Sine-Gordon equation, I can obtain *the negative (m)KdV hierarchy*, acting on the fields $\phi(x_-,x_+)$ and

$$u^+(x_-,x_+) = -(\partial_+\phi(x_-,x_+))^2 - \partial_+^2\phi(x_-,x_+), \tag{26}$$

in the same way as above but with the change of rôles $x_- \to x_+$ (and consequently $\partial_- \to \partial_+$). Similarly, I obtain the other half of a Witt algebra by using the same construction already showed, but with x_- substituted by x_+.

Of course, it is not obvious at all that the two different halves will recombine into a unique Witt algebra. Actually, even the first Witt vector field in the original construction (24) needs a *symmetrising improvement* to leave exactly invariant the zero curvature form of SG equation (16):

$$\delta_0^V\phi = -x_-\partial_-\phi - x_+\partial_+\phi. \tag{27}$$

Nevertheless, I have checked this statement brute force in the case

$$\left[\delta_2^V, \delta_{-2}^V\right]\phi = 4\delta_0^V\phi, \tag{28}$$

and it works in a peculiar manner, simply using the transformation definitions (27) and the second of (24). I would like to leave for future publication the detailed explanation of how a complete proof of this proposition may be elaborated along smart lines [11].

In conclusion, I have found an entire Witt algebra of transformations acting as gauge symmetries on SG equation (16). Moreover, I sketch now how the restriction of the action on soliton solution phase space yields the result argued in [12] following a slightly different procedure.

3 The Witt symmetry acting on the soliton solution variables

I start with a brief description of the well known soliton solutions of SG equation and (m)KdV hierarchy in the infinite *times* formalism. To see how a N-soliton solution can be parametrised, I need to go through the expression of the so-called *tau-function*. This can be written as a determinant

$$\tau(X_1,\ldots,X_N|B_1,\ldots,B_N) = \det(1+V), \tag{29}$$

where V is a NxN matrix

$$V_{ij} = 2\frac{B_iX_i}{B_i+B_j}, \qquad i,j=1,\ldots,N, \tag{30}$$

and $X_i(\{t_{2k+1}\}|x_i,B_i)$ are exponential functions of of all the *times* $\{t_{2k+1}\}$, $k\in\mathbb{Z}$ (e.g. in the previous notation $t_{-1}=x_+$, $t_1=x_-$, $t_3=t$)

$$X_i(\{t_{2k+1}\}|x_i,B_i) = x_i\exp\left(2\sum_{k=-\infty}^{+\infty} B_i^{2k+1}t_{2k+1}\right). \tag{31}$$

The constant parameters B_i and x_i describe the soliton velocities and positions respectively. Now the SG or the mKdV field solution is expressed in a beautiful unitary way as

$$e^{\phi} = \frac{\tau_-}{\tau_+}, \tag{32}$$

where simply

$$\tau_{\pm} = \tau(\{\pm X_i\}|\{B_i\}), \tag{33}$$

in the sense that after putting all the negative (positive) times to zero, I end up with the N-soliton solution of the mKdV hierarchy (the negative mKdV hierarchy), whereas after the position to zero of all the times but $t_{-1} = x_+$, $t_1 = x_-$, I end up with the N-soliton solution of the SG equation.

The main goal of this Section is to find the action of the Witt symmetry on the N-soliton solution and this is more conveniently achieved introducing other variables $\{A_i, B_i\}$, expressed implicitly by the old variables $\{X_i, B_i\}$ through the implicit formulae

$$X_j \prod_{k\neq j} \frac{B_j - B_k}{B_j + B_k} = \prod_{k=1}^{N} \frac{B_j - A_k}{B_j + A_k}, \qquad j = 1, \dots, N. \tag{34}$$

In fact, the $\{A_i, B_i\}$ are the soliton limit of certain variables describing the more general quasi-periodic finite-zone solutions of (m)KdV [17], being the B_i the limit of the branch points of the hyperelliptic curve describing a particular solution and the A_i the limit of the zeroes of the so-called Baker–Akhiezer function defined on the curve. Actually, even for the description of the quantum physics of Form Factors these variables are apparently more natural and suitable [18]. Although, in terms of these variables the tau functions have still a cumbersome form

$$\begin{aligned}
\tau_+ &= 2^N \prod_{j=1}^{N} B_j \left\{ \frac{\prod_{i<j} (A_i + A_j) \prod_{i<j} (B_i + B_j)}{\prod_{i,j} (B_i + A_j)} \right\}, \\
\tau_- &= 2^N \prod_{j=1}^{N} A_j \left\{ \frac{\prod_{i<j} (A_i + A_j) \prod_{i<j} (B_i + B_j)}{\prod_{i,j} (B_i + A_j)} \right\},
\end{aligned} \tag{35}$$

the SG (mKdV) field (32) enjoys a simple expression

$$e^{\phi} = \prod_{j=1}^{N} \frac{A_j}{B_j}. \tag{36}$$

In consequence, the two components of the stress-energy tensor (25) and (26) take a wieldy form as well

$$u^- = -2 \left(\sum_{j=1}^{N} A_j^2 - \sum_{j=1}^{N} B_j^2 \right), \qquad u^+ = -2 \left(\sum_{j=1}^{N} A_j^{-2} - \sum_{j=1}^{N} B_j^{-2} \right). \tag{37}$$

Now I am in the position to restrict the Witt symmetry of SG equation developed in the previous Section to the case of soliton solutions. Although these transformations have been derived in [12], here I will follow a more intuitive path, which underlines the geometrical meaning of this symmetry. In other words our starting point consists in the transformations of the rapidities under the Witt symmetry: I do expect that they change the conformal structure of the Riemann surface describing the finite-zone solutions. Actually, in the (m)KdV theory the soliton limit of the Witt action on the Riemann surface reads simply [13]

$$\delta_{2n} B_i = B_i^{2n+1}, \qquad n \geq 0, \tag{38}$$

where I have forgotten the superscript V for indicating the action on soliton variables. Further, the action of negative transformations should not be different

$$\delta_{-2n} B_i = -B_i^{-2n+1}, \qquad n > 0, \tag{39}$$

save an additional $-$ sign in the r.h.s. [12] which takes into account the Witt algebra commutation relations. I have to show now the transformations of the A_i variables as consequences of (38) and (39) once applied to the implicit map (34) by using the expression (31) of X_i in terms of B_i. The problem is simplified by the fact that I know from the field theory that the symmetry algebra is a Witt algebra, and hence I need to compute only the transformations δ_0, $\delta_{\pm 2}$ and $\delta_{\pm 4}$, for the higher vector fields are then furnished by commuting. In this way it is evident why the Witt transformations become *local* when restricted on the soliton solutions, though the transformations of ϕ and $u^{\pm}$ in the SG theory are quasi-local. Actually, I think more natural and more compact to express the Witt action on A_i by using the *equations of motion* of A_i derived from (31) and (34), like for instance

$$\begin{aligned}
\delta_{-1} A_i &= \partial_+ A_i = \prod_{j=1}^{N} \frac{\left(A_i^2 - B_j^2\right)}{B_j^2} \prod_{j \neq i} \frac{A_j^2}{\left(A_i^2 - A_j^2\right)}, \\
\delta_1 A_i &= \partial_- A_i = \prod_{j=1}^{N} \left(A_i^2 - B_j^2\right) \prod_{j \neq i} \frac{1}{\left(A_i^2 - A_j^2\right)}, \\
\delta_3 A_i &= 3 \left(\sum_{j=1}^{N} B_j^2 - \sum_{k \neq i} A_k^2 \right) \partial_- A_i, \\
\frac{1}{5} \delta_5 A_i &= \left(\sum_{j=1}^{N} B_j^4 - \sum_{k \neq i} A_k^4 \right) \partial_- A_i - \sum_{j \neq i} \left(A_i^2 - A_j^2\right) \partial_- A_i \partial_- A_j.
\end{aligned} \tag{40}$$

In conclusion, the direct calculation is quite tiresome and I present here only few results:

$$\delta_{-2} A_i = \frac{1}{3} x_+ \delta_{-3} A_i - A_i^{-1} - \partial_+ A_i \sum_{j=1}^{N} A_j^{-1} - x_- \partial_+ A_i, \tag{41}$$

$$\delta_{-4} A_i = \frac{1}{5} x_+ \delta_{-5} A_i - A_i^{-3} - \left\{ \sum_{j \neq i}^{N} \frac{1}{A_i} \left(\frac{1}{A_i^2} - \frac{1}{A_j^2} \right) + \sum_{j=1}^{N} \frac{1}{A_j} \sum_{k=1}^{N} \frac{1}{B_k^2} \right\} \partial_+ A_i - x_- \delta_{-3} A_i$$

and for non-negative vector fields

$$\delta_0 A_i = (x_- \partial_- - x_+ \partial_+ + 1) A_i, \qquad \delta_2 A_i = \frac{1}{3} x_- \delta_3 A_i + A_i^3 - \left(\sum_{j=1}^{N} A_j \right) \partial_- A_i - x_+ \partial_- A_i,$$

$$\delta_4 A_i = \frac{1}{5} x_- \delta_5 A_i + A_i^5 - \left\{ \sum_{j \neq i} A_i \left(A_i^2 - A_j^2\right) + \sum_{j=1}^{N} A_j \sum_{k=1}^{N} B_k^2 \right\} \partial_- A_i - x_+ \delta_3 A_i. \tag{42}$$

At this point, I need to carry out two important checks. First, I have to calculate the commutators of the δ_{2m} (with $m \in \mathbb{Z}$) with the light-come SG flow $\partial_\pm$, acting on A_i. These are always zero and represent an equivalent way to express the symmetry action. Second, I have to verify the algebra of the δ_{2m} (with $m \in \mathbb{Z}$) on A_i and this is a very non trivial check for I have derived all the transformations (41) and (42) from the Witt algebra on B_i, written in (38) and (39), and from the implicit map (34). Nevertheless the action on A_i is again a representation of the Witt algebra:

$$[\delta_{2n}, \delta_{2m}]A_i = (2n - 2m)\delta_{2n+2m}A_i, \qquad n, m \in \mathbb{Z}. \tag{43}$$

4 Comments about quantisation

Of course, I might be interested in the quantum Sine-Gordon theory. In the case of solitons there is a standard procedure: the canonical quantisation of the N-soliton solutions. Indeed, let me introduce the variables canonically conjugated to the A_i:

$$P_j = \prod_{k=1}^{N} \frac{B_k - A_j}{B_k + A_j}, \qquad j = 1, \ldots, N. \tag{44}$$

In these variables one can perform the canonical quantisation of the N-soliton system introducing the deformed commutation relations between the operators $\hat{A}_i$ and $\hat{P}_i$:

$$\begin{aligned} &\hat{P}_j \hat{A}_j = q \hat{A}_j \hat{P}_j, \\ &\hat{P}_k \hat{A}_j = \hat{A}_j \hat{P}_k, \qquad \text{for} \quad k \neq j, \end{aligned} \tag{45}$$

where $q = \exp(i\xi)$, $\xi = \frac{\pi\gamma}{\pi-\gamma}$ and γ is the coupling constant of the SG theory. Understanding how the Witt symmetry is deformed after quantisation is a very seductive problem.

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Integrable Structures for 2D Euler Equations of Incompressible Inviscid Fluids

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In this article, I will report a Lax pair structure, a Bäcklund–Darboux transformation, and the investigation of homoclinic structures for 2D Euler equations of incompressible inviscid fluids.

1 Introduction

The governing equation of turbulence, that we are interested in, is the incompressible 2D Navier–Stokes equation under periodic boundary conditions. We are particularly interested in investigating the dynamics of 2D Navier–Stokes equation in the infinite Reynolds number limit and of 2D Euler equation. Our approach is different from many other studies on 2D Navier–Stokes equation in which one starts with Stokes equation to prove results on 2D Navier–Stokes equation for small Reynolds number. In our studies, we start with 2D Euler equation and view 2D Navier–Stokes equation for large Reynolds number as a (singular) perturbation of 2D Euler equation. 2D Euler equation is a Hamiltonian system with infinitely many Casimirs. To understand the nature of turbulence, we start with investigating the hyperbolic structure of 2D Euler equation. We are especially interested in investigating the possible homoclinic structures.

In [1], we studied a linearized 2D Euler equation at a fixed point. The linear system decouples into infinitely many one-dimensional invariant subsystems. The essential spectrum of each invariant subsystem is a band of continuous spectrum on the imaginary axis. Only finitely many of these invariant subsystems have point spectra. The point spectra can be computed through continued fractions. Examples show that there are indeed eigenvalues with positive and negative real parts. Thus, there is linear hyperbolicity.

In [2] and [3], a Lax pair and a Bäcklund–Darboux transformation were found for the 2D Euler equation. Typically, Bäcklund–Darboux transformation can be used to generate homoclinic orbits [4].

The 2D Euler equation can be written in the vorticity form,

$$\partial_t \Omega + \{\Psi, \Omega\} = 0, \tag{1}$$

where the bracket $\{\ ,\ \}$ is defined as

$$\{f, g\} = (\partial_x f)(\partial_y g) - (\partial_y f)(\partial_x g),$$

where Ψ is the stream function given by,

$$u = -\partial_y \Psi, \qquad v = \partial_x \Psi,$$

u and v are the velocity components, and the relation between vorticity Ω and stream function Ψ is,

$$\Omega = \partial_x v - \partial_y u = \Delta \Psi.$$

2 A Lax pair and a Darboux transformation

Theorem 1 (Li, [2]). *The Lax pair of the 2D Euler equation (1) is given as*

$$L\varphi = \lambda\varphi,$$
$$\partial_t\varphi + A\varphi = 0, \tag{2}$$

where

$$L\varphi = \{\Omega, \varphi\}, \qquad A\varphi = \{\Psi, \varphi\},$$

and λ is a complex constant, and φ is a complex-valued function.

Consider the Lax pair (2) at $\lambda = 0$, i.e.

$$\{\Omega, p\} = 0, \tag{3}$$
$$\partial_t p + \{\Psi, p\} = 0, \tag{4}$$

where we replaced the notation φ by p.

Theorem 2 (Li and Yurov, [3]). *Let $f = f(t, x, y)$ be any fixed solution to the system (3), (4), we define the Gauge transform G_f:*

$$\tilde{p} = G_f p = \frac{1}{\Omega_x}[p_x - (\partial_x \ln f)p], \tag{5}$$

and the transforms of the potentials Ω and Ψ:

$$\tilde{\Psi} = \Psi + F, \qquad \tilde{\Omega} = \Omega + \Delta F, \tag{6}$$

where F is subject to the constraints

$$\{\Omega, \Delta F\} = 0, \qquad \{\Omega + \Delta F, F\} = 0. \tag{7}$$

Then $\tilde{p}$ solves the system (3), (4) at $(\tilde{\Omega}, \tilde{\Psi})$. Thus (5) and (6) form the Darboux transformation for the 2D Euler equation (1) and its Lax pair (3), (4).

3 Preliminaries on linearized 2D Euler equation

We consider the two-dimensional incompressible Euler equation written in vorticity form (1) under periodic boundary conditions in both x and y directions with period 2π. We also require that both u and v have means zero,

$$\int_0^{2\pi}\int_0^{2\pi} u \, dxdy = \int_0^{2\pi}\int_0^{2\pi} v \, dxdy = 0.$$

We expand Ω into Fourier series,

$$\Omega = \sum_{k \in \mathbb{Z}^2/\{0\}} \omega_k \, e^{ik\cdot X},$$

where $\omega_{-k} = \overline{\omega_k}$, $k = (k_1, k_2)^T$, $X = (x, y)^T$. In this paper, we confuse 0 with $(0, 0)^T$, the context will always make it clear. By the relation between vorticity Ω and stream function Ψ, the system (1) can be rewritten as the following kinetic system,

$$\dot{\omega}_k = \sum_{k=p+q} A(p, q) \, \omega_p \omega_q, \tag{8}$$

where $A(p,q)$ is given by,

$$A(p,q) = \frac{1}{2}\left[|q|^{-2} - |p|^{-2}\right](p_1q_2 - p_2q_1), \tag{9}$$

where $|q|^2 = q_1^2 + q_2^2$ for $q = (q_1, q_2)^T$, similarly for p.

We denote $\{\omega_k\}_{k\in\mathbb{Z}^2/\{0\}}$ by ω. For any fixed $p \in \mathbb{Z}^2/\{0\}$, we consider the simple fixed point ω^*:

$$\omega_p^* = \Gamma, \qquad \omega_k^* = 0, \quad \text{if} \quad k \neq p \quad \text{or} \quad -p, \tag{10}$$

of the 2D Euler equation (8), where Γ is an arbitrary complex constant. The *linearized two-dimensional Euler equation* at ω^* is given by,

$$\dot{\omega}_k = A(p, k-p)\,\Gamma\,\omega_{k-p} + A(-p, k+p)\,\bar{\Gamma}\,\omega_{k+p}. \tag{11}$$

Definition 1 (Classes). For any $\hat{k} \in \mathbb{Z}^2/\{0\}$, we define the class $\Sigma_{\hat{k}}$ to be the subset of $\mathbb{Z}^2/\{0\}$:

$$\Sigma_{\hat{k}} = \left\{\hat{k} + np \in \mathbb{Z}^2/\{0\} \;\middle|\; n \in \mathbb{Z},\; p \text{ is specified in (10)}\right\}.$$

See Fig. 1 for an illustration. According to the classification defined in Definition 1, the linearized two-dimensional Euler equation (11) decouples into infinitely many *invariant subsystems*:

$$\dot{\omega}_{\hat{k}+np} = A(p, \hat{k}+(n-1)p)\,\Gamma\,\omega_{\hat{k}+(n-1)p} + A(-p, \hat{k}+(n+1)p)\,\bar{\Gamma}\,\omega_{\hat{k}+(n+1)p}\,. \tag{12}$$

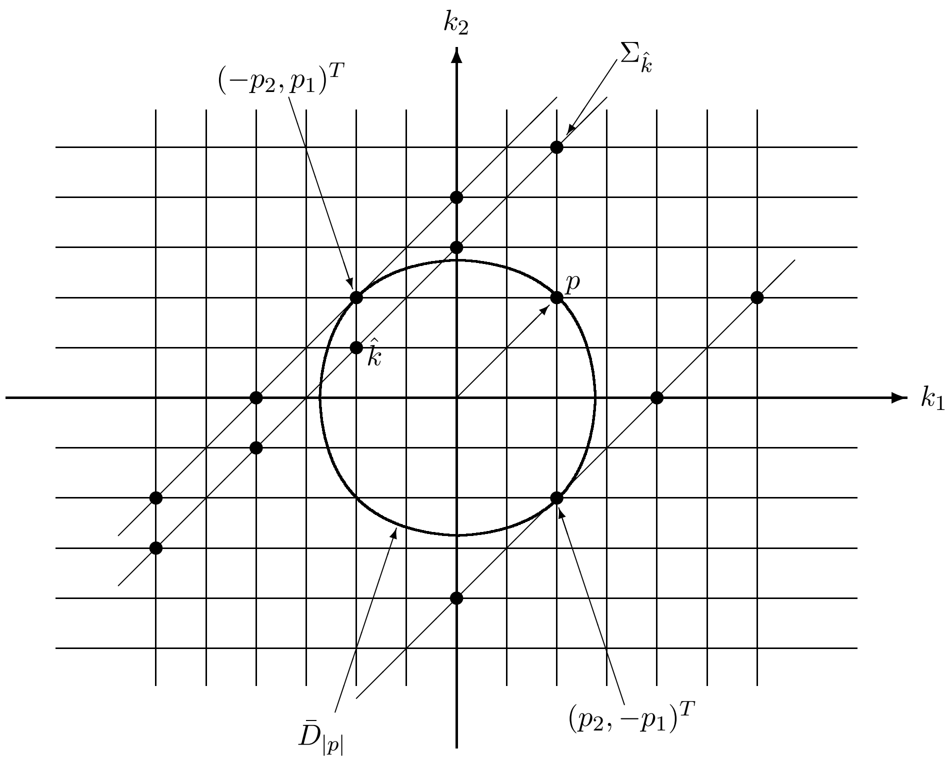

Figure 1. An illustration of the classes $\Sigma_{\hat{k}}$ and the disk $\bar{D}_{|p|}$.

Theorem 3. *The eigenvalues of the linear operator $\mathcal{L}_{\hat{k}}$ defined by the right hand side of (12), are of four types: real pairs $(c, -c)$, purely imaginary pairs $(id, -id)$, quadruples $(\pm c \pm id)$, and zero eigenvalues.*

The eigenvalues can be computed through continued fractions.

Definition 2 (The Disk). The disk of radius $|p|$ in $\mathbb{Z}^2/\{0\}$, denoted by $\bar{D}_{|p|}$, is defined as

$$\bar{D}_{|p|} = \left\{ k \in \mathbb{Z}^2/\{0\} \;\middle|\; |k| \leq |p| \right\}.$$

Theorem 4 (The Spectral Theorem). *We have the following claims on the spectra of the linear operator $\mathcal{L}_{\hat{k}}$:*

1. *If $\Sigma_{\hat{k}} \cap \bar{D}_{|p|} = \emptyset$, then the entire ℓ_2 spectrum of the linear operator $\mathcal{L}_{\hat{k}}$ is its continuous spectrum. See Fig. 2, where $b = -\frac{1}{2}|\Gamma||p|^{-2} \begin{vmatrix} p_1 & \hat{k}_1 \\ p_2 & \hat{k}_2 \end{vmatrix}$.*
2. *If $\Sigma_{\hat{k}} \cap \bar{D}_{|p|} \neq \emptyset$, then the entire essential ℓ_2 spectrum of the linear operator $\mathcal{L}_{\hat{k}}$ is its continuous spectrum. That is, the residual spectrum of $\mathcal{L}_{\hat{k}}$ is empty, $\sigma_r(\mathcal{L}_{\hat{k}}) = \emptyset$. The point spectrum of $\mathcal{L}_{\hat{k}}$ is symmetric with respect to both real and imaginary axes. See Fig. 2.*

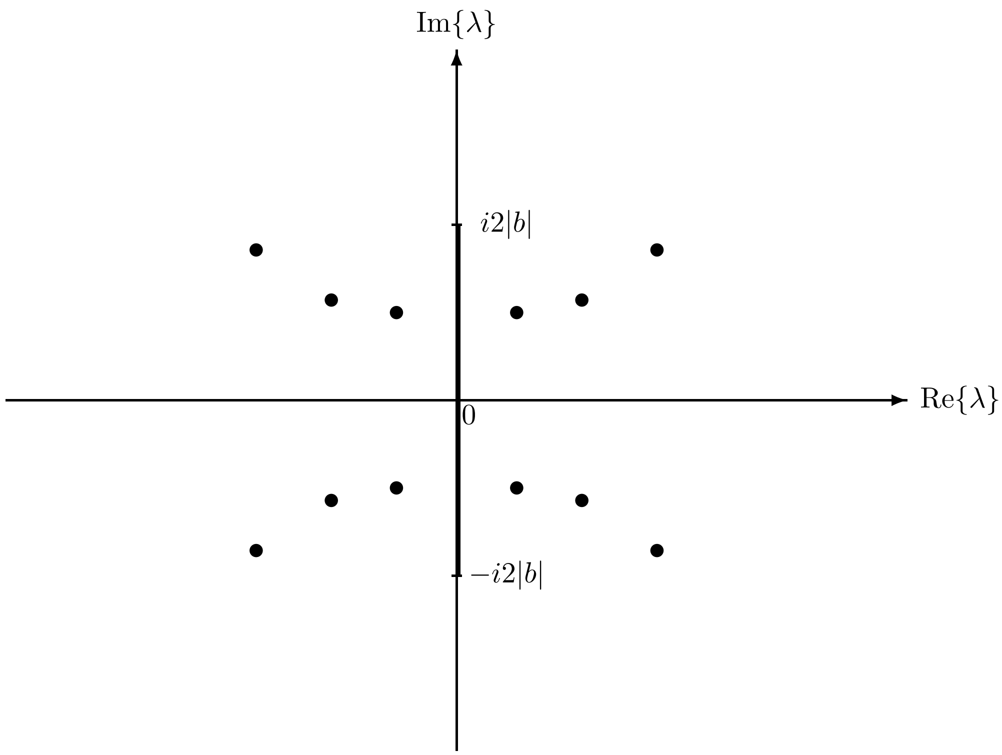

Figure 2. The spectrum of $\mathcal{L}_{\hat{k}}$.

4 A Galerkin truncation

To simplify our study, we study only the case when ω_k is real, $\forall\, k \in \mathbb{Z}^2/\{0\}$, i.e. we only study the cosine transform of the vorticity,

$$\Omega = \sum_{k \in \mathbb{Z}^2/\{0\}} \omega_k \cos(k \cdot X),$$

and the 2D Euler equation (1), (8) preserves the cosine transform. To further simplify our study, we will study a concrete line of fixed points (10) with the mode $p = (1,1)^T$ parametrized by Γ. When $\Gamma \neq 0$, each fixed point has 4 eigenvalues which form a quadruple. These four eigenvalues

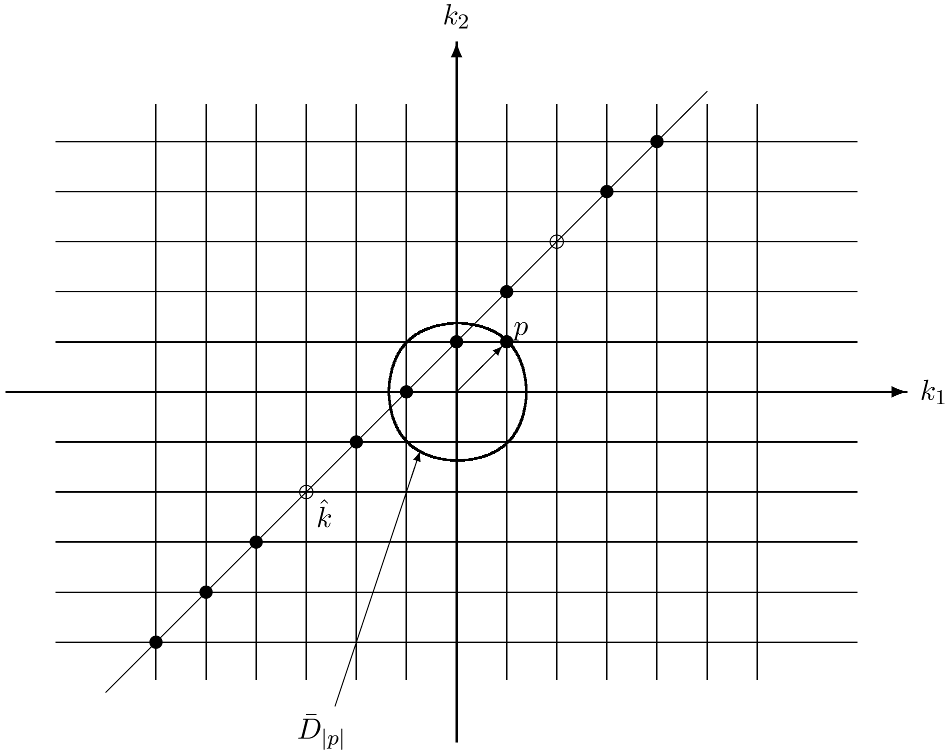

Figure 3. The collocation of the modes in the Galerkin truncation.

appear in the only unstable invariant linear subsystem labeled by $\hat{k} = (-3, -2)^T$. See Fig. 3 for an illustration.

We computed the eigenvalues through continued fractions, one of them is [1]:

$$\tilde{\lambda} = 2\lambda/|\Gamma| = 0.24822302478255 + i\, 0.35172076526520. \tag{13}$$

We hope that a Galerkin truncation with a small number of modes including those inside the disk $\bar{D}_{|p|}$ can capture the eigenvalues. We propose the Galerkin truncation to the linear system (12) with the four modes $\hat{k} + p$, $\hat{k} + 2p$, $\hat{k} + 3p$, and $\hat{k} + 4p$,

$$\begin{aligned} &\dot{\omega}_1 = -A_2\Gamma\omega_2, \qquad \dot{\omega}_2 = A_1\Gamma\omega_1 - A_3\Gamma\omega_3, \\ &\dot{\omega}_3 = A_2\Gamma\omega_2 - A_4\Gamma\omega_4, \qquad \dot{\omega}_4 = A_3\Gamma\omega_3. \end{aligned}$$

From now on, the abbreviated notations,

$$\omega_n = \omega_{\hat{k}+np}, \qquad A_n = A(p, \hat{k} + np), \qquad A_{m,n} = A(\hat{k} + mp, \hat{k} + np), \tag{14}$$

will be used. The eigenvalues of this four dimensional system can be easily calculated. It turns out that this system has a quadruple of eigenvalues:

$$\lambda = \pm \frac{\Gamma}{2\sqrt{10}} \sqrt{1 \pm i\sqrt{35}} \ \doteq \ \pm \left(\frac{\Gamma}{2}\right) \times 0.7746 \times e^{\pm i\theta_1}, \tag{15}$$

where $\theta_1 = \arctan(0.845)$, in comparison with the quadruple of eigenvalues (13), where

$$\lambda \doteq \pm \left(\frac{\Gamma}{2}\right) \times 0.43 \times e^{\pm i\theta_2},$$

and $\theta_2 = \arctan(1.418)$. Thus, *the quadruple of eigenvalues of the original system is recovered by the four-mode truncation.* We further study the corresponding Galerkin truncation of 2D Euler equation:

$$\begin{aligned}
&\dot{\omega}_1 = -A_2\, \omega_p\, \omega_2, \qquad \dot{\omega}_2 = A_1\, \omega_p\, \omega_1 - A_2\, \omega_p\, \omega_3, \qquad \dot{\omega}_3 = A_2\, \omega_p\, \omega_2 - A_1\, \omega_p\, \omega_4,\\
&\dot{\omega}_4 = A_2\, \omega_p\, \omega_3, \qquad \dot{\omega}_p = A_{1,2}\, (\omega_3\, \omega_4 - \omega_1\, \omega_2),
\end{aligned} \tag{16}$$

and the equations for the decoupled variables ω_0 and ω_5 are given by,

$$\dot{\omega}_0 = -A_1\, \omega_p\, \omega_1, \qquad \dot{\omega}_5 = A_1\, \omega_p\, \omega_4,$$

where

$$\begin{aligned}
&A_1 = -\frac{3}{10}, \qquad A_2 = \frac{1}{2}, \qquad A_3 = A_2, \qquad A_4 = A_1,\\
&A_{1,2} = A_1 - A_2 = -\frac{4}{5}, \qquad A_{2,3} = 0, \qquad A_{3,4} = -A_{1,2}.
\end{aligned}$$

There are three invariants for the system (16):

$$I = 2A_{1,2}(\omega_1\omega_3 + \omega_2\omega_4) + A_2\omega_p^2, \tag{17}$$

$$U = A_1\left(\omega_1^2 + \omega_4^2\right) + A_2\left(\omega_2^2 + \omega_3^2\right), \tag{18}$$

$$J = \omega_p^2 + \omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2. \tag{19}$$

J is the enstrophy, and U is a linear combination of the kinetic energy and the enstrophy. I is an extra invariant which is peculiar to this invariant subsystem. With I, the explicit formula for the hyperbolic structure can be computed.

The common level set of these three invariants which is connected to the fixed point (10) determines the stable and unstable manifolds of the fixed point and its negative $-\omega^*$:

$$\omega_p = -\Gamma, \qquad \omega_n = 0 \qquad (n \in \mathbb{Z}). \tag{20}$$

Using the polar coordinates:

$$\omega_1 = r\cos\theta, \qquad \omega_4 = r\sin\theta; \qquad \omega_2 = \rho\cos\vartheta, \qquad \omega_3 = \rho\sin\vartheta$$

we have the following explicit expressions for the stable and unstable manifolds of the fixed point (10) and its negative (20) represented through the homoclinic orbits asymptotic to the line of fixed points:

$$\begin{aligned}
&\omega_p = \Gamma\tanh\tau, \qquad r = \sqrt{\frac{A_2}{A_2 - A_1}}\ \Gamma\ \mathrm{sech}\ \tau,\\
&\theta = -\frac{A_2}{2\kappa}\ln\cosh\tau + \theta_0, \qquad \rho = \sqrt{\frac{-A_1}{A_2}}\ r,\\
&\theta + \vartheta = \begin{cases} -\arcsin\left[\frac{1}{2}\sqrt{\frac{A_2}{-A_1}}\right], & (\kappa > 0),\\ \pi + \arcsin\left[\frac{1}{2}\sqrt{\frac{A_2}{-A_1}}\right], & (\kappa < 0),\end{cases}
\end{aligned} \tag{21}$$

where A_1 and A_2 are given in (16), $\tau = \kappa\Gamma t + \tau_0$, (τ_0, θ_0) are the two parameters parametrizing the two-dimensional stable (unstable) manifold, and

$$\kappa = \sqrt{-A_1A_2}\cos(\theta + \vartheta) = \pm\sqrt{-A_1A_2}\sqrt{1 + \frac{A_2}{4A_1}}\ .$$

The two auxilliary variables ω_0 and ω_5 have the expressions:

$$\omega_0 = \frac{\alpha\beta}{1+\beta^2}\,\mathrm{sech}\,\tau \left\{ \sin[\beta \ln\cosh\tau + \theta_0] - \frac{1}{\beta}\cos[\beta\ln\cosh\tau + \theta_0] \right\},$$
$$\omega_5 = \frac{\alpha\beta}{1+\beta^2}\,\mathrm{sech}\,\tau \left\{ \cos[\beta \ln\cosh\tau + \theta_0] + \frac{1}{\beta}\sin[\beta\ln\cosh\tau + \theta_0] \right\},$$

where

$$\alpha = -A_1\Gamma\kappa^{-1}\sqrt{\frac{A_2}{A_2 - A_1}}, \qquad \beta = -\frac{A_2}{2\kappa}.$$

The graphs of these homoclinic orbits are spirals on a 2D ellipsoid, with turning points.

5 Conclusion

Certain newly developed results on 2D Euler equation have been discussed, which include a Lax pair, a Darboux transformation, and the investigation on homoclinic structures.

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High-Frequency Absorption by a Soliton Gas in One-Dimensional Magnet

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The additional mechanism of the super-high-frequency power absorption in the quasy-one-dimensional antiferromagnet was considered. This absorption works due to the generation of the stationary nonlinear excitations of kinks type. The estimation of the effect is obtained for the real physical system. The shape of the signal of absorption is analyzed for some values of the external magnetic field. The quantity of the effect is detectable.

1 Introduction

We investigate theoretical problem of the super-high-frequency (SHF) field absorption by a gas of kink type solitons in the model of one-dimensional easy axis antiferromagnet (AFM) to show that the effect of linear response can be visible. The frequencies of solitons are comparable to the frequencies of magnons, and can be in intersection with the second ones. The external stationary homogeneous magnetic field, applied along the easy axis, causes the phase of system state, the eigenvalue frequencies spectrum of this phase and the value of the gap, particularly. The external microwave magnetic field is applied at the same direction. The shape of the expected absorption signal has the marked intense and is analyzed for some values of the external constant magnetic field.

The paper has the following structure. In the Section 2, the known results about the magnetization created by one kink [1, 2, 3], we obtain using the method of adiabatic approximation. This way turns out very convenient for the following calculation of the contribution of the weak uniform magnetic field into the energy of interaction between the kink and magnetic field, and to the magnetization also (see (1), (6) and (7)). The Section 3 is devoted to the calculation of the average energy absorbed from the external field over one period of our system. The theoretical investigations are illustrated by the numerical calculated curves of the dependence of the absorbed capacity on the frequency for some values of external magnetic field. The calculation of the quantity of the effect is based on the data of computer simulation, and was done with papameters of well-investigated quasi-one-dimensional AFM system, which admits the existence of the soliton excitations [4, 5]. We discuss obtained results in the Conclusion.

2 "Mechanical" aspects of solitons

Familiar model of one-dimensional two-sublattice AFM in an external magnetic field was considered in the paper of Bar'yakhtar and Ivanov [2]. This system was described in terms of weak FM $\vec{m}$ and AFM $\vec{l}$ vectors, such that $(\vec{m},\vec{l})=0$, $\vec{m}^2+\vec{l}^2=1$ (here $\vec{m}=\frac{\vec{M}_1+\vec{M}_2}{2M_0}$, $\vec{l}=\frac{\vec{M}_1-\vec{M}_2}{2M_0}$, and $\vec{M}_1$, $\vec{M}_2$ are the sublattices magnetizations, $|\vec{M}_1|=|\vec{M}_2|=M_0$). This formulation of the effective equations for magnetizations of the two subblattices was obtained for the natural assuming for AFM that the energy of relativistic interaction is small comparing to the exchange energy. The magnetization $\vec{m}$, created by one kink, was expressed in terms of $\vec{l}$ and $\partial\vec{l}/\partial t$

$$\vec{m}(x,t) = \frac{2}{\omega_0\delta}\left[\frac{\partial\vec{l}}{\partial t},\vec{l}\right] + \frac{2}{\delta}\left[\vec{h} - (\vec{h},\vec{l})\vec{l}\right], \tag{1}$$

where $\omega_0 = 2\mu_0 M_0/\hbar$, μ_0 is the Bohr magneton, and M_0 is an equilibrium magnetization. In angular variables θ and φ for vector $\vec{l}$, $|\vec{l}| = 1$, $l_z = \cos\theta$, $l_x + l_y = \sin\theta\exp(\imath\varphi)$,

The well-known [2] nonperturbed kink type solutions were obtained

$$\begin{aligned}&\cos\theta = \sigma\tanh B(x - vt - x_0),\\ &\varphi = \omega t - \varphi_0 + \Delta(x - vt - x_0),\end{aligned} \tag{2}$$

where $\sigma = \pm 1$, $B = \frac{\kappa(v;\omega)}{1-v^2}$, $\kappa^2(v;\omega) = \gamma^2(1-v^2) - (\omega - h_3)^2$, $\gamma = (c/\omega_0)\sqrt{(\beta/\alpha)}$, $\Delta = \frac{v}{1-v^2}(h_3 - \omega)$.

The method of adiabatic approximation proposed in our paper allows to justify the results obtained earlier [1, 2, 3] as well as to study the further applications.

The dynamics of free kink in 4-dimensional phase-space (X, Φ, I_1, I_2) is defined by the Hamiltonian equations

$$\frac{dI_1}{dt} = -\frac{\partial\mathcal{H}_0}{\partial X}, \qquad \frac{dI_2}{dt} = -\frac{\partial\mathcal{H}_0}{\partial\Phi}, \qquad v = \frac{dX}{dt} = \frac{\partial\mathcal{H}_0}{\partial I_1}, \qquad \omega = \frac{d\Phi}{dt} = \frac{\partial\mathcal{H}_0}{\partial I_2}. \tag{3}$$

Here X and Φ are the parameters defining the kink spatial arrangement and form, correspondingly, I_2 is the adiabatic invariant of two-parameter solution of the Landau–Lifshitz equations, I_1 is the field impact, $\mathcal{H}_0$ is an unperturbed Hamiltonian. For the first two variables we can write the definitions $\Phi(t) = \int dt\omega(t)$, $X = \int dt v(t)$. The variables $I_1 = I_1(v,\omega)$, $I_2 = I_2(v,\omega)$ will be defined later. The solution (2) can be rewritten in the form

$$\varphi = \Phi + \Delta(v,\omega)(x - X), \qquad \cos\theta = \sigma\tanh B(x - X).$$

The Lagrange function L_0 in new variables can be obtained as follows:

$$L_0 = I_1\frac{dX}{dt} + I_2\frac{d\Phi}{dt} - \mathcal{E}_0(I_1, I_2),$$

where $I_1 = 2\gamma^2 v/\kappa(v,\omega)$, $I_2 = 2(\omega - h_3)/\kappa(v,\omega)$.

In [11] it was shown that with a week uniform magnetic field $h(\vec{t}) = \vec{h_0}\cos 2\Omega t$, $(\vec{h_0},\vec{M}) \ll \mathcal{H}_0$, which is polarized along the easy magnetization axis, the breather dynamics equations (the adiabatic approximation equations) will have the form (3) as before, where $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\mathrm{int}}$. Moreover, it was investigated the FM linear response to the SHF field with the frequency the same with the initial soliton one. It was expected that under the weak uniform magnetic field the resonance interaction between the breathers having frequency $\omega = \Omega$ and the external magnetic field can exist, but the numerical calculations showed that the effect was small sufficiently.

The relation for $\mathcal{H}_{\mathrm{int}}$ for our AFM system (the energy of interaction between the kink and magnetic field) obtained by us is the following:

$$\mathcal{H}_{\mathrm{int}} = \mathcal{H}_1\exp(\imath\Phi) + \bar{\mathcal{H}}_1\exp(\imath\Phi), \tag{4}$$

where

$$\mathcal{H}_1 = h_0\frac{i\pi\sigma}{2B^2}\frac{v}{(1-v^2)}\cosh^{-1}\left(\frac{\pi\Delta}{2B}\right)\left\{\gamma^2 + h_3(\omega - h_3)\right\}. \tag{5}$$

The whole magnetization created by one kink can be written as

$$M_x = \int dx\, m_x = M_1 \exp(\imath\Phi) + \bar{M}_1 \exp(-\imath\Phi). \tag{6}$$

The calculating of M_1 leads to the following relation

$$M_1 = -\gamma^2 \left(\frac{\imath\pi\sigma}{\delta B^2}\right) \frac{v}{(1-v^2)} \cosh^{-1}\left(\frac{\pi\Delta}{2B}\right). \tag{7}$$

3 High-frequency properties of solitons

Investigations of the contribution of the solitons of different types to the specific heat, magnetization and the dynamical structure factor, defining non-elastic neutron dissipation and so on, are actual problems of the solid state physics during last years [6–10].

The magnet state involving a great number of kinks, the average distance between which is much larger than the average size of the kinks (a "gas" of kinks), can be described by the distribution function $\rho(X, \Phi, I_1, I_2)$, determining the number of quasiparticles per an element of phase volume $\triangle\Gamma$. It would appear reasonable that in the thermodynamic equilibrium state $\rho = \rho_0 = \exp(-\tilde{\beta}\mathcal{E}_0)$, where the inverse energetic temperature $\tilde{\beta} = (M_0 a)^2 \alpha\omega_0 / c\mathcal{T}$, $\mathcal{T}$ is the temperature.

The kinetic equation in general case has the form $\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\vec{u}) = -\frac{\rho-\rho_0}{\tau}$, where τ is the relaxation time. The kinetic equation determining the small nonequilibrium correction ρ_1, owing to the presence of a weak external magnetic field which varies in time, has the following form in the linear approximation:

$$\frac{\partial\rho_1}{\partial t} + \frac{\partial\mathcal{E}_0}{\partial I_2}\frac{\partial\rho_1}{\partial\Phi} + \frac{\rho_1}{\tau} = \frac{\partial\rho_0}{\partial I_2}\cos(\Omega\tau)\frac{\partial\mathcal{H}_{\mathrm{int}}}{\partial\Phi}. \tag{8}$$

The average energy $\bar{Q}$ absorbed from the external field over one period is given by the expression

$$\bar{Q} = -\frac{1}{TLa^2}\int dt \int d^3x \frac{\partial\vec{h}}{\partial t}\vec{\mathcal{M}}, \qquad \mathcal{M}_x = \int \frac{d\Gamma}{(2\pi\hbar)^2}\rho(\Gamma)\, M_x,$$

where L is the length, $\vec{\mathcal{M}}$ is the total magnetization of a sample, $T = 2\pi/\Omega$ which is determined by the kink distribution function ρ, and M_x is the whole magnetization, created by one kink.

The equation (8) can be solved by means of relations (4)–(7). The term with ρ_0 will vanish by averaging t over.

It is essentially to point out that the range of the kinks existence defined by the inequalities $v^2 + \frac{(\omega-h_3)^2}{\gamma^2} < 1$ $(0 < h_3 < \gamma)$ is the ellipse. It is important that there exists the region of negative frequencies. We can accept that the external SHF field can have negative frequencies also. So, the appropriate frequencies of field are defined by the inequality $|\Omega| < \gamma + h_3$. By this means we have three domains of frequencies:

$$\begin{aligned} &\mathrm{I.}\ \Omega_2 < \Omega < \Omega_1, && q_{\mathrm{I}} = J(\Omega), \\ &\mathrm{II.}\ \Omega_4 < \Omega < \Omega_2, && q_{\mathrm{II}} = J(\Omega) + J(-\Omega), \\ &\mathrm{III.}\ \Omega_5 < \Omega < \Omega_4, && q_{\mathrm{III}} = J(-\Omega), \end{aligned} \tag{9}$$

where $\Omega_1 = \gamma + h_3$, $\Omega_2 = \gamma - h_3$, $\Omega_4 = -(\gamma - h_3)$, $\Omega_5 = -(\gamma + h_3)$.

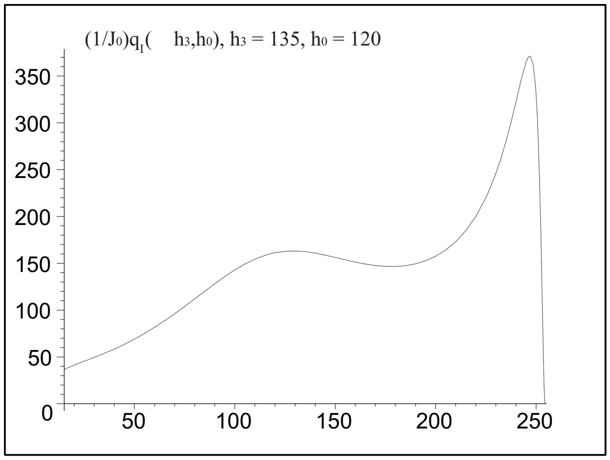

Figure 1.

Now we introduce the expression for the absorbed energy in the linear-response approximation. Here for convenience we use the substitution $x = \gamma v/\kappa(\tilde{v})$, $dx = (\gamma^3 \tilde{v} v)dv/\kappa^3$, in which $J(\Omega)$ has the form:

$$J = J(\Omega; h_3, h_0) = J_0(h_0)\,|\Omega|\,\frac{F(\Omega; h_3)}{\tilde{v}} \int_1^\infty \sqrt{x^2 - 1}\left[1 + \frac{(\Omega - h_3)^2}{\gamma^2\,\tilde{v}^2} x^2\right]^2 \\ \times \cosh^{-2}\left[\frac{\pi}{2}\left(\frac{\Omega - h_3}{\gamma}\right)\sqrt{x^2 - 1}\right] \exp\left[-2\gamma\tilde{\beta}\frac{F(\Omega; h_3)}{\tilde{v}} x\right] dx, \tag{10}$$

where

$$J_0 = 2h_0^2\pi^3\varepsilon^2\tilde{\beta}/\delta, \qquad F(\Omega; h_3) = 1 + \frac{h_3}{\gamma}\left(\frac{\Omega - h_3}{\gamma}\right), \qquad \tilde{v} = \sqrt{1 - \frac{(\Omega - h_3)^2}{\gamma^2}},$$

Note that J depends on one variable Ω, but thereafter we will include the parameters h_0 and h_3 as arguments of J to emphasize their important role.

Conceptually, the parameter $2\gamma\tilde{\beta}$ (see (2)) causes the quantity of the effect. For this one to be nonzero, it is essential to $2\gamma\tilde{\beta} \leq 1$.

4 Conclusion

The numerical calculated curves of $(1/J_0)q_{\mathrm{I,II}}(\Omega; h_3, h_0)$ (see (9), (10) for definition) for several values of external field h_3 are obtained. For this calculation we used the parameters of exchange $J = 3K$ and anisotropy $H_A = 500\,\mathrm{Oe}$ of the well investigated quasi-one-dimensional AFM system $CsMnCl_3 \cdot 2H_2O$, which admits the existence of the soliton excitations [5]. This choice of parameters of the crystal leads to the following values of the dimensionless parameters: $\varepsilon = 0.62$, $J_0 = 1.5 \cdot 10^{-13}$, $\tilde{\beta} = 3.46 \cdot 10^{-3}$, $\delta = 3.63 \cdot 10^3$. The result is represented on Fig. 1 for the most interesting case of h_3 close to the spin-flop field h_{sf}, i.e. $h_3 = 120$, $h_{sf} = 135$ (take into account that the frequency, field and other variables and parameters are dimensionless).

The fluent peak observed at the $\Omega_m = 120$ with the capacity value $Q_m = 10^3\,\mathrm{Erg/sec \cdot cm^3}$ is the most important for our investigation. This peak is the respective signal connected with the additional contribution of the kinks into our system SHF absorption [4]. The sharp increasing of capacity observed at the frequency $\Omega = 255$, coincided closely with the upper AFM resonance frequencies, is not taken into account. The kinks in this region of frequencies are out of the physical interest. The value of the imaginary part of the susceptibility is $\chi''_m = 0.22$. Thus, the maximum at the frequency $\Omega_m = 120$ is of interest, since it is renegated out of the uniform resonance line, has the marked intensily and therefore can be analyzed in the experiments aimed at the finding of the additional line form in the absorption spectrum, the frequency-field, angle, temperature and other dependencies, followed from the theory, presented in this paper.

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Boussineq-Type Equations and “Switching” Solitons

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It is well known that the Boussinesq equation is the *bidirectional* equivalent of the celebrated Korteweg-de Vries equation. Here we consider Boussinesq-type versions of two classical unidirectional integrable equations. A procedure is presented for deriving multisoliton solutions of one of these equations – a bidirectional Kaup–Kupershmidt equation. These solitons have the unusual property that they “switch” shape on switching their direction of propagation.

1 Introduction

In a recent article [1], we constructed a *bidirectional* version of the well-known Kaup–Kupershmidt (KK) equation [2, 3]

$$u_t + 45u^2 u_x - \frac{75}{2} u_x u_{xx} - 15uu_{3x} + u_{5x} = 0, \tag{1}$$

which has the nonlocal form

$$5\,\partial_x^{-1} u_{tt} + 5u_{xxt} - 15uu_t - 15u\,\partial_x^{-1} u_t - 45u^2 u_x + \frac{75}{2} u_x u_{xx} + 15uu_{3x} - u_{5x} = 0. \tag{2}$$

In Ref. [1], equation (2) was designated the bidirectional Kaup–Kupershmidt (bKK) equation. A second nonlinear evolution equation (NEE) that is also of interest here,

$$5\,\partial_x^{-1} u_{tt} + 5u_{xxt} - 15uu_t - 15u_x \partial_x^{-1} u_t - 45u^2 u_x + 15u_x u_{xx} + 15uu_{3x} - u_{5x} = 0, \tag{3}$$

was formulated in [1] as a bidirectional counterpart of the classical Sawada–Kotera (SK) equation [4, 5]

$$u_t + 45u^2 u_x - 15u_x u_{xx} - 15uu_{3x} + u_{5x} = 0. \tag{4}$$

The integrability of equations (2) and (3) was assured by finding their Lax pairs [1]. Indeed, by obtaining the bilinear form of equation (3), we were able to identify this bidirectional equation with the well-known Ramani equation [6] (see equation (7) below). The latter equation has been studied extensively – though only in its more familiar bilinear form (7) – and is now deemed to be completely integrable [6, 7, 8, 9]; we shall refer to equation (3) as the “bSK-Ramani” equation. On the other hand, the bKK equation (2) has received little attention of note in the literature (although the equation in its normal form (2) is listed in the Jimbo–Miwa classification of integrable systems [10]). In Ref. [1] we reported its Lax pair, along with an infinity of conservation laws. We also derived there the solitary-wave solution which has the remarkable property that it “switches” shape on switching its direction of propagation.

In this paper, a procedure is described for obtaining multisoliton solutions of the bKK equation (2). The preliminary results presented here build on the work of the prior study [1] where it was shown that the ‘anomalous’ character of these solitons arises quite naturally within Hirota’s bilinear transform theory [11, 12]. Yet our approach also makes use of the strategy pursued by one of us (A.P.) to obtain the soliton solutions of its unidirectional cousin, the KK equation (1) [13]. However, the current problem is complicated by the need to take account of the bidirectional nature of the bKK solitons; like the solitary wave, they too are found to be *directionally dependent.*

2 Bilinear forms and solitary waves

Following Hirota [11], we make a change of dependent variable

$$u\left(x,t\right) = \alpha\, \partial_x^2 \ln f\left(x,t\right), \qquad \alpha = \text{const.} \tag{5}$$

where ∂_x^n denotes the nth partial derivative with respect to x. Under this transformation, we find that the bSK-Ramani equation (3) has *two* bilinear forms [1]: when $\alpha = -1$ we get

$$\begin{aligned} &\left(80\mathrm{D}_t^2 + 20\mathrm{D}_x^3\mathrm{D}_t - \mathrm{D}_x^6\right) f\cdot f - \left(120\mathrm{D}_x\mathrm{D}_t - 30\mathrm{D}_x^4\right) f\cdot g = 0,\\ &\mathrm{D}_x^2 f\cdot f + 2f\cdot g = 0, \end{aligned} \tag{6}$$

where D_x, D_t are the usual Hirota derivatives [12]

$$\mathrm{D}_x\mathrm{D}_t\, a\left(x,t\right)\cdot b\left(x,t\right) = \left(\partial_x - \partial_{x'}\right)\left(\partial_t - \partial_{t'}\right) a\left(x,t\right) b\left(x',t'\right)\big|_{x'=x,\ t'=t}$$

and $g\left(x,t\right)$ is an auxiliary function. The second bilinear form has $\alpha = -2$ and is given by

$$\left(5\mathrm{D}_t^2 + 5\mathrm{D}_x^3\mathrm{D}_t - \mathrm{D}_x^6\right) f\cdot f = 0. \tag{7}$$

The single bilinear equation (7) identifies the bSK-Ramani equation (3) with Ramani's equation [6], whereas the less well-known coupled system (6) appeared somewhat later [10].

Similarly, under the transformation (5), the bKK equation (2) admits two bilinear representations [1]: $\alpha = -1$

$$\begin{aligned} &\left(80\mathrm{D}_t^2 + 20\mathrm{D}_x^3\mathrm{D}_t - \mathrm{D}_x^6\right) f\cdot f - 120\mathrm{D}_x\mathrm{D}_t f\cdot g + 30\mathrm{D}_x^2 f\cdot h = 0,\\ &\mathrm{D}_x^2 f\cdot f + 2f\cdot g = 0,\\ &\mathrm{D}_x^4 f\cdot f + 2f\cdot h = 0; \end{aligned} \tag{8}$$

$\alpha = -2$:

$$\begin{aligned} &16\left(5\mathrm{D}_t^2 + 5\mathrm{D}_x^3\mathrm{D}_t - \mathrm{D}_x^6\right) f\cdot f - 30\mathrm{D}_x^4 f\cdot g + 30\mathrm{D}_x^2 f\cdot h = 0,\\ &\mathrm{D}_x^2 f\cdot f + f\cdot g = 0,\\ &\mathrm{D}_x^4 f\cdot f + f\cdot h = 0. \end{aligned} \tag{9}$$

Equations (8) and (9), in which g and h are auxiliary functions, are derived in Ref. [1].

Finding the multisoliton solutions of the bSK-Ramani equation (3) is straightforward since we may solve the *single* bilinear form (7) rather than the coupled system (6). Thus, the N-soliton solution of equation (7) is given by Hirota's ansatz [11]

$$f\left(x,t\right) = \sum_{\mu=0,1} \exp\left[\sum_{i=1}^{N} \mu_i\theta_i + \sum_{1\le i<j\le N} \mu_i\mu_j \ln A_{ij}\right], \tag{10}$$

where $\theta_i = p_i x + \omega_i t + \eta_i$ $(i = 1,\ldots,N)$ and p_i, ω_i, η_i are constant parameters. Following Ref. [13], we will call the generic solution (10) the *regular* N-soliton: observe that it is described by a single interaction coefficient A_{ij}. The solitary wave is given by setting $N = 1$ in equation (10) and yields the familiar sech^2 pulse [1]

$$u\left(x,t\right) = -\frac{1}{2}\, p^2 \mathrm{sech}^2 \frac{1}{2}\left(px + \omega t + \eta\right), \tag{11}$$

where $\omega(p)$ satisfies the quadratic dispersion relation

$$5\omega^2 + 5\omega p^3 - p^6 = 0. \tag{12}$$

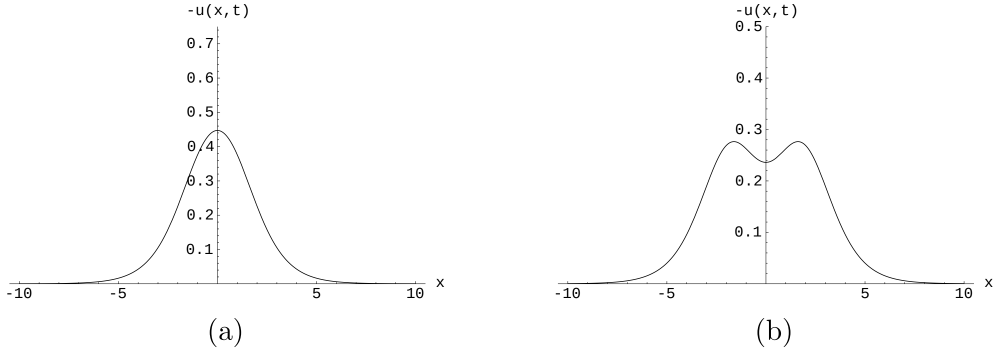

Figure 1. Solitary-wave solutions of the bKK equation: (a) a right-travelling single-humped wave, (b) a left-travelling double-humped wave.

For the bKK equation, no reduction of the bilinear forms (8) and (9) to a single bilinear equation, akin to the Ramani equation (7), is possible. We must therefore solve one or other of the coupled systems (8) or (9) for which no prescribed ansatz, comparable to the regular N-soliton (10), is available. However, we may exploit the close connection between the bKK and bSK-Ramani equations – that is evident from equations (2) and (3) – to argue as follows. Since the bilinear forms (6) and (7) of the bSK-Ramani equation are equivalent under $f^2 \leftrightarrow f$ [1], the N-soliton solution of the coupled bilinear form (6) is the *square* of the regular N-soliton (10). But then the duality of the bKK and bSK-Ramani equations suggests the following hypothesis: the N-soliton solution of the bKK bilinear form (8) will *mimic* its counterpart for the corresponding bSK-Ramani system (6). For example, if we apply this reasoning to the regular solitary wave (set $N = 1$ in (10)), we obtain the solution of equation (8) [1],

$$f\,(x,t) = 1 + \mathrm{e}^{\theta} + \frac{1}{16}a^2\mathrm{e}^{2\theta}, \qquad \theta = px + \omega t + \eta, \tag{13}$$

where

$$a^2 = \frac{4\omega - p^3}{\omega - p^3} \qquad (a > 0) \tag{14}$$

and $\omega(p)$ satisfies the (bSK-Ramani) dispersion relation (12). Then, using $u = -\partial_x^2 \ln f$ (equation (5) with $\alpha = -1$), we obtain the 'anomalous' solitary wave of the bKK equation (2)

$$u\,(x,t) = -ap^2\frac{a + 2\cosh\theta}{(a\cosh\theta + 2)^2}, \tag{15}$$

which was first reported in Ref. [1]. The most significant feature of this solitary wave is that *it "switches" its shape on switching direction* (cf. the bSK-Ramani solitary wave (11) that propagates to the left or right with the *same* bell-shaped profile). The right-travelling single-humped solitary wave is shown in Fig. 1(a), whilst the left-running wave has the double-humped profile pictured in Fig. 1(b) (where here, and in subsequent figures, we plot the physical wave $-u\,(x,t)$). Extending the argument, we conjecture that the N-soliton of the bKK equation (8) has the structure – though not the precise analytical form – of the *squared* regular N-soliton (10). We shall use this duality hypothesis – which was formulated in Ref. [1] – to obtain higher-order soliton solutions of the bKK equation; in effect, we choose to solve the coupled bilinear form (8) rather than the alternate system (9).

3 Two-soliton solution of the bKK equation

Before proceeding, it will be helpful to introduce the following notation: if $F(\mathrm{D}_x, \mathrm{D}_t)$ is any bilinear operator, then we define $F(\boldsymbol{p}) = F(p, \omega)$. Now, the regular two-soliton solution of the bSK-Ramani equation is given by (set $N = 2$ in equation (10))

$$f(x,t) = 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + A_{12}\mathrm{e}^{\theta_1+\theta_2}, \qquad \theta_i = p_i x + \omega_i t + \eta_i, \qquad i = 1,2, \tag{16}$$

which solves equation (7) if

$$A_{12} = -\frac{F_{\mathrm{R}}(\boldsymbol{p}_1 - \boldsymbol{p}_2)}{F_{\mathrm{R}}(\boldsymbol{p}_1 + \boldsymbol{p}_2)} = -\frac{5(\omega_1-\omega_2)^2 + 5(\omega_1-\omega_2)(p_1-p_2)^3 - (p_1-p_2)^6}{5(\omega_1+\omega_2)^2 + 5(\omega_1+\omega_2)(p_1+p_2)^3 - (p_1+p_2)^6} \tag{17}$$

and $\omega_i(p_i)$ satisfies the dispersion relation (cf. equation (12))

$$F_{\mathrm{R}}(\boldsymbol{p}_i) = 5\omega_i^2 + 5\omega_i p_i^3 - p_i^6 = 0, \qquad i = 1,2, \tag{18}$$

where $F_{\mathrm{R}}(\mathrm{D}_x, \mathrm{D}_t) = 5\mathrm{D}_t^2 + 5\mathrm{D}_x^3\mathrm{D}_t - \mathrm{D}_x^6$ is the Ramani bilinear operator. According to our duality hypothesis, the two-soliton solution of the bKK equation will mimic f^2 (a solution of the bSK-Ramani bilinear form (6)). We therefore seek a solution of the bilinear form (8) with

$$\begin{aligned} f(x,t) = {} & 1 + \mathrm{e}^{\theta_1} + \mathrm{e}^{\theta_2} + \frac{1}{16}a_1^2\,\mathrm{e}^{2\theta_1} + \frac{1}{16}a_2^2\,\mathrm{e}^{2\theta_2} + b_{12}\,\mathrm{e}^{\theta_1+\theta_2} \\ & + \frac{A}{16}\left(a_1^2\,\mathrm{e}^{2\theta_1+\theta_2} + a_2^2\,\mathrm{e}^{\theta_1+2\theta_2}\right) + \left(\frac{A}{16}\right)^2 a_1^2\,a_2^2\,\mathrm{e}^{2(\theta_1+\theta_2)}, \end{aligned} \tag{19}$$

where (cf. equation (14))

$$a_i^2 = \frac{4\omega_i - p_i^3}{\omega_i - p_i^3}, \qquad i = 1,2, \tag{20}$$

and $\omega_i(p_i)$ satisfies the (bSK-Ramani) dispersion law (18). The expression (19) merits further comment: firstly, it has been *normalised* by setting the coefficients of the terms e^{θ_i} to unity (η_i are arbitrary). Further, f is symmetrical under the exchange $\theta_1 \leftrightarrow \theta_2$. Finally, by applying the "elastic" interaction property of colliding solitons [14, 15] – whereby (19) separates asymptotically into two distinct 'solitary' waves of the form (13)–(14) – we are left with just the two unknown constants b_{12} and A. The parameter A arises quite naturally as a measure of the post-interaction phase shifts of the constituent solitary waves, and so plays the same rôle as A_{12} in the bSK-Ramani two-soliton (16).

We now substitute the putative bKK two-soliton (19) into the bilinear form (8) and make use of the standard result [12]

$$F(\mathrm{D}_x, \mathrm{D}_t)\,\mathrm{e}^{\theta_1} \cdot \mathrm{e}^{\theta_2} = F(\boldsymbol{p}_1 - \boldsymbol{p}_2)\,\mathrm{e}^{\theta_1+\theta_2}, \qquad \theta_i = p_i x + \omega_i t + \eta_i, \quad i = 1,2.$$

Following some routine but lengthy algebra (that is best carried out using a symbolic manipulation programme such as Mathematica), we find that $A = A_{12}$, equation (17), and

$$b_{12} = \frac{\Delta_{12}}{2F_{\mathrm{R}}(\boldsymbol{p}_i + \boldsymbol{p}_j)} = \frac{\Delta_{12}}{2D_{12}}, \tag{21}$$

where

$$\Delta_{12} = 20\omega_1\omega_2 + 10\omega_1 p_2\left(3p_1^2 + p_2^2\right) + 10\omega_2 p_1\left(p_1^2 + 3p_2^2\right) - p_1 p_2\left(12p_1^4 - 5p_1^2p_2^2 + 12p_2^4\right) \tag{22}$$

and

$$\begin{aligned} D_{12} = {} & 10\omega_1\omega_2 + 5\omega_1 p_2\left(3p_1^2 + 3p_1p_2 + p_2^2\right) + 5\omega_2 p_1\left(p_1^2 + 3p_1p_2 + 3p_2^2\right) \\ & - p_1p_2\left(6p_1^4 + 15p_1^3p_2 + 20p_1^2p_2^2 + 15p_1p_2^3 + 6p_2^4\right). \end{aligned} \tag{23}$$

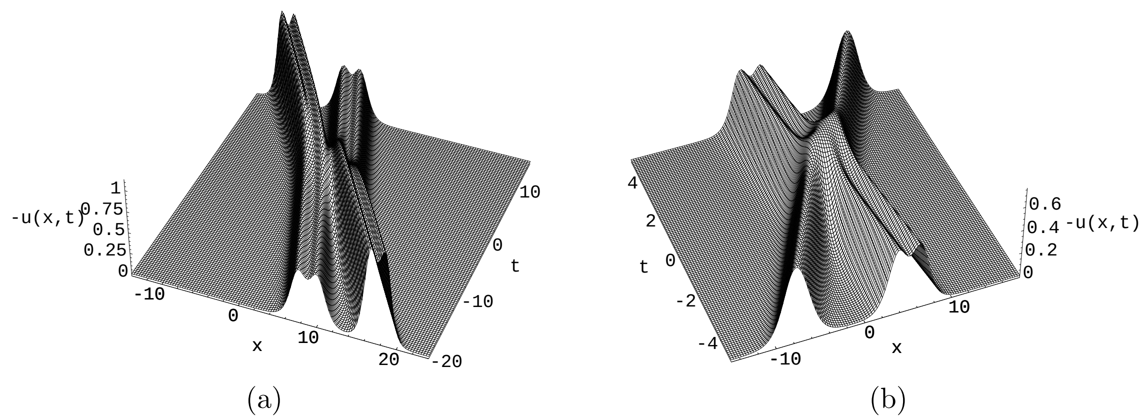

Figure 2. A perspective view of the bKK two-soliton: (a) the interaction of two left-travelling double-humped solitary waves, (b) the head-on collision of a single- and a double-humped pulse.

This completes the derivation of the two-soliton solution $u\,(x,t)$ of the bKK equation (2) which is obtained explicitly from (19) (with $A \to A_{12}$) and the relation $u = -\partial_x^2 \ln f$. Fig. 2(a) shows a two-soliton comprised of a pair of double-humped 'solitary' waves propagating to the left, whilst Fig. 2(b) pictures the head-on collision between a single-peaked and a double-peaked 'solitary' wave. Typically, the soliton pulses emerge from the interactions intact, except for the clearly visible phase shifts. The bKK two-soliton (19) bears further comment. It shares the same wave dynamics as the bSK-Ramani two-soliton, equation (16): their colliding solitary waves undergo identical phase shifts that are determined by the common interaction coefficient A_{12}, equation (17). This bears out the intimate connection between the bKK and bSK-Ramani equations that is already apparent through the shared dispersion relations (12) and (18), and justifies the duality hypothesis on which our solution procedure is based. Another important feature of (19) is the 'new' parameter b_{12}, equation (21), which cannot be expressed in terms of A_{12} alone (cf. the bSK-Ramani two-soliton (16)). It is instructive to compare this key parameter with its counterpart for the bSK-Ramani equation. Squaring (and normalising) the regular two-soliton (16), and extracting the coefficient of $\mathrm{e}^{\theta_1+\theta_2}$, we find

$$b_{12}^{\mathrm{R}} = \frac{1}{2}(A_{12}+1) = \frac{\Delta_{12}^{\mathrm{R}}}{2D_{12}}$$

with

$$\Delta_{12}^{\mathrm{R}} = 20\omega_1\omega_2 + 10\omega_1 p_2\left(3p_1^2+p_2^2\right) + 10\omega_2 p_1\left(p_1^2+3p_2^2\right) - p_1p_2\left(12p_1^4+40p_1^2p_2^2+12p_2^4\right).$$

Thus, b_{12} mimics b_{12}^{R} (they differ only in the $p_1^3p_2^3$ term in their numerators) and suggests that our duality hypothesis can be extended to include this crucial parameter. This further conjecture will help us when we seek solitons of higher order.

4 Further soliton solutions of the bKK equation

For the sake of brevity, we must content ourselves with describing the main results. We will leave a more complete presentation of these preliminary results – giving a full account of the technical details – to a future work.

According to our duality hypothesis, to obtain the bKK three-soliton solution we start with the regular three-soliton (put $N = 3$ in equation (10)). We then square (and normalise) this expression, introducing a minimal number of undetermined coefficients to give the form of the ansatz f (consistent with the symmetry in θ_i, $i = 1, 2, 3$). Rather than solve the coupled bilinear form (8) directly, we proceed by iteration on the solitons of lower order. (This soliton reduction procedure was first developed in Ref. [13] to solve the related KK equation (1)). Once f has been reduced to a solitary wave, equation (13), and then a two-soliton, equation (19), we arrive at the three-soliton

$$\begin{aligned}
f = {} & 1 + \sum_{i=1}^{3} \mathrm{e}^{\theta_i} + \frac{1}{16}\sum_{i=1}^{3} a_i^2 \mathrm{e}^{2\theta_i} + \sum_{1\le i<j\le 3} b_{ij}\,\mathrm{e}^{\theta_i+\theta_j} + \frac{1}{16}\sum_{1\le i<j\le 3} A_{ij}\left(a_i^2\mathrm{e}^{2\theta_i+\theta_j} + a_j^2\mathrm{e}^{\theta_i+2\theta_j}\right) \\
& + b_{123}\mathrm{e}^{\theta_1+\theta_2+\theta_3} + \frac{1}{16^2}\sum_{1\le i<j\le 3} A_{ij}^2 a_i^2 a_j^2\,\mathrm{e}^{2(\theta_i+\theta_j)} \\
& + \frac{1}{16}\left[a_1^2 b_{23}A_{12}A_{13}\mathrm{e}^{2\theta_1+\theta_2+\theta_3} + a_2^2 b_{13}A_{12}A_{23}\mathrm{e}^{\theta_1+2\theta_2+\theta_3} + a_3^2 b_{12}A_{13}A_{23}\mathrm{e}^{\theta_1+\theta_2+2\theta_3}\right] \\
& + \frac{1}{16^2}A_{12}A_{13}A_{23}\left[a_1^2a_2^2A_{12}\mathrm{e}^{2(\theta_1+\theta_2)+\theta_3} + a_1^2a_3^2A_{13}\mathrm{e}^{2\theta_1+\theta_2+2\theta_3} + a_2^2a_3^2A_{23}\mathrm{e}^{\theta_1+2(\theta_2+\theta_3)}\right] \\
& + \frac{1}{16^3}a_1^2a_2^2a_3^2A_{12}^2A_{13}^2A_{23}^2\,\mathrm{e}^{2(\theta_1+\theta_2+\theta_3)}
\end{aligned} \tag{24}$$

in which all but one of the coefficients have been fixed. The parameters A_{ij} and b_{ij} in (24) generalise equations (17) and (21), respectively, in the obvious way. The only unknown is the 'new' parameter b_{123} which cannot be found by reducing f to a soliton of lower order. However, we can deduce the following useful reductions in this way: with $\boldsymbol{p}_i = (p_i, \omega_i)$, we have

$$b_{123}(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{0}) = b_{12}(\boldsymbol{p}_1,\boldsymbol{p}_2), \quad b_{123}(\boldsymbol{p}_1,\boldsymbol{0},\boldsymbol{0}) = b_{23}(\boldsymbol{0},\boldsymbol{0}), \quad b_{123}(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{p}_2) = \frac{1}{8}a_2^2A_{12}. \tag{25}$$

We now invoke our further conjecture that b_{123} will mimic its counterpart b_{123}^{R}: this yields

$$b_{123} = \frac{\Delta_{123}}{4D_{123}}, \qquad D_{123} = D_{12}D_{13}D_{23}, \tag{26}$$

where D_{ij} generalises (23) and

$$\begin{aligned}
\Delta_{123} = {} & 18 \ll p_i^2p_j^2p_k^2\left(5\omega_i - 2p_i^3\right)\left(5\omega_j - 2p_j^3\right)\Delta_{ij} \gg +810 \ll p_i^{10}p_j^4p_k^4 \gg +324p_1^6p_2^6p_3^6 \\
& - 4050 \ll \omega_i p_i^7p_j^4p_k^4 \gg +405 \ll \omega_i p_i^5p_j^6p_k^4 \gg -2430 \ll \omega_i p_i^3p_j^6p_k^6 \gg \\
& + 1620 \ll \omega_i p_i p_j^{10}p_k^4 \gg -8100 \ll \omega_i\omega_j p_i^7p_jp_k^4 \gg -4050 \ll \omega_i\omega_j p_i^5p_j^3p_k^4 \gg \\
& + 810 \ll \omega_i\omega_j p_i^5p_jp_k^6 \gg +16200 \ll \omega_i\omega_j p_i^3p_j^3p_k^6 \gg +3240 \ll \omega_i\omega_j p_ip_jp_k^{10} \gg \\
& - 16200 \ll \omega_i\omega_j\omega_k p_i^7p_jp_k \gg -8100 \ll \omega_i\omega_j\omega_k p_i^5p_j^3p_k \gg -81000\omega_1\omega_2\omega_3p_1^3p_2^3p_3^3.
\end{aligned} \tag{27}$$

The symbol $\ll\gg$ denotes the sum over all distinct permutations of $(1,2,3)$ assigned to the subscripts (i,j,k) of the enclosed product, and Δ_{ij} generalises (22). All but two of the coefficients in (27) are fixed by the reductions (25); the remaining two coefficients are obtained by using the bilinear equation in (8) once more (though with a much simplified ansatz in place of (24)). The explicit three-soliton solution $u(x,t)$ of the bKK equation (2) follows from (24) and $u = -\partial_x^2 \ln f$. Fig. 3 shows a three-soliton solution in which two left-running double-humped 'solitary' waves collide head-on with a single-peaked pulse propagating to the right. Though we shall not do so here, we could continue in the same way to derive the four-soliton solution by iterating on the first three known solitons. In principle, we are now able to generate the N-soliton solution of the bKK equation (2) by iteration on the solitons of lower order; however, the practical difficulties should not be underestimated. The sheer complexity of the algebraic expressions involved will present severe difficulties beyond the first few multisolitons (even with the aid of symbolic software such as Mathematica).

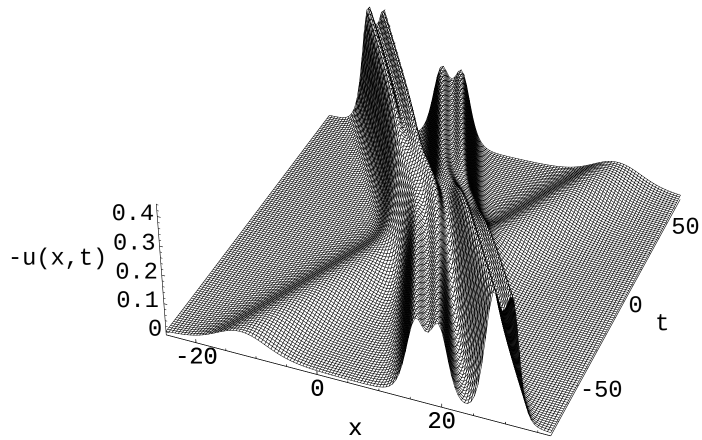

Figure 3. A three-soliton solution of the bKK equation showing the head-on interaction of a right-running single-peaked pulse with two left-running double-humped solitary waves.

5 Concluding remarks

A direct method has been presented for obtaining explicit multisoliton solutions of the bidirectional Kaup-Kupershmidt equation (2). Not surprisingly, these solitons possess the same remarkable property as the 'anomalous' solitary wave found in Ref. [1]; namely, their wave profiles are *directionally dependent*. As far as we know, this type of soliton behaviour has not been observed before now and these "switching" solitons are reported here for the first time. The 'anomalous' character of the bKK solitons – whose description requires the introduction of a new parameter at each order – arises quite naturally within the bilinear formalism as a *squared* regular N-soliton. This canonical form, in conjunction with the duality of the bKK and bSK-Ramani equations, provides the basis for the iterative procedure that is used to obtain the solitons of higher order. From a wave perspective, this formulation – couched in terms of the common interaction parameters A_{ij} and shared dispersion laws (18) – would seem to be the natural one. For not only does it make explicit the dynamical duality of the soliton solutions of the bKK and bSK-Ramani equations, but it also underlines the intimacy between these fundamentally different integrable bidirectional equations. This mirrors the deep connection between their better known undirectional cousins the KK equation (1) and the SK equation (4), respectively [2, 13, 16, 17]. We intend to report a more comprehensive discussion of these preliminary findings, together with the derivation of further multisolitons, in the near future.

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Transformation Operators for Integrable Hierarchies with Additional Reductions

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New integrable reductions of the modified Kadomtsev–Petviashvili (mKP) hierarchy was obtained. We solve the so-called $\mathcal{D}$-Hermitian constrained mKP ($\mathcal{D}$HcmKP) hierarchy by using the dressing transformation technique. The dressing (transformation) operator for the $\mathcal{D}$HcmKP hierarchy is defined, and multicomponent derivative nonlinear Schrödinger equation was integrated as an example.

1 Introduction

We consider Lax–Zakharov–Schabat equations

$$\beta U_t - \alpha V_y + UV - VU = 0 \quad \Leftrightarrow \quad [\alpha\partial_y - U,\ \beta\partial_t - V] = 0,$$
$$\alpha, \beta \in \mathbb{C}, \quad \partial_y := \frac{\partial}{\partial y}, \quad \partial_t := \frac{\partial}{\partial t}, \tag{1}$$

in the algebra ζ of the microdifferential operators (MDO) [1].

$$U, V \in \zeta := \left\{ L = \sum_{i=-\infty}^{n(L)} a_i \mathcal{D}^i : \ a_i = a_i(x,y,t);\ i, n(L) \in \mathbb{Z} \right\}, \tag{2}$$

where MDO U, V satisfy additional constraints, which are concretely defined in the following section, and coefficients a_i are, in general, smooth $(N \times N)$ matrix-valued functions of $x, y, t, \in \mathbb{R}$. In the algebra MDO ζ (2) operation of multiplication is induced by the generalized Leibnitz rule

$$\mathcal{D}^n f := \sum_{j=0}^{\infty} \binom{n}{j} f^{(j)} \mathcal{D}^{n-j}, \quad n \in \mathbb{Z}, \qquad \mathcal{D}^m(f) := \frac{\partial^m f}{\partial x^m} = f^{(m)}, \quad m \in \mathbb{Z}_+, \tag{3}$$

where

$$\binom{n}{j} := \frac{n(n-1)\dots(n-j+1)}{j!}, \qquad \mathcal{D}^n \mathcal{D}^m := \mathcal{D}^{n+m}, \qquad n, m \in \mathbb{Z},$$

and f is the operator of multiplication by function $f(x,y,t)$, which belongs to the same functional space as the coefficients of microdifferential operators $L \in \zeta$ do. Lie's commutator in algebra ζ is defined as $[U,V] := UV - VU$, and Hermitian-conjugated operator $L^* := \sum\limits_{i=-\infty}^{n(L)} (-1)^i \mathcal{D}^i a_i^*$, $a_i^* = \bar{a}^\top$, $(\alpha\partial_y)^* := -\bar{\alpha}\partial_y$, $(\beta\partial_t)^* := -\bar{\beta}\partial_t$.

2 Reduction of $\mathcal{D}$-Hermitian conjugation

Definition 1. We say that an operator $L \in \zeta$ is $\mathcal{D}$-Hermitian ($\mathcal{D}$-skew-Hermitian) if

$$L^* = \mathcal{D} L \mathcal{D}^{-1} \quad \left(L^* = -\mathcal{D} L \mathcal{D}^{-1}\right).$$

Definition 2. We say that an integral operator $W \in \zeta_{<1} := \left\{ L_{<1} := \sum_{i=-\infty}^{0} u_i \mathcal{D}^i \right\}$ is $\mathcal{D}$-unital if $W^{-1} = \mathcal{D}^{-1} W^* \mathcal{D}$.

Lemma 1. *Let $L^* = \mu \mathcal{D} L \mathcal{D}^{-1}$, $\mu = \pm 1$, and $W^{-1} = \mathcal{D}^{-1} W^* \mathcal{D}$. Then $\hat{L}^* = \mu \mathcal{D} \hat{L} \mathcal{D}^{-1}$, where $\hat{L} := WLW^{-1}$.*

Proof. $\hat{L}^* := \left(WLW^{-1}\right)^* = \left(W^{-1}\right)^* L^* W^* = \mu \mathcal{D} W \mathcal{D}^{-1} \mathcal{D} L \mathcal{D}^{-1} \mathcal{D} W^{-1} \mathcal{D}^{-1} = \mu \mathcal{D} \hat{L} \mathcal{D}^{-1}$. ■

Lemma 2. *Let h_i, g_i be smooth $(N \times K)$ matrix-valued functions of real variable $x \in \mathbb{R}$, $i = 1, 2$; $A = (a_{mn}) = \mathrm{const} \in \mathrm{Mat}_{K \times K}(\mathbb{C})$ and $a \in \mathbb{R} \cup \{\pm\infty\}$. Then*

$$h_1 \mathcal{D}^{-1} g_1^\top h_2 \mathcal{D}^{-1} g_2^\top = h_1 \left(A + \int_a^x g_1^\top h_2 dx \right) \mathcal{D}^{-1} g_2^\top - h_1 \mathcal{D}^{-1} \left(A + \int_a^x g_1^\top h_2 dx \right) g_2^\top.$$

Proof. By direct calculation from the Leibnitz rule (3) for $n = -1$ we obtain:

$$h_1 \mathcal{D}^{-1} g_1^\top h_2 \mathcal{D}^{-1} g_2^\top = h_1 \sum_{i=0}^{\infty} (-1)^i (g_1^\top h_2)^{(i)} \mathcal{D}^{-i-2} g_2^\top,$$

$$h_1 \left(A + \int_a^x g_1^\top h_2 dx \right) \mathcal{D}^{-1} g_2^\top - h_1 \sum_{i=0}^{\infty} (-1)^i \left(A + \int_a^x g_1^\top h_2 dx \right)^{(i)} \mathcal{D}^{-i-1} g_2^\top$$
$$= -h_1 \sum_{i=1}^{\infty} (-1)^i \left(A + \int_a^x g_1^\top h_2 dx \right)^{(i)} \mathcal{D}^{-i-1} g_2^\top = h_1 \sum_{i=0}^{\infty} (-1)^i (g_1^\top h_2)^{(i)} \mathcal{D}^{-i-2} g_2^\top. \quad ■$$

Lemma 3. *Let $C^* = -C = \mathrm{const} \in \mathrm{Mat}_{K \times K}(\mathbb{C})$, $\varphi = \varphi(x)$ be a matrix $(N \times K)$ function and $\varphi \in L_2(-\infty, s)$ $\forall\, s \in \mathbb{R}$. Then $w_0^{-1} = w_0^*$, where*

$$w_0 := I - \varphi \left(C + \int_{-\infty}^x \varphi^* \varphi_x dx \right)^{-1} \varphi^* := I - \varphi \Omega^{-1} \varphi^*. \tag{4}$$

Proof.

$$w_0^* = I - \varphi \Omega^{*-1} \varphi^* = I - \varphi \left(\varphi^* \varphi - C - \int_{-\infty}^{x} \varphi^* \varphi_x dx \right)^{-1} \varphi^*, \tag{5}$$

$$w_0 w_0^* = I - \varphi \left[\left(C + \int_{-\infty}^x \varphi^* \varphi_x dx \right)^{-1} + \left(\varphi^* \varphi - C - \int_{-\infty}^x \varphi^* \varphi_x dx \right)^{-1} \right.$$
$$\left. - \left(C + \int_{-\infty}^{x} \varphi^* \varphi_x dx \right)^{-1} \varphi^* \varphi \left(\varphi^* \varphi - C - \int_{-\infty}^{x} \varphi^* \varphi_x dx \right)^{-1} \right] \varphi^*$$
$$= I - \varphi \Omega^{-1} \left[I + (\Omega - \varphi^* \varphi) \Omega^{*-1} \right] \varphi^* = I. \quad ■$$

Theorem 1. *Let $W := w_0 + \varphi \Omega^{-1} \mathcal{D}^{-1} \varphi_x^*$ (see conditions of Lemma 3). Then $W^{-1} = \mathcal{D}^{-1} W^* \mathcal{D}$ (i.e. W is a D-unital operator).*

Proof.

1. $W = I - \varphi \Omega^{-1} \varphi^* + \varphi \Omega^{-1} \mathcal{D}^{-1} \varphi_x^* = I - \varphi \Omega^{-1} \mathcal{D}^{-1} \varphi^* \mathcal{D}$,
   $W^* = I - \mathcal{D} \varphi \mathcal{D}^{-1} \Omega^{*-1} \varphi^*$,
   $\mathcal{D}^{-1} W^* \mathcal{D} = I - \varphi \mathcal{D}^{-1} \Omega^{*-1} \varphi^* \mathcal{D} = w_0^{-1} + \varphi \mathcal{D}^{-1} \left(\Omega^{*-1} \varphi^* \right)_x$.

2. $$W\mathcal{D}^{-1}W^*\mathcal{D} = \left(w_0 + \varphi\Omega^{-1}\mathcal{D}^{-1}\varphi_x^*\right)\left(w_0^{-1} + \varphi\mathcal{D}^{-1}(\Omega^{*-1}\varphi^*)_x\right)$$
$$= I + w_0\varphi\mathcal{D}^{-1}\left(\Omega^{*-1}\varphi^*\right)_x + \varphi\Omega^{-1}\mathcal{D}^{-1}\varphi_x^* w_0^{-1} + \varphi\Omega^{-1}\mathcal{D}^{-1}\varphi_x^*\varphi\mathcal{D}^{-1}\left(\Omega^{*-1}\varphi^*\right)_x, \quad (6)$$

and using Lemma 2, the definitions of Ω and Ω^* (4)–(5) by direct calculation we obtain that the sum of integral operators in (6) is equal to zero, i.e. $W\mathcal{D}^{-1}W^*\mathcal{D} = I$. ■

3 Lax equation invariant under reductions of $\mathcal{D}$-Hermitian conjugation

In this paper we restrict ourselves by the scalar cases ($N = 1$) of the algebra (2).

We consider the modified Kadomtsev–Petviashvili (mKP) hierarchy [2]

$$\alpha_n \frac{\partial Z}{\partial t_n} = -\left(Z\mathcal{D}^n Z^{-1}\right)_{<1} Z, \qquad \alpha_n \in \mathbb{C}, \quad n \in \mathbb{N}, \quad t_1 := x, \tag{7}$$

where integral operator Z is given by

$$\zeta_{<1} \ni Z = z_0 + z_1\mathcal{D}^{-1} + z_2\mathcal{D}^{-2} + \cdots \quad \left(z_0^{-1} \text{ exists}\right). \tag{8}$$

With the use of the MDO $L := Z\mathcal{D}Z^{-1} := L_{\mathrm{mKP}} = \mathcal{D} + U_0 + U_1\mathcal{D}^{-1} + U_2\mathcal{D}^{-2} + \cdots$, system (7) can be rewritten in the form of the Lax representation

$$\alpha_m L_{t_m} = [B_m, L] := B_m L - L B_m, \tag{9}$$

where $B_m := (L^m)_{>0}$, $m \in \mathbb{N}$.

The mKP hierarchy (7) can be transformed into Zakharov–Schabat equations

$$\alpha_n B_{m_{t_n}} - \alpha_m B_{n_{t_m}} + [B_m, B_n] = 0, \qquad m, n \in \mathbb{N}, \quad \alpha_m, \alpha_n \in \mathbb{C}. \tag{10}$$

Note that the subscripts mean partial differentiations with respect to the indicated variables (evolutionary parameters t_j, $j \in \mathbb{N}$). If we eliminate $U_0, U_1, U_2, \ldots$ from (9), the remaining equations for the function $U := U_0$ in (9) (or in (10)) for $t_1 := x$, $t_2 := y$, $t_3 := t$ would represent the mKP equation

$$\alpha_3 U_t = \frac{1}{4}U_{xxx} - \frac{3}{2}U^2 U_x + \frac{3}{4}\alpha_2^2\partial_x^{-1}U_{yy} + \frac{3}{2}\alpha_2 U_x \partial_x^{-1}U_y, \tag{11}$$

where $\partial_x^{-1} f := \int^x f dx$, and its hierarchy flows.

W. Oevel and W. Strampp have also introduced so-called constrained modified Kadomtsev–Petviashvili (cmKP) [3], apart from the cKP (constrained KP) hierarchy [4, 5, 6, 7] (see, also [8]). The Lax operator of the cmKP hierarchy is defined by

$$L_{\mathrm{cmKP}} = \mathcal{D}^n + u_{n-1}\mathcal{D}^{n-1} + \cdots + u_1\mathcal{D} + u_0 + \mathcal{D}^{-1}s, \tag{12}$$

or

$$L_{\mathrm{cmKP}} := (L_{\mathrm{mKP}}^n)_{\geq 0} + \mathcal{D}^{-1}s,$$

and the hierarchy flows are described by

$$\alpha_m \frac{\partial L_{\mathrm{cmKP}}}{\partial t_m} = \left[\left(L_{\mathrm{cmKP}}^{m/n}\right)_{>0}, \; L_{\mathrm{cmKP}}\right], \qquad \alpha_m s_{t_m} = -\left(L_{\mathrm{cmKP}}^{m/n}\right)_{>0}^* (s). \tag{13}$$

We proposed another restriction of mKP hierarchy, so-called $\mathcal{D}$-Hermitian cmKP ($\mathcal{D}$HcmKP) hierarchy in the form

$$L_{\mathcal{D}\mathrm{HcmKP}} := L_n = \mathcal{D}^n + u_{n-1}\mathcal{D}^{n-1} + \cdots + u_1\mathcal{D} - V, \tag{14}$$

where $\zeta_{<1} \ni V$ is $\mathcal{D}$-Hermitian ($\mathcal{D}$-skew-Hermitian) integral degenerated Volterra operator, defined as

$$V = \boldsymbol{q}\mathcal{M}\mathcal{D}^{-1}\boldsymbol{q}^*\mathcal{D} = (\boldsymbol{q}\mathcal{M}\boldsymbol{q}^*) - \boldsymbol{q}\mathcal{M}\mathcal{D}^{-1}\boldsymbol{q}^*_x, \qquad \text{if} \quad n = 2k,$$

where $\mathcal{M}^* = \mathcal{M}$,(or $V = i\boldsymbol{q}\mathcal{M}\mathcal{D}^{-1}\boldsymbol{q}^*\mathcal{D}$, if $n = 2k-1$); $\boldsymbol{q} = (q_1, \ldots, q_l)$, $k, l \in \mathbb{N}$, and additional reduction for operator L_n:

$$L_n^* = \mu\mathcal{D}L_n\mathcal{D}^{-1}, \qquad \mu = \pm 1.$$

In this case, the $\mathcal{D}$-unital operator $Z := W$ (the definition of integral operator W see below in Theorem 1) is the transformation (dressing) operator for mKP hierarchy (7)–(10). We now work out a few examples of restrictions of the mKP hierarchy connected with $\mathcal{D}$-Hermitian Lax operators of the form (14). We consider the evolution equations

$$\alpha_m L_{n_{t_m}} = [B_m, L_n], \tag{15}$$

where $L_n := L_{\mathcal{D}\mathrm{HcmKP}}$ (14), and B_m are fractional powers m/n of L_n; $n, m, \in \mathbb{N}$. The "basic root" $L_n^{\frac{1}{n}} = \mathcal{D} + a_0 + a_{-1}\mathcal{D}^{-1} + \cdots$ is calculated by requiring $(L_n^{\frac{1}{n}})^n = L_{\mathcal{D}\mathrm{HcmKP}}$. This leads to straightforward recursive scheme for the coefficients $a_0, a_{-1}, \ldots$ of $L_n^{\frac{1}{n}}$, from which these coefficients can be calculated as differential expressions of $u_{n-1}, u_{n-2}, \ldots, u_1, \boldsymbol{q}, \boldsymbol{q}^*$. Higher fractional powers $L_n^{m/n}$ of L_n are then calculated as powers $L_n^{m/n} = (L_n^{1/n})^m$ of this "basic root". By construction, the first question with $m = 1$ in the hierarchy (14) is given by $L_{n_{t_1}} = [\mathcal{D}, L_n] = \frac{\partial L_n}{\partial x}$, so that the first time variable t_1 may be identified with the underlying space variable x.

4 Some examples of equations from the $\mathcal{D}$HcmKP flow

Let us $n = 1$. For $L_1 = \mathcal{D} - i\boldsymbol{q}\mathcal{M}\mathcal{D}^{-1}\boldsymbol{q}^*\mathcal{D}$ the first nontrivial equations in (15) are given by ($\alpha_2 = i$)

$$i\boldsymbol{q}_{t_2} = \boldsymbol{q}_{xx} - 2i\boldsymbol{q}\mathcal{M}\boldsymbol{q}^*\boldsymbol{q}_x, \tag{16}$$

which are the first equations in the multicomponent modified nonlinear Schrödinger hierarchy discussed in [9].

$n = 2$. For $L_2 = \mathcal{D}^2 + iu\mathcal{D} - \boldsymbol{q}\mathcal{M}\mathcal{D}^{-1}\boldsymbol{q}^*\mathcal{D}$ we obtain

$$i\boldsymbol{q}_{t_2} = \boldsymbol{q}_{xx} + iu\boldsymbol{q}_x, \qquad u_{t_2} = 2(\boldsymbol{q}\mathcal{M}\boldsymbol{q}^*)_x, \qquad \alpha_2 = i, \tag{17}$$

$$\boldsymbol{q}_{t_3} = \boldsymbol{q}_{xxx} + \frac{3}{2}iu\boldsymbol{q}_{xx} - \left(\frac{3}{8}u^2 + \frac{3}{2}\boldsymbol{q}\mathcal{M}\boldsymbol{q}^* - \frac{3}{4}iu_x\right)\boldsymbol{q}_x,$$

$$u_{t_3} = \frac{1}{4}u_{xxx} + \frac{3}{8}u^2u_x - \frac{3}{2}\left(\boldsymbol{q}\mathcal{M}\boldsymbol{q}^*u\right)_x, \qquad \alpha_3 = 1. \tag{18}$$

This represents the modified KdV hierarchy coupled with its eigenfunctions. The system (17) is the new multicomponent integrable model of Yajima–Oikawa type [9, 10]. The next higher flow in this hierarchy has the following form ($\alpha_4 = -i$):

$$u_{t_4} = 2\left[\boldsymbol{q}\mathcal{M}\boldsymbol{q}^*_{xx} + \boldsymbol{q}_{xx}M\boldsymbol{q}^* + (\boldsymbol{q}\mathcal{M}\boldsymbol{q}^*)^2\right]_x, \qquad i\boldsymbol{q}_{t_4} = \left(L_2^2\right)_{>0}(\boldsymbol{q}). \tag{19}$$

5 Method of integration of the Lax equation from the $\mathcal{D}$HcmKP hierarchy

There are many mathematical and physical problems associated with the $\mathcal{D}$HcmKP hierarchy. However, the most important one, may be, is finding the soliton solutions for the equations from this hierarchy. We have shown (see Lemma 1) that the $\mathcal{D}$-unital M$\mathcal{D}$O W transforms $\mathcal{D}$-Hermitian operator L into $\mathcal{D}$-Hermitian operator $\hat{L}$ by the dressing transformation $L \to WLW^{-1} := \hat{L}$. Now, we want to extend the previous results to the equations from the $\mathcal{D}$HcmKP hierarchy.

Theorem 2. *Let $\varphi = (\varphi_1, \dots, \varphi_K)$, $K \in \mathbb{N}$ be a smooth fast decreasing on the $-\infty$ complex value K-component vector-function of variable $x \in \mathbb{R}$ and an evolution parameter $t_2 \in \mathbb{R}$ which satisfy additional conditions:*

a) φ be a solution of the equation $i\varphi_{t_2} = \varphi_{xx}$,

b) $\varphi_x = \varphi\Lambda$, where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_K) = \mathrm{const}$; $\lambda_j := \lambda_{j1} + i\lambda_{j2} \in \mathbb{C}$; $\lambda_{j1} > 0$, $j = \overline{1, K}$.

Then the vector-function

$$\boldsymbol{q} := \varphi\Omega^{-1} = \varphi\left(C + \int_{-\infty}^{x} \varphi^*\varphi dx\Lambda\right)^{-1} \tag{20}$$

is a solution of the mNSE (16) with the matrix $\mathcal{M} = -i(C\Lambda + \Lambda^* C)$, where $C^* = -C$ is a skew-Hermitian $(K \times K)$ complex matrix.

Proof. The proof is constructed by direct calculation. Using the lemmas we get

$$\begin{aligned} &L_0 := \mathcal{D} \to L := W\mathcal{D}W^{-1} = \mathcal{D} - \varphi\Omega^{-1}(C\Lambda - \Lambda^* C^*)\mathcal{D}^{-1}\Omega^{*-1}\varphi^*\mathcal{D}, \\ &M_0 := i\partial_{t_2} - \mathcal{D}^2 \to M := WM_0W^{-1} = \left(L^2\right)_{>0}, \end{aligned} \tag{21}$$

and from the trivial equation $[L_0, M_0] = 0$ we obtain that $[L, M] = 0$. ■

Corollary 1. *Let $K \geq l \in \mathbb{N}$ and matrix $C = \frac{i}{2}\,\mathrm{diag}\left(\frac{\mu_1}{\lambda_{11}}, \dots, \frac{\mu_l}{\lambda_{l1}}, 0, \dots, 0\right) \in \mathrm{Mat}_{K\times K}(i\mathbb{R})$.*

Then the function $\boldsymbol{q} = (q_1, \dots, q_l)(x, t_2)$, where $\boldsymbol{q} := \varphi\Omega^{-1}$ and

$$q_j = (-1)^{K-j}\frac{\begin{vmatrix} \Omega_{(j)} \\ \varphi \end{vmatrix}}{\Omega}, \qquad j = 1, \dots, l \tag{22}$$

is a solution of the l-component mNSE

$$i\boldsymbol{q}_{t_2} = \boldsymbol{q}_{xx} - 2i\sum_{j=1}^{l} \mu_j |q_j|^2 \boldsymbol{q}_x. \tag{23}$$

Here $\Omega_{(j)}$ is obtained from Ω by deletion of j-line and $|\Omega| := \det\Omega$. In order to prove (22) we use the well-known algebraic equality for framed determinant

$$\det\begin{pmatrix} \Omega & \varphi^* \\ \varphi & \alpha \end{pmatrix} := \begin{vmatrix} \Omega & \varphi^* \\ \varphi & \alpha \end{vmatrix} = \alpha\det\Omega - \varphi\Omega^c\varphi^*, \qquad \alpha \in \mathbb{C}, \tag{24}$$

where Ω^c is the matrix of cofactors, and then

$$q_j = \varphi\Omega^{-1}e_j^T, \qquad e_k := (e_{k_1}, e_{k_2}, \dots, e_{k_K}), \qquad e_{k_m} = \delta_k^m.$$

For $l = 1$ we obtain from the formula (22) the K-soliton solution for the scalar mNSE [11, 12]

$$q = (-1)^{K+1} \frac{\begin{vmatrix} \Omega_{(1)} \\ \varphi \end{vmatrix}}{|\Omega|}, \qquad \Omega = (w_{mn}), \qquad m, n = \overline{1, K},$$
$$\varphi_m = \varphi_{m_0} \exp\left\{\lambda_m x - i\lambda_m^2 t_2\right\}, \qquad \varphi_{m_0} = \mathrm{const},$$
$$w_{mn} = \frac{i\mu}{2\lambda_{11}} \delta_1^{mn} + \frac{\lambda_n}{\bar{\lambda}_m + \lambda_n} \bar{\varphi}_{m0} \varphi_{n0} \exp\left\{\left(\bar{\lambda}_m + \lambda_n\right) x + i\left(\bar{\lambda}_m^2 - \lambda_n^2\right) t_2\right\},$$
$$iq_{t_2} = q_{xx} - 2i\mu |q|^2 q_x. \tag{25}$$

In particular, if $K = 1$, then the previous formulas represent a one-soliton solution for the (25):

$$q = \frac{2\lambda_{11}\varphi_0 \exp\{\lambda x - i\lambda^2 t_2\}}{i\mu + |\varphi_0|^2 \exp\{2\lambda_{11} x + 4\lambda_{11}\lambda_{12} t_2\}},$$

where $\varphi_0 := \varphi_{10}$, $\lambda := \lambda_1 = \lambda_{11} + i\lambda_{12} := \operatorname{Re}\lambda + i \operatorname{Im}\lambda$.

6 Conclusion

In conclusion, we hope that the method of integration of Lax equations with $\mathcal{D}$-Hermitian reductions presented here will be generalized to other nonlinear models from the $\mathcal{D}$HcmKP hierarchy (and in the matrix case too). Similar generalizations for Hermitian reductions in cKP hierarchy [7, 8] were considered in the papers [13, 14], but these results were obtained using the methods from the article [15].

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On Integrable Quantum System of Particles with Chern–Simons Interaction

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The Gibbs (grand-canonical) reduced density matrices (RDMs) are calculated in the thermodynamic weak-coupling limit for the non-relativistic spinless system of particles, interacting via collective electromagnetic pair vector Chern–Simons potential, and characterized by the Maxwell–Boltzmann (MB) statistics.

1 Introduction

The Chern–Simons 2-d quantum system of n nonrelativistic spinless identical particles of unit mass is described by the Hamiltonian $\dot{H}_n$, defined on $C^\infty(\mathbb{R}^{2n}\setminus \cup_{j,k}\,(x_j = x_k))$

$$\dot{H}_n = \frac{1}{2}\sum_{j=1}^{n} ||p_j - a_j(X_n)||^2, \qquad a_j^\nu(X_n) = \epsilon^{\nu\mu}\partial_{\mu,j}U_C(X_n), \quad \nu,\mu = 1,2, \tag{1}$$

$$U_C(X_n) = g\sum_{1\le k<j\le n} e_j e_k \ln|x_j - x_k|, \quad X_n = (x_1,\ldots,x_n)\in\mathbb{R}^{2n}, \quad p_j^\mu = i^{-1}\partial_{\mu,j} = i^{-1}\frac{\partial}{\partial x_j^\mu},$$

where $||v||^2 = \left(v^1\right)^2 + \left(v^2\right)^2$, ϵ is the skew symmetric tensor and the repeating index implies a summation, real number e_j (a charge) takes values in a finite set $E_{\{c\}}$ from $\mathbb{R}$.

Differentiating the equality $f\left(f^{-1}(x)\right) = x$ we derive the formula

$$\frac{df^{-1}(x)}{dx} = \left(\frac{df(y)}{dy}\right)^{-1}, \qquad y = f^{-1}(x).$$

From this equality for $f(x) = \tan x$ and the equality $\frac{d}{dx}\tan x = 1+\tan^2(x)$ the following relation is derived

$$\frac{\partial}{\partial x^\nu}\arctan\frac{x^2}{x^1} = \epsilon^{\nu\mu}x^\mu|x|^{-2} = \epsilon^{\nu\mu}\frac{\partial}{\partial x^\mu}\ln|x|.$$

This means that CS system is almost(quasi-)integrable, that is

$$\dot{H}_n = e^{i\hat{U}}\dot{H}_n^0 e^{-i\hat{U}},$$

where $-2\dot{H}_n^0$ is the 2n-dimensional Laplacian $-2H_n^0$ restricted to $C^\infty(\mathbb{R}^{2n}\setminus\cup_{j,k}(x_j = x_k))$ and $\hat{U}$ is the operator of multiplication by $U(X_n)$,

$$U(X_n) = g\sum_{1\le k<j\le n} e_j e_k \phi(x_j - x_k), \qquad \phi(x_j - x_k) = \arctan\frac{x_j^2 - x_k^2}{x_j^1 - x_k^1}$$

As a result, there exists the simplest self-adjoint extension H_n of $\dot{H}_n$

$$H_n = e^{i\hat{U}}H_n^0 e^{-i\hat{U}}. \tag{2}$$

with the domain $D(H_n) = e^{i\hat{U}} D\left(H_n^0\right)$. Another self-adjoint extension of $\dot{H}_n$ is given by (2) in which, instead of H_n^0, another self-adjoint extension of $\dot{H}_n^0$ is considered.

The CS system of particles with different statistics has been studied by many authors [1, 2, 3, 4] since it can be derived (formally) in 3-d topological electrodynamics (its Lagrangian contains CS term) in the limit of the vanishing Maxwell term (the same is true for its relativistic version). There is a hope it can give a new mechanism of superconductivity, superfluidity and P, T violation.

2 Main result

Let $\Lambda \in \mathbb{R}^2$ be a compact set and assume the Dirichlet boundary conditions on the boundary $\partial\Lambda$. For the inverse temperature β, and the activity z_{e_s} of the particles with the charge e_s, the Gibbs (grand-canonical, equilibrium) RDMs are given by

$$\begin{aligned} &\rho^\Lambda(X_m|Y_m) \\ &= Z_{(e)_m} \Xi_\Lambda^{-1} \sum_{n\geq 0} \prod_{s=1}^{n} \sum_{e'_s} \frac{z_{e'_s}^{n_s}}{n_s!} \int_{\Lambda^n} dX'_n \exp\{i[U(X_m, X'_n) - U(Y_m, X'_n)]\} P_{0(\Lambda)}^\beta (X_m, X'_n|Y_m, X'_n), \end{aligned}$$

where Ξ_Λ is the grand partition function (it coincides with the numerator in the r.h.s. of this equality for the case $m = 0$, i.e. when there are no X_m and Y_m), $P_{0(\Lambda)}^\beta(X_m|Y_m)$ is the kernel of the semigroup, whose infinitesimal generator concides with $H_{n,\Lambda}^0$ ($-2H_{n,\Lambda}^0$ is the Laplacian in Λ^n with the Dirichlet boundary condition on the boundary $\partial\Lambda^n$), $Z_{(e)_m} = \prod_{s=1}^{m} z_{e_s}$, and the summation in e_s is performed over $E_{\{c\}}$.

$$P_{0(\Lambda)}^\beta(X_n|Y_n) = \prod_{j=1}^{n} P_{0(\Lambda)}^\beta(x_j|y_j), \qquad X_m = \left(X_m^1, X_m^2\right), \; Y_m = \left(Y_m^1, Y_m^2\right) \in \mathbb{R}^{2m}, \tag{3}$$

$P_{0(\Lambda)}^\beta(x|y)$ is the transition probability of the 2-dimensional free diffusion proccess with the Dirichlet boundary condition on $\partial\Lambda$.

$$P_{0(\Lambda)}^\beta(x|y) = \int P_{x,y}^\beta(d\omega)\chi_\Lambda(\omega),$$

$P_{x,y}(d\omega)$ is the conditional Wiener measure and $\chi_\Lambda(\omega)$ is the characteristic function of the paths that are inside Λ.

From the equality

$$U(X_m, X'_n) = U(X_m) + \sum_{j=1}^{m}\sum_{k=1}^{n} \phi(x_j - x'_k) e_j e'_k + U(X'_n)$$

we obtain

$$\begin{aligned} \rho^\Lambda(X_m|Y_m) &= Z_{(e)_m} \Xi_\Lambda^{-1} \exp\{i[U(X_m) - U(Y_m)]\} P_{0(\Lambda)}^\beta(X_m|Y_m) \\ &\times \sum_{n\geq 0} \sum_{e'_s} \frac{z_{e'_s}^{n_s}}{n_s!} \int_{\Lambda^n} dX'_n \prod_{k=1}^{n} \exp\left\{ i \sum_{j=1}^{m} e_j e_k (\phi(x_j - x'_k) - \phi(y_j - x'_k)) \right\} P_{0(\Lambda)}^\beta(x'_k|x'_k). \end{aligned}$$

As a result

$$\rho^\Lambda(X_m|Y_m) = Z_{(e)_m} \exp\{i[U(X_m) - U(Y_m)] + G_\Lambda(X_m|Y_m)\} P_{0(\Lambda)}^\beta(X_m|Y_m), \tag{4}$$

$$G_\Lambda(X_m|Y_m) = \sum_e z_e \int\limits_\Lambda P^\beta_{0(\Lambda)}(x|x) \left[\exp\left\{ i\sum_{j=1}^m ee_j(\phi(x_j-x)-\phi(y_j-x))\right\} - 1\right] dx. \qquad (5)$$

Here we used the equality

$$\Xi_\Lambda = \exp\left\{ \sum_e z_e \int\limits_\Lambda P^\beta_{0(\Lambda)}(x|x)dx \right\}.$$

With the help of the equality

$$\exp\left\{ i \arctan \frac{x^2}{x^1}\right\} = \frac{x}{|x|} = \left(\frac{x}{x^*}\right)^{\frac{1}{2}}, \qquad x = x^1 + ix^2,$$

we derive

$$\exp\left\{ i\sum_{j=1}^m ee_j(\phi(x_j-x)-\phi(y_j-x))\right\} = \prod_{j=1}^m \left(\frac{(x-x_j)(x^*-y_j^*)}{(x^*-x_j^*)(x-y_j)}\right)^{\frac{1}{2}gee_j} = G_x(X_m|Y_m).$$

We have to use the Taylor expansions for $|x|<1$

$$(1-x)^g = 1 - gx + \frac{g(g-1)}{2}x^2 + \sum_{n\geq 3} C_n^g x^n = \sum_{n\geq 0} C_n^g x^n,$$

$$(1-x)^{-g} = 1 + gx + \frac{g(g+1)}{2}x^2 + \sum_{n\geq 3} C_n^{-g} x^n = \sum_{n\geq 0} C_n^{-g} x^n,$$

As a result for $g' = \frac{1}{2}gee_j$, $g' \notin \mathbb{Z}$, $\left|\frac{x_j}{x}\right| < 1$, $\left|\frac{y_j}{x}\right| < 1$

$$\left(\frac{x-x_j}{x^*-x_j^*}\right)^{g'} = \left(\frac{x}{x^*}\right)^{g'} \left(\frac{1-\frac{x_j}{x}}{1-\frac{x_j^*}{x^*}}\right)^{g'} = \left(\frac{x}{x^*}\right)^{g'} \left\{ 1 + g'\left(-\frac{x_j}{x} + \frac{x_j^*}{x^*}\right) + \frac{g'2}{2}\left(\frac{x_j^2}{x^2} + \frac{x_j^{*2}}{x^{*2}}\right)\right.$$
$$\left. + \frac{g'}{2}\left(\frac{x_j^{*2}}{x^{*2}} - \frac{x_j^2}{x^2}\right) - g'^2\left|\frac{x_j}{x}\right|^2 + \sum_{n_1^+ n_1^- \geq 3} C_{n_1^+}^{-g'} C_{n_1^-}^{g'} \left(\frac{x_j^*}{x^*}\right)^{n_1^+} \left(\frac{x_j}{x}\right)^{n_1^-}\right\}.$$

Applying this formula we deduce

$$\left(\frac{(x-x_j)(x^*-y_j^*)}{(x^*-x_j^*)(x-y_j)}\right)^{g'} = 1 + G'_x(x_j|y_j)$$
$$+ \sum_{n_1^+ + \dots + n_2^- \geq 3} C_{n_1^+}^{-g'} C_{n_1^-}^{g'} C_{n_2^+}^{-g'} C_{n_2^-}^{g'} \left(\frac{x_j^*}{x^*}\right)^{n_1^+} \left(\frac{x_j}{x}\right)^{n_1^-} \left(\frac{y_j^*}{x^*}\right)^{n_2^-} \left(\frac{y_j}{x}\right)^{n_2^+},$$

where

$$G'_x(x_j|y_j) = g'\left(-\frac{x_j-y_j}{x} + \frac{x_j^*-y_j^*}{x^*}\right) - g'^2\left(\frac{x_j}{x} - \frac{x_j^*}{x^*}\right)\left(\frac{y_j}{x} - \frac{y_j^*}{x^*}\right)$$
$$+ \frac{g'^2}{2}\left(\frac{x_j^2+y_j^2}{x^2} + \frac{x_j^{*2}+y_j^{*2}}{x^{*2}}\right) + \frac{g'}{2}\left(\frac{y_j^2-x_j^2}{x^2} - \frac{y_j^{*2}-x_j^{*2}}{x^{*2}}\right) - g'^2\left[\left|\frac{x_j}{x}\right|^2 + \left|\frac{y_j}{x}\right|^2\right].$$

As a result

$$G_x(X_m|Y_m) = 1 + G^0_{x,e}(X_m|Y_m) + \sum_{j=1}^{m} G'_x(x_j|y_j), \tag{6}$$

where

$$G^0_{x,e}(X_m|Y_m) = \sum_{n^+_{1,1}+\dots+n^-_{2,m}\geq 3}\prod_{j=1}^{m} C^{-g'}_{n^+_{1,j}} C^{g'}_{n^-_{1,j}} C^{-g'}_{n^+_{2,j}} C^{g'}_{n^-_{2,j}} \left(\frac{x^*_j}{x^*}\right)^{n^+_{1,j}} \left(\frac{x_j}{x}\right)^{n^-_{1,j}} \left(\frac{y^*_j}{x^*}\right)^{n^-_{2,j}} \left(\frac{y_j}{x}\right)^{n^+_{2,j}}.$$

It can be checked that

$$-g'^2\left(\frac{x_j}{x} - \frac{x^*_j}{x^*}\right)\left(\frac{y_j}{x} - \frac{y^*_j}{x^*}\right) + \frac{g'^2}{2}\left(\frac{x^2_j + y^2_j}{x^2} + \frac{x^{*2}_j + y^{*2}_j}{x^{*2}}\right) - g'^2\left[\left|\frac{x_j}{x}\right|^2 + \left|\frac{y_j}{x}\right|^2\right]$$
$$= -g'^2\left|\frac{x_j - y_j}{x}\right|^2 + \frac{g'^2}{2}\left(\frac{(x_j - y_j)^2}{x^2} + \frac{(x^*_j - y^*_j)^2}{x^{*2}}\right). \tag{7}$$

This yields

$$G'_x(x_j|y_j) = -g'^2\left|\frac{x_j - y_j}{x}\right|^2 + G^-_x(x_j|y_j), \qquad G^-_x(x_j|y_j) = g'\left(-\frac{x_j - y_j}{x} + \frac{x^*_j - y^*_j}{x^*}\right)$$
$$+ \frac{g'^2}{2}\left(\frac{(x_j - y_j)^2}{x^2} + \frac{(x^*_j - y^*_j)^2}{x^{*2}}\right) + \frac{g'}{2}\left(\frac{y^2_j - x^2_j}{x^2} - \frac{y^{*2}_j - x^{*2}_j}{x^{*2}}\right). \tag{8}$$

Let $l^+_m = \max(|x_j|, |y_j|, j = 1, \dots, m)$. Let $\Lambda = B_L$ then

$$G_\Lambda(X_m|Y_m) = \sum_e z_e \left\{ \int\limits_{|x|\leq 2l^+_m} P^\beta_{0(\Lambda)}(x|x)\left[\prod_{j=1}^{m}\left(\frac{x^* - x^*_j}{x - x_j}\frac{x - y_j}{x^* - y^*_j}\right)^{\frac{1}{2}gee_j} - 1\right]dx \right.$$
$$\left. + \int\limits_{2l^+_m\leq|x|\leq L} P^\beta_{0(\Lambda)}(x|x)G^0_x(X_m|Y_m)dx + \int\limits_{2l^+_m\leq|x|\leq L} P^\beta_{0(\Lambda)}(x|x)G'_x(X_m|Y_m)dx \right\}. \tag{9}$$

For $|x| \geq 2l^+_m$ we have the bound

$$G^0_{x,e}(X_m|Y_m) \leq \frac{(2l^+_m)^3}{|x|^3}2^{4|g'|m}.$$

Here we used the inequalities

$$\left|\left(\frac{x^*_j}{x^*}\right)^{n^+_{1,j}} \left(\frac{x_j}{x}\right)^{n^-_{1,j}} \left(\frac{y^*_j}{x^*}\right)^{n^-_{2,j}} \left(\frac{y_j}{x}\right)^{n^+_{2,j}}\right| \leq \frac{(2l^+_m)^3}{|x|^3}2^{-(n^+_{1,j}+n^-_{1,j}+n^+_{2,j}+n^-_{2,j})},$$
$$\left|C^{+(-)g}_n\right| \leq C^{-|g|}_n > 0.$$

After applying them we enlarge the sum in the expression for $G^0_{x,e}$ to the sum over $(\mathbb{Z}^+)^{4m}$.

Since $P_\Lambda(x|x)$ tends to $(2\pi\beta)^{-1}$ the first and the second terms in (9) have limits when L tends to infinity. We have only to calculate the third term. Let us show that

$$\int\limits_{r\leq|x|\leq L} G^-_x(x'|y')dx = 0. \tag{10}$$

For arbitrary r, L, $v = v^1 + iv^2$, $B = B_L \backslash B_r$ we have

$$\int_B \left(\frac{v}{x} - \frac{v^*}{x^*} \right) dx = -2i \left[v^1 \int_B \frac{x^2}{|x|^2} dx - v^2 \int_B \frac{x^1}{|x|^2} dx \right] = 0,$$

$$\int_B \left(\frac{v}{x^2} + \frac{v^*}{x^{*2}} \right) dx = 2 \left\{ v^1 \int_B \frac{\left(x^1\right)^2 - \left(x^2\right)^2}{|x|^4} dx + 2v^2 \int_B \frac{x^1 x^2}{|x|^4} dx \right\} = 0,$$

$$\int_B \left(\frac{v}{x^2} - \frac{v^*}{x^{*2}} \right) dx = 2i \left\{ v^2 \int_B \frac{\left(x^1\right)^2 - \left(x^2\right)^2}{|x|^4} dx - 2v^1 \int_B \frac{x^1 x^2}{|x|^4} dx \right\} = 0.$$

All the above integrals are zero since the all the functions change signs when either a sign of one of the variables is changed, or a permutation is done.

As a result

$$\int_{r<|x|\le L} G'_x(X_m|Y_m)dx = -\frac{1}{4} g^2 \left(\sum_e z_e e^2 \right) N_{r,L} \sum_{j=1}^m e_j^2 |x_j - y_j|^2. \tag{11}$$

The integral in the right-hand-side of this equality diverges as $2 \ln L$. For $g' = k$, $k \in \mathbb{Z}$ we obtain (6) in which we have to put $C_n^k = 0$ for $n > k$, $C_n^k = \frac{k!}{(n-k)!n!}$.

From (9), (11) we derive

$$G_\Lambda(X_m|Y_m) = \sum_e z_e \left\{ \int_{|x|\le 2l_m^+} P^\beta_{0(\Lambda)}(x|x) \left[\prod_{j=1}^m \left(\frac{x^* - x_j^*}{x - x_j} \frac{x - y_j}{x^* - y_j^*} \right)^{\frac{1}{2} g e e_j} - 1 \right] dx \right.$$

$$\left. + \int_{2l_m^+ \le |x| \le L} P^\beta_{0(\Lambda)}(x|x) G^0_x(X_m|Y_m) dx - \frac{1}{4} g^2 \left(\sum_e z_e e^2 \right) N_{2l_m^+,L} \sum_{j=1}^m e_j^2 |x_j - y_j|^2 \right\}. \tag{12}$$

Using the equalities

$$\lim_{L\to\infty} P^\beta_{0(B_L)}(x|x) = (2\pi\beta)^{-1}, \qquad \lim_{L\to\infty} (\ln L)^{-1} N_{r,L} = \beta^{-1}$$

and the fact that $G^0_{x,e}$ is an integrable function in $\Lambda \backslash \{0\}$ and we derive the following result.

Theorem 1. *Let Λ coincide with B_L.*

I. If $g^2 = g_0^2 (\ln L)^{-1}$ then

$$\lim_{L\to\infty} \rho^{B_L}(X_m|Y_m) = Z_{(e)_m} \exp\left\{ i[U(X_m) - U(Y_m)] \right\} P_0^\beta(X_m|Y_m)$$

$$\times \exp\left\{ -\frac{1}{4\beta} g_0^2 \left(\sum_e z_e e^2 \right) \sum_{j=1}^m e_j^2 |x_j - y_j|^2 \right\}.$$

II. If $\lim_{L\to\infty} g^2 \ln L = 0$ then

$$\lim_{L\to\infty} \rho^{B_L}(X_m|Y_m) = Z_{(e)_m} \exp\{ i[U(X_m) - U(Y_m)] \} P_0^\beta(X_m|Y_m).$$

III. If $\lim_{L\to\infty} g^2 \ln L = \infty$ then

$$\lim_{L\to\infty} \rho^{B_L}(X_m|Y_m) = 0, \qquad x_j \ne y_j.$$

The mean-field type limit for the quantum CS system does not exit contrary to the classical CS particle system [8]. But there is a similarity in the behavior of the RDMs in the weak coupling limit and the Gibbs correlation functions in the mean-field type limit.

Description of integrable systems with magnetic interaction in the thermodynamic limit can be found in [5, 6, 7, 9, 10].

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Integrable Hamiltonian Systems via Quasigraded Lie Algebras

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In the present paper we construct integrable Hamiltonian systems of the Euler–Arnold type associated with infinite-dimensional quasigraded Lie algebras of matrix valued functions on higher genus curves. In details is considered the case when underlying matrix Lie algebra coincide with $gl(n)$. Corresponding generalizations of Steklov integrable systems as long as $gl(n)$ analogues of Clebsh integrable systems are obtained.

1 Introduction

The main purpose of the present paper is to construct new integrable Hamiltonian systems of the Euler–Arnold type. Our approach to the solution of this problem is based on the usage of infinite-dimensional Lie algebras. Traditionally Lie theoretical explanation of the integrability of Euler–Arnold equations on finite-dimensional Lie algebras is based on on the loop algebras and Kostant–Adler scheme [1, 2]. In the papers [3, 4] it was shown, that in similar way integrable Euler–Arnold equations on the algebra $so(3)$ and some its extensions could be obtained from the infinite-dimensional Lie algebras of the special elliptic functions with the values in $so(3)$. In our previous papers [5, 6] we generalized construction described in [4] for the case of classical matrix algebras of higher ranks. Growth of the rank of algebra requires automatic growth of the genus of the curve. In the result we have obtained algebras of $gl(n)$-, $so(n)$- and $sp(n)$-valued functions on the algebraic curves of higher genus. The most important property of the discovered algebras is that they admit Kostant–Adler scheme, and hence, could be used to construct new integrable systems. Using them we have constructed new integrable Hamiltonian systems on the Lie algebras $so(n) \oplus so(n)$, $so(n) + so(n)$, $e(n)$ that generalize integrable systems of Steklov–Veselov, Steklov–Liapunov, and Clebsh [5, 6, 7].

In the present paper we consider the case when underlying matrix Lie algebra coincides with $\mathfrak{g} = gl(n)$. We show that there exist precise integrable $gl(n)$-analogues of Steklov–Veselov and Steklov–Liapunov systems on $gl(n) \oplus gl(n)$, $gl(n) + gl(n)$ along with $gl(n)$ analogue of the Clebsh system on $gl(n-1) + \mathbb{R}^{2n}$. It is necessary to notice that same results are valid for the case of $\mathfrak{g} = sp(n)$. We do not adduce them here due to the restricted size of the article.

2 Quasi-graded algebras on higher genus curves

2.1 Construction

1. Higher genus curve embedded in $\mathbb{C}^n$. Let us consider in the space $\mathbb{C}^n$ with the coordinates $w_1, w_2, \ldots, w_n$ the following system of quadrics:

$$w_i^2 - w_j^2 = a_j - a_i, \qquad i, j = \overline{1, n}, \tag{1}$$

where a_i are arbitrary complex numbers. Rank of this system is $n-1$, so substitution:

$$w_i^2 = w - a_i, \qquad y = \prod_{i=1}^{n} w_i, \qquad y^2 = \prod_{i=1}^{n} w_i^2$$

solves these equations and defines the equation of the hyperelliptic curve $\mathcal{H}$.

2. Classical Lie algebras. Let $\mathfrak{g}$ denotes one of the classical matrix Lie algebras $gl(n)$, $so(n)$ and $sp(n)$ over the field of the complex numbers. We will need explicit form of their bases. Let $I_{i,j} \in \mathrm{Mat}\,(n, C)$ be a matrix defined as:

$$(I_{ij})_{ab} = \delta_{ia}\delta_{jb}.$$

Evidently, a basis in the algebra $gl(n)$ could be built from the matrices $X_{ij} \equiv I_{ij}$, $i, j \in 1, \ldots, n$. The commutation relations in $gl(n)$ will have the standard form:

$$[X_{i,j}, X_{k,l}] = \delta_{k,j}X_{i,l} - \delta_{i,l}X_{k,j}.$$

The basis in the algebra $so(n)$ could be chosen as: $X_{ij} \equiv I_{ij} - I_{i,j}$, $i, j \in 1, \ldots, n$, with "skew-symmetry" property $X_{ij} = -X_{ji}$ and the following commutation relations:

$$[X_{i,j}, X_{k,l}] = \delta_{k,j}X_{i,l} - \delta_{i,l}X_{k,j} + \delta_{j,l}X_{k,i} - \delta_{k,i}X_{j,l}.$$

The basis in the algebra $sp(n)$ we choose as $X_{ij} = I_{ij} - \epsilon_i\epsilon_j I_{-i,-j}$, $|i|, |j| \in 1, \ldots, n$, with the property $X_{i,j} = -\epsilon_i\epsilon_j X_{-j,-i}$, where $\epsilon_j = \operatorname{sign} j$ and commutation relations:

$$[X_{i,j}, X_{k,l}] = \delta_{k,j}X_{i,l} - \delta_{i,l}X_{k,j} + \epsilon_i\epsilon_j(\delta_{j,-l}X_{k,-i} - \delta_{k,-i}X_{-j,l}).$$

3. Algebras on the curve. For the basic elements X_{ij} of all three algebras $gl(n)$, $so(n)$ and $sp(n)$ and arbitrary $n \in \mathbb{Z}$ we introduce the following algebra-valued functions on the curve $\mathcal{H}$, or to be more precise on its ramified covering:

$$X_{ij}^n = X_{ij} \otimes w^n w_i w_j.$$

The next theorem holds true:

Theorem 1. *(i) Elements X_{ij}^n form $n \in \mathbb{Z}$ quasi-graded Lie algebra $\widetilde{\mathfrak{g}}_{\mathcal{H}}$ with the following commutation relations:*

$$1)\ [X_{ij}^n, X_{kl}^m] = \delta_{kj}X_{il}^{n+m+1} - \delta_{il}X_{kj}^{n+m+1} + a_i\delta_{il}X_{kj}^{n+m} - a_j\delta_{kj}X_{il}^{n+m} \quad \textit{for the}\ \ gl(n); \tag{2a}$$

$$\begin{aligned}2)\ [X_{ij}^n, X_{kl}^m] &= \delta_{kj}X_{il}^{n+m+1} - \delta_{il}X_{kj}^{n+m+1} + \delta_{jl}X_{ki}^{n+m+1} - \delta_{ik}X_{jl}^{n+m+1} \\ &+ a_i\delta_{il}X_{kj}^{n+m} - a_j\delta_{kj}X_{il}^{n+m} + a_i\delta_{ik}X_{jl}^{n+m} - a_j\delta_{jl}X_{ki}^{n+m} \quad \textit{for the}\ \ so(n);\end{aligned} \tag{2b}$$

$$\begin{aligned}3)\ [X_{ij}^n, X_{kl}^m] &= \delta_{kj}X_{il}^{n+m+1} - \delta_{il}X_{kj}^{n+m+1} + \epsilon_i\epsilon_j\left(\delta_{j-l}X_{k-i}^{n+m+1} - \delta_{i-k}X_{j-l}^{n+m+1}\right) \\ &+ a_i\delta_{il}X_{kj}^{n+m} - a_j\delta_{kj}X_{il}^{n+m} + a_i\epsilon_i\epsilon_j\left(a_i\delta_{i-k}X_{j-l}^{n+m} - a_j\delta_{j-l}X_{k-i}^{n+m}\right) \quad \textit{for the}\ sp(n).\end{aligned} \tag{2c}$$

(ii) Algebra $\widetilde{\mathfrak{g}}_{\mathcal{H}}$ as a linear space admits a decomposition into the direct sum of two subalgebras: $\widetilde{\mathfrak{g}}_{\mathcal{H}} = \widetilde{\mathfrak{g}}_{\mathcal{H}}^+ + \widetilde{\mathfrak{g}}_{\mathcal{H}}^-$, where subalgebras $\widetilde{\mathfrak{g}}_{\mathcal{H}}^+$ and $\widetilde{\mathfrak{g}}_{\mathcal{H}}^-$ are generated by the elements X_{ij}^0, and X_{ij}^{-1} correspondingly.

Example 1. Let $\mathfrak{g} = so(3)$. In this case putting $X_k \equiv \epsilon_{ijk}X_{ij}$, we obtain the following commutation relations:

$$[X_i^n, X_j^m] = \epsilon_{ijk}X_k^{n+m+1} + \epsilon_{ijk}a_kX_k^{n+m}.$$

Remark 1. From the item (i) of the theorem it follows that in the rational degeneration, i.e. when $a_i = 0$, $\widetilde{\mathfrak{g}}_{\mathcal{H}} = \widetilde{\mathfrak{g}}$, where $\widetilde{\mathfrak{g}}$ is an ordinary loop algebra.

2.2 Coadjoint representation

To define the coadjoint representation we have to define $\widetilde{\mathfrak{g}}^*_{\mathcal{H}}$. For our purposes it will be convenient to identify $\widetilde{\mathfrak{g}}^*_{\mathcal{H}}$ with $\widetilde{\mathfrak{g}}_{\mathcal{H}}$ as linear spaces. In order to do this we will define pairing between $L(w) \in \widetilde{\mathfrak{g}}^*_{\mathcal{H}}$ and $X(w) \in \widetilde{\mathfrak{g}}_{\mathcal{H}}$ in the following way:

$$\langle X(w), L(w)\rangle_f = c_n \operatorname*{res}_{w=0} y^{-2}(w)(X(w)|L(w)), \tag{3}$$

where $f(w)$ is arbitrary function on the curve $\mathcal{H}$. It is easy to show that element dual to X_{ij}^{-m} with respect to this pairing is $Y_{ij}^m \equiv (X_{ij}^{-m})^* = \frac{w^{m-1}y^2(w)}{w_i w_j} X^*_{ij}$. Hence the general element of the dual space has the following form:

$$L(w) = \sum_{m\in Z}\sum_{i,j=1}^{n} l_{ij}^{(m)} \frac{w^{m-1}y^2(w)}{w_i w_j} X^*_{ij}. \tag{4}$$

Coadjoint action of algebra $\widetilde{\mathfrak{g}}_{\mathcal{H}}$ on its dual space $\widetilde{\mathfrak{g}}^*_{\mathcal{H}}$ coincides with commutator:

$$\mathrm{ad}^*_{X(w)} L(w) = [L(w), X(w)]. \tag{5}$$

From the explicit form of coadjoint action (5) follows the next statement:

Proposition 1. *Functions $I^k_m(L(w)) = \operatorname*{res}_{w=0} w^{-m-1} \operatorname{Tr} L(w)^k$, where $m \in \mathbb{Z}$, are invariants of coadjoint representation.*

3 Integrable systems from hyperelliptic algebras

3.1 Poisson structures and Poisson subspaces

1. First Lie–Poisson structure. In the space $\widetilde{\mathfrak{g}}^*_{\mathcal{H}}$ we can define Lie–Poisson brackets using introduced above pairing (3). It defines brackets on $P(\widetilde{\mathfrak{g}}^*_{\mathcal{H}})$ in the following way:

$$\{F(L), G(L)\} = \sum_{l,m\in Z}\sum_{i,j,p,s=1}^{n} \langle L(w), [X_{ij}^{-l}, X_{ps}^{-m}]\rangle \frac{\partial G}{\partial l_{ij}^{(l)}} \frac{\partial F}{\partial l_{ps}^{(m)}}. \tag{6}$$

From the Proposition 1 follows the next statement:

Proposition 2. *Functions $I^k_m(L(w))$ are central for brackets $\{\ ,\ \}$.*

Let us explicitly calculate Poisson brackets (6). Taking into account that $l_{ij}^{(m)} = \langle L(w), X_{ij}^{-m}\rangle$, it is easy to show, that for the coordinate functions $l_{ij}^{(m)}$ these brackets have the following form:

$$1)\ \left\{l_{ij}^{(n)}, l_{kl}^{(m)}\right\} = \delta_{kj} l_{il}^{(n+m-1)} - \delta_{il} l_{kj}^{(n+m-1)} + a_i\delta_{il} l_{kj}^{(n+m)} - a_j \delta_{kj} l_{il}^{(n+m)} \quad \text{for the } gl(n); \tag{7a}$$

$$\begin{aligned} 2)\ \left\{l_{ij}^{(n)}, l_{kl}^{(m)}\right\} &= \delta_{kj} l_{il}^{(n+m-1)} - \delta_{il} l_{kj}^{(n+m-1)} + \delta_{jl} l_{ki}^{(n+m-1)} - \delta_{ik} l_{jl}^{(n+m-1)} \\ &+ a_i\delta_{il} l_{kj}^{(n+m)} - a_j\delta_{kj} l_{il}^{(n+m)} + a_i\delta_{ik} l_{jl}^{(n+m)} - a_j\delta_{jl} l_{ki}^{(n+m)} \quad \text{for the } so(n); \end{aligned} \tag{7b}$$

$$\begin{aligned} 3)\ \left\{l_{ij}^{(n)}, l_{kl}^{(m)}\right\} &= \delta_{kj} l_{il}^{(n+m-1)} - \delta_{il} l_{kj}^{(n+m-1)} + \epsilon_i\epsilon_j \left(\delta_{j-l} l_{k-i}^{(n+m-1)} - \delta_{i-k} l_{j-l}^{(n+m-1)}\right) \\ &+ a_i\delta_{il} l_{kj}^{(n+m)} - a_j\delta_{kj} l_{il}^{(n+m)} + \epsilon_i\epsilon_j \left(a_i\delta_{i-k} l_{j-l}^{(n+m)} - a_j\delta_{j-l} l_{k-i}^{(n+m)}\right) \quad \text{for the } sp(n). \end{aligned} \tag{7c}$$

2. Second Lie–Poisson structure. Let us introduce into the space $\widetilde{\mathfrak{g}}^*_{\mathcal{H}}$ new Poisson brackets $\{\ ,\ \}_0$, which are a Lie–Poisson brackets for the algebra $\widetilde{\mathfrak{g}}^0_{\mathcal{H}}$, where $\widetilde{\mathfrak{g}}^0_{\mathcal{H}} = \widetilde{\mathfrak{g}}^-_{\mathcal{H}} \ominus \widetilde{\mathfrak{g}}^+_{\mathcal{H}}$. Explicitly, this brackets have the following form:

$$\{l^{(n)}_{ij}, l^{(m)}_{kl}\}_0 = -\{l^{(n)}_{ij}, l^{(m)}_{kl}\}, \quad n,m \in \mathbb{Z}_+, \qquad \{l^{(n)}_{ij}, l^{(m)}_{kl}\}_0 = \{l^{(n)}_{ij}, l^{(m)}_{kl}\}, \quad n,m \in \mathbb{Z}_- \cup 0,$$
$$\{l^{(n)}_{ij}, l^{(m)}_{kl}\}_0 = 0, \quad m \in \mathbb{Z}_- \cup 0, \quad n \in \mathbb{Z}_+ \quad \text{or} \quad n \in \mathbb{Z}_- \cup 0, \quad m \in \mathbb{Z}_+.$$

Let subspace $\mathcal{M}_{s,p} \subset \widetilde{\mathfrak{g}}^*_{\mathcal{H}}$ be defined as follows:

$$\mathcal{M}_{s,p} = \sum_{m=-s+1}^{p} (\widetilde{\mathfrak{g}}^*_{\mathcal{H}})_m.$$

Brackets $\{\ ,\ \}_0$ could be correctly restricted to $\mathcal{M}_{s,p}$. It follows from the next proposition:

Proposition 3. *Subspaces* $\mathcal{J}_{p,s} = \sum\limits_{m=-\infty}^{-p-1} (\widetilde{\mathfrak{g}}_{\mathcal{H}})_m + \sum\limits_{m=s}^{\infty} (\widetilde{\mathfrak{g}}_{\mathcal{H}})_m$ *are ideals in* $\widetilde{\mathfrak{g}}^0_{\mathcal{H}}$.

3.2 Algebras of integrals and Hamiltonian equations

To construct integrable Hamiltonian systems we need a large family of mutually commuting functions (integrals of motion). It is provided by the following theorem:

Theorem 2. *Let functions* $\{I^k_m(L)\}$ *be defined as in Proposition 1. Their restriction to* $\mathcal{M}_{s,p}$ *generate commutative algebra with respect to the restriction of the brackets* $\{\ ,\ \}_0$ *on* $\mathcal{M}_{s,p}$.

Dynamical equations we will consider here are Hamiltonian equations of the form:

$$\frac{dl^{(k)}_{ij}}{dt} = \left\{ l^{(k)}_{ij}, H\left(l^{(m)}_{kl}\right)\right\}_0, \tag{8}$$

where Hamiltonian H is one of the functions I^k_m or their linear combination. These equations could be written in the form of Lax type equations [2]:

$$\frac{dL(w)}{dt} = P_{M_{s,p}}([L(w), M(w)]), \tag{9}$$

where $P_{M_{s,p}}$ denotes operator that project dual space onto subspace $M_{s,p}$ $L(w) \in \mathcal{M}_{s,p}$, and second operator is defined as follows: $M(w) = (P_- - P_+)\nabla H(L(w))$. Here $P_\pm$ are projection operators on the subalgebra $\widetilde{\mathfrak{g}}^\pm_{\mathcal{H}}$,

$$\nabla H(L(w)) = \sum_{k=-p}^{s-1} \sum_{ij=1}^{n} \frac{\partial H}{\partial l^{(k)}_{ij}} X^{-k}_{ij} \tag{10}$$

is an algebra-valued gradient of H.

Thus we have constructed Hamiltonian systems possessing (Theorem 2) a lot of mutually commuting integrals of motion. In the next section we will consider examples of such systems.

4 Integrable systems in finite-dimensional quotients

The most interesting from the physical point of view examples usually arise in the spaces $\mathcal{M}_{s,p}$ with small s and p. We will assume, that curve $\mathcal{H}$ is nondegenerated, i.e. $a_i \neq a_j$ for $i \neq j$.

4.1 Generalized $gl(n)$ tops

Let us consider subspace $\mathcal{M}_{0,1}$. It is evident that $\mathcal{M}_{0,1} = (\widetilde{\mathfrak{g}}^+_{\mathcal{H}}/\mathcal{J}_{1,0})^* = \mathfrak{g}^*$. Corresponding Lax operator $L(w) \in \mathcal{M}_{0,1}$ has the following form:

$$L(w) = \sum_{i,j=1,k} l^{(1)}_{ij} \frac{y^2(w)}{w_i w_j} X^*_{ij}.$$

Let us consider the case $\mathfrak{g} = gl(n)$. In this case we have: $X^*_{ij} = X_{ji}$. Lie–Poisson brackets between the coordinate functions $l_{ij} \equiv l^{(1)}_{ij}$ have standard form:

$$\{l_{ij}, l_{kl}\} = \delta_{kj} l_{il} - \delta_{il} l_{kj}.$$

Commuting integrals are constructed using expansions in the powers of w of the functions: $I_k(w) = \mathrm{Tr}\,(L(w))^k$. We are especially interested in the quadratic Hamiltonians. Let

$$h(w) \equiv I_2(w) = \sum_{s=0}^{2n-2} h_s(l_{ij}) w^s = \sum_{ij} \left(\prod_{k \neq i,j} (w - a_k)^2 \right) l_{ij} l_{ji}.$$

We obtain:

$$h_0 = \left(\prod_{k=1}^{n} a_k^2 \right) \sum_{i,j=1}^{n} \frac{l_{ij} l_{ji}}{a_i a_j},$$
$$h_1 = -\left(\prod_{k=1}^{n} a_k^2 \right) \sum_{i,j=1}^{n} \left(2 \sum_{k=1}^{n} a_k^{-1} - \left(a_i^{-1} + a_j^{-1} \right) \right) \frac{l_{ij} l_{ji}}{a_i a_j},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$h_{2n-3} = -\sum_{i,j=1}^{n} \left(2 \sum_{k=1}^{n} a_k - (a_i + a_j) \right) l_{ij} l_{ji},$$
$$h_{2n-2} = \sum_{i,j=1}^{n} l_{ij} l_{ji}.$$

Last function in this set is a Casimir function, previous $2n-3$ define nontrivial flows on each coadjoint orbit in $\mathfrak{g}^*$. For the Hamiltonian of the generalized $gl(n)$ rigid body we can take $H(l_{ij}) \equiv 1/2 h_{n-1}(l_{ij})$ or $H(l_{ij}) \equiv 1/2 h_0(l_{ij})$. They are transformed to the standard Hamiltonian of the Euler top in the case $n = 3$ after reduction to $so(n)$ subalgebra.

4.2 Generalized $gl(n-1)$ Clebsh systems

Let us consider subspace $\mathcal{M}_{1,0}$. Corresponding Lax matrix $L(w) \in \mathcal{M}_{1,0}$ has the following form:

$$L(w) = w^{-1} \sum_{i,j=1,n} l^{(0)}_{ij} \frac{y^2(w)}{w_i w_j} X_{ji}.$$

In the space $\mathcal{M}_{1,0}$ Poisson structure $\{\ ,\ \}$ has the following form:

$$\left\{ l^{(0)}_{ij}, l^{(0)}_{kl} \right\} = a_i \delta_{il} l^{(0)}_{kj} - a_j \delta_{kj} l^{(0)}_{il}. \tag{11}$$

The Lie algebraic structure that is defined by these brackets strongly depends on the constants a_i. Let us consider the case of the simplest "degeneration" $a_n \to 0$, $a_i \neq 0$, where $i < n$. In this case we will have the following commutation relations:

$$\left\{l_{ij}^{(0)}, l_{kl}^{(0)}\right\} = a_i\delta_{il}l_{kj}^{(0)} - a_j\delta_{kj}l_{il}^{(0)}, \qquad \left\{l_{ij}^{(0)}, l_{kn}^{(0)}\right\} = -a_j\delta_{kj}l_{in}^{(0)}, \qquad \left\{l_{ij}^{(0)}, l_{nk}^{(0)}\right\} = a_i\delta_{ik}l_{nj}^{(0)},$$
$$\left\{l_{in}^{(0)}, l_{jn}^{(0)}\right\} = \left\{l_{ni}^{(0)}, l_{nj}^{(0)}\right\} = \left\{l_{ij}^{(0)}, l_{nn}^{(0)}\right\} = 0, \qquad \left\{l_{in}^{(0)}, l_{nj}^{(0)}\right\} = a_i\delta_{ij}l_{nn}^{(0)},$$

where $i, j, k < n$. Making the following change of the variables:

$$l_{ij} = \frac{l_{ij}^{(0)}}{b_i b_j}, \quad x_k = \frac{l_{kn}^{(0)}}{b_k}, \quad y_k = \frac{l_{nk}^{(0)}}{b_k}, \quad z = l_{nn}^{(0)}, \quad \text{where } \ b_i = a_i^{1/2}, \quad i, j, k < n \tag{12}$$

we obtain commutation relations for the Lie algebra $gl(n-1) + H^{2n+1}$:

$$\{l_{ij}, l_{kl}\} = \delta_{il}l_{kj} - \delta_{kj}l_{il}, \qquad \{x_i, y_j\} = z, \qquad \{l_{ij}, x_k\} = -\delta_{kj}x_i,$$
$$\{l_{ij}, y_k\} = \delta_{ik}y_k, \qquad \{x_i, x_j\} = \{y_i, y_j\} = \{l_{ij}, z\} = 0,$$

where H^{2n+1} is a Heisenberg algebra in the space $\mathbb{R}^{2n+1}$. It is evident, that z is a central element in this algebra, so we can put $z = 0$. Corresponding Poisson algebra will coincide with semi-direct sum $gl(n-1) + \mathbb{R}^{2n}$. We will call corresponding integrable Hamiltonian system "*$gl(n)$ Clebsh system*".

Let us calculate commuting integrals of the $gl(n)$ Clebsh system. They are constructed using expansions in the powers of w of the functions: $H_k(w) = \mathrm{Tr}(L(w))^k$. Let us calculate explicitly second order integrals:

$$h(w) \equiv H_2(w) = \sum_{s=-2}^{2n-4} h_s\left(l_{ij}^{(0)}\right) w^s = w^{-2}\sum_{i,j=1,n}\left(\prod_{k\neq i,j}(w-a_k)^2\right) l_{ij}^{(0)}l_{ji}^{(0)}.$$

It is not difficult to notice that in the case $a_n \neq 0$, Hamiltonians have essentially the same form as in the previous example of the generalized tops (modulo the shift the indices $h_k \to h_{k-2}$ and replacing of variables: $l_{ij} \to l_{ij}^{(0)}$). Let us now calculate these Hamiltonians in the limit $a_n \to 0$, $z \to 0$. Taking into account coordinate transformation (12) we obtain:

$$h_{-2} = 2\left(\prod_{k=1}^{n-1} a_k^2\right)\sum_{k=1}^{n-1} x_k y_k,$$
$$h_{-1} = (-1)\left(\prod_{k=1}^{n-1} a_k^2\right)\left(\sum_{i,j=1}^{n-1}\left(l_{ij}l_{ji} - 2a_i^{-1}x_i y_i\right) - h_0\right),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$h_{2n-5} = (-1)\left(\sum_{i,j=1}^{n-1}(a_i+a_j)a_i a_j l_{ij}l_{ji} + 2a_i^2 x_i y_i\right) - 2\left(\sum_{k=1}^{n-1} a_k\right) h_{2n-4},$$
$$h_{2n-4} = \left(\sum_{i,j=1}^{n-1} a_i a_j l_{ij}l_{ji} + 2a_i x_i y_i\right).$$

Function h_{-2} is a Casimir function. For the Hamiltonian of the Clebsh system one can take, for example, h_{-1} or h_{2n-4}. They are transformed to the standard integrals of the Clebsh system in the case $n = 3$ after reduction to $so(n)$ subalgebra.

4.3 Generalized interacting $gl(n)$ tops

Let us consider subspace $\mathcal{M}_{1,1}$. In the case $a_i \neq 0$, as it follows from the explicit form of the brackets given below, $\mathcal{M}_{1,1} = (\mathfrak{g} \oplus \mathfrak{g})^*$. Corresponding Lax operator $L(w) \in \mathcal{M}_{1,1}$ has the following form:

$$L(w) = \sum_{i,j=1}^{n} \left(w^{-1} l_{ij}^{(0)} + l_{ij}^{(1)} \right) \frac{y^2(w)}{w_i w_j} X_{ij}^*.$$

In the of $gl(n)$ case we may put $X_{ij}^* = X_{ji}$. Lie–Poisson brackets between the coordinate functions $l_{ij}^{(1)}$ are the following:

$$\left\{ l_{ij}^{(0)}, l_{kl}^{(0)} \right\} = -a_i \delta_{il} l_{kj}^{(0)} + a_j \delta_{kj} l_{il}^{(0)}, \qquad \left\{ l_{ij}^{(1)}, l_{kl}^{(1)} \right\} = \delta_{kj} l_{il}^{(1)} - \delta_{il} l_{kj}^{(1)}, \qquad \left\{ l_{ij}^{(0)}, l_{kl}^{(1)} \right\} = 0.$$

Putting $b_i = a_i^{1/2}$ and making the change of variables: $l_{ij} = l_{ij}^{(1)}$, $m_{ij} = \frac{l_{ij}^{(0)}}{b_i b_j}$, we obtain canonical coordinates of the direct sum of two algebras $gl(n)$:

$$\{m_{i,j}, m_{k,l}\} = \delta_{kj} m_{il} - \delta_{il} m_{kj}, \qquad \{l_{ij}, l_{kl}\} = \delta_{kj} l_{il} - \delta_{il} l_{kj}, \qquad \{l_{ij}, m_{kl}\} = 0.$$

Commuting integrals are constructed using expansion in the powers of w of the functions: $I_k(w) = \operatorname{Tr}(L(w))^k$. We are interested in the quadratic integrals:

$$h(w) \equiv I_2(w) = \sum_{s=-2}^{2n-2} h_s \left(l_{ij}^{(1)} \right) w^s = \sum_{ij} \left(\prod_{k \neq i,j} (w - a_k)^2 \right) \left(l_{ij}^{(0)} + w l_{ij}^{(1)} \right)^2.$$

By direct calculations making the described above change of variables we obtain:

$$h_{-2} = \left(b_1^4 b_2^4 \cdots b_n^4 \right) \sum_{i,j=1}^{n} m_{ij} m_{ji},$$

$$h_{-1} = -\left(b_1^4 b_2^4 \cdots b_n^4 \right) \left(\sum_{i,j=1}^{n} \left(2 \sum_{k=1,n} b_k^{-2} - \left(b_i^{-2} + b_j^{-2} \right) \right) m_{ij} m_{ji} - 2 b_i^{-1} b_j^{-1} m_{ij} l_{ji} \right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$h_{2n-3} = -\left(\sum_{i,j=1}^{n} \left(2 \sum_{k=1}^{n} b_k^2 - \left(b_i^2 + b_j^2 \right) \right) l_{ij} l_{ji} - 2 b_i b_j m_{ij} l_{ji} \right),$$

$$h_{2n-2} = \sum_{i,j=1}^{n} l_{ij} l_{ji}.$$

It is evident that functions h_{-2} and h_{2n-2} are invariants. For the Hamiltonian of the generalized interacting rigid bodies we can take either h_{n-1} or h_1. Operator M and Lax equations for these Hamiltonians are calculated using formulas (9), (10).

4.4 Steklov–Liapunov system on $gl(n) + gl(n)$

Let us consider subspace $\mathcal{M}_{0,2} = (\widetilde{\mathfrak{g}}_{\mathcal{H}}^{+} / \mathcal{J}_{2,0})^*$. It is easy to show that $\mathcal{M}_{0,2} = (\mathfrak{g} + \mathfrak{g})^*$. Corresponding Lax operator $L(w) \in \mathcal{M}_{0,2}$ has the following form:

$$L(w) = \sum_{i,j=1}^{n} \left(l_{ij}^{(1)} + w l_{ij}^{(2)} \right) \frac{y^2(w)}{w_i w_j} X_{ij}^*.$$

We will again be concentrated on $\mathfrak{g} = gl(n)$ case and put $X^*_{ij} = X_{ji}$. Lie–Poisson brackets between coordinate functions are the following:

$$\left\{l^{(1)}_{ij}, l^{(1)}_{kl}\right\} = \delta_{kj} l^{(1)}_{il} - \delta_{il} l^{(1)}_{kj} + a_i \delta_{il} l^{(2)}_{kj} - a_j \delta_{kj} l^{(2)}_{il},$$
$$\left\{l^{(1)}_{ij}, l^{(2)}_{kl}\right\} = \delta_{kj} l^{(2)}_{il} - \delta_{il} l^{(2)}_{kj}, \qquad \left\{l^{(2)}_{ij}, l^{(2)}_{kl}\right\} = 0.$$

Change of variables: $l^{(1)}_{ij} = l_{ij} - a_i p_{ij}$, $l^{(2)}_{ij} = p_{ij}$ transforms described above brackets to the standard brackets on the semi-direct sum $gl(n) + gl(n)$:

$$\{l_{ij}, l_{kl}\} = \delta_{kj} l_{il} - \delta_{il} l_{kj}, \qquad \{l_{ij}, p_{kl}\} = \delta_{kj} p_{il} - \delta_{il} p_{kj}, \qquad \{p_{ij}, p_{kl}\} = 0.$$

Commuting integrals are constructed using expansion in the powers of w of the functions: $I_k(w) = \operatorname{Tr}(L(w))^k$. We are again interested mainly in quadratic integrals:

$$h(w) \equiv I_2(w) = \sum_{s=0}^{2n-2} h_{s+2}\left(l^{(1)}_{ij}\right) w^s = w^2 \sum_{ij} \left(\prod_{k\neq i,j} (w - a_k)^2\right) \left(l^{(1)}_{ij} + w l^{(2)}_{ij}\right)^2.$$

By direct calculations, making the described above change of variables we obtain the following set of Hamiltonians:

$$h_0 = \left(a_1^2 a_2^2 \cdots a_n^2\right) \sum_{i,j=1}^{n} \frac{(l_{ij} - a_i p_{ij})(l_{ji} - a_j p_{ji})}{a_i a_j},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$h_{2n-1} = (-1)\left(2\sum_{k=1}^{n} a_k\right) \sum_{i,j=1}^{n} p_{ij} p_{ji} + 2 l_{ij} p_{ji},$$

$$h_{2n} = \sum_{i,j=1}^{n} p_{ij} p_{ji}.$$

Last two functions are invariant functions. If we choose function $H = h_0$ for the Hamiltonian function we obtain precise $gl(n)$ generalization of Steklov–Liapunov system.

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nth Discrete KP Hierarchy

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We report an infinite class of discrete hierarchies which naturally generalize familiar discrete KP one.

1 Introduction

The interrelation between discrete and differential integrable hierarchies plays crucial role in obtaining solutions to the discrete multi-matrix models [1, 2]. At a level of KP-type differential hierarchies the discrete structure of multi-matrix models is captured by the Darboux–Bäcklund (DB) transformations. In turn partition functions of multi-matrix models turns out to be τ-functions of differential hierarchies and are constructed as DB orbits of certain simple initial conditions [2]. The well known discrete KP (1-Toda lattice) hierarchy [3] together with its reductions can be viewed as a container for a set of KP-type differential hierarchies whose solutions are generated by DB transformations.

This paper is designed to exhibit certain class of discrete hierarchies which generalize discrete KP and show the relationship with general (unconstrained) differential KP. This relationship yields bi-infinite sequences of differential KP equipped with two compatible gauge transformations. We believe that these results might be of potential interest from the physical point of view.

2 nth discrete KP

Given the shift operator $\Lambda = (\delta_{i,j-1})_{i,j\in\mathbb{Z}}$ one considers the Lie algebra of pseudo-difference operators

$$\mathcal{D} = \left\{ \sum_{-\infty<k\ll\infty} \ell_k \Lambda^k \right\} = \mathcal{D}_- + \mathcal{D}_+$$

with usual splitting into “negative” and “positive” parts:

$$\mathcal{D}_- = \left\{ \sum_{-\infty<k\leq -1} \ell_k \Lambda^k \right\} \qquad \text{and} \qquad \mathcal{D}_+ = \left\{ \sum_{0<k\ll\infty} \ell_k \Lambda^k \right\}.$$

We assume that entries of bi-infinite diagonal matrices $\ell_k \equiv (\ell_k(i))_{i\in\mathbb{Z}}$ may depend on “spectral” parameter z and multi-time $t \equiv (t_1 \equiv x, t_2, t_3, \ldots)$. In what follows $\partial \equiv \partial/\partial x$ and $\partial_p \equiv \partial/\partial t_p$.

Let us define[1]

$$Q = \Lambda + a_0 z^{n-1}\Lambda^{1-n} + a_1 z^{2(n-1)}\Lambda^{1-2n} + \cdots \in \mathcal{D}, \qquad n \in \mathbb{N} \tag{1}$$

with $a_k = (a_k(i))_{i\in\mathbb{Z}}$ being functions on t only.

[1]Here z acts as component-wise multiplication.

Proposition 1. *Lax equations of Q-deformations*

$$z^{p(n-1)}\partial_p Q = \left[Q_+^{pn}, Q\right], \qquad p = 1, 2, \ldots \tag{2}$$

make sense.

Proof. One needs to use standard simple arguments to prove correctness of equations (2). It is enough to show that $\left[Q_+^{pn}, Q\right] = -\left[Q_-^{pn}, Q\right]$ is of the same form as l.h.s. of (2). ■

We will refer to (2) as nth discrete KP hierarchy. Let us represent Q as a dressing up of Λ by a "wave" operator as $Q = W\Lambda W^{-1}$ where

$$W = I + w_1 z^{n-1}\Lambda^{-n} + w_2 z^{2(n-1)}\Lambda^{-2n} + w_3 z^{3(n-1)}\Lambda^{-3n} + \cdots \in I + \mathcal{D}_-.$$

Then Q-deformations are induced by W-deformations

$$\begin{aligned} &z^{p(n-1)}\partial_p W = Q_+^{pn} W - W\Lambda^{pn}, \\ &z^{p(n-1)}\partial_p \left(W^{-1}\right)^T = \left(W^{-1}\right)^T \Lambda^{-pn} - \left(Q_+^{pn}\right)^T \left(W^{-1}\right)^T. \end{aligned} \tag{3}$$

Define $\chi(t,z) = \left(z^i e^{\xi(t,z)}\right)_{i\in\mathbb{Z}}$, $\chi^*(t,z) = \left(z^{-i}e^{-\xi(t,z)}\right)_{i\in\mathbb{Z}}$ with $\xi(t,z) \equiv \sum\limits_{p=1}^{\infty} t_p z^p$ and wave vectors

$$\Psi(t,z) = W\chi(t,z), \qquad \Psi^*(t,z) = \left(W^{-1}\right)^T \chi^*(t,z). \tag{4}$$

Discrete linear system

$$\begin{aligned} &Q\Psi(t,z) = z\Psi(t,z), \qquad Q^T\Psi^*(t,z) = z\Psi^*(t,z), \\ &z^{p(n-1)}\partial_p\Psi = Q_+^{pn}\Psi, \qquad z^{p(n-1)}\partial_p\Psi^* = -(Q_+^{pn})^T\Psi^* \end{aligned} \tag{5}$$

are evident consequence of (3) and (4). Making use of obvious relations $z\chi = \Lambda\chi$ and $\chi_i = \partial^{i-j}\chi_j$ with i and j being arbitrary integers, we deduce

$$\begin{aligned} \Psi_i(t,z) &= z^i\left(1 + w_1(i)z^{-1} + w_2(i)z^{-2} + \cdots\right)e^{\xi(t,z)} \\ &= z^i\left(1 + w_1(i)\partial^{-1} + w_2(i)\partial^{-2} + \cdots\right)e^{\xi(t,z)} \equiv z^i\hat{w}_i(\partial)e^{\xi(t,z)} \equiv z^i\psi_i(t,z). \end{aligned}$$

What we are going to do next is to establish equivalence of nth discrete KP to bi-infinite sequence of differential KP copies "glued" together by two compatible gauge transformations one of which can be recognized as DB transformation mapping $\mathcal{Q}_i \equiv \hat{w}_i\partial\hat{w}_i^{-1}$ to $\mathcal{Q}_{i+n} \equiv \hat{w}_{i+n}\partial\hat{w}_{i+n}^{-1}$. By straightforward calculations one can prove

Proposition 2. *The following three statements are equivalent*

i) the wave vector $\Psi(t,z)$ satisfies discrete linear system

$$Q\Psi(t,z) = z\Psi(t,z), \qquad z^{n-1}\partial\Psi = Q_+^n\Psi; \tag{6}$$

ii) the components ψ_i of a vector $\psi \equiv (\psi_i = z^{-i}\Psi_i)_{i\in\mathbb{Z}}$ satisfy

$$G_i\psi_i(t,z) = z\psi_{i+n-1}(t,z), \qquad H_i\psi_i(t,z) = z\psi_{i+n}(t,z) \tag{7}$$

with $H_i \equiv \partial - \sum\limits_{s=1}^{n} a_0(i+s-1)$ and

$$G_i \equiv H_i + a_0(i+n-1) + a_1(i+n-1)H_{i-n}^{-1} + a_2(i+n-1)H_{i-2n}^{-1}H_{i-n}^{-1} + \cdots;$$

iii) for sequence of dressing operators $\hat{w}_i$ following equations

$$G_i\hat{w}_i = \hat{w}_{i+n-1}\partial, \qquad H_i\hat{w}_i = \hat{w}_{i+n}\partial \tag{8}$$

hold.

Consistency condition of (6) is given by Lax equation

$$z^{n-1}\partial Q = \left[Q_+^n, Q\right] \tag{9}$$

which in explicit form looks as

$$\begin{aligned}\partial a_k(i) &= a_{k+1}(i+n) - a_{k+1}(i)\\ &\quad + a_k(i)\left(\sum_{s=1}^{n} a_0(i+s-1) - \sum_{s=1}^{n} a_0(i+s-(k+1)n)\right), \qquad k \geq 0.\end{aligned} \tag{10}$$

Remark 1. One-field reductions of the systems (10) lead to Bogoyavlenskii lattices [4]

$$\partial r_i = r_i\left(\sum_{s=1}^{n-1} r_{i+s} - \sum_{s=1}^{n-1} r_{i-s}\right), \qquad r_i \equiv a_0(i)$$

including well known Volterra lattice $\partial r_i = r_i(r_{i+1} - r_{i-1})$ in the case $n = 2$.

Consistency condition of (8) is given by relations

$$G_{i+n}H_i = H_{i+n-1}G_i, \qquad i \in \mathbb{Z} \tag{11}$$

which in fact are equivalent to (9).

Proposition 3. *By virtue of (8) and its consistency condition, Lax operators $\mathcal{Q}_i$ are connected with each other by two invertible compatible gauge transformations*

$$\mathcal{Q}_{i+n-1} = G_i\mathcal{Q}_iG_i^{-1}, \qquad \mathcal{Q}_{i+n} = H_i\mathcal{Q}_iH_i^{-1}. \tag{12}$$

Proof. By virtue of (8), we have

$$\mathcal{Q}_{i+n-1} = \hat{w}_{i+n-1}\partial\hat{w}_{i+n-1}^{-1} = \left(G_i\hat{w}_i\partial^{-1}\right)\partial\left(\partial\hat{w}_i^{-1}G_i^{-1}\right) = G_i\hat{w}_i\partial\hat{w}_i^{-1}G_i^{-1} = G_i\mathcal{Q}_iG_i^{-1}.$$

The similar arguments are applied to show second relation in (12). The mapping $\mathcal{Q}_i \to \tilde{\mathcal{Q}}_i = \mathcal{Q}_{i+n-1}$ we denote as s_1, while s_2 stands for transformation $\mathcal{Q}_i \to \overline{\mathcal{Q}}_i = \mathcal{Q}_{i+n}$. As for compatibility of s_1 and s_2, by virtue of (11), we have

$$\begin{aligned}\mathcal{Q}_{i+2n-1} &= G_{i+n}\mathcal{Q}_{i+n}G_{i+n}^{-1} = G_{i+n}H_i\mathcal{Q}_iH_i^{-1}G_{i+n}^{-1}\\ &= H_{i+n-1}G_i\mathcal{Q}_iG_i^{-1}H_{i+n-1}^{-1} = H_{i+n-1}\mathcal{Q}_{i+n-1}H_{i+n-1}^{-1}.\end{aligned}$$

So we can write $s_1 \circ s_2 = s_2 \circ s_1$. The inverse maps s_1^{-1} and s_2^{-1} are well defined by the formulas $\mathcal{Q}_{i-n+1} = G_{i-n+1}^{-1}\mathcal{Q}_iG_{i-n+1}$ and $\mathcal{Q}_{i-n} = H_{i-n}^{-1}\mathcal{Q}_iH_{i-n}$. ■

It is obvious that relation $s_1^n = s_2^{n-1}$ holds. Indeed the l.h.s. and r.h.s. of this relation correspond to the same mapping $\mathcal{Q}_i \to \mathcal{Q}_{i+n(n-1)}$. The Abelian group generated by s_1 and s_2 we denote by symbol $\mathcal{G}$.

Rewrite second equation in (7) as $z^{n-1}H_i\Psi_i(t,z) = \Psi_{i+n}(t,z) = (\Lambda^n\Psi)_i$. From this we derive

$$\begin{aligned}&z^{k(1-n)}(\Lambda^{kn}\Psi)_i = H_{i+(k-1)n}\cdots H_{i+n}H_i\Psi_i,\\ &z^{k(n-1)}(\Lambda^{-kn}\Psi)_i = H_{i-kn}^{-1}\cdots H_{i-2n}^{-1}H_{i-n}^{-1}\Psi_i.\end{aligned}$$

These relations make connection between matrices of the form $P = \sum_{k\in\mathbb{Z}} z^{k(1-n)}p_k(t)\Lambda^{kn}$ and sequences of pseudo-differential operators $\{\mathcal{P}_i,\ i \in \mathbb{Z}\}$ mapping the upper triangular part of given matrix (including main diagonal) into the differential parts of $\mathcal{P}_i$'s and the lower triangular part of the matrix to the purely pseudo-differential parts. More exactly, we have $(P\Psi)_i = \mathcal{P}_i\Psi_i$, $(P_-\Psi)_i = (\mathcal{P}_i)_-\Psi_i$ and $(P_+\Psi)_i = (\mathcal{P}_i)_+\Psi_i$, where

$$\mathcal{P}_i = \sum_{k>0} p_{-k}(i,t)H_{i-kn}^{-1}\cdots H_{i-2n}^{-1}H_{i-n}^{-1} + \sum_{k\geq 0} p_k(i,t)H_{i+(k-1)n}\cdots H_{i+n}H_i = (\mathcal{P}_i)_- + (\mathcal{P}_i)_+.$$

Proposition 4. *Equations* $z^{p(n-1)}\partial_p\Psi = Q_+^{pn}\Psi$, $p = 2, 3, \ldots$ *lead to* $\partial_p\psi_i = (\mathcal{Q}_i^p)_+\psi_i$, $p = 2, 3, \ldots$.

Proof. We have

$$z^{p(1-n)}(Q^{pn}\Psi)_i = z^p\Psi_i = z^{i+p}\hat{w}_i e^{\xi(t,z)} = z^i\hat{w}_i\partial^p e^{\xi(t,z)} = z^i\hat{w}_i\partial^p\hat{w}_i^{-1}\psi_i = z^i\mathcal{Q}_i^p\psi_i = \mathcal{Q}_i^p\Psi_i.$$

Thus

$$z^{p(n-1)}\partial_p\Psi_i = z^{i+p(n-1)}\partial_p\psi_i = (Q_+^{pn}\Psi)_i = z^{p(n-1)}(\mathcal{Q}_i^p)_+\Psi_i = z^{i+p(n-1)}(\mathcal{Q}_i^p)_+\psi_i.$$

The latter proves proposition. ■

Let us establish equations managing G_i- and H_i-evolutions with respect to KP flows. Differentiating l.h.s. and r.h.s. of (8) by virtue of Sato–Wilson equations $\partial_p\hat{w}_i = (\mathcal{Q}_i^p)_+\hat{w}_i - \hat{w}_i\partial^p$ formally leads to evolution equations

$$\begin{aligned}
&\partial_p G_i = \left(\mathcal{Q}_{i+n-1}^p\right)_+ G_i - G_i\left(\mathcal{Q}_i^p\right)_+, \\
&\partial_p H_i = \left(\mathcal{Q}_{i+n}^p\right)_+ H_i - H_i\left(\mathcal{Q}_i^p\right)_+ .
\end{aligned} \tag{13}$$

Standard arguments can be used to show that equations (13) are properly defined individually. Let us show that permutation relations (11) are invariant under the flows given by equations (13). We have

$$\begin{aligned}
\partial_p(H_{i+n-1}G_i) &= \left\{\left(\mathcal{Q}_{i+2n-1}^p\right)_+ H_{i+n-1} - H_{i+n-1}\left(\mathcal{Q}_{i+n-1}^p\right)_+\right\} G_i \\
&+ H_{i+n-1}\left\{\left(\mathcal{Q}_{i+n-1}^p\right)_+ G_i - G_i\left(\mathcal{Q}_i^p\right)_+\right\} = \left(\mathcal{Q}_{i+2n-1}^p\right)_+ H_{i+n-1}G_i - H_{i+n-1}G_i\left(\mathcal{Q}_i^p\right)_+ \\
&= \left(\mathcal{Q}_{i+2n-1}^p\right)_+ G_{i+n}H_i - G_{i+n}H_i\left(\mathcal{Q}_i^p\right)_+ = \left\{\left(\mathcal{Q}_{i+2n-1}^p\right)_+ G_{i+n} - G_{i+n}\left(\mathcal{Q}_{i+n}^p\right)_+\right\} H_i \\
&+ G_{i+n}\left\{\left(\mathcal{Q}_{i+n}^p\right)_+ H_i - H_i\left(\mathcal{Q}_i^p\right)_+\right\} = \partial_p(G_{i+n}H_i).
\end{aligned}$$

Hence we proved that evolution equations (13) are consistent.

Define $\Phi_i = \Phi_i(t)$ via $H_i\Phi_i = 0$ or equivalently through equation $\partial\Phi_i = \Phi_i\sum\limits_{s=1}^{n} a_0(i+s-1)$. Taking into consideration second equation in (13), we have

$$\partial_p(H_i\Phi_i) = \left(\mathcal{Q}_{i+n}^p\right)_+ H_i\Phi_i - H_i\left(\mathcal{Q}_i^p\right)_+\Phi_i + H_i\partial_p\Phi_i = 0.$$

From this we derive $\partial_p\Phi_i = \left(\mathcal{Q}_i^p\right)_+\Phi_i + \alpha_i\Phi_i$ where α_i's are some constants. Commutativity condition $\partial_p\partial_q\Phi_i = \partial_q\partial_p\Phi_i$ leads to evolution equations for KP eigenfunctions $\partial_p\Phi_i = \left(\mathcal{Q}_i^p\right)_+\Phi_i$, i.e. $\alpha_i = 0$. Thus the relations $\mathcal{Q}_{i+n} = H_i\mathcal{Q}_iH_i^{-1}$ defines DB transformations with eigenfunctions $\Phi_i = \tau_{i+n}/\tau_i$. It should perhaps to recall that arbitrary eigenfunction of Lax operator $\mathcal{Q}$ contains information about DB transformation $\tau \to \overline{\tau} = \Phi\tau$ while the identity[2]

$$\left\{\tau\left(t - \left[z^{-1}\right]\right), \overline{\tau}(t)\right\} + z\left(\tau\left(t - \left[z^{-1}\right]\right)\overline{\tau}(t) - \overline{\tau}\left(t - \left[z^{-1}\right]\right)\tau(t)\right) = 0$$

holds.

So, we have shown that nth discrete KP is equivalent to sequence of differential KP linked with each other by two compatible gauge transformations one of which, namely, $s_2 : \mathcal{Q}_i \to \mathcal{Q}_{i+n}$ are nothing but Darboux–Bäcklund transformation. The problem which can be addressed is to describe nth discrete KP in the language of bilinear identities by analogy as was done for ordinary discrete KP [5].

[2]Here conventional notations $\{f, g\} = \partial f \cdot g - \partial g \cdot f$ and $\left[z^{-1}\right] = \left(1/z, 1\left(2z^2\right), \ldots\right)$ are used.

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The Construction of Alternative Modified KdV Equation in $(2+1)$ Dimensions

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A typical and effective way to construct a higher dimensional integrable equation is to extend the Lax pair for a $(1+1)$ dimensional equation known as integrable to higher dimensions. Here we construct an alternative modified KdV equation in $(2+1)$ dimensions by the higher-dimensional extension of a Lax pair. And it is shown that this higher dimensional modified KdV equation passes the Painlevé test (WTC method).

1 Introduction

A central and so active topic in the theory of integrable systems is to construct as many higher dimensional integrable systems as possible. The Lax representation is a powerful tool for constructing integrable equations in $(2+1)$ dimensions. In this paper we will derive a $(2+1)$ dimensional equation of the modified KdV (mKdV) equation. Let us first recall here that the mKdV equation in $(1+1)$ dimensions reads

$$v_t + \frac{1}{4}v_{xxx} + \frac{3}{2}v^2 v_x = 0. \tag{1}$$

Higher dimensional integrable equations are not usually unique, in the sense that there exist several equations that reduce to a given one under dimensional reduction. It is widely known, for instance, that

$$v_t + \frac{1}{4}v_{xxz} + v^2 v_z + \frac{1}{2}v_x \partial_x^{-1}\left(v^2\right)_z = 0 \tag{2}$$

and

$$v_t + \frac{1}{4}v_{xxx} + \frac{3}{4}v_x\partial_z^{-1}\left\{v\left(\partial_z^{-1}v_x\right)_x\right\} + \frac{3}{4}\left(\partial_z^{-1}v_x\right)\left(v\partial_z^{-1}v_x\right)_x - \frac{3}{4}v_x\left(\partial_z^{-1}v_x\right)^2 = 0 \tag{3}$$

are the higher-dimensional mKdV equations [1, 2, 3]. It is easy to check equation (2) and (3) are reduced to equation (1), setting $z = x$. Our goal in this paper is to add into them an alternative one derived from the higher-dimensional extension of a Lax pair.

It is well-known that the Lax representation [4] describes $(1+1)$ dimensional integrable equations as follows. Consider two operators L and T which are called the Lax pair and given by

$$L = L_0 - \lambda, \tag{4}$$
$$T = \partial_x(L_0) + T_0' + \partial_t, \tag{5}$$

with λ being a spectral parameter independent upon t. Then the commutator

$$[L, T] = 0 \tag{6}$$

contains a nonlinear evolution equation for suitably chosen L and T. Equation (6) is so-called the Lax equation. For example if we take

$$L_0 = L_{\mathrm{mKdV}} = \partial_x^2 + 2\sigma v \partial_x, \tag{7}$$

$$T_0' = T'_{\mathrm{mKdV}} = \sigma v \partial_x^2 - \left(\frac{3}{2}v^2 + \frac{1}{2}\sigma v_x\right)\partial_x, \tag{8}$$

with $\sigma = \pm i$, then L_{mKdV} and T'_{mKdV} satisfy the Lax equation (6) provided that $v(x,t)$ satisfies the mKdV equation (1). By operator L_{mKdV1} (7), the mKdV equation (1) can be extended to the higher-dimensional ones (2) and (3). In this paper we will extend the mKdV equation (1) to an alternative $(2+1)$ dimensional one by taking a different L_0 operator

$$L_0 = \partial_x^2 + v\partial_x + \frac{1}{4}v^2 + \frac{1}{2}v_x. \tag{9}$$

This paper is organised as follows. In Section 2, we shall begin with verifying a different L_0 operator (9) gives the mKdV equation in $(1+1)$ dimensions. In the process, we use the Painlevé test. Next an alternative mKdV equation in $(2+1)$ dimensions is introduced by the extension of T operator of the Lax pair. In this process, we also need to perform the Painlevé test. Section 5 contains our summary.

2 The modified KdV equation

In this section, let us show the mKdV equation (1) can be constructed by the operator L_0

$$L_0 = L_{\mathrm{mKdV}'} = \partial_x^2 + v\partial_x + \frac{1}{4}v^2 + \frac{1}{2}v_x. \tag{10}$$

The Lax pair (4) and (5) are given by

$$L = L_{\mathrm{mKdV}'} - \lambda, \tag{11}$$

$$T = \partial_x(L_{\mathrm{mKdV}'}) + T'_{\mathrm{mKdV}'} + \partial_t, \tag{12}$$

where $T'_{\mathrm{mKdV}'}$ is an unknown operator. And then the Lax equation (6) gives

$$[L_{\mathrm{mKdV}'} - \lambda, \partial_x(L_{\mathrm{mKdV}'}) + \partial_t] + [L_{\mathrm{mKdV}'} - \lambda, T'_{\mathrm{mKdV}'}] = 0. \tag{13}$$

The first term in the left-hand side of equation (13) gives

$$\begin{aligned}[L_{\mathrm{mKdV}'} - \lambda, \partial_x(L_{\mathrm{mKdV}'}) + \partial_t] &= -v_x\partial_x^3 - \left(\frac{3}{2}vv_x + \frac{1}{2}v_{xx}\right)\partial_x^2 \\ &- \left(\frac{3}{2}v_x^2 + \frac{3}{4}v^2v_x + \frac{1}{2}v_{xx} + v_t\right)\partial_x + \left(\frac{3}{4}v_xv_{xx} + \frac{1}{2}vv_{xxx} + \cdots\right),\end{aligned} \tag{14}$$

where note that $\partial_x(L_{\mathrm{mKdV}'}) = \partial_x^3 + v\partial_x^2 + \left(\frac{1}{4}v^2 + \frac{3}{2}v_x\right)\partial_x + \frac{1}{2}vv_x + \frac{1}{2}v_{xx}$. So we choose here the form of the operator $T'_{\mathrm{mKdV}'}$ so that it involves, at least, a second-order differential operator,

$$T'_{\mathrm{mKdV}'} = U\partial_x^2 + V\partial_x + W, \tag{15}$$

where U, V and W are functions of x and t. Then the second term in the left-hand side of equation (13) gives

$$\begin{aligned}[L_{\mathrm{mKdV}'} - \lambda, T'_{\mathrm{mKdV}'})] &= 2U_x\partial_x^3 + (U_{xx} + 2V_x + vU_x - 2Uv_x)\partial_x^2 \\ &+ (V_{xx} + 2W_x + vV_x - Vv_x - Uvv_x - 2Uv_{xx})\partial_x + (W_{xx} + vW_x + \cdots).\end{aligned} \tag{16}$$

By comparing the first term (14) to the second one (16),

$$U = \frac{v}{2}, \tag{17}$$

$$V = \frac{v^2}{2}, \tag{18}$$

$$2W_x = v_t + \frac{1}{2}vv_{xx} + \frac{1}{2}v_x^2 + \frac{3}{4}v^2v_x \tag{19}$$

and

$$W_{xx} + vW_x + \frac{3}{4}v_xv_{xx} + \frac{1}{2}vv_{xxx} + \cdots = 0, \tag{20}$$

where equation (20) is an identity by U, V and W_x. The exact forms of U and V have obtained and one of W has not yet.

Now let us get it by applying the Painlevé test in the sense of Weiss–Tabor–Carnevale (WTC) method [5]. For that, let us compute the degree of variables in equation (19). Equation (19) demands that, if taking $[\partial_x] = 1$,

$$[v] = 1, \tag{21}$$

$$[\partial_t] = 3, \tag{22}$$

$$[W_x] = 4, \tag{23}$$

where $[*]$ means the degree of a variable $*$. These degrees lead us to take as unknown function W_x

$$-2W_x = \alpha v_{xxx} + \beta vv_{xx} + \gamma v_x^2 + \delta v^2v_x, \tag{24}$$

where α, β, γ and δ are real constants. Equation (19) reads

$$v_t + \alpha v_{xxx} + \left(\beta + \frac{1}{2}\right)vv_{xx} + \left(\gamma + \frac{1}{2}\right)v_x^2 + \left(\delta + \frac{3}{4}\right)v^2v_x. \tag{25}$$

Now we show four constants in equation (25) are obtained such as passing the Painlevé test (WTC method). The solution to equation (1) has the form

$$v \sim v_0\phi^\eta. \tag{26}$$

Here ϕ is single valued about an arbitrary movable singular manifold. In η is a negative integer (leading order). By using leading order analysis, we obtain

$$\eta = -1. \tag{27}$$

Substituting

$$v(x,t) = \sum_{j=0} v_j(x,t)\phi(x,t)^{j-1} \tag{28}$$

leads to the resonances, after trivial algebra,

$$j = -1,\ 3,\ 4, \tag{29}$$

in the condition

$$\beta = \gamma = -\frac{1}{2}. \tag{30}$$

To simplify the calculations, we use the reduced manifold ansatz of Kruskal:

$$\phi(x,t) = x + \rho(t), \tag{31}$$

$$v_j(x,t) = v_j(t). \tag{32}$$

The resonance $j = -1$ in (29) corresponds to the arbitrary singularity manifold ϕ. We used *MATHEMATICA* to handle the computation for the existence of arbitrary functions corresponding to the resonances except $j = -1$. We find that v_3 and v_4 are arbitrary for equation (25). Thus the general solution v to equation (25) admits the sufficient number of arbitrary functions, thus passing the Painlevé test with the condition (30). Then equation (25) is reduced to the mKdV equation

$$v_t + \alpha v_{xxx} + \left(\delta + \frac{3}{4}\right) v^2 v_x = 0, \tag{33}$$

where α and δ are still arbitrary. Hereafter let us choose

$$\alpha = \frac{1}{4} \qquad \text{and} \qquad \delta = \frac{3}{4}. \tag{34}$$

This choice, of course, is meaningless. From condition (30) and (34), W and $T'_{\mathrm{mKdV}'}$ are given, respectively, by

$$W = -\frac{1}{8}v_{xx} + \frac{1}{4}vv_x - \frac{1}{8}v^3, \tag{35}$$

$$T'_{\mathrm{mKdV}'} = \frac{1}{2}v\partial_x^2 + \frac{1}{2}v^2\partial_x - \frac{1}{8}v_{xx} + \frac{1}{4}vv_x - \frac{1}{8}v^3. \tag{36}$$

Namely it has been shown that the operator $L_{\mathrm{mKdV}'}$ can give the mKdV equation (1) by the Lax equation (6) and the Painlevé test.

3 An extension the modified KdV equation to $(2+1)$ dimensions

It is well known that the Lax differential operator plays a key role in constructing higher dimensional equations from lower dimensional ones. We extend only T operator to $(2+1)$ dimensions as follows [1, 6, 7]

$$T = \partial_z(L_{\mathrm{mKdV}'}) + \widetilde{T}_{\mathrm{mKdV}'} + \partial_t. \tag{37}$$

Here z is a new spatial coordinate. Then the Lax pair is given by

$$L = L_{\mathrm{mKdV}'} - \lambda, \tag{38}$$

$$T = \partial_z\left(L_{\mathrm{mKdV}'}\right) + \widetilde{T}_{\mathrm{mKdV}'} + \partial_t, \tag{39}$$

where note that $\partial_z(L_{\mathrm{mKdV}'}) = \partial_x^2\partial_z + v\partial_x\partial_z + v_z\partial_x + \left(\frac{1}{4}v^2 + \frac{1}{2}v_x\right)\partial_z + \frac{1}{2}vv_z + \frac{1}{2}v_{xz}$ and $\widetilde{T}_{\mathrm{mKdV}'}$ is an unknown operator. So we obtain

$$[L_{\mathrm{mKdV}'} - \lambda, \partial_z(L_{\mathrm{mKdV}'}) + \partial_t] + [L_{\mathrm{mKdV}'} - \lambda, \widetilde{T}_{\mathrm{mKdV}'}] = 0, \tag{40}$$

from the Lax equation (6) of the pair (38) and (39). The first term in the left-hand side of equation (40) gives

$$\begin{aligned}
[L_{\mathrm{mKdV}'} - \lambda, \partial_z(L_{\mathrm{mKdV}'}) + \partial_t] &= -v_z\partial_x^3 - \left(\frac{3}{2}vv_z + \frac{1}{2}v_{xz}\right)\partial_x^2 \\
&- \left(\frac{3}{2}v_xv_z + \frac{3}{4}v^2v_z + \frac{1}{2}v_{xz} + v_t\right)\partial_x + \left(\frac{3}{4}v_xv_{xz} + \frac{1}{2}vv_{xxz} + \cdots\right).
\end{aligned} \tag{41}$$

As in $(1+1)$ dimensions, we assume the form of the operator $\widetilde{T}_{\mathrm{mKdV}'}$

$$\widetilde{T}_{\mathrm{mKdV}'} = U\partial_x^2 + V\partial_x + W, \tag{42}$$

where U, V and W are functions of x, z and t. Then the second term in the left-hand side of equation (40) gives

$$\begin{aligned}[L_{\mathrm{mKdV}'} - \lambda, \widetilde{T}_{\mathrm{mKdV}'})] &= 2U_x\partial_x^3 + (U_{xx} + 2V_x + vU_x - 2Uv_x)\,\partial_x^2 \\ &\quad + (V_{xx} + 2W_x + vV_x - Vv_x - Uvv_x - 2Uv_{xx})\,\partial_x + (W_{xx} + vW_x + \cdots).\end{aligned} \tag{43}$$

By comparing the first term (41) to the second one (43),

$$U = \frac{1}{2}\partial_x^{-1}v_z, \tag{44}$$

$$V = \frac{1}{2}v\left(\partial_x^{-1}v_z\right), \tag{45}$$

$$2W_x = v_t + \frac{1}{4}v^2v_z + \frac{1}{2}vv_x\left(\partial_x^{-1}v_z\right) + \frac{1}{2}v_xv_z + \frac{1}{2}v_{xx}\left(\partial_x^{-1}v_z\right) \tag{46}$$

and

$$W_{xx} + vW_x + \frac{3}{4}v_xv_{xz} + \frac{1}{2}vv_{xxz} + \cdots = 0, \tag{47}$$

where equation (47) is an identity by U, V and W_x. The exact forms of U and V have obtained and one of W has not yet as in $(1+1)$ dimensions.

Let us compute the degree of variables in equation (46), if taking $[\partial_x] = 1$,

$$[v] = 1, \tag{48}$$

$$[\partial_t] = 2 + [\partial_z], \tag{49}$$

$$[W_x] = 3 + [\partial_z], \tag{50}$$

with $[\partial_z]$ being arbitrary. These degrees lead us to take as unknown function W_x

$$\begin{aligned}-2W_x &= av_{xxz} + bv_{xx}\left(\partial_x^{-1}v_z\right) + cvv_{xz} + dv_xv_z \\ &\qquad + evv_x\left(\partial_x^{-1}v_z\right) + fv^2v_z + gv^3\left(\partial_x^{-1}v_z\right),\end{aligned} \tag{51}$$

where all from a to g is real constant. Substituting W_x into equation (46) gives

$$\begin{aligned}&v_t + av_{xxz} + \left(b + \frac{1}{2}\right)v_{xx}\left(\partial_x^{-1}v_z\right) + cvv_{xz} + \left(d + \frac{1}{2}\right)v_xv_z \\ &\quad + \left(e + \frac{1}{2}\right)vv_x\left(\partial_x^{-1}v_z\right) + \left(f + \frac{1}{4}\right)v^2v_z + gv^3\left(\partial_x^{-1}v_z\right) = 0.\end{aligned} \tag{52}$$

Here we perform the Painlevé test for equation (52) to get real constants in it. For that, we need to rewrite equation (52) for taking away the term of ∂_x^{-1}. That exact form, however, is very complicated for writing down here. We would like to write down the result. That is,

$$\text{leading order}: \quad -1 \tag{53}$$

$$\text{resonances}: \quad -1,\ 1,\ 3,\ 4 \tag{54}$$

$$\begin{aligned}\text{real constants}: \quad & b = d = -\frac{1}{2}, \quad c = g = 0, \\ & \text{and other constants are arbitrary.}\end{aligned} \tag{55}$$

Thus equation (52) gives

$$v_t + \frac{1}{4} v_{xxz} + \left(\frac{1}{2} + e\right) v v_x \left(\partial_x^{-1} v_z\right) + (1 - e)\, v^2 v_z = 0, \tag{56}$$

where we take $a = \frac{1}{4}$ and $f = \frac{3}{4} - e$ for being reduced to the mKdV equation (1) setting $z = x$. This equation (56) is quite different from the higher dimensional mKdV equation (2) and (3).

That is, the alternative mKdV equation (56) in $(2+1)$ dimensions was given by the Lax equation (6) and the Painlevé test.

4 Summary

A natural problem in the integrable systems is whether we can find new $(2+1)$ dimensional integrable equations from already known $(1+1)$ dimensional integrable ones. The Lax representation is a powerful tool to do so. The method used in this paper is based on works by Calogero et al.

Our results in this paper are as follows.

(i) The $(1+1)$ dimensional mKdV equation (1) has been obtained by the Lax pair

$$L = \partial_x^2 + v\partial_x + \frac{1}{4}v^2 + \frac{1}{2}v_x - \lambda, \tag{57}$$

$$T = \partial_x^3 + \frac{3}{2}v\partial_x^2 + \left(\frac{3}{4}v^2 + \frac{3}{2}v_x\right)\partial_x + \frac{3}{4}vv_x + \frac{3}{8}v_{xx} - \frac{1}{8}v^3 + \partial_t. \tag{58}$$

(ii) By extending the Lax pair (57) and (58) to $(2+1)$ dimensions, the higher dimensional mKdV equation (56) has been introduced. And then the Lax pair is given by

$$L = \partial_x^2 + v\partial_x + \frac{1}{4}v^2 + \frac{1}{2}v_x - \lambda, \tag{59}$$

$$\begin{aligned} T = {} & \partial_x^2\partial_z + \frac{1}{2}\left(\partial_x^{-1}v_z\right)\partial_x^2 + v\partial_x\partial_z + \left(v_z + \frac{1}{2}v\partial_x^{-1}v_z\right)\partial_x + \left(\frac{1}{4}v^2 + \frac{1}{2}v_x\right)\partial_z + \frac{3}{8}v_{xz} \\ & + \frac{1}{4}v_x\partial_x^{-1}v_z + \frac{1}{2}vv_z - \frac{e}{4}v^2\partial_x^{-1}v_z + \left(\frac{3e}{4} - \frac{3}{8}\right)\partial^{-1}\left(v^2 v_z\right) + \partial_t \end{aligned} \tag{60}$$

This equation is integrable in the sense of the existence of the Lax pair and passing the Painlevé test.

Next let us mention our further works.

(i) The higher dimensional mKdV equations (2) and (3) have various exact solutions (soliton solution and so on) [2, 3]. They constructed via Bilinear approach or Hirota method. We have not been able to constructed exact solutions to equation (56) yet.

(ii) We will extend the Lax pair (57) and (58) to (2+1) dimensions by using other method [7, 8].

We believe that higher dimensional integrable equations can be obtained from lower dimensional integrable ones by extending the Lax pairs to higher dimensions. We have a dream such as constructing $(3+1)$ dimensional integrable equations (if there exist). Further study on this topic continues.

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A Novel Nonlinear Evolution Equation Integrable by the Inverse Scattering Method

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In this report we consider the nonlinear evolution equation $(u_t + uu_x)_x + u = 0$ (Vakhnenko equation – VE) that can be integrated by the inverse scattering transform (IST) method. This equation arose as a result describing the high-frequency perturbations in a relaxing medium. The VE has two families of travelling wave solutions, both of which are stable to long wavelength perturbations. In particular, the VE has a loop-like soliton solution. The interaction of two solitons by both Hirota's method and the IST method are considered. The associated eigenvalue problem has been formulated. This has been achieved by finding a Bäcklund transformation. The inverse scattering method has a third order eigenvalue problem. Under the interaction of solitons there are features that are not typical for the KdV equation.

1 Introduction

Describing real media under the action of intense waves is often unsuccessful in the framework of equilibrium models of continuum mechanics. To develop physical models for wave propagation through media with complicated inner kinetics, the notions based on the relaxational nature of a phenomenon are regarded to be promising. A nonlinear evolution equation is suggested to describe the propagation of waves in a relaxing medium [1]. It is shown that for the low-frequency approach this equation is reduced to the Korteweg-de Vries (KdV) equation. In contrast to the low-frequency perturbations, the high-frequency perturbations satisfy a new nonlinear equation [2]

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)u + u = 0. \tag{1}$$

The equation (1) has been studied in various Refs. [2, 3, 4, 5, 6, 7, 8, 9]. Hereafter, as was initiated in [3], this equation is referred to as the Vakhnenko equation (VE). There is a certain analogy between the KdV equation and the VE. They have the same hydrodynamic nonlinearity and do not contain dissipative terms; only the dispersive terms are different. It turns out that the VE possesses, at least partially, the remarkable properties inherent to the KdV equation. The study of the VE has scientific interest both from the viewpoint of the existence of stable wave formations and from the viewpoint of the general problem of integrability of nonlinear equations.

2 Physical processes described by the Vakhnenko equation

From the nonequilibrium thermodynamics standpoint, the models of a relaxing medium are more general than the equilibrium models. Thermodynamic equilibrium is disturbed owing to the propagation of fast perturbations. There are processes of the interaction that tend to return the equilibrium. In essence, the change of macroparameters caused by the changes of inner parameters is a relaxation process.

To analyze the wave motion, we use the hydrodynamic equations in Lagrangian coordinaties:

$$\frac{\partial V}{\partial t} - \frac{1}{\rho_0}\frac{\partial u}{\partial x} = 0, \qquad \frac{\partial u}{\partial t} + \frac{1}{\rho_0}\frac{\partial p}{\partial x} = 0. \tag{2}$$

The following dynamic state equation is applied to account for the relaxation effects:

$$d\rho = c_f^{-2} dp + \tau_p^{-1}(\rho - \rho_e)\, dt. \tag{3}$$

We note that the mechanisms of the exchange processes are not defined concretely when deriving the dynamic state equation (3). In this equation the thermodynamic and kinetic parameters appear only as sound velocities c_e, c_f and relaxation time τ_p. These characteristics can be found experimentally.

Let us consider a small nonlinear perturbation $p' < p_0$. Combining the relationships (2), (3) we obtain the nonlinear evolution equation in one unknown p (the dash in p' is omitted) [1]

$$\tau_p \frac{\partial}{\partial t}\left(\frac{\partial^2 p}{\partial x^2} - c_f^{-2}\frac{\partial^2 p}{\partial t^2} + \alpha_f \frac{\partial^2 p^2}{\partial t^2}\right) + \left(\frac{\partial^2 p}{\partial x^2} - c_e^{-2}\frac{\partial^2 p}{\partial t^2} + \alpha_e \frac{\partial^2 p^2}{\partial t^2}\right) = 0. \tag{4}$$

A similar equation has been obtained by Clarke [10], but without nonlinear terms.

In [1] it is shown that for low-frequency perturbations ($\tau_p\omega \ll 1$) the equation (4) is reduced to the Korteweg-de Vries – Burgers (KdVB) equation

$$\frac{\partial p}{\partial t} + c_e \frac{\partial p}{\partial x} + \alpha_e c_e^3 p \frac{\partial p}{\partial x} - \beta_e \frac{\partial^2 p}{\partial x^2} + \gamma_e \frac{\partial^3 p}{\partial x^3} = 0,$$

while for high-frequency waves ($\tau_p\omega \gg 1$) we have obtained the new equation

$$\frac{\partial^2 p}{\partial x^2} - c_f^{-2}\frac{\partial^2 p}{\partial t^2} + \alpha_f c_f^2 \frac{\partial^2 p^2}{\partial x^2} + \beta_f \frac{\partial p}{\partial x} + \gamma_f p = 0. \tag{5}$$

The nonlinear equation (5) has dissipative $\beta_f \partial p/\partial x$ and dispersive $\gamma_f p$ terms. Without nonlinear and dissipative terms, we have a linear Klein–Gordon equation.

In the general case the last equation has been investigated insufficiently. It is likely that this is connected with the fact, noted by Whitham [11], that the high-frequency perturbations attenuate very fast. However in Whitham's monograph, the evolution equation without nonlinear and dispersive terms was considered. Certainly, the lack of such terms restricts the class of solutions. At least, there is no solution in the form of a solitary wave, which is caused by nonlinearity and dispersion.

3 Evolution equation for high-frequency perturbations

The equation (5), which we are interested in, is written down in dimensionless form. In the moving coordinates system with velocity c_f, the equation has the form in dimensionless variables $\tilde{x} = \sqrt{\frac{\gamma_f}{2}}(x - c_f t)$, $\tilde{t} = \sqrt{\frac{\gamma_f}{2}} c_f t$, $\tilde{u} = \alpha_f c_f^2 p$ (tilde over variables $\tilde{x}$, $\tilde{t}$, $\tilde{u}$ is omitted)

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right) u + \alpha \frac{\partial u}{\partial x} + u = 0. \tag{6}$$

The constant $\alpha = \beta_f/\sqrt{2\gamma_f}$ is always positive. Equation (6) without the dissipative term has the form of the nonlinear equation [2, 3] (see equation (1))

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right) u + u = 0. \tag{7}$$

The travelling-wave solutions of the VE (7) were derived in [2, 3], and its symmetry properties were studied in [5]. A remarkable feature of the VE is that it has a soliton solution which has loop-like form, i.e. it is a multi-valued function (see Fig. 1 in [2]). Whilst loop soliton solutions are rather intriguing, it is the solution to the initial value problem that is of more interest in a physical context.

The physical interpretation of the multi-valued functions that describe the loop-like soliton solutions was given in [1]. The problem is whether the ambiguity has a physical nature or is related to the incompleteness of the mathematical model, in particular to the lack of dissipation. It is significant that the loop-like solutions are stable to long-wavelength perturbations [3], and that the introduction of a dissipative term (see equation (6)), with dissipation parameter less than some limiting value, does not destroy these loop-like solutions [1]. Since the solution has a parametric form [2, 3], there is a space of variables in which the solution is a single-value function. Consequently, the ambiguity of solution does not relate to the incompleteness of the mathematical model. Thus in the framework of this model approach, the high-frequency perturbation can be described by the multi-valued functions [1].

We have succeeded in finding new coordinates (X, T), in terms of which the solution of equation (1) is given by single-valued parametric relations. New independent coordinates X, T are defined as [4]

$$x = x_0 + T + W(X, T), \qquad t = X, \qquad W = \int_{-\infty}^{X} U(X', T)\, dX'. \tag{8}$$

Here $u(x, t) = U(X, T)$, and x_0 is a constant. We also assume that, as $X \to -\infty$, the derivatives of W vanish and W tends to a constant. Equation (1) then has the form [4, 8]

$$W_{XXT} + (1 + W_T)W_X = 0. \tag{9}$$

If the solution $U(X, T) = W_X$ of the transformed VE (9) has been obtained, the original independent space coordinate x can be found by means of the formula (8). This relationship together with $u(x, t) = U(X, T)$ enables us to define the solution of the VE in parametric form with T as parameter. We note that the transformation (9) between old and new coordinates is similar to the transformation between Eulerian variables (x, t) and Lagrangian variables (T, X) [8].

Finally, by taking $W = 6(\ln f)_X$, where f is a function of X and T, we observe that the transformed VE (9) may be written as the bilinear equation [4]

$$\left(D_T D_X^3 + D_X^2\right) f \cdot f = 0, \tag{10}$$

where D is the Hirota binary operator [12].

4 Bäcklund transformation for the transformed Vakhnenko equation

We present a Bäcklund transformation for equation (10), following the method developed in [13]. It is well known that the Bäcklund transformation is one of the analytical tools for dealing with soliton problems and has a close relationship to the IST method [12, 13, 14]. First we define P as follows:

$$P := \left[\left(D_T D_X^3 + D_X^2\right) f' \cdot f'\right] ff - f'f' \left[\left(D_T D_X^3 + D_X^2\right) f \cdot f\right].$$

We aim to find a pair of equations such that each equation is linear in each of the dependent variables f and f', and such that together f and f' satisfy $P = 0$. The pair of equations is the required Bäcklund transformation.

Combining this relationship we can rewrite P in the following form [8]:

$$P = 2D_T\left(\left\{D_X^3 - \lambda(X)\right\} f' \cdot f\right) \cdot (f'f) - 2D_X(\{3D_T D_X + 1 + \mu(T)D_X\}f' \cdot f) \cdot (D_X f' \cdot f).$$

Thus we have proved [8] that the Bäcklund transformation is given by the two equations

$$\left(D_X^3 - \lambda\right) f' \cdot f = 0, \tag{11}$$

$$(3D_X D_T + 1 + \mu D_X) f' \cdot f = 0, \tag{12}$$

where $\lambda = \lambda(X)$ is an arbitrary function of X and $\mu = \mu(T)$ is an arbitrary function of T. In original form with $\mu = 0$ we have

$$(W' - W)_{XX} + \frac{1}{2}(W' - W)(W' + W)_X + \frac{1}{36}(W' - W)^3 - 6\lambda = 0, \tag{13}$$

$$\begin{aligned} &(W' - W)\left[3(W' + W)_{XT} + \frac{1}{2}(W' - W)(W' - W)_T\right] \\ &\qquad - 6(W' - W)_X\left[1 + \frac{1}{2}(W' + W)_T\right] = 0. \end{aligned} \tag{14}$$

Separately the two equations (11), (12) appear as part of the Bäcklund transformation for other nonlinear evolution equations. For example, equation (11) is the same as one of the equations that is part of the Bäcklund transformation for a higher order KdV equation (see equation (5.139) in [12]), and equation (12) is similar to (5.132) in [12] that is part of the Bäcklund transformation for a model equation for shallow water waves [15].

5 Interaction of the solitons

The transformation into new coordinates (8) is the key to solving the problem of the interaction of the solitons. The exact N-soliton solutions are obtained by use of (i) Hirota's method [4, 7]; (ii) elements of the inverse scattering transform procedure for the KdV equation (spectral equation of second order – Schrödinger equation) [6]; (iii) the inverse scattering transform procedure (spectral equation of third order) [9].

Since the equation (1) can be written in bilinear form (10), Hirota's method enables us to find soliton solutions. These solutions have been obtained in [4, 7], for example, for the one-soliton solution

$$f = 1 + \exp(2\eta), \qquad W = 6(\ln f)_X, \qquad \eta = kX - \omega T + \alpha, \qquad U = W_X = 6k^2 \operatorname{sech}^2\eta,$$

and for the two-soliton solution

$$f = 1 + \exp(2\eta_1) + \exp(2\eta_2) + b^2 \exp(2\eta_1 + 2\eta_2), \qquad W = 6(\ln f)_X,$$

$$b^2 = \frac{F\left[2(k_1 - k_2), -2(\omega_1 - \omega_2)\right]}{F\left[2(k_1 + k_2), -2(\omega_1 + \omega_2)\right]} = \frac{(k_2 - k_1)^2}{(k_2 + k_1)^2}\frac{k_1^2 + k_2^2 - k_1k_2}{k_1^2 + k_2^2 + k_1k_2},$$

$$\eta_i = k_iX - \omega_iT + \alpha_i, \qquad F(D_X, D_T) := D_T D_X^3 + D_X^2.$$

Now we present IST method for finding the solution of the VE. The IST is the most appropriate way of tackling the initial value problem. The results of applying the IST method would be useful in solving the Cauchy problem for the VE. In order to use the IST method one first has to formulate the associated eigenvalue problem.

Introducing the function $\psi = f'/f$, we find that equations (11), (12) reduce to

$$\psi_{XXX} + U\psi_X - \lambda\psi = 0, \tag{15}$$
$$3\psi_{XT} + (W_T + 1)\psi + \mu\psi_X = 0, \tag{16}$$

respectively. It may be shown

$$[W_{XXT} + (1 + W_T)W_X]_X\psi + \lambda_X(3\psi_T + \mu\psi) = 0.$$

Hence equation (9) is the condition for $\lambda_X = 0$, and hence for λ to be constant. Constant λ (spectral parameter) is what is required in the IST problem.

Thus the IST problem is directly related to a spectral equation of third order (15). The third order eigenvalue problem is similar to the one associated with a higher order KdV equation [16, 17], a Boussinesq equation [16, 18], and a model equation for shallow water waves [12, 19].

Kaup [16], Caudrey [18, 20] and Deift et al. [21] studied the inverse problem for certain third order spectral equations. We adapt the results obtained by these authors to the present problem and describe a procedure for using the IST to find the N-soliton solution to the transformed VE, and hence to the VE itself.

We proved that the T-evolution of the scattering data is given by the relationships [9] ($k = 1, 2, \ldots, 2N$)

$$\begin{aligned} &\zeta_j^{(k)}(T) = \zeta_j^{(k)}(0), \\ &\gamma_{1j}^{(k)}(T) = \gamma_{1j}^{(k)}(0)\exp\left\{\left[-\left(3\lambda_j\left(\zeta_1^{(k)}\right)\right)^{-1} + \left(3\lambda_1\left(\zeta_1^{(k)}\right)\right)^{-1}\right]T\right\}. \end{aligned} \tag{17}$$

Here $\lambda_j(\zeta) = \omega_j\zeta$, $\lambda_j^3(\zeta) = \lambda$, and $\omega_j = e^{\mathrm{i}2\pi(j-1)/3}$ are the cube of roots of 1.

The final result for the N-soliton solution of the transformed VE is given by the relation [9]

$$U(X,T) = 3\frac{\partial^2}{\partial X^2}\ln\left(\det M(X,T)\right), \tag{18}$$

where M is the $2N\times 2N$ matrix given by

$$\begin{aligned} &M_{kl} = \delta_{kl} - \\ &\sum_{j=2}^{3}\gamma_{1j}^{(k)}(0)\,\frac{\exp\left\{\left[-\left(3\lambda_j\left(\zeta_1^{(k)}\right)\right)^{-1} + \left(3\lambda_1\left(\zeta_1^{(k)}\right)\right)^{-1}\right]T + \left(\lambda_j\left(\zeta_1^{(k)}\right) - \lambda_1\left(\zeta_1^{(l)}\right)\right)X\right\}}{\lambda_j\left(\zeta_1^{(k)}\right) - \lambda_1\left(\zeta_1^{(l)}\right)}, \end{aligned} \tag{19}$$

and the scattering data is calculated from constants ξ_m, β_m as

$$\begin{array}{llll} n = 1,\ 2, \ldots,\ N, & m = 2n-1, & & \\ \lambda_1\left(\zeta_1^{(m)}\right) = \mathrm{i}\omega_2\xi_m, & \lambda_2\left(\zeta_1^{(m)}\right) = \mathrm{i}\omega_3\xi_m, & \gamma_{12}^{(m)}(0) = \omega_2\beta_m, & \gamma_{13}^{(m)}(0) = 0, \\ \lambda_1\left(\zeta_1^{(m+1)}\right) = -\mathrm{i}\omega_3\xi_m, & \lambda_3\left(\zeta_1^{(m+1)}\right) = -\mathrm{i}\omega_2\xi_m, & \gamma_{12}^{(m+1)}(0) = 0, & \gamma_{13}^{(m+1)}(0) = \omega_3\beta_m. \end{array}$$

For the N-soliton solution there are N arbitrary constants ξ_m and N arbitrary constants β_m.

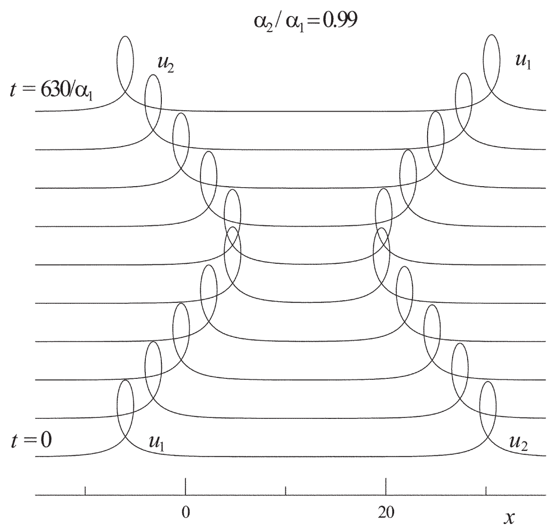

Figure 1. Interaction of two solitons in moving coordinates at time interval $\Delta t = 70/\alpha_1$.

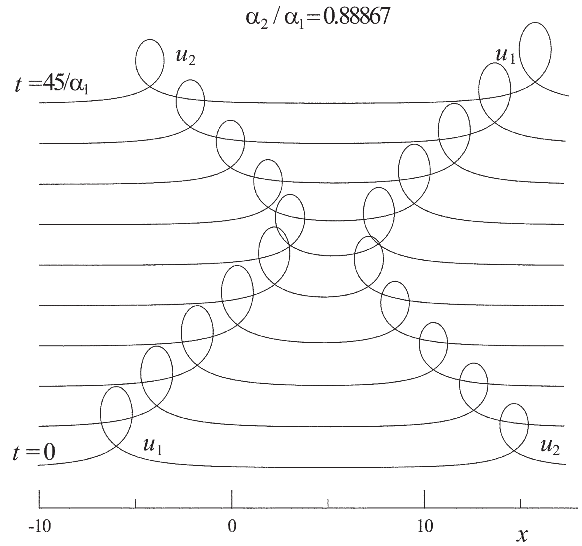

Figure 2. The phaseshifts of the smaller soliton is zero. Time interval is $\Delta t = 5/\alpha_1$.

For example, the matrix for one-soliton solution has a form

$$\begin{pmatrix} 1-\dfrac{\omega_2\beta_1}{\sqrt{3}\,\xi_1}\exp\left[\sqrt{3}\,\xi_1 X-(\sqrt{3}\,\xi_1)^{-1}T\right] & \dfrac{\mathrm{i}\omega_3\beta_1}{2\xi_1}\exp\left[2\mathrm{i}\omega_3\xi_1 X-(\sqrt{3}\,\xi_1)^{-1}T\right] \\ \dfrac{-\mathrm{i}\omega_2\beta_1}{2\xi_1}\exp\left[-2\mathrm{i}\omega_2\xi_1 X-(\sqrt{3}\,\xi_1)^{-1}T\right] & 1-\dfrac{\omega_3\beta_1}{\sqrt{3}\,\xi_1}\exp\left[\sqrt{3}\,\xi_1 X-(\sqrt{3}\,\xi_1)^{-1}T\right] \end{pmatrix}. \qquad (20)$$

Calculating the determinant

$$\det M = \left\{1+\frac{\beta_1}{2\sqrt{3}\,\xi_1}\exp\left[\sqrt{3}\,\xi_1\left(X-\frac{T}{3\xi_1^2}\right)\right]\right\}^2,$$

we have from (18) the one-soliton solution of the transformed VE as obtained by the IST method

$$U = 3\frac{\partial^2}{\partial X^2}\ln\left(\det M(X,T)\right) = \frac{9}{2}\,\xi_1^2\,\mathrm{sech}^2\left[\frac{\sqrt{3}}{2}\,\xi_1\left(X-\frac{T}{3\xi_1^2}\right)+\alpha_1\right], \qquad (21)$$

where $\alpha_1 = \frac{1}{2}\ln(\beta_1/2\sqrt{3}\xi_1)$ is an arbitrary constant.

The determinant of the matrix for two-soliton solution has a form

$$\det M = \left(1+q_1^2+q_2^2+b^2q_1^2q_2^2\right)^2, \qquad (22)$$

where

$$q_i = \exp\left[\frac{\sqrt{3}}{2}\,\xi_i\left(X-\frac{T}{3\xi_i^2}\right)+\alpha_i\right], \qquad b^2 = \left(\frac{\xi_2-\xi_1}{\xi_2+\xi_1}\right)^2\frac{\xi_1^2+\xi_2^2-\xi_1\xi_2}{\xi_1^2+\xi_2^2+\xi_1\xi_2},$$

and $\alpha_i = \frac{1}{2}\ln(\beta_i/2\sqrt{3}\xi_i)$ are arbitrary constants.

In the interaction of two solitons for the VE [4, 7, 6] there are features that are not typical for the KdV equation (see Figs. 1–3). The larger soliton moving with larger velocity catches up with the smaller soliton moving in the same direction. For convenience in the figures, the interactions of solitons are shown in coordinates moving with the speed of the centre mass.

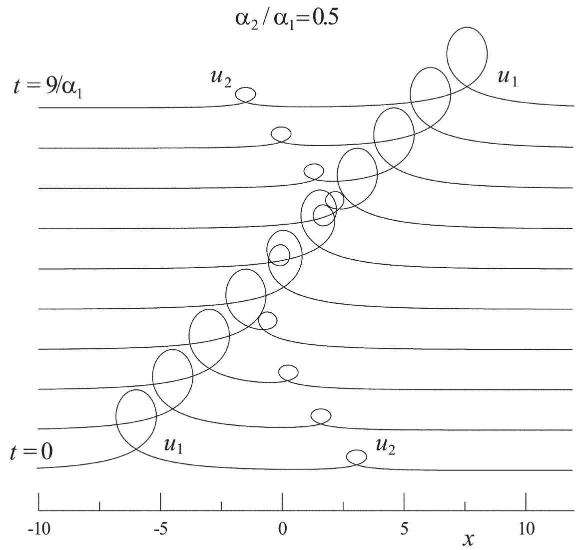

Figure 3. Both solitons have phaseshifts in the same direction. Time interval is $\Delta t = 1/\alpha_1$.

After the nonlinear interaction the solitons separate, their forms are restored, but phaseshifts arise. The larger soliton always has a forward phaseshift, while the smaller soliton can have three kinds of phaseshift. Note that this property is not typical for the KdV equation. There is a special value of the ratio $(\alpha_2/\alpha_1)^* = 0.88867$. The different kinds of phaseshift are illustrated in Figs. 1–3.

- For $\alpha_2/\alpha_1 > (\alpha_2/\alpha_1)^*$ the phaseshift of smaller soliton is in the opposite direction to the phaseshift of the larger soliton (Fig. 1).
- For $\alpha_2/\alpha_1 = (\alpha_2/\alpha_1)^*$ the smaller soliton has no phaseshift (Fig. 2).
- For $\alpha_2/\alpha_1 < (\alpha_2/\alpha_1)^*$ less critical value both solitons have phaseshifts in the same direction (Fig. 3).

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Цей том "Праць Інституту математики НАН України" є збірником статей учасників Четвертої міжнародної конференції "Симетрія у нелінійній математичній фізиці". Збірник складається з двох частин, кожна з яких видана окремою книгою.

Дане видання є другою частиною і включає праці, присвячені проблемам алгебри, суперсиметрії та застосуванню теоретико-групових методів у сучасній математичній та теоретичній фізиці та інших природничих науках. Основні розділи, відображені у цій частині, – квантові групи, деформації алгебр, розширені та узагальнені суперсиметрії та інші прикладні проблеми, які можуть бути дослідженні з використанням симетрійних методів.

Розраховано на наукових працівників, аспірантів, які цікавляться сучасними проблемами алгебри та узагальнених симетрій математичних моделей.

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This volume of the Proceedings of Institute of Mathematics of NAS of Ukraine includes papers of participants of the Fourth International Conference "Symmetry in Nonlinear Mathematical Physics". The collection consists of two parts which are published as separate issues.

This issue is the second part which is devoted to problems of algebra, supersymmetry and applications of group theoretical methods in modern mathematical and theoretical physics and other natural sciences. The main topics covered are quantum groups, deformations of algebras, extended and generalized supersymmetry, and some applied problems which can be investigated using symmetry methods.

The book may be useful for researchers and post graduate students who are interested in modern problems of algebra and generalized symmetries of mathematical models.

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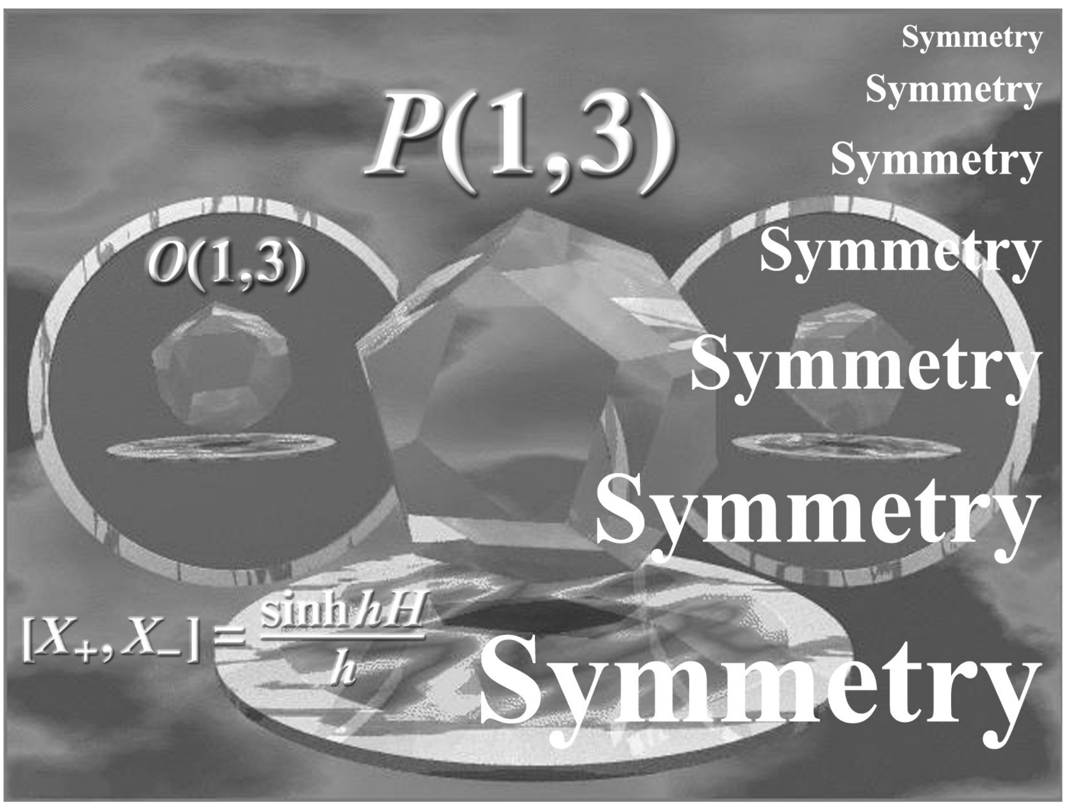

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On the Heisenberg–Lie Algebra and Some Non-Hermitian Operators in Oscillatorlike Developments

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After a few generalities we compare fundamental quantum mechanics applied to the harmonic oscillator with unusual oscillatorlike developments dealing with non-hermitian operators, this latter aspect exploiting in particular the property of subnormality. In this last context we can also restore the hermiticity of the Hamiltonian operator and discover a nice property of the new scalar product. General constructions of oscillatorlike Hamiltonians are also considered and new ideas connected with Heisenberg relations are presented.

1 Introduction

In every treatise on Quantum Mechanics [1] we learn that physical observables are represented by linear and self-adjoint operators acting on states belonging to Hilbert spaces characterized by a well defined scalar product. It is the case for important observables such as position, momentum and energy in particular, the last one being in correspondence with the (quantum) Hamiltonian operator (ensuring by its self-adjointness to have a real spectrum). The operators generate between themselves a Lie algebra which in the context of position and momentum is called the Heisenberg algebra

$$[x,p] = iI, \qquad [x,x] = [p,p] = 0 \qquad (\hbar = 1).$$

When the one-dimensional harmonic oscillator is described, this algebra can be put on the form

$$[a,a^\dagger] = I, \qquad [a,a] = [a^\dagger,a^\dagger] = 0 \qquad (\omega = 1), \tag{1}$$

where

$$a = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right), \qquad a^\dagger = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right), \qquad (a^\dagger)^\dagger = a,$$

a being known as the annihilation operator and $a^\dagger$ as the creation one, acting on Fock states $\{|n\rangle, n = 0,1,2,\ldots\}$. Within such developments, the Hamiltonian

$$H_{\mathrm{H.O.}} = \frac{1}{2}\{a,a^\dagger\} = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}x^2$$

has the *real* spectrum

$$E_n = n + \frac{1}{2} \tag{2}$$

and the set of eigenfunctions

$$h_n(x) = \frac{\pi^{-\frac{1}{4}}}{\sqrt{2^n n!}} e^{-\frac{x^2}{2}} H_n(x),$$

where H_n are the well known (classical) Hermite polynomials. Let us also recall that we have the so-called intertwining relations

$$[H_{\mathrm{H.O.}}, a] = -a, \qquad [H_{\mathrm{H.O.}}, a^\dagger] = a^\dagger \tag{3}$$

characteristic of these developments. It is important to notice here that all these elements are ad-hoc ones for getting interesting constructions of the famous coherent states [2] but not for visiting squeezed states [3].

Let us now enter a recent property called "subnormality of unbounded operators" [4] and let us mention its definition: "a densely defined Hilbert space operator S is said to be subnormal if there is a normal operator N (acting possibly in a larger space) such that S is included in N". Mathematicians have shown that "the best behaving unbounded subnormal operator is the famous creation operator $a^\dagger$ of the harmonic oscillator acting in $L^2(\mathbb{R})$". It is equivalent to the one defined by

$$a^\dagger_\lambda = a^\dagger + \lambda I, \qquad \lambda \in \mathbb{R},$$

an interesting way to introduce a new supplementary parameter in oscillatorlike developments.

By noticing that the previous commutation relations (1) and (3) are unchanged by the substitution $a^\dagger \to a^\dagger_\lambda$, we point out that the new Hamiltonian H_λ becomes

$$H_\lambda = H_{\mathrm{H.O.}} + \lambda a.$$

It is no more self-adjoint (notice that $(a^\dagger_\lambda)^\dagger \neq a$) *but* still has a real spectrum (2) and eigenfunctions depending now on λ. Indeed we get [5] a Fock basis characterized by

$$|n\rangle_\lambda \equiv \psi_n(\lambda, x) = \frac{\pi^{-\frac{1}{4}}}{\sqrt{2^n n!}} \frac{1}{\sqrt{L_n^{(0)}(-\lambda^2)}} e^{-\frac{x^2}{2}} H_n\left(x + \frac{\lambda}{\sqrt{2}}\right)$$

which has the new interesting property to lead here to meaningful squeezed states [5]. We have thus deformed the states but without deforming the Lie algebra.

The lost of the self-adjointness of H_λ is evident with respect to the original scalar product but we have noticed that this well accepted property can be restored if we modify [6] the scalar product by asking that

$$\left(a^\dagger_\lambda \psi_n(\lambda, x), \psi_m(\lambda, x)\right) = (\psi_n(\lambda, x), a\psi_m(\lambda, x)) = \delta_{nm}.$$

Such a property evidently corresponds to $(a^\dagger_\lambda)^\dagger = a$ in the new Hilbert space characterized by a measure depending on λ. It is not difficult to prove [6] that the new measure is given in a unique way by

$$\rho(\lambda, x)dx = \exp\left[-\sqrt{2}\lambda x - \frac{\lambda^2}{2}\right] dx,$$

where we recognize the generating function of the Hermite polynomials.

Unhappily this context only gives new families of coherent states but no information on squeezed states.

In order to include these $\lambda \neq 0$ and $\lambda = 0$ contexts, we have proposed [7] a general construction of oscillatorlike Hamiltonians by maintaining $(a^\dagger)^\dagger = a$ but permitting $H^\dagger = H$ or $H^\dagger \neq H$. Let us now construct the following operators

$$b = (1 + c_1)a + c_2 a^\dagger + c_3 \qquad \text{and} \qquad b^+ = c_4 a + (1 + c_5)a^\dagger + c_6$$

and require that

$$H = \frac{1}{2}\{b, b^+\}, \qquad [b, b^+] = 1, \qquad [H, b] = -b, \qquad [H, b^+] = b^+.$$

We point out that the usual harmonic oscillator corresponds to all the c_i's equal to zero and our deformed context to all the c_i's equal to zero except the sixth one ($c_6 = \lambda$). Moreover by asking for Schrödinger Hamiltonians of the type

$$H = A\frac{d^2}{dx^2} + (Bx + C)\frac{d}{dx} + Dx^2 + Ex + F,$$

we can easily get A, B, C, D, E, F as functions of the c_i's parameters and see that if $A = -D = -\frac{1}{2}$, $B = C = E = F = 0$, we recover the harmonic oscillator and if $A = -D = -\frac{1}{2}$, $C = E = \frac{\lambda}{\sqrt{2}}$, $B = F = 0$, we recover our λ-context. Moreover we notice that $H^\dagger = H$ iff $B = C = 0$ while $H^\dagger \neq H$ in the other cases. All these developments are subtended by new Fock spaces $\{|n\rangle_c, n = 0, 1, 2, \ldots\}$ characterized by square integrable eigenfunctions (when $A < 0$ and $B < 1$) associated to real eigenvalues $E_n = n + \frac{1}{2}$ and the action of the operators given by

$$\begin{aligned}
&b|n\rangle_c = \frac{n}{\sqrt{-A}}\frac{N_n}{N_{n-1}}(1 + c_1 - c_2)|n-1\rangle_c,\\
&b^+|n\rangle_c = \frac{1}{2\sqrt{-A}}\frac{N_n}{N_{n+1}}(1 + c_5 - c_4)|n+1\rangle_c,\\
&b^+b|n\rangle_c = n|n\rangle_c, \qquad bb^+|n\rangle_c = (n+1)|n\rangle_c.
\end{aligned}$$

These results give once more new results [7] in quantum optics through coherence and squeezing developments.

Let us end this communication by two comments dealing once again with non-hermitian operators and some surprising results.

The *first* comment is connected to the choice of unusual annihilation and creation operators illustrating the $(a^\dagger)^\dagger \neq a$ context. Following Ushveridze [8], we can choose

$$a = \frac{1}{\alpha'(x)}\left(\frac{d}{dx} + \beta(x)\right), \qquad a^\dagger = \alpha(x) \tag{4}$$

which ensure the Heisenberg algebra (2) and lead to

$$H = a^\dagger a + \frac{1}{2} \to E_n = n + \frac{1}{2}, \qquad f_n(x) = N_n \exp\left[-\int \beta(x)dx\right] A^n(x). \tag{5}$$

With the Ushveridze specific values

$$a = \frac{d}{dx} + cx^3, \qquad a^\dagger = x, \qquad c > 0, \tag{6}$$

one finds square integrable eigenfunctions which are normalizable but not orthogonal. Our surprise was that the construction of the orthogonal ones by the Schmidt procedure leads to eigenfunctions containing the so-called Freud polynomials [9] permitting once again squeezing developments [10] in this special matter.

The *second* comment is on the position and momentum operators discussed from equations (4)–(6). In fact we immediately have new operators given by

$$X = \frac{1}{\sqrt{2}}\left(a + a^\dagger\right) = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + cx^3 + x\right) \tag{7}$$

and

$$P = \frac{i}{\sqrt{2}}\left(a - a^{\dagger}\right) = \frac{i}{\sqrt{2}}\left(-\frac{d}{dx} - cx^3 + x\right) \tag{8}$$

which evidently are such that $X^{\dagger} \neq X$ and $P^{\dagger} \neq P$. Our proposal [11] is to rewrite (7) and (8) in the following forms

$$X = \operatorname{Re} X + i \operatorname{Im} X = \frac{1}{\sqrt{2}}\left(x + cx^3\right) + i\left(-\frac{i}{\sqrt{2}}\frac{d}{dx}\right)$$

and

$$P = \operatorname{Re} P + i \operatorname{Im} P = \left(-\frac{i}{\sqrt{2}}\frac{d}{dx}\right) + \frac{i}{\sqrt{2}}\left(x - cx^3\right)$$

putting in evidence four "new" operators which are self-adjoint while X and P are not. Moreover we have

$$[X, P] = iI = [\operatorname{Re} X, \operatorname{Re} P] - [\operatorname{Im} X, \operatorname{Im} P],$$

a trivial generalization of the usual Heisenberg characteristics. If we come back on $X = x$ and $P = -i\frac{d}{dx}$, we get

$$\operatorname{Im} X = 0 = \operatorname{Im} P$$

and the usual commutation relation takes place in correspondence with

$$\Delta x \Delta p \geq \frac{1}{2}.$$

Different questions are now open and have to be discussed in the future.

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On Classification of Irreducible Representations of q-Deformed Algebra $U_q'(\mathrm{so}_n)$ Related to Quantum Gravity

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Classification of finite dimensional irreducible representations of nonstandard q-deformation $U_q'(\mathrm{so}_n)$ of the universal enveloping algebra $U(\mathrm{so}(n,\mathbb{C}))$ of the Lie algebra $\mathrm{so}(n,\mathbb{C})$ (which does not coincide with the Drinfeld–Jimbo quantized universal enveloping algebra $U_q(\mathrm{so}_n)$) is given for the case when q is not a root of unity. It is shown that such representations are exhausted by representations of the classical and nonclassical types. Examples of the algebras $U_q'(\mathrm{so}_3)$ and $U_q'(\mathrm{so}_4)$ are considered in detail. Notions of weights, highest weights, highest weight vectors are introduced. Raising and lowering operators for irreducible finite dimensional representations of $U_q'(\mathrm{so}_n)$ are introduced. They depend on weight upon which they act. Explicit formulas for these operators are given.

1 Introduction

Quantum orthogonal groups, quantum Lorentz groups and their quantized universal enveloping algebras are of special interest for modern mathematics and physics. M. Jimbo [1] and V. Drinfeld [2] defined q-deformations (quantized universal enveloping algebras) $U_q(g)$ for all simple complex Lie algebras g by means of Cartan subalgebras and root subspaces (see also [3] and [4]). However, these approaches do not give a satisfactory presentation of the quantized algebra $U_q(\mathrm{so}(n,\mathbb{C}))$ from a viewpoint of some problems in quantum physics and mathematics. Considering irreducible representations of the quantum groups $SO_q(n+1)$ and $SO_q(n,1)$ we are interested in reducing them onto the quantum subgroup $SO_q(n)$. This reduction would give an analogue of the Gel'fand–Tsetlin basis for these representations. However, definitions of quantized universal enveloping algebras, mentioned above, do not allow the inclusions $U_q(\mathrm{so}(n+1,\mathbb{C})) \supset U_q(\mathrm{so}(n,\mathbb{C}))$ and $U_q(\mathrm{so}(n,1)) \supset U_q(\mathrm{so}(n))$. To be able to exploit such reductions we have to consider q-deformation of the universal enveloping algebra of the Lie algebra $\mathrm{so}(n+1,\mathbb{C})$ defined in terms of the generators $I_{k,k-1} = E_{k,k-1} - E_{k-1,k}$ (where E_{is} is the matrix with elements $(E_{is})_{rt} = \delta_{ir}\delta_{st}$) rather than by means of Cartan subalgebras and root elements. To construct such deformations we have to deform trilinear relations for elements $I_{k,k-1}$ instead of Serre's relations (used in the case of quantized universal enveloping algebras of Drinfeld and Jimbo). As a result, we obtain an associative algebra which will be denoted as $U_q'(\mathrm{so}_n)$.

This q-deformation was first constructed in [5]. It permit us to construct the reductions of $U_q'(\mathrm{so}_{n,1})$ and $U_q'(\mathrm{so}_{n+1})$ onto $U_q'(\mathrm{so}_n)$. The q-deformed algebra $U_q'(\mathrm{so}_n)$ leads for $n=3$ to the q-deformed algebra $U_q'(\mathrm{so}_3)$ defined by D. Fairlie [6]. The cyclically symmetric algebra, similar to Fairlie's one, was also considered somewhat earlier by Odesskii [7]. The algebra $U_q'(\mathrm{so}_4)$ is a q-deformation of the algebra $U(\mathrm{so}(4,\mathbb{C}))$ given by means of commutation relations between the elements I_{ji}, $1 \le i < j \le 4$. For the Lie algebra $\mathrm{so}(4,\mathbb{C})$ we have $\mathrm{so}(4,\mathbb{C}) = \mathrm{so}(3,\mathbb{C}) + \mathrm{so}(3,\mathbb{C})$, while in the case of our q-deformation $U_q'(\mathrm{so}_4)$ this is not the case (see e.g. [8]).

In the classical case, the imbedding $SO(n) \subset SU(n)$ (and its infinitesimal analogue) is of great importance for nuclear physics and in the theory of Riemannian symmetric spaces. It is well known that in the framework of Drinfeld–Jimbo quantum groups and algebras one cannot construct the corresponding embedding. The algebra $U'_q(\mathrm{so}_n)$ allows to define such an embedding [9], that it is possible to define the embedding $U'_q(\mathrm{so}_n) \subset U_q(\mathrm{sl}_n)$, where $U_q(\mathrm{sl}_n)$ is a Drinfeld–Jimbo quantum algebra.

As a disadvantage of the algebra $U'_q(\mathrm{so}_n)$ we have to mention the difficulties with Hopf algebra structure. Nevertheless, $U'_q(\mathrm{so}_n)$ turns out to be a coideal in $U_q(\mathrm{sl}_n)$ (see [9]) and this fact allows us to consider tensor products of finite dimensional irreducible representations of $U'_q(\mathrm{so}_n)$ for many interesting cases (see [10]).

Finite dimensional irreducible representations of the algebra $U'_q(\mathrm{so}_n)$ for q being not a root of unity were constructed in [5]. The formulas of action of the generators of $U'_q(\mathrm{so}_n)$ upon the basis (which is a q-analogue of the Gel'fand–Tsetlin basis) are given there. A proof of these formulas and some their corrections were given in [11]. However, finite dimensional irreducible representations described in [5] and [11] are representations of the classical type. They are q-deformations of the corresponding irreducible representations of the Lie algebra so_n, that is, at $q \to 1$ they turn into representations of so_n.

If q is not a root of unity, the algebra $U'_q(\mathrm{so}_n)$ has other classes of finite dimensional irreducible representations which have no classical analogues. These representations are singular at the limit $q \to 1$. They are described in [12]. A detailed description of these representations for the algebra $U'_q(\mathrm{so}_3)$ is given in [13]. A classification of irreducible $*$-representations of real forms of the algebra $U'_q(\mathrm{so}_3)$ is given in [14].

The aim of this paper is to give classification theorem for finite dimensional irreducible representations of the algebra $U'_q(\mathrm{so}_n)$ on complex vector spaces when q is not a root of unity. We show that in this case all irreducible finite dimensional representations of $U'_q(\mathrm{so}_n)$ are exhausted by representations of the classical and nonclassical types. Detailed proofs of propositions and theorems, given in this paper, will be given separately.

Everywhere below we assume that q is not a root of unity.

2 Definition of the q-deformed algebra $U'_q(\mathrm{so}_n)$

An existence of a q-deformation of the universal enveloping algebra $U(\mathrm{so}(n,\mathbb{C}))$, different from the Drinfeld–Jimbo quantized universal enveloping algebra $U_q(\mathrm{so}_n)$, is explained by the following reason. The Lie algebra $\mathrm{so}(n,\mathbb{C})$ has two structures:

(a) The structure related to existing in $\mathrm{so}(n,\mathbb{C})$ a Cartan subalgebra and root elements. A quantization of this structure leads to the Drinfeld–Jimbo quantized universal enveloping algebra $U_q(\mathrm{so}_n)$.

(b) The structure related to realization of $\mathrm{so}(n,\mathbb{C})$ by skew-symmetric matrices. In the Lie algebra $\mathrm{so}(n,\mathbb{C})$ there exists a basis consisting of the matrices I_{ij}, $i > j$, defined as $I_{ij} = E_{ij} - E_{ji}$, where E_{ij} is the matrix with entries $(E_{ij})_{rs} = \delta_{ir}\delta_{js}$. These matrices are not root elements.

Using the structure (b), we may say that the universal enveloping algebra $U(\mathrm{so}(n,\mathbb{C}))$ is generated by the elements I_{ij}, $i > j$. But in order to generate the universal enveloping algebra $U(\mathrm{so}(n,\mathbb{C}))$, it is enough to take only the elements $I_{21}, I_{32}, \ldots, I_{n,n-1}$. It is a minimal set of elements necessary for generating $U(\mathrm{so}(n,\mathbb{C}))$. These elements satisfy the relations

$$
\begin{aligned}
&I_{i,i-1}^2 I_{i+1,i} - 2I_{i,i-1}I_{i+1,i}I_{i,i-1} + I_{i+1,i}I_{i,i-1}^2 = -I_{i+1,i},\\
&I_{i,i-1}I_{i+1,i}^2 - 2I_{i+1,i}I_{i,i-1}I_{i+1,i} + I_{i+1,i}^2 I_{i,i-1} = -I_{i,i-1},\\
&I_{i,i-1}I_{j,j-1} - I_{j,j-1}I_{i,i-1} = 0 \quad \text{for} \quad |i-j| > 1.
\end{aligned}
$$

The following theorem is true [15] for the universal enveloping algebra $U(\mathrm{so}(n,\mathbb{C}))$:

Theorem 1. *The universal enveloping algebra $U(\mathrm{so}(n,\mathbb{C}))$ is isomorphic to the complex associative algebra (with a unit element) generated by the elements $I_{21}, I_{32}, \ldots, I_{n,n-1}$ satisfying the above relations.*

We make the q-deformation of these relations by fulfilling the deformation of the integer 2 in these relations as

$$2 \to [2]_q := \left(q^2 - q^{-2}\right) / \left(q - q^{-1}\right) = q + q^{-1}.$$

As a result, we obtain the complex unital (that is, with a unit element) associative algebra generated by elements $I_{21}, I_{32}, \ldots, I_{n,n-1}$ satisfying the relations

$$I_{i,i-1}^2 I_{i+1,i} - \left(q + q^{-1}\right) I_{i,i-1} I_{i+1,i} I_{i,i-1} + I_{i+1,i} I_{i,i-1}^2 = -I_{i+1,i}, \tag{1}$$
$$I_{i,i-1} I_{i+1,i}^2 - \left(q + q^{-1}\right) I_{i+1,i} I_{i,i-1} I_{i+1,i} + I_{i+1,i}^2 I_{i,i-1} = -I_{i,i-1}, \tag{2}$$
$$I_{i,i-1} I_{j,j-1} - I_{j,j-1} I_{i,i-1} = 0 \quad \text{for} \quad |i-j| > 1. \tag{3}$$

This algebra was introduced by us in [5] and is denoted by $U_q'(\mathrm{so}_n)$.

The analogue of the elements I_{ij}, $i > j$, can be introduced into $U_q'(\mathrm{so}_n)$ (see [16]). In order to give them we use the notation $I_{k,k-1} \equiv I_{k,k-1}^+ \equiv I_{k,k-1}^-$. Then for $k > l+1$ we define recursively

$$I_{kl}^+ := [I_{l+1,l}, I_{k,l+1}]_q \equiv q^{1/2} I_{l+1,l} I_{k,l+1} - q^{-1/2} I_{k,l+1} I_{l+1,l}, \tag{4}$$
$$I_{kl}^- := [I_{l+1,l}, I_{k,l+1}]_{q^{-1}} \equiv q^{-1/2} I_{l+1,l} I_{k,l+1} - q^{1/2} I_{k,l+1} I_{l+1,l}.$$

The elements I_{kl}^+, $k > l$, satisfy the commutation relations

$$[I_{ln}^+, I_{kl}^+]_q = I_{kn}^+, \quad [I_{kl}^+, I_{kn}^+]_q = I_{ln}^+, \quad [I_{kn}^+, I_{ln}^+]_q = I_{kl}^+ \quad \text{for} \quad k > l > n, \tag{5}$$
$$[I_{kl}^+, I_{nr}^+] = 0 \quad \text{for} \quad k > l > n > r \quad \text{and} \quad k > n > r > l, \tag{6}$$
$$[I_{kl}^+, I_{nr}^+]_q = \left(q - q^{-1}\right)\left(I_{lr}^+ I_{kn}^+ - I_{kr}^+ I_{nl}^+\right) \quad \text{for} \quad k > n > l > r. \tag{7}$$

For I_{kl}^-, $k > l$, the commutation relations are obtained from these relations by replacing I_{kl}^+ by I_{kl}^- and q by q^{-1}.

The algebra $U_q'(\mathrm{so}_n)$ can be defined as a unital associative algebra generated by I_{kl}^+, $1 \le l < k \le n$, satisfying the relations (5)–(7). In fact, using the relations (4) we can reduce the relations (5)–(7) to the relations (1)–(3) for $I_{21}, I_{32}, \ldots, I_{n,n-1}$.

The Poincaré–Birkhoff–Witt theorem for the algebra $U_q'(\mathrm{so}_n)$ can be formulated as follows (a proof of this theorem is given in [17]): *The elements*

$${I_{21}^+}^{m_{21}} {I_{31}^+}^{m_{31}} \cdots {I_{n1}^+}^{m_{n1}} {I_{32}^+}^{m_{32}} {I_{42}^+}^{m_{42}} \cdots {I_{n2}^+}^{m_{n2}} \cdots {I_{n,n-1}^+}^{m_{n,n-1}}, \quad m_{ij} = 0, 1, 2, \ldots, \tag{8}$$

form a basis of the algebra $U_q'(\mathrm{so}_n)$. This assertion is true if I_{ij}^+ are replaced by the corresponding elements I_{ij}^-.

Example 1. Let us consider the case of the algebra $U_q'(\mathrm{so}_3)$. It is generated by two elements I_{21} and I_{32}, satisfying the relations

$$I_{21}^2 I_{32} - \left(q - q^{-1}\right) I_{21} I_{32} I_{21} + I_{32} I_{21}^2 = -I_{32}, \tag{9}$$
$$I_{21} I_{32}^2 - \left(q + q^{-1}\right) I_{32} I_{21} I_{32} + I_{32}^2 I_{21} = -I_{21}. \tag{10}$$

Introducing the element $I_{31}^+ \equiv I_{31} = q^{1/2} I_{21} I_{32} - q^{-1/2} I_{32} I_{21}$ we have for I_{21}, I_{32}, I_{31} the relations

$$[I_{21}, I_{32}]_q = I_{31}, \quad [I_{32}, I_{31}]_q = I_{21}, \quad [I_{31}, I_{21}]_q = I_{32}, \tag{11}$$

where the q-commutator $[\cdot,\cdot]_q$ is defined as $[A,B]_q = q^{1/2} AB - q^{-1/2} BA$.

Note that the algebra $U'_q(\mathrm{so}_3)$ has a large automorphism group. In fact, it is seen from (9) and (10) that these relations do not change if we permute I_{21} and I_{32}. From relations (11) we see that the set of these relations do not change under cyclic permutation of the elements I_{21}, I_{32}, I_{31}. The change of a sign at I_{21} or at I_{32} also does not change the relations (9) and (10). Generating a group by these automorphisms, we may find that they generate the group isomorphic to the modular group $SL(2,\mathbb{Z})$. It is why the algebra $U'_q(\mathrm{so}_3)$ is interesting for algebraic geometry and quantum gravity (see, for example, [18] and [19]).

Example 2. Let us consider the case of the algebra $U'_q(\mathrm{so}_4)$. It is generated by the elements I_{21}, I_{32} and I_{43}. We create the elements

$$I_{31} = [I_{21}, I_{32}]_q, \quad I_{42} = [I_{32}, I_{43}]_q, \quad I_{41} = [I_{21}, I_{42}]_q. \tag{12}$$

Then the elements I_{ij}, $i > j$, satisfy the following set of relations

$$\begin{aligned}
&[I_{21}, I_{32}]_q = I_{31}, \quad [I_{32}, I_{31}]_q = I_{21}, \quad [I_{31}, I_{21}]_q = I_{32},\\
&[I_{32}, I_{43}]_q = I_{42}, \quad [I_{43}, I_{42}]_q = I_{32}, \quad [I_{42}, I_{32}]_q = I_{43},\\
&[I_{31}, I_{43}]_q = I_{41}, \quad [I_{43}, I_{41}]_q = I_{31}, \quad [I_{41}, I_{31}]_q = I_{43},\\
&[I_{21}, I_{42}]_q = I_{41}, \quad [I_{42}, I_{41}]_q = I_{21}, \quad [I_{41}, I_{21}]_q = I_{42},\\
&[I_{21}, I_{43}] = 0, \quad [I_{32}, I_{41}] = 0, \quad [I_{42}, I_{31}] = \left(q - q^{-1}\right)(I_{21}I_{43} - I_{32}I_{41})
\end{aligned}$$

which completely determine the algebra $U'_q(\mathrm{so}_4)$. At $q = 1$ these relations define just the Lie algebra $\mathrm{so}(4,\mathbb{C})$. Each of the sets (I_{21}, I_{32}, I_{31}), (I_{32}, I_{43}, I_{42}), (I_{31}, I_{43}, I_{41}), (I_{21}, I_{42}, I_{41}) determine a subalgebra isomorphic to $U'_q(\mathrm{so}_3)$.

The algebra $U'_q(\mathrm{so}_4)$ is also important for quantum gravity and algebraic geometry (see [20] and [21]). The algebra $U'_q(\mathrm{so}_n)$ for general n is also used in quantum gravity [22].

Let us describe the automorphism group G of the algebra $U'_q(\mathrm{so}_n)$. It is clear from the defining relations of the algebra $U'_q(\mathrm{so}_n)$ that for each i ($i = 2, 3, \ldots, n$) this algebra admits an automorphism τ_i given by the formulas

$$\tau_i : I_{j,j-1} \to I_{j,j-1}, \quad j \neq i, \qquad \tau_i : I_{i,i-1} \to -I_{i,i-1}.$$

These automorphisms generate a group of automorphisms which will be denoted by G. Elements of G can be denoted by $g = (\epsilon_2, \epsilon_3, \ldots, \epsilon_n)$, where ϵ_j runs independently the values $+1$ and -1. Namely, if under action of g generating elements $I_{j_1,j_1-1}, \ldots, I_{j_s,j_s-1}$ change a sign, then in $g = (\epsilon_2, \epsilon_3, \ldots, \epsilon_n)$ $\epsilon_{j_1} = \cdots = \epsilon_{j_s} = -1$ and other ϵ_i are equal to 1. It is clear that the group G has 2^{n-1} elements.

If $n = 3$, then the group G does not coincide with the group of all automorphisms of $U'_q(\mathrm{so}_3)$. It is not known if this assertion is true for $n > 3$.

3 Representations of classical and nonclassical types

The elements of the set

$$I_{21}, I_{43}, \ldots, I_{2k,2k-1}, \tag{13}$$

where $n = 2k$ if n is even and $n = 2k + 1$ if n is odd, commute pairwise.

Proposition 1. (a) *If T is a finite dimensional irreducible representation of the algebra $U'_q(\mathrm{so}_n)$, then the operators*

$$T(I_{21}), T(I_{43}), \ldots, T(I_{2k,2k-1})$$

are simulteneously diaginalizable.

(b) *Possible eigenvalues of any of these operators can be as* $\mathrm{i}[m]_q$, $m \in \frac{1}{2}\mathbb{Z}$, *or as* $[m]_+$, $m \in \frac{1}{2}\mathbb{Z}$, $m \notin \mathbb{Z}$, *where*

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}, \qquad [m]_+ = \frac{q^m + q^{-m}}{q - q^{-1}}.$$

Eigenvalues of the form $\mathrm{i}[m]_q$ are called *eigenvalues of the classical type*. Eigenvalues of the form $[m]_+$ are called *eigenvalues of the nonclassical type*.

The following proposition is important for construction of weight theory for finite dimensional representations of $U_q'(\mathrm{so}_n)$.

Proposition 2. *Let T be a finite dimensional irreducible representation of $U_q'(\mathrm{so}_n)$. Then*

(a) *Eigenvalues of any operator $T(I_{2i,2i-1})$ are all of the classical type or all of the nonclassical type.*

(b) *Moreover, all operators $T(I_{2i,2i-1})$, $i = 1, 2, \ldots, k$, have eigenvalues of the same type.*

This proposition is proved by restricting the representation T to the subalgebras $U_q'(\mathrm{so}_4)$ generated by the elements $I_{j,j-1}, I_{j+1,j}, I_{j+2,j+1}$, $j = 2, 3, \ldots, n-2$ and using the results of the paper [8].

Definition 1. A finite dimensional irreducible representation T of the algebra $U_q'(\mathrm{so}_n)$ is called a representation of *classical* (*nonclassical*) *type* if the operators $T(I_{2i,2i-1})$, $i = 1, 2, \ldots, k$ have eigenvalues of the classical (of the nonclassical) type.

Proposition 3. *Let T be a finite dimensional irreducible representation of $U_q'(\mathrm{so}_n)$ of the classical (nonclassical) type. Then a restriction of T to the subalgebra $U_q'(\mathrm{so}_{n-1})$ decomposes into a direct sum of irreducible representations of this subalgebra belonging to the same type.*

4 Weights of representations

In this section we construct a q-analogue of weights for finite dimensional irreducible representations of the algebra $U_q'(\mathrm{so}_n)$. Note that this algebra has no elements which can be treated as root elements (similar to root elements of semisimple Lie algebras or quantized universal enveloping algebras of Drinfeld and Jimbo). For this reason, we do not have a weight theory for finite dimensional representations of $U_q'(\mathrm{so}_n)$ similar to that for semisimple Lie algebras. However, we can construct the theory which can replace the weight theory of representations of semisimple Lie algebras.

Definition 2. Let T be a finite dimensional representation of the algebra $U_q'(\mathrm{so}_n)$. Eigenvectors $\boldsymbol{v}$ of operators $T(I_{2j,2j-1})$, $j = 1, 2, \ldots, k$, are called *weight vectors* of the representation T. If $T(I_{2j,2j-1})\boldsymbol{v} = m_j \boldsymbol{v}$, then the the set of numbers $\boldsymbol{m} = (m_1, m_2, \ldots, m_k)$, where $n = 2k+1$ or $n = 2k$, is called a weight of the vector $\boldsymbol{v}$.

The set of all weights of an irreducible representation T of $U_q'(\mathrm{so}_n)$ is called a *weight diagram* of the representation T.

Proposition 4. *A weight diagram of a finite dimensional irreducible representation T of the classical type is invariant with respect to the Weyl group W of the Lie algebra* $\mathrm{so}(n, \mathbb{C})$.

This proposition is proved by restriction of the representation T to the subalgebras $U_q'(\mathrm{so}_3)$ generated by the pairs of generators $I_{2j,2j-1}, I_{2j+1,2j}$, $j = 1, 2, \ldots$, and using the results of the paper [23].

Note that a weight diagram of a finite dimensional irreducible representation of the nonclassical type is not invariant with respect to the Weyl group W.

5 Raising and lowering operators

Recall that in the Lie algebra $\mathrm{so}(n,\mathbb{C})$ there exist root elements $E_{\alpha_1}, E_{\alpha_2}, \ldots, E_{\alpha_k}$, corresponding to simple roots, and root elements $F_{\alpha_1}, F_{\alpha_2}, \ldots, F_{\alpha_k}$, corresponding to simple roots taken with sign minus. If T' is a finite dimensional irreducible representation of $\mathrm{so}(n,\mathbb{C})$ and $|\boldsymbol{m}\rangle$ is its weight vector, then

$$T'(E_{\alpha_i})|\boldsymbol{m}\rangle = \beta_{\boldsymbol{m}}|\boldsymbol{m}+\alpha_i\rangle, \qquad T'(F_{\alpha_i})|\boldsymbol{m}\rangle = \gamma_{\boldsymbol{m}}|\boldsymbol{m}-\alpha_i\rangle,$$

where $\beta_{\boldsymbol{m}}$ and $\gamma_{\boldsymbol{m}}$ are complex numbers. In the algebra $U_q'(\mathrm{so}_n)$ there exist no elements similar to E_{α_j} and F_{α_j}. However, in finite dimensional representations of $U_q'(\mathrm{so}_n)$ there exist operators having properties of the operators $T'(E_{\alpha_i})$ and $T'(F_{\alpha_i})$. These operators depend on a weight on which they act and are called *raising and lowering operators* of the representation. They are described as follows.

Let T be a finite dimensional irreducible representation of $U_q'(\mathrm{so}_n)$ of the classical type and let $|\boldsymbol{m}\rangle$ be its weight vector. If $n = 2k$ we create the operators

$$\begin{aligned} R^{\boldsymbol{m}}_{\alpha_i} &= -T(I_{2i+2,2i-1}) + q^{-(m_i+m_{i+1})/2}T(I_{2i+1,2i}) - \mathrm{i}q^{-m_i+1/2}T(I_{2i+2,2i}) \\ &\quad - \mathrm{i}q^{-m_{i+1}-1/2}T(I_{2i+1,2i-1}), \qquad i = 1,2,\ldots,k-1, \end{aligned} \tag{14}$$

$$\begin{aligned} L^{\boldsymbol{m}}_{\alpha_i} &= -T(I_{2i+2,2i-1}) + q^{(m_i+m_{i+1})/2}T(I_{2i+1,2i}) + \mathrm{i}q^{m_i+1/2}T(I_{2i+2,2i}) \\ &\quad + \mathrm{i}q^{m_{i+1}-1/2}T(I_{2i+1,2i-1}), \qquad i = 1,2,\ldots,k-1, \end{aligned} \tag{15}$$

and the operators

$$\begin{aligned} R^{\boldsymbol{m}}_{\alpha_k} &= T(I_{2k,2k-3}) + q^{(-m_{k-1}+m_k)/2}T(I_{2k-1,2k-2}) + \mathrm{i}q^{-m_{k-1}+1/2}T(I_{2k,2k-2}) \\ &\quad - \mathrm{i}q^{m_k-1/2}T(I_{2k-1,2k-3}), \end{aligned} \tag{16}$$

$$\begin{aligned} L^{\boldsymbol{m}}_{\alpha_k} &= -T(I_{2k,2k-3}) + q^{(m_{k-1}-m_k)/2}T(I_{2k-1,2k-2}) + \mathrm{i}q^{m_{k-1}+1/2}T(I_{2k,2k-2}) \\ &\quad + \mathrm{i}q^{-m_k-1/2}T(I_{2k-1,2k-3}). \end{aligned} \tag{17}$$

If $n = 2k+1$, then we create the operators (14), (15) and the operators

$$R^{\boldsymbol{m}}_{\alpha_k} = T(I_{2k+1,2k-1}) + \mathrm{i}q^{-m_k+1/2}T(I_{2k+1,2k}), \tag{18}$$

$$L^{\boldsymbol{m}}_{\alpha_k} = T(I_{2k+1,2k-1}) - \mathrm{i}q^{m_k+1/2}T(I_{2k+1,2k}). \tag{19}$$

If T is a finite dimensional representation of $U_q'(\mathrm{so}_n)$ of the nonclassical type and $|\boldsymbol{m}\rangle$ is its weight vector, then we create the operators

$$\begin{aligned} R^{\boldsymbol{m}}_{\alpha_i} &= -T(I_{2i+2,2i-1}) + q^{-(m_i+m_{i+1})/2}T(I_{2i+1,2i}) - q^{-m_i+1/2}T(I_{2i+2,2i}) \\ &\quad - q^{-m_{i+1}-1/2}T(I_{2i+1,2i-1}), \qquad i = 1,2,\ldots,k-1, \end{aligned} \tag{20}$$

$$\begin{aligned} L^{\boldsymbol{m}}_{\alpha_i} &= -T(I_{2i+2,2i-1}) + q^{(m_i+m_{i+1})/2}T(I_{2i+1,2i}) - q^{m_i+1/2}T(I_{2i+2,2i}) \\ &\quad - q^{m_{i+1}-1/2}T(I_{2i+1,2i-1}), \qquad i = 1,2,\ldots,k-1, \end{aligned} \tag{21}$$

and the operators

$$\begin{aligned} R^{\boldsymbol{m}}_{\alpha_k} &= T(I_{2k,2k-3}) + q^{(-m_{k-1}+m_k)/2}T(I_{2k-1,2k-2}) + q^{-m_{k-1}+1/2}T(I_{2k,2k-2}) \\ &\quad + q^{m_k-1/2}T(I_{2k-1,2k-3}), \end{aligned} \tag{22}$$

$$\begin{aligned} L^{\boldsymbol{m}}_{\alpha_k} &= -T(I_{2k,2k-3}) - q^{(m_{k-1}-m_k)/2}T(I_{2k-1,2k-2}) + q^{m_{k-1}+1/2}T(I_{2k,2k-2}) \\ &\quad + q^{-m_k-1/2}T(I_{2k-1,2k-3}) \end{aligned} \tag{23}$$

if $n = 2k$. If $n = 2k + 1$, then we create the operators (20), (21) and the operators

$$R^{\boldsymbol{m}}_{\alpha_k} = T(I_{2k+1,2k-1}) + q^{-m_k+1/2}T(I_{2k+1,2k}), \tag{24}$$

$$L^{\boldsymbol{m}}_{\alpha_k} = T(I_{2k+1,2k-1}) - q^{m_k+1/2}T(I_{2k+1,2k}). \tag{25}$$

The operators $R^{\boldsymbol{m}}_{\alpha_k}$ and $L^{\boldsymbol{m}}_{\alpha_k}$ correspond to the operators $T'(E_{\alpha_i})$ and $T'(F_{\alpha_i})$ of a representation T' of the Lie algebra $\mathrm{so}(n,\mathbb{C})$, respectively. We have

$$R^{\boldsymbol{m}}_{\alpha_k}|\boldsymbol{m}\rangle = \beta_i|\boldsymbol{m} + \alpha_i\rangle, \qquad L^{\boldsymbol{m}}_{\alpha_k}|\boldsymbol{m}\rangle = \gamma_i|\boldsymbol{m} - \alpha_i\rangle, \tag{26}$$

where α_i and γ_i are complex numbers, which depend on the representation of $U'_q(\mathrm{so}_n)$. Note that the relations (26) are not true if we replace the vector $|\boldsymbol{m}\rangle$ by some other weight vector $|\boldsymbol{m}'\rangle$, since in such a case in the right hand side we shall obtain, beside the vectors $|\boldsymbol{m}' + \alpha_i\rangle$ and $|\boldsymbol{m}' - \alpha_i\rangle$, other weight vectors.

Formulas (14)–(17) and (20)–(23) for raising and lowering operators follow from formulas of section 8 of the paper [8] if to restrict the representation T of $U'_q(\mathrm{so}_n)$ to the subalgebras $U'_q(\mathrm{so}_4)$ generated by the elements $I_{2j,2j-1}, I_{2j+1,2j}, I_{2j+2,2j+1}$, $j = 1, 2, \ldots, k-1$.

Formulas (18), (19), (24) and (25) for raising and lowering operators follow from formulas for raising and lowering operators for irreducible representations of the algebra $U'_q(\mathrm{so}_3)$ of the paper [8] if to restrict the representation T to the subalgebra $U'_q(\mathrm{so}_3)$ generated by the elements $I_{2k+1,2k}$ and $I_{2k,2k-1}$.

Definition 3. If T is a finite dimensional irreducible representation of the algebra $U'_q(\mathrm{so}_n)$, then a weight $\boldsymbol{m}$ of this representation is called a *highest weight* if $R^{\boldsymbol{m}}_{\alpha_i}|\boldsymbol{m}\rangle = 0$, $i = 1, 2, \ldots, k$. The corresponding vector $|\boldsymbol{m}\rangle$ is called a *highest weight vector.*

Let us give a form of highest weights of irreducible representations of the classical and of the nonclassical types. In order to determine such a form we restrict the corresponding irreducible representations of $U'_q(\mathrm{so}_n)$ to the subalgebras $U'_q(\mathrm{so}_4)$ and $U'_q(\mathrm{so}_3)$ and use the results of the papers [8] and [23]. As a result, we find that if a weight $\boldsymbol{m} \equiv (m_1, m_2, \ldots, m_k)$ of an irreducible representation T of the classical type is a highest weight, then the numbers m_j are all integral or all half-integral (but not integral) and satisfy the conditions

$$m_1 \geq m_2 \geq \cdots \geq m_k \quad \text{if} \quad n = 2k+1$$

and the conditions

$$m_1 \geq m_2 \geq \cdots \geq m_{k-1} \geq |m_k| \quad \text{if} \quad n = 2k.$$

The set of these highest weights coincides with the set of highest weights of irreducible finite dimensional representations of the Lie algebra $\mathrm{so}(n,\mathbb{C})$. These highest weights will be called *highest weights of the classical type.*

If a weight $\boldsymbol{m} \equiv (m_1, m_2, \ldots, m_k)$ of an irreducible representation T of the nonclassical type is a highest weight, then the numbers m_j are all half-integral (but not integral). In order to formulate the classification theorem for representations of the nonclassical type we shall need only highest weights $\boldsymbol{m}$ for which all m_j are positive. Such highest weights must satisfy the conditions

$$m_1 \geq m_2 \geq \cdots \geq m_k \geq 1/2.$$

These highest weights will be called *highest weights of the nonclassical type.*

It is well known that the root elements E_{α_i} and F_{α_i} of the Lie algebra $\mathrm{so}(n,\mathbb{C})$ satisfy the relations

$$[E_{\alpha_i}, F_{\alpha_i}] = 2H_{\alpha_i}, \qquad [E_{\alpha_i}, F_{\alpha_j}] = 0, \qquad i \neq j.$$

Instead of these relations for raising and lowering operators of representations of the classical and nonclassical type of the algebra $U'_q(\mathrm{so}_{2k})$ we have the relations

$$\left(R_{\alpha_i}^{\boldsymbol{m}-\alpha_i} L_{\alpha_i}^{\boldsymbol{m}} - L_{\alpha_i}^{\boldsymbol{m}+\alpha_i} R_{\alpha_i}^{\boldsymbol{m}}\right) |\boldsymbol{m}\rangle = [2l]_q \left\{ \left(q - q^{-1}\right)^2 C_4 - \left(q^{2l} + q^{-2l}\right) \left(q^2 - q^{-2}\right) \right\} |\boldsymbol{m}\rangle, \quad (27)$$

$$\left(R_{\alpha_i}^{\boldsymbol{m}-\alpha_j} L_{\alpha_j}^{\boldsymbol{m}} - L_{\alpha_j}^{\boldsymbol{m}+\alpha_i} R_{\alpha_i}^{\boldsymbol{m}}\right) |\boldsymbol{m}\rangle = 0, \quad (28)$$

where $l = (m_i - m_{i+1})/2$ if $i \neq k$ and $l = (m_i + m_{i+1})/2$ if $i = k$ and C_4 is the Casimir operator of the subalgebra $U'_q(\mathrm{so}_4)$ generated by the elements $I_{2i,2i-1}, I_{2i+1,2i}, I_{2i+2,2i+1}$, which is given as

$$C_4 = q^{-1} I_{2i,2i-1} I_{2i+2,2i+1} - I_{2i+1,2i-1} I_{2i+2,2i} + q I_{2i+1,2i} I_{2i+2,2i-1}.$$

For the algebra $U'_q(\mathrm{so}_{2k+1})$ we have the relations (27) for $i \neq k$, (28) for $i \neq j$ and the relations

$$\left(R_{\alpha_k}^{\boldsymbol{m}-\alpha_k} L_{\alpha_k}^{\boldsymbol{m}} - L_{\alpha_k}^{\boldsymbol{m}+\alpha_k} R_{\alpha_k}^{\boldsymbol{m}}\right) |\boldsymbol{m}\rangle = q[m_k]_q [m_k]_+ \left(q - q^{-1}\right)^2 |\boldsymbol{m}\rangle. \quad (29)$$

6 Classification theorems

For finite dimensional irreducible representations of the classical type the following theorem is true.

Theorem 2. (a) *Each irreducible finite dimensional representation of the classical type has a highest weight. A highest weight is unique (up to a constant).*

(b) *Irreducible finite dimensional representations with different highest weights are not equivalent. Conversely, nonequivalent irreducible finite dimensional representations of $U'_q(\mathrm{so}_n)$ have different highest weights.*

Existing of a highest weight is proved in the same way as in the case of irreducible representations of the Lie algebra $\mathrm{so}(n, \mathbb{C})$ by using Propositions 1 and 2. A proof of uniqueness of highest weight is not simple. The relations (27)–(29) are used in this proof.

The assertion (b) is proved by using a proof of the similar assertion for irreducible representations of the algebra $U'_q(\mathrm{so}_4)$ from paper [8]. Namely, if $|\boldsymbol{m}\rangle$ is a highest weight vector, then we act upon $|\boldsymbol{m}\rangle$ successively by the corresponding operators $L_{\alpha_i}^{\boldsymbol{m}'}$, $i = 1, 2, \ldots, k$. Then, as in [8], we can find how the operators $R_{\alpha_i}^{\boldsymbol{m}'}$ act upon weight vectors $|\boldsymbol{m}'\rangle$. Therefore, by the method of the paper [8] we evaluate uniquely how the operators $T(I_{2i+2,2i-1}), T(I_{2i+1,2i}), T(I_{2i+2,2i}), T(I_{2i+2,2i+1})$ act upon the corresponding weight vectors. Thus, a highest weight determines uniquely (up to equivalence) the operators $T(I_{j,j-1})$, $j = 2, 3, \ldots, n$.

Thus, in order to obtain a classification of irreducible finite dimensional representations of the classical type of the algebra $U'_q(\mathrm{so}_n)$ we have to determine highest weights, described in the previous section, to which such irreducible representations with these highest weights correspond.

It can be proved that the irreducible representation $T_{\boldsymbol{m}}$ of $U'_q(\mathrm{so}_n)$ from the paper [5] are of the classical type and has highest weight $\boldsymbol{m}$. If we take all these irreducible representations $T_{\boldsymbol{m}}$, then they give all highest weights $\boldsymbol{m}$, described in previous section for irreducible representations of the classical type. That is, for each highest weight $\boldsymbol{m}$ of the classical type from the previous section there corresponds an irreducible representation of $U'_q(\mathrm{so}_n)$. Thus, we obtain the following classification of irreducible representations of the classical type.

Theorem 3. *Irreducible finite dimensional representations of the classical type of the algebra $U'_q(\mathrm{so}_n)$ are in one-to-one correspondence with highest weights of the classical type, described in the previous section.*

Thus, irreducible finite dimensional representations of the classical type of the algebra $U_q'(\mathrm{so}_n)$ are in one-to-one correspondence with irreducible finite dimensional representations of the Lie algebra so_n. The corresponding irreducible representations of $U_q'(\mathrm{so}_n)$ and of so_n act on the same vector space. Moreover, when $q \to 1$, then operators of an irreducible representation of $U_q'(\mathrm{so}_n)$ tend to the corresponding operators of the corresponding irreducible representation of so_n. This is a reason why the representations of Theorem 3 are called representations of the classical type.

An analogue of Theorem 2 for irreducible representations of the nonclassical type is formulated as follows.

Theorem 4. (a) *Each irreducible finite dimensional representation of the nonclassical type has a highest weight. A highest weight is unique (up to a constant).*

(b) *Irreducible finite dimensional representations with different highest weights are not equivalent.*

This theorem is proved in the same way as Theorem 2.

In order to formulate the classification theorem for irreducible representations of the nonclassical type we first formulate the following proposition.

Proposition 5. *If T is an irreducible representation of the nonclassical type and G is the automorphism group of $U_q'(\mathrm{so}_n)$ from section 2, then the composition $T^{(g)} := T \circ g$, $g \in G$, $g \neq e$, is a representation of the nonclassical type which is not equivalent to T.*

This proposition is proved by showing that spectrum of the operator $T(I_{2i,2i-1})$ ($i = 1, 2, \ldots, k$) coincides with the set $[\frac{1}{2}]_+, [\frac{3}{2}]_+, \ldots, [\frac{s}{2}]_+$ or with the set $-[\frac{1}{2}]_+, -[\frac{3}{2}]_+, \ldots, -[\frac{s}{2}]_+$, where s is some positive integer. In order to show this we use the method of mathematical induction. For $U_q'(\mathrm{so}_4)$ this assertion is true (see [8]). The induction is proved by using Wigner–Eckart theorem for irreducible representations of the nonclassical type derived by N. Iorgov (this theorem will be published).

Thus, with every irreducible representation T of the nonclassical type we associate a set of irreducible representations $\{T^{(g)} \mid g \in G\}$, consisting of 2^{n-1} pairwise nonequivalent irreducible representations of the nonclassical type. In this set there exists exactly one irreducible representation with highest weight $\boldsymbol{m} = (m_1, m_2, \ldots, m_k)$ such that $m_1 \geq m_2 \geq \cdots \geq m_k \geq \frac{1}{2}$.

For every highest weight of the nonclassical type $\boldsymbol{m}$ with $m_1 \geq m_2 \geq \cdots \geq m_k \geq \frac{1}{2}$ there exists an irreducible representation of the nonclassical type having $\boldsymbol{m}$ as its highest weight. Therefore, from above reasoning we derive the following classification of irreducible representations of the nonclassical type.

Theorem 5. *Irreducible representations of the nonclassical type of the algebra $U_q'(\mathrm{so}_n)$ are in one-to-one correspondence with pairs $(\boldsymbol{m}, g)$, where $\boldsymbol{m}$ is a highest weight of the nonclassical type with $m_1 \geq m_2 \geq \cdots \geq m_k \geq \frac{1}{2}$ and g is an element of the automorphism group G.*

Note that irreducible representations of the nonclassical type have no classical analogue. Namely, operators of representations of the nonclassical type are singular at the point $q = 1$.

7 Irreducible representations of $U_q'(\mathrm{so}_3)$

This and the next sections are devoted to examples of the theory described above. In this section we describe irreducible finite dimensional representations of the algebra $U_q'(\mathrm{so}_3)$.

Irreducible finite dimensional representations of the classical type of this algebra are given by nonnegative integral or half-integral number l. The irreducible representation T_l, given by such a number l, acts on $(2l+1)$-dimensional vector space $\mathcal{H}_l$ with a basis $|l,m\rangle$, $m=-l,-l+1,\ldots,l$. The operators $T_l(I_{21})$ and $T_l(I_{32})$ are given by the formulas

$$T_l(I_{21})|l,m\rangle = \mathrm{i}[m]_q|l,m\rangle,$$
$$T_l(I_{32})|l,m\rangle = \frac{1}{q^m+q^{-m}}\left([l-m]_q|l,m+1\rangle - [l+m]_q|l,m-1\rangle\right),$$

where $[a]_q$ denotes a q-number. Note that for these representations we have

$$\mathrm{Tr}\ T_l(I_{21}) = 0, \qquad \mathrm{Tr}\ T_l(I_{32}) = 0.$$

Irreducible representations $T_n^{\epsilon_1,\epsilon_2}$ of the nonclassical type are given by the numbers $\epsilon_i=\pm 1$ (they determine elements of the automorphism group G) and by the integer $n=1,2,\ldots$. (According to Section 6, these representations are given by half-integral number l, but we replaced l by $n=l+1/2$.) The representation $T_n^{\epsilon_1,\epsilon_2}$ acts on n-dimensional vector space with the basis $|k\rangle$, $k=1,2,\ldots,n$. The operators $T_n^{\epsilon_1,\epsilon_2}(I_{21})$ and $T_n^{\epsilon_1,\epsilon_2}(I_{32})$ are given by the formulas

$$T_n^{\epsilon_1,\epsilon_2}(I_{21})|k\rangle = \epsilon_1\frac{q^{k-1/2}+q^{-k+1/2}}{q-q^{-1}}|k\rangle,$$
$$T_n^{\epsilon_1,\epsilon_2}(I_{32})|1\rangle = \frac{1}{q^{1/2}-q^{-1/2}}\left(\epsilon_2[n]_q|1\rangle + \mathrm{i}[n-1]_q|2\rangle\right),$$
$$T_n^{\epsilon_1,\epsilon_2}(I_{32})|k\rangle = \frac{1}{q^{k-1/2}-q^{-k+1/2}}\left(\mathrm{i}[n-k]_q|k+1\rangle) + +\mathrm{i}[n+k-1]_q|k-1\rangle\right),$$

These representations have the properties

$$\mathrm{Tr}\ T_n^{\epsilon_1,\epsilon_2}(I_{21}) \neq 0, \qquad \mathrm{Tr}\ T_n^{\epsilon_1,\epsilon_2}(I_{32}) \neq 0.$$

There exist 4 one-dimensional irreducible representations of the nonclassical type. They are equivalent to $T_1^{\epsilon_1,\epsilon_2}$, $\epsilon_i=\pm 1$.

Note that a proof of the fact that these representations of $U'_q(\mathrm{so}_3)$ exhaust all irreducible representations of this algebra is given in [23].

8 Irreducible representations of $U'_q(\mathrm{so}_4)$

Irreducible finite dimensional representations of the classical type of the algebra $U'_q(\mathrm{so}_4)$ are given by two integral or two half-integral (but not integral) numbers r and s such that $r\geq |s|$. These numbers constitute the highest weight of the representation. We define the numbers $j=(r+s)/2$ and $j'=(r-s)/2$ and denote the representation by $T_{jj'}$. This representation acts on the vector space with the basis

$$|k,l\rangle, \qquad k=-j,-j+1,\ldots,j, \qquad l=-j',-j'+1,\ldots,j'.$$

The operators $T_{jj'}(I_{i,i-1})$, $i=2,3,4$, act upon these vectors by the formulas

$$T_{jj'}(I_{21})|k,l\rangle = \mathrm{i}[k+l]_q|k,l\rangle, \qquad T_{jj'}(I_{43})|k,l\rangle = \mathrm{i}[k-l]_q|k,l\rangle,$$
$$T_{jj'}(I_{32})|k,l\rangle = \frac{1}{(q^{k+l}+q^{-k-l})(q^{k-l}+q^{-k+l})}$$
$$\times\left\{-\left(q^{j-l}+q^{-j+l}\right)[j'-l]_q|k,l+1\rangle + \left(q^{j+l}+q^{-j-l}\right)[j'+l]_q|k,l-1\rangle\right.$$
$$\left.+\left(q^{j'-k}+q^{-j'+k}\right)[j-k]_q|k+1,l\rangle - \left(q^{j'+k}+q^{-j'-k}\right)[j+k]_q|k-1,l\rangle\right\}.$$

Irreducible finite dimensional representations of the nonclassical type of the algebra $U_q'(\mathrm{so}_4)$ are given by two half-integral (but not integral) numbers r, s such that $r \ge s > 0$ and by the numbers $\epsilon_1, \epsilon_2, \epsilon_3$, $\epsilon_i = \pm 1$, which determine elements of the automorphism group G. The numbers r and s constitute a highest weight of the representation if $\epsilon_1 = \epsilon_2 = \epsilon_3 = 1$. We define the numbers $j = (r+s)/2$ and $j' = (r-s)/2$ and denote the corresponding representations by $T_{jj'}^{\epsilon_1\epsilon_2,\epsilon_3}$.

If (r,s) runs over all highest weights of the nonclassical type with $r \ge s > 0$, then j and j' run over the values

$$j = 0, 1, 2, \ldots, \quad j' = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{3}, \ldots \qquad \text{or} \qquad j = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{3}, \ldots, \quad j' = 0, 1, 2, \ldots.$$

The representation $T_{jj'}^{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ acts on the vector space $\mathcal{H}$ with the basis

$$|k,l\rangle, \qquad k = j, j-1, \ldots, \tfrac{1}{2}, \qquad l = j', j'-1, \ldots, -j',$$

if j' is integral and with the basis

$$|k,l\rangle, \qquad k = j, j-1, \ldots, -j, \qquad l = j', j'-1, \ldots, \tfrac{1}{2},$$

if j is integral. The representations are given by the formulas

$$T_{jj'}^{\varepsilon_1,\varepsilon_2,\varepsilon_3}(I_{21})|k,l\rangle = \varepsilon_1 [k+l]_+ |k,l\rangle, \qquad T_{jj'}^{\varepsilon_1,\varepsilon_2,\varepsilon_3}(I_{43})|k,l\rangle = \varepsilon_2 [k-l]_+ |k,l\rangle,$$
$$T_{jj'}^{\varepsilon_1,\varepsilon_2,\varepsilon_3}(I_{32})|k,l\rangle = \frac{1}{[k+l]_q[k-l]_q(q-q^{-1})}\{-\mathrm{i}[j'-l]_q[j-l]_q|k,l+1\rangle$$
$$+ \mathrm{i}[j'+l]_q[j+l]_q|k,l-1\rangle - \mathrm{i}[j'-k]_q[j-k]_q|k+1,l\rangle + \mathrm{i}[j'+k]_q[j+k]_q|k-1,l\rangle\},$$

where $k \ne \frac{1}{2}$ if j is half-integral and $l \ne \frac{1}{2}$ if j' is half-integral, and by

$$T_{jj'}^{\varepsilon_1,\varepsilon_2,\varepsilon_3}(I_{32})|\tfrac{1}{2},l\rangle = \frac{1}{[l+\frac{1}{2}][l-\frac{1}{2}](q-q^{-1})}\{-\mathrm{i}[j-l]_q[j'-l]_q|\tfrac{1}{2},l+1\rangle$$
$$+ \mathrm{i}[j+l]_q[j'+l]_q|\tfrac{1}{2},l-1\rangle - \mathrm{i}[j'-\tfrac{1}{2}]_q[j-\tfrac{1}{2}]_q|\tfrac{3}{2},l\rangle + \mathrm{i}[j'+\tfrac{1}{2}]_q[j+\tfrac{1}{2}]_q\varepsilon_3(-1)^l|\tfrac{1}{2},-l\rangle\}$$

if j is half-integral and by

$$T_{jj'}^{\varepsilon_1,\varepsilon_2,\varepsilon_3}(I_{32})|k,\tfrac{1}{2}\rangle = \tfrac{1}{[k+\frac{1}{2}]_q[k-\frac{1}{2}]_q(q-q^{-1})}\{-\mathrm{i}[j-\tfrac{1}{2}]_q[j'-\tfrac{1}{2}]_q|k,\tfrac{3}{2}\rangle + \mathrm{i}[j+\tfrac{1}{2}]_q[j'+\tfrac{1}{2}]_q$$
$$\times \varepsilon_3(-1)^k|-k,\tfrac{1}{2}\rangle - \mathrm{i}[j'-k]_q[j-k]_q|k+1,\tfrac{1}{2}\rangle + \mathrm{i}[j'+k]_q[j+k]+q|k-1,\tfrac{1}{2}\rangle\}$$

if j' is half-integral.

Note that a proof of the fact that these representations of $U_q'(\mathrm{so}_4)$ exhaust all irreducible representations of this algebra is given in [8].

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Invariant Differential Operators and Representations with Spherical Orbits

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It is known that the algebra $\mathcal{D}(V)^G$ of G-invariant differential operators corresponding to a G-module V of a complex reductive group G is commutative if and only if V is a spherical G-module. In the present work we study the structure of $\mathcal{D}(V)^G$ for G-modules with spherical orbits. It is proved that the centralizer $\mathcal{Z}(V)^G$ of the subalgebra $k[V]^G$ in $\mathcal{D}(V)^G$ is commutative. Also a characterization of actions with spherical orbits in terms of the reduced action is obtained.

1 Multiplicity-free representations and spherical varieties

Let G be a connected reductive algebraic group defined over an algebraically closed field of zero characteristic, and (ρ, V) be a finite-dimensional representation of the group G. The induced representation of G on the algebra of polynomials $k[V]$ is given by the formula $(g * f)(v) := f\left(\rho\left(g^{-1}\right)v\right)$ for any $g \in G$, $f \in k[V]$, $v \in V$. It is well known that $k[V]$ as a G-module has the isotypic decomposition

$$k[V] = \oplus_{\lambda \in \Xi_+(G)} k[V]_\lambda,$$

where $\Xi_+(G)$ is the semigroup of dominant weights of G and $k[X]_\lambda$ is the sum of all irreducible G-submodules in $k[V]$ with the highest weight λ.

Definition 1. A representation (ρ, V) is called *multiplicity-free* if for any $\lambda \in \Xi_+(G)$ such that $k[V]_\lambda \neq 0$ the G-module $k[V]_\lambda$ is irreducible. We say in this case that the G-module $k[V]$ is *multiplicity-free.*

A complete list of multiplicity-free irreducible linear actions of connected reductive groups obtained by V. Kac [1, Theorem 3] is as follows:

(1) SL_n, Sp_n, $SO_n \otimes k^*$, S^2GL_n, $\Lambda^2 SL_n$ (for n odd), $\Lambda^2 GL_n$ (for n even), $SL_m \otimes SL_n$ (for $m \neq n$), $GL_n \otimes SL_n$, $GL_2 \otimes Sp_n$, $GL_3 \otimes Sp_3$, $GL_4 \otimes Sp_4$, $SL_n \otimes Sp_4$ (for $n > 4$), $Spin_7 \otimes k^*$, $Spin_9 \otimes k^*$, $Spin_{10}$, $G_2 \otimes k^*$, $E_6 \otimes k^*$.
(2) $G \otimes k^*$ for all semisimple groups G from list (1).

Here k^* is the multiplicative group of the field k considered as a one-dimensional algebraic group. The linear group $\Lambda^2 SL_n$ is the image of SL_n under the representation in the second exterior power of the tautological representation, and $S^2 SL_n$ is the same thing with respect to the second symmetric power.

A classification of reducible multiplicity-free representations was obtained independently by C. Benson and G. Ratkliff [2], and by A. Leahy [3].

Multiplicity-free representations form a very restricted class of representations. Nevertheless they are very important due to Roger Howe's philosophy that every "nice" result in the invariant theory of particular representations can be traced back to a multiplicity-free representation. For example, all of Weyl's first and second fundamental theorems can be explained by some multiplicity freeness results. Some other examples we shall discuss below.

Let B be a Borel subgroup of G.

Definition 2. A normal algebraic variety X with regular G-action (and the action $G : X$ itself) is said to be *spherical* if there exists a point $x \in X$ such that the orbit Bx is open in X.

Denote by $k(X)$ the field of rational functions on a variety X and by $k(X)^L$ (resp. $k[X]^L$) the subfield (resp. the subalgebra) of L-fixed elements for any subgroup $L \subset G$. By Rosenlicht's theorem [4, 2.3], the G-variety X is spherical if and only if $k(X)^B = k$.

Theorem 1 ([5]). *Suppose that X is a normal affine variety. Then an action $G : X$ is spherical if and only if the G-module $k[X]$ is multiplicity-free.*

In particular, multiplicity-free representations are in the natural one-to-one correspondence with spherical linear actions.

For more information on interconnections between spherical actions and representation theory, symplectic geometry, classical mechanics and so on, see the recent survey [6].

2 Representations with spherical orbits

In this section we consider a generalization of the notion of spherical action.

Definition 3. Let X be an irreducible algebraic variety. An action $G : X$ is called *an action with spherical orbits* if there exists an open susbet $X_0 \subset X$ such that for any $x \in X_0$ the orbit Gx is a spherical G-variety.

Below we list some basic facts about actions with spherical orbits.

(1) Any spherical actions is an action with sperical orbits.

(2) Any trivial G-actions is an action with spherical orbits.

(3) Rosenlicht's theorem implies that an action $G : X$ is an action with spherical orbits if and only if $k(X)^G = k(X)^B$.

(4) It is shown in [7, Corollary 1] that for an action with spherical orbits any G-orbit is spherical.

(5) Let $G_1 : X_1$ and $G_2 : X_2$ be actions with spherical orbits. Then the action $(G_1 \times G_2) : (X_1 \times X_2)$ is an action with spherical orbits.

Now we consider a fragment of a classification of representations with spherical orbits [8].

Definition 4. A G-module V is *indecomposable* if there exist no proper decompositions $G = G_1 \times G_2$ and $V = V_1 \oplus V_2$ such that $(g_1, g_2) * (v_1, v_2) = (g_1v_1, g_2v_2)$ for any $g = (g_1, g_2) \in G$ and any $v = (v_1, v_2) \in V$.

By property (5), it is sufficient to classify indecomposable representations with spherical orbits. In Tables 1 and 2 all indecomposable representations with spherical orbits (but non-spherical!) for connected semisimple groups are indicated. Table 1 contains representations with a one-dimensional quotient (i.e., $k[V]^G = k[q_1]$), and Table 2 contains representations with a two-dimensional quotient (i.e., $k[V]^G = k[q_1, q_2]$). (Here q_i are basic invariants.) There is no indecomposable representations with spherical orbits and a higher-dimensional quotient, for more details see [8].

Comments to the Tables. In the column "weights" the highest weights of the G-module are indicated. For the group $G_1 \times G_2$ the weight $\phi \otimes \psi$ corresponds to the tensor product of simple G_1- and G_2-modules with highest weights ϕ and ψ respectively. The symbol + denotes a direct sum of modules. If G is the product of several simple groups, then their fundamental weights are denoted successively by letters ϕ_i, ψ_i and τ_i.

Table 1.

	G	weights	$\dim V$
0	$\{e\}$	0	1
1	$\Lambda^2 SL_{2n}$	ϕ_2	$2n^2 - n$
2	$S^2 SL_n$	$2\phi_1$	$n(n+1)/2$
3	SO_n, $n > 2$	ϕ_1	n
4	$Spin_7$	ϕ_3	8
5	$Spin_9$	ϕ_4	16
6	G_2	ϕ_1	7
7	E_6	ϕ_1	27
8	SL_n, $n > 2$	$\phi_1 + \phi_{n-1}$	$2n$
9	SL_{2n+1}	$\phi_1 + \phi_2$	$(2n+1)(n+1)$
10	SL_{2n}	$\phi_1 + \phi_2$ $\phi_1 + \phi_{2n-2}$	$n(2n+1)$
11	$SL_n \times SL_n$	$\phi_1 \otimes \phi_1$	n^2
12	$SL_2 \times Sp_{2n}$	$\phi_1 \otimes \phi_1$	$4n$
13	$SL_4 \times Sp_4$	$\phi_1 \otimes \phi_1$	16
14	$SL_n \times SL_2 \times Sp_{2m}$, $n > 2$, $m \geq 1$	$\phi_1 \otimes \psi_1 + \psi_1 \otimes \tau_1$	$2(n+2m)$

Table 2.

	G	weights	$\dim V$
1	SO_8	$\phi_1 + \phi_3$	16
2	$Sp_{2n} \times SL_2 \times Sp_{2m}$, $n, m \geq 1$	$\phi_1 \otimes \psi_1 + \psi_1 \otimes \tau_1$	$4(m+n)$

3 Invariant differential operators

Let X be an affine variety, and set $A = k[X]$. We define the algebra of (algebraic) differential operators on A and X as follows: If $P \in \mathrm{End}_k(A)$ and $a \in A$, then $[P, a]$ denotes the usual commutator: $[P, a](b) = P(ab) - a(P(b))$, $b \in A$. Define $D^n(A) = 0$ for $n < 0$, and for $n \geq 0$ inductively define:

$$D^n(A) = \{P \in \mathrm{End}_k(A) \mid [P, a] \in D^{n-1}(A) \text{ for all } a \in A\}.$$

Clearly, $D^0(A) \cong A$ acting on itself by multiplication. Note that $D^n(A) \subset D^{n+1}(A)$ for all n, and we define $D(A) := \cup_n D^n(A)$. Now we set $\mathcal{D}^n(X) := D^n(A)$, and similarly for $\mathcal{D}(X)$. We call $\mathcal{D}(X)$ *the algebra of differential operators on* X.

Suppose that $X = k^n$, so that $A = k[x_1, \ldots, x_k]$. Then $\mathcal{D}(X)$ is the *kth Weyl algebra* W_k, i.e., the noncommutative algebra $k < x_1, \ldots, x_k, \partial_1, \ldots, \partial_k >$ generated by the x_i and the $\partial_j := \partial/\partial x_j$ with there usual commutation relations.

Now let X be an affine G-variety, where G is complex reductive. The group G acts rationally on $k[X]$ and $\mathcal{D}(X)$ [9, § 3]. Denote by $\mathcal{D}(X)^G$ the algebra of G-invariant differential operators.

We shall need the following well-known result.

Proposition 1. *If X is a spherical G-variety, then the algebra $\mathcal{D}(X)^G$ is commutative.*

Proof. By Schur's Lemma, any endomorphism T of $k[X]$ which commutes with G must preserve each isotypic component $k[V]_\lambda$. Further, the restriction of T to a given component must be a scalar, again by Schur's lemma. Hence $\mathcal{D}(X)^G$ is a subalgebra of the multiplication algebra on the set of isotypic components, and so is abelian. ■

There is a beautiful characterization of multiplicity-free representations in terms of invariant differential operators.

Theorem 2 ([10, Proposition 7.1]). *The algebra $\mathcal{D}(V)^G$ is commutative if and only if the representation (ρ, V) is multiplicity-free.*

Moreover, for multiplicity-free representations the algebra $\mathcal{D}(V)^G$ is isomorphic to a polynomial algebra, see [11] and [12].

The main purpose of this work is to obtain an analogous characterization for representations with spherical orbits.

4 Reduced actions and the algebra $\mathcal{Z}(X)^G$

Let X be an affine variety and G be a reductive algebraic group. The algebra $k[X]^G$ is finitely generated, and there is a canonical morphism $\pi_{X,G}$ (or just π_G) : $X \to X//G$, where $X//G$ is the affine variety corresponding to $k[X]^G$ and π_G^* is the inclusion $k[X]^G \subset k[X]$. The morphism π_G is surjective and induces a one-to-one correspondence between the closed G-orbits in X and the points of $X//G$, see [4, 4.4].

To any action $G : X$ one can canonically associate an action without non-constant invariants over some field of algebraic functions [13].

Namely, denote by K the field of quotients $Qk[X]^G$ of $k[X]^G$ and by $\overline{K}$ its algebraic closure. Let X^{red} be the spectrum of the $\overline{K}$-algebra $\overline{K}[X^{\mathrm{red}}] = \overline{K} \otimes_{k[X]^G} k[X]$. This is an irreducible affine variety over $\overline{K}$ defined over K, with $K[X^{\mathrm{red}}] = K \otimes_{k[X]^G} k[X]^G$. Its dimension equals

$$\dim X^{\mathrm{red}} = \dim X - \dim X//G, \tag{1}$$

which is the dimension of a generic fiber of the quotient morphism $\pi_G : X \to X//G$.

The action of G on $k[X]$ is $k[X]^G$-linear and hence can be extended to an action of $G(K)$ on $K[X^{\mathrm{red}}]$, which, in its turn, can be extended to an action of $G(\overline{K})$ on $\overline{K}[X^{\mathrm{red}}]$. This gives rise to an action of $G(\overline{K})$ on X^{red} defined over K. This action is called *reduced action.*

Proposition 2. *The reduced action is spherical if and only if the following conditions hold:*

1) the action $G : X$ is an action with spherical orbits;

2) there exists an open dense subset $X_0 \subset X$ such that for any points $x_1, x_2 \in X_0$ with $Gx_1 \neq Gx_2$ there is $f \in k[X]^G$ such that $f(x_1) \neq f(x_2)$ (i.e. generic G-orbits can be separated by invariants).

Proof. We follow the proof of [13, Proposition 4]. Elements of $\overline{K}$ can be thought as algebraic functions on $Y = X//G$, and points of X^{red} as algebraic mappings $\phi : Y \to X$ such that $\pi_G \circ \phi = \mathrm{id}$. We may assume that $G \subset GL_n(k)$ and X is a G-invariant closed subvariety of k^n (see, e.g., [4]). Denote by $\mathfrak{b}$ a Borel subalgebra in the Lie algebra $\mathfrak{g}$ of the group G. Let us think elements of $\mathfrak{b}(\overline{K})$ as algebraic mappings $\xi : Y \to \mathfrak{b}$. The tangent algebra of the stabilizer $B(\overline{K})_\phi$ is defined by the linear equations

$$\xi(y)\phi(y) = 0 \tag{2}$$

over $\overline{K}$. For a generic point $y \in Y$ they turn into linear equations defining the tangent algebra of the stabilizer $B_{\phi(y)}$ over k.

Obviously, the functional rank of system (2) is the maximum of the ranks of its specializations. Since all $\phi(Y)$ do not belong to a proper closed subvariety of X, we obtain that the dimension of a generic stabilizer for the action $B(\overline{K}) : X^{\mathrm{red}}$ is equal to that for the action $B : X$.

By Rosenlicht's theorem, the dimension of a generic B-orbit on X is equal to $\dim X -$ tr.deg $k(X)^B$. By (1), the action $G(\overline{K}) : X^{\mathrm{red}}$ is spherical if and only if

$$\text{tr.deg } k(X)^B = \dim X//G = \text{tr.deg } Qk[X]^G. \tag{3}$$

Note that $Qk[X]^G \subseteq k(X)^G \subseteq k(X)^B$. Hence (3) is equivalent to tr.deg $Qk[X]^G =$ tr.deg $k(X)^G$ $=$ tr.deg $k(X)^B$. The second equality is the condition 1) of Proposition 2. By [4, 3.2], the first equality means that generic G-orbits can be separated by invariants. ■

Corollary 1. *Suppose that $G : V$ is a linear action of a semisimple group G. Then the reduced action $G(\overline{K}) : V^{\mathrm{red}}$ is spherical if and only if $G : V$ is an action with spherical orbits.*

Proof. By [4, Theorem 3.3], for a semisimple group action on a factorial variety the condition $Qk[X]^G = k(X)^G$ holds automatically. ■

Consider the centralizer of $k[X]^G$ in $\mathcal{D}(X)^G$:

$$\mathcal{Z}(X)^G = \left\{ D \in \mathcal{D}(X)^G \mid D(ab) = aD(b) \text{ for any } a \in k[X]^G,\ b \in k[X] \right\}.$$

Clearly, $k[X]^G \subset \mathcal{Z}(X)^G$. We are going to show that $\mathcal{Z}(X)^G$ contains differential operators of positive order.

There is a canonical morphism $(\pi_G)_* : \mathcal{D}(X)^G \to \mathcal{D}(X//G)$, where $(\pi_G)_*(P)$ is the restriction of $P \in \mathcal{D}(X)^G$ to $k[X]^G = k[X//G]$. We let $\mathcal{K}^n(X)$ denote the elements of $\mathcal{D}^n(X)$ which annihilate $k[X]^G$. Then, by definition, $\mathcal{K}^n(X)^G$ is the kernel of $(\pi_G)_*$ restricted to $\mathcal{D}^n(X)^G$, and $\mathcal{K}(X)^G := \cup_n \mathcal{K}^n(X)^G$ is the kernel of $(\pi_G)_*$. We have

$$0 \longrightarrow \mathcal{K}(X)^G \hookrightarrow \mathcal{D}(X)^G \xrightarrow{(\pi_G)_*} \mathcal{D}(X//G).$$

Note that $\mathcal{D}^{n-1}(X)\tau(\mathsf{g}) \subset \mathcal{K}^n(X)$, where $\tau(C)$ denotes the action of $C \in \mathsf{g}$ on $k[X]$ as a derivation.

Define a positive integer n_0 by

$$\mathcal{K}^{n_0}(X)^G \neq 0 \text{ and } \mathcal{K}^m(X)^G = 0 \text{ for any } m < n_0.$$

Lemma 1. *The space $\mathcal{K}^{n_0}(X)^G$ is contained in $\mathcal{Z}(X)^G$.*

Proof. For any $a, b \in k[X]^G$ and $P \in \mathcal{K}^{n_0}(X)^G$ one has $[P, a](b) = P(ab) - bP(a) = 0$. Hence $[P, a] \in \mathcal{K}^{n_0-1}(X)^G$. By definition, this implies $[P, a] = 0$. ■

Now we are able to prove the main result of this note.

Theorem 3. *Let $G : X$ be an action with spherical orbits of a reductive group G on an affine variety X. Suppose that generic G-orbits can be separated by invariants. Then the algebra $\mathcal{Z}(X)^G$ is commutative.*

Proof. Elements of $\mathcal{Z}(X)^G$ commute with the $k[X]^G$-action on $k[X]$ and can be considered as differential operators on $K \otimes_{k[X]^G} k[X]$ or on $\overline{K} \otimes_{k[X]^G} k[X]$. Thus one has the embedding $\mathcal{Z}(X)^G \hookrightarrow \mathcal{D}(X^{\mathrm{red}})^{G(\overline{K})}$. By Propositions 1 and 2, the last algebra is commutative. ■

The algebra $\mathcal{D}(V)^G$ is the centralizer of its scalar subalgebra k. This algebra is commutative in spherical case. By Theorem 3, for representations of Table 1 (resp. Table 2) the commutativity holds if one replaces scalars by $k[q_1]$ (resp. $k[q_1, q_2]$).

Example 1. Let $G = (k^*)^s$ be an algebraic torus acting on $V = k^n$, $n \geq s$ by

$$(t_1, \ldots, t_s) * (x_1, \ldots, x_n) := (t_1 x_1, \ldots, t_s x_s, x_{s+1}, \ldots, x_n).$$

It is clear that any torus action is an action with spherical orbits. For this particular action generic orbits can be separated by invariants. One has $k[V]^G = k[x_{s+1}, \ldots, x_n]$ and $\mathcal{Z}(V)^G = k[x_1\partial_1, \ldots, x_s\partial_s, x_{s+1}, \ldots, x_n]$.

Example 2. Consider the action $k^* : k^2$, $t * (x_1, x_2) = (tx_1, tx_2)$. This is an action with spherical orbits, but generic orbits can not be separated by invariants. Here $k[V]^G = k$ and $\mathcal{Z}(V)^G = \mathcal{D}(V)^G = k\langle x_1\partial_1, x_1\partial_2, x_2\partial_1, x_2\partial_2\rangle$. The last algebra is not commutative.

Finishing this section, we would like to state the following

Conjecture. *The following conditions are equivalent:*

1) an action $G : X$ is an action with spherical orbits and generic orbits can be separated by invariants;

2) the algebra $\mathcal{Z}(X)^G$ is commutative.

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C^*-Algebras Associated with $\mathcal{F}_{2^n}$ Zero Schwarzian Unimodal Mappings

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In this paper we consider C^*-algebras connected with a simple unimodal non-bijective dynamical system (f, I) with zero Schwarzian. We associate with f a C^*-algebra $C^*(\mathcal{A}_f)$. In the first part we describe the dynamics of (f, I). In the second part we describe the set of irreducible representation of $C^*(\mathcal{A}_f)$ for a special subclass of mappings (Theorem 3) and give realization (Theorem 4) of this algebra as C^*-algebra generated by continuous fields of C^*-algebras on the spectrum of $C^*(\mathcal{A}_f)$. As a result we find out when two such C^*-algebras are isomorphic.

1 Zero Schwarzian unimodal mappings

Many important examples of C^*-algebras arising in physical models are connected with dynamical systems. In particular, the two-parameter unit quantum disk algebra [1] is generated by the relation

$$qzz^* - z^*z = q - 1 + \mu(1 - zz^*)(1 - z^*z),$$
$$0 \le \mu \le 1, \qquad 0 \le q \le 1, \qquad (\mu, q) \ne (0, 1),$$

which can be rewritten [2] in the form $XX^* = F(X^*X)$, where

$$F(\lambda) = \frac{(q+\mu)\lambda + 1 - q - \mu}{\mu\lambda + 1 - \mu}.$$

In present paper we investigate unimodal deformation of the above relation. Consider a continuous unimodal map $f : [0, 1] \to [0, 1]$ with zero Schwarzian that consists of two hyperbolae:

$$f(x) = \begin{cases} f_1(x) = \dfrac{\alpha_1 x + \beta_1}{\gamma_1 x + \delta_1}, & x \in [0, \rho], \\[2mm] f_2(x) = \dfrac{\alpha_2 x + \beta_2}{\gamma_2 x + \delta_2}, & x \in (\rho, 1]. \end{cases}$$

Let $\mathrm{Orb}_+(f)$ be a set of all non-cyclic positive orbits [7]. Considering mappings up to topological conjugacy [3] we can assume that $\gamma_2 = 0$, $\delta_2 = 1$. In the present paper we restrict ourselves with the following types of f (see Fig. 1):

Type 1: $f_2(1) = 1$, $f_1(\rho) = f_2(\rho) = 0$,

$$f(x) = \begin{cases} f_1(x) = \dfrac{\alpha x - \alpha\rho}{\gamma x + \delta}, & x \in [0, \rho], \\[2mm] f_2(x) = \dfrac{x - \rho}{1 - \rho}, & x \in (\rho, 1]; \end{cases}$$

Type 2: $f_2(1) = 0$, $f_1(\rho) = f_2(\rho) = 1$,

$$f(x) = \begin{cases} f_1(x) = \dfrac{\alpha(x - \rho) + \delta + \gamma\rho}{\gamma x + \delta}, & x \in [0, \rho], \\[2mm] f_2(x) = \dfrac{x - 1}{\rho - 1}, & x \in (\rho, 1]. \end{cases}$$

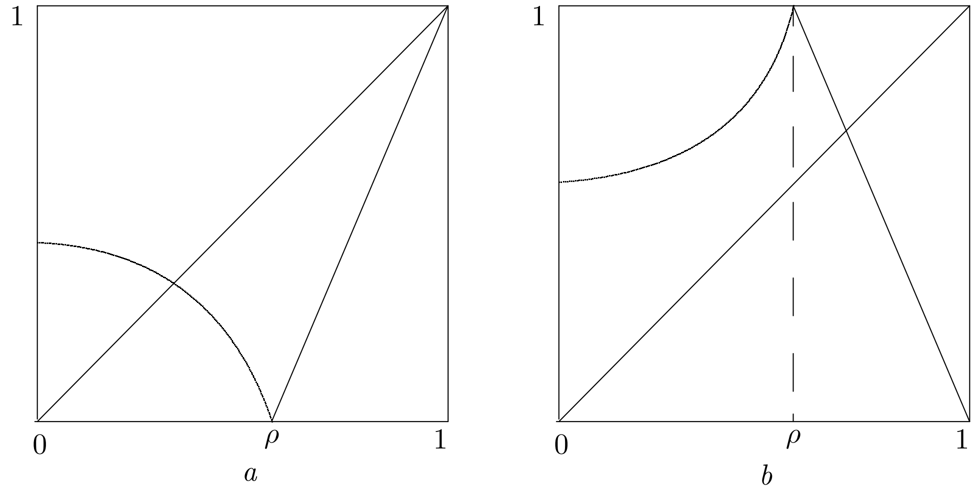

Figure 1.

Lemma 1. *Let (f, I) be $\mathcal{F}_{2^n}$ dynamical system. Then $n \leqslant 3$ and for each $i \in \{0, 1, 2\}$ only two following cases are possible.*

1. There exists only one attractive cycle of the period 2^i, smaller cycles are repellent and no cycles of larger periods.

2. There exists an interval of periodic points such that the middle point of the interval has period 2^i, other points of the interval have period 2^{i+1}, smaller cycles are repellent and no cycles of larger periods.

Cases for $i = 0$ correspond only to type 1. Cases for $i = 1$ correspond either to type 1 or to type 2. Cases for $i = 2$ correspond only to type 2.

Proof. First consider type 1 mappings. Define ρ and ρ_1 as $f_1(0) = \rho_1$ and $f_1(\rho) = 0$. Let x_0 be a stable point of $f_1(x)$ that lies between 0 and ρ. It can be easily checked that: $\mathrm{Sign}\,(1-|f'(x_0)|) = \mathrm{Sign}\,(\rho - \rho_1)$.

When f has an attracting stable point, $|f_1'(x_0)|$ is less than 1. Hence $\rho > \rho_1$. Therefore $\forall\, x \in [0, x_0)$ $f^{(2)}(x) > x$, $\forall\, x \in (x_0, 1)$ $f^{(2)}(x) < x$ and mapping f has no cycles of the period two. We will observe the same situation until $|f_1'(x_0)|$ equals 1. When $|f_1'(x_0)| = 1$ the left hyperbola is symmetric with respect to diagonal. Therefore each point of the interval $[0, \rho]$ except x_0 has period two. As follows from a simple geometrical considerations in this case mapping f has no cycles of period four. If $|f_1'(x_0)|$ is more than 1 or equivalently the stable point becomes repellent, then two following cases are possible: 1) any cycle of period 2^n exists, 2) there exists either attracting cycle of period two or an interval of periodic points such that the middle point of the interval has period two and other points have period four. Therefore in the first case the dynamical system is not $\mathcal{F}_{2^n}$. In the second case the dynamical system has obviously no cycles of larger periods.

Let us prove the latter statement.

1. The proof is by induction on n. The base of induction is existence of repellent cycles of periods one and two. Let $\gamma \neq 0$. Hence we can put $\gamma = 1$. When $\gamma = 0$, the proof is trivial because if cycle of period 2^n exists, then one is obviously repellent. Let x_n be first from the left stable point of $f^{(2^n)}$. It is really uninteresting work to show that if a cycle of period two is repellent, then for all $x \in [0, x_2]$ $|{f^{(4)}}'(x)| > 1$. Hence we can add this fact to the base of induction. Let x' be a point such that $f_1(x') = \rho$. It is also very boring to show that for any f_1 such that x_0 is repellent point the derivative $(f_1(f_1(x)))'$, $x \in [x', x_0]$ is more than one.

Now we prove that if the cycles of periods 2^{n-1} and 2^n are repellent for some $n \geqslant 1$ and $\forall\, x \in [0, x_n]$ $|{f^{(2^n)}}'(x)| > 1$, then there exists a cycle of period 2^{n+1} which is repellent and $\forall\, x \in [0, x_{n+1}]$ $|{f^{(2^{n+1})}}'(x)| > 1$.

1) First we prove that if a stable point x_0 is repellent, then there exists a cycle of period two. If point x_0 is repellent, then $\rho < \rho_1$. Therefore $f^{(2)}(0) > 0$ and exists $x_1 \in [0, x_0]$ such that $f(x_1) = \rho$ and $f^{(2)}(x_1) = 0$. Hence there exists $x_2 \in [0, x_1]$ such that $f^{(2)}(x_2) = x_2$.

Now consider $f^{(2^n)}$. It is evident that $(f^{(2^n)}, J)$, where $J = [0, x_{n-1}]$ and x_{n-1} is the first from the left stable point of $f^{(2^{n-1})}$, is equivalent to type 1 mapping. Therefore $f^{(2^n)}$ has cycle of period two or equivalently f has cycle of period 2^{n+1}.

2) Now let repellent cycles of the periods 2^{n-1} and 2^n exist. Consider a dynamical system $(f^{(2^n)}, J)$, where $J = [0, x_{n-1}]$. It is clearly type one system for some $\tilde{\rho}$, $\tilde{f}_1$ and $\tilde{f}_2$. By induction hypothesis $\forall\ x \in [0, x_n]\ |\tilde{f}_1'(x)| > 1$. Also $\forall\ x \in [\tilde{\rho}, x_{n-1}]\ \tilde{f}_2'(x) > 1$. Therefore, for all $x \in [0, x_{n+1}]\ |f^{(2^{(n+1)})'}(x)| = |(\tilde{f}_2(\tilde{f}_1(x)))'| > 1$.

Now consider type 2 mapping. Let $g(x) = f(f(x))$ (see Fig. 2). We will consider $g(x)$ in $[0, s] \times [0, s]$, where s is a stable point of f, bearing in mind parallel considerations for the right corner. Like in case one we define ρ and ρ_1 as $g(0) = \rho_1$ and $g(\rho) = 0$. Let x_0 be a stable point of $g(x)$ that lies between 0 and ρ. Thus we obtain a situation of the type 1. Therefore if (f, I) is $\mathcal{F}_{2^n}$, then for each $i = 1, 2$ mapping f has either an attracting cycle of period 2^i or an interval of periodic points such that the middle point of the interval has period 2^i and other points have period 2^{i+1}. ■

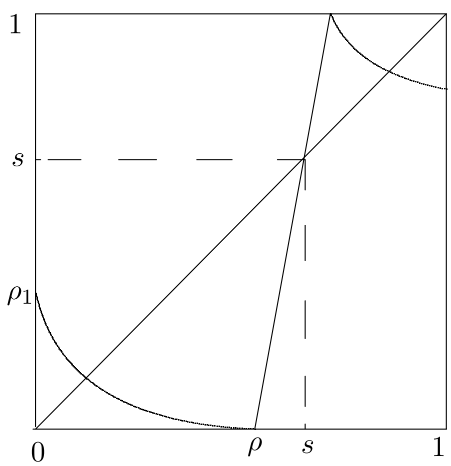

Figure 2.

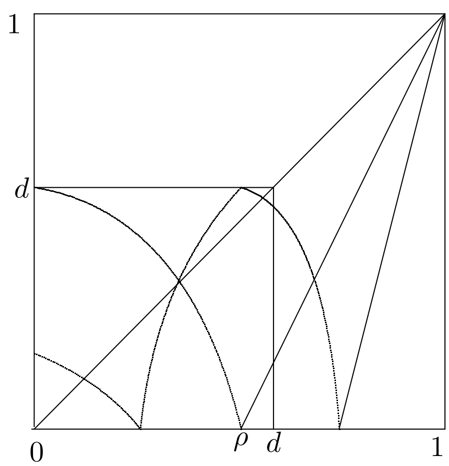

Figure 3.

Theorem 1. *Let (f, I) be type 1 or type 2 dynamical system. Let s be its stable repellent point and β_j its repellent cycle of period 2^j (if it exists). For type one mapping $j = 0$ and $\beta_0 \neq s$. For type two mapping $j = 1$. Define $P_s = \{\delta | \delta \in \mathrm{Orb}_+(f),\ \alpha(\delta) = s\}$ and $P_{\beta_j} = \{\delta | \delta \in \mathrm{Orb}_+(f),\ \alpha(\delta) = \beta_j\}$. Let i be defined as in Lemma 1. Then*

I. For dynamical system of type 1 and $i = 0$ or for dynamical system of type 2 and $i = 1$:

1) $\mathrm{Orb}_+(f) = P_s$;

2) there exists $I_s = [t_1, t_2)$ and one-to-one mapping $\phi : I_s \to P_s$ such that $t \in \phi(t)$ for every $t \in I_s$;

3) I_s can be chosen to lie in arbitrary neighborhood of s.

II. For dynamical system of type 1 and $i = 1$ or for dynamical system of type 2 and $i = 2$:

1) $\mathrm{Orb}_+(f) = P_s \dot{\cup} P_{\beta_j}$;

2) there exists $I_{\beta_j} = [t_1, t_2)$ and one-to-one mapping $\phi : I_{\beta_j} \to P_{\beta_j}$ such that $t \in \phi(t)$ for every $t \in I_{\beta_j}$. There exists $I_s = [t_1, t_2)$ and one-to-one mapping $\phi : I_s \to P_s$ such that $t \in \phi(t)$ for every $t \in I_s$;

3) $I_s \cap I_{\beta_j} = \varnothing$. Moreover I_{β_j} can be chosen to lie in arbitrary neighborhood of β_j.

Proof. Let us first consider type 1 mapping. Define ρ and ρ_1 as in lemma: $f_1(0) = \rho_1$ and $f_1(\rho) = 0$. By the lemma $\rho \geqslant \rho_1$. Define intervals $I_n = f_2^{(-n)}([0,\rho))$, $n \geqslant 1$. Note that $I_i \cap I_j = \varnothing$ $i \neq j$, $\forall$ n $I_n \cap [0,\rho) = \varnothing$ and $\cup_{n\geqslant 1} I_n \cup [0,\rho) = [0,1)$. It is easy to see that for $x \in \delta \in \mathrm{Orb}_+(f)$, $x \in I_n$, $\quad f^{-1}(x) = f_2^{-1}(x) \in I_{n+1}$ and $\alpha(x) = s$. Now prove that any of the intervals I_n can be chosen as I_s. Since $\forall\, x \in I_n$ $f^{(-n)}(x) \notin I_n$ we obtain that different points of I_n correspond to different trajectories. It is clear to see that if $x \in \delta \in \mathrm{Orb}_+(f)$ and $x \in I_i$, then for any j there exists point $y \in \delta$ such that $y \in I_j$. Now let $\exists$ $x \in \delta \in \mathrm{Orb}_+(f)$, $x \in [0,\rho)$ and $x \notin \mathrm{Per}(f)$. The stable point $x_0 \in [0,\rho)$ is repellent for f_1^{-1} and therefore there exists a natural number m such that $f_1^{(-m)}(x) \in [\rho_1,\rho)$ and $f^{(-1)}(f_1^{(-m)})(x) = f_2^{-1} f_1^{(-m)}(x) \in I_1$. By $f^{-1}(x)$ we mean either $f_1^{-1}(x)$ or $f_2^{-1}(x)$ and by $f^{-1}(x) = f_2^{-1}(x)$ we mean that there is only one possibility. If $x \in \delta \in \mathrm{Orb}_+(f)$, $x \in [0,\rho)$ and $x \in \mathrm{Per}(f)$, then for $f^{-1}(x) \in \delta$ we obtain $f^{-1}(x) = f_2^{-1}(x) \in I_1$.

Now consider type 2 mapping. Let $g(x) = f(f(x))$. Dynamical system $(g,[0,s])$ satisfies all conditions for the case 1. Let $\delta \in \mathrm{Orb}_+(f)$, $\delta = \{x_k\}$, $k \in \mathbb{Z}$ and $x_0 \in [0,s)$. It is clear to see that such point x_0 always exists. By the case two subsequences $\{x_{2n}\}$, $n \in \mathbb{Z}$ can be parametrized by I_s. Since f^{-1} is one to one on $[0,s)$, then I_s parametrizes all $\delta \in \mathrm{Orb}_+(f)$. Consider type 1 mapping. It has a cycle of period two which is attracting by Lemma 1. In this case $s = 1$ and β_0 are stable repellent points. Consider mapping $g(x) = f(f(x))$ (see Fig. 3). Define $d = f_1(0)$. It is clear that $(g(x),[0,d])$ is equivalent to type 2 mapping for $i = 1$. Therefore exists interval I_{β_0} that parameterizes orbits $\delta \in \mathrm{Orb}_+(f)$ $\delta = \{x_k\}$ such that $x_k < d$ for all k ($\delta \in P_{\beta_0}$). Define $I_1 = [d, f_2^{-1}(d)]$ and $I_{j+1} = f_2^{-1}(I_j)$, $j \geqslant 1$. Let's prove that any of the intervals I_j can be chosen as I_s. Indeed if $t_1, t_2 \in I_j$ and $t_1 \neq t_2$, then $f_2^{(-n)}(t_1) \neq f_2^{(-n)}(t_2)$ for all $n \geqslant 1$. Therefore different points of the interval correspond to different orbits. If $x_k \in \delta \in \mathrm{Orb}_+(f)$, $x_k > d$, then $x_k \in I_l$ for some l and for all $j \geqslant 1$ there exists $n \in \mathbb{Z}$ such that $x_{k+n} \in I_j$. Hence $P_s \dot\cup P_{\beta_0}$ parameterizes all orbits in $\mathrm{Orb}_+(f)$.

The proof for type 2 and $i = 2$ is absolutely analogous to the proof for type 2 and $i = 1$. ■

Proposition 1. *Let (f,I) be either type 1 dynamical system and $i = 0$ or type 2 dynamical system and $i = 1$; then it has only one anti-Fock orbit δ, $|\alpha(\delta)| = 1$.*

Proof. For type 1 mapping we can simply write it: $\Big\{0,\ \rho_1,\ f_2^{-1}(\rho_1),\ f_2^{(-2)}(\rho_1),\ \dots\Big\}$. It is clear that $\lim\limits_{n\to\infty} f_2^{(-n)}(\rho_1) = 1$ exists. Hence $|\alpha(\delta)| = 1$.

For type 2 mapping we consider a sequence $\{0,\ \rho_1,\ g^{-1}(\rho_1),\ g^{(-2)}(\rho_1),\ \dots\}$, where $g(x) = f(f(x))$. Like for type 1 this sequence is a unique anti-Fock orbit for g. Since f^{-1} is one to one on $[0,s]$ we obtain that a unique anti-Fock orbit for g corresponds to a unique anti-Fock orbit for f. ■

2 Enveloping C^*-algebra

By $C^*(A_f)$ we mean a C^*-algebra obtained from free $*$-algebra $\mathcal{F}(X,X^*)$ generated by X with sub-norm $\|b\| = \sup\limits_{\pi} \|\pi(b)\|$ where supremum is taken over all $\pi \in \mathrm{Rep}(\mathcal{F}(X,X^*))$ such that $\pi(XX^*) = f(\pi(X^*X))$ by standard factorization and completion procedure. The following theorem (see [2]) connects representations of C^*-algebra $C^*(A_f)$ with certain orbits of dynamical system $(f,\mathbb{R}_+)$.

Theorem 2. *Let f be partially monotone continuous map and $(f,\mathbb{R})$ be $\mathcal{F}_{2^m}$ dynamical system. Let $\mathcal{A} = \mathbb{C}^*(A_f)$ be corresponding C^*-algebra.*

1. To every positive non-cyclic orbit $\omega(x_k)_{k\in\mathbb{Z}}$ there corresponds an irreducible representation π_ω in Hilbert space $l_2(\mathbb{Z})$ given by the formulae: $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k \in \mathbb{Z}$ and $X = UC$ is a polar decomposition.

2. To positive non-cyclic Fock-orbit $\omega = (x_k)_{k\in\mathbb{N}}$ there corresponds an irreducible representation π_ω in Hilbert space $l_2(\mathbb{N})$ given by the formulae: $Ue_0 = 0$, $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k > 1$ and $X = UC$.

3. To positive non-cyclic anti-Fock-orbit $\omega = (x_{-k})_{k\in\mathbb{N}}$ there corresponds an irreducible representation π_ω in Hilbert space $l_2(\mathbb{N})$ given by the formulae: $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k > 1$ and $X = UC$.

4. To cyclic positive orbit $\omega = (x_k)_{k\in\mathbb{N}}$ of length m there corresponds a family of m-dimensional irreducible representation $\pi_{\omega,\phi}$ in Hilbert space $l_2(\{1,\dots,m\})$ given by the formulae: $Ue_0 = e^{i\phi}e_{m-1}$, $Ue_k = e_{k-1}$, $Ce_k = \sqrt{x_k}e_k$ for $k = 1,\dots,m$; $0 \le \phi \le 2\pi$ and $X = UC$.

This is a complete list of unequivalent irreducible representation of a given $*$-algebra.

As follows from [6] C^*-algebras generated by operators of irreducible representations are either $Z \times_\delta C(\overline{\delta})$, where $\overline{\delta} = \delta\cup\omega(\delta)\cup\alpha(\delta)$ for non-cyclic bilateral orbit or $M_m(\mathcal{T}(C(\mathbb{T})))$, where $\mathcal{T}(C(\mathbb{T}))$ is algebra of the Toeplitz operators for Fock and anti-Fock orbits.

Consider T as a topological space with topology induced from $\mathbb{R}$. Let H be a Hilbert space with orthonormal basis $(e_k)_{k\in\mathbb{Z}}$. Let U be unitary shift operator $Ue_k = e_{k+1}$ for all $k \in \mathbb{Z}$. We know that for any $t \in T$ $\phi(t) = (x_k)_{k\in\mathbb{Z}} \ni t$ futher on we will assume, without loss of generality, that $x_0 = t$. Denote by $C_{\phi(t)}$ diagonal operator $C_{\phi(t)}e_k = x_ke_k$ for all $k \in \mathbb{Z}$. Algebra $C^*(\pi_{\phi(t)})$ is generated by operator $X_{\phi(t)} = U(C_{\phi(t)})^{1/2}$. Denote by $\Psi : C^*(\pi_{\phi(t)}) \to B(H)^T$ the $*$-homomorphism defined on the generator as $\Psi(X)(t) = X_{\phi(t)}$. Further on we will denote by $\pi_{\phi(t)}$ the (irreducible for $\phi(t) \in \mathrm{Orb}_+(f)$ and reducible when $t \in \overline{T} \setminus T$) representation associated with non-cyclic orbit $\phi(t)$ by formulas of the Theorem 2 and by $\pi_{\beta,\psi}$ the finite dimensional representation associated with cycle β and parameter $\psi \in [0, 2\pi]$. In the following theorem we give the description of all irreducible representations of $C^*(A_f)$ in cases 1 and 3 of Lemma 1 as well as fix some notations.

Theorem 3. *Let (f, I) be either type 1 mapping and cycle of period one is attracting or type two mapping and cycle of period two is attracting then*

1. In the first case $C^*(A_f)$ has only one-dimensional irreducible finite dimensional representations parameterized by $\phi, \psi \in [0, 2\pi)$. They are given by the following formulas: $\pi_0(X) = \sqrt{x_0}e^{i\phi}$, $\pi_1(X) = e^{i\psi}$. In the second case $C^*(A_f)$ has only one-dimensional and two-dimensional irreducible finite dimensional representations parameterized by $\phi \in [0, 2\pi)$ they are of the form $\pi_{s,\phi}$ and $\pi_{\beta_1,\phi}$.

2. $C^*(A_f)$ has irreducible Fock representation π_f and one irreducible anti-Fock representation π_{af}. Both of them in case 1 and π_{af} in case 2 generate algebras of Toeplitz operators. In case 2 π_f generate algebra $M_2(\mathcal{T}(C(\mathbb{T})))$, where $\mathcal{T}(C(\mathbb{T}))$ is the algebra of Toeplitz operators.

3. In the first case for each $t \in T = I_1$ there is irreducible infinite-dimensional representations $\pi_{\phi(t)}$ of $C^*(A_f)$. For all $t \in T$ operators of π_t generate isomorphic C^*-algebras. Denote this algebra by $\mathcal{A}$. Algebra $\mathcal{A}$ is a cross-product algebra $C(X) \times \mathbb{Z}$ where X is a closure of any orbit $\phi(t)$. Algebra $\mathcal{A}$ has only one infinite-dimensional representation and two circles of one dimensional representations denote γ_s, γ_1 two arbitrary such representations from different circles. In the second case for each $t \in T = I_s$ there is irreducible infinite-dimensional representations $\pi_{\phi(t)}$ of $C^*(A_f)$. For all $t \in I_s$ operators of $\pi_{\phi(t)}$ generate isomorphic C^*-algebras. Denote this algebra by $\mathcal{B}$. Algebra $\mathcal{B}$ has only one infinite-dimensional representation one circle of one dimensional representations (denote η_s any of them) and one circle of two-dimensional representations (denote η_{β_1} any of them).

4. $*$-algebra $C^*(A_f)$ has no other irreducible representations.

5. For any $a \in C^*(A_f)$ the mapping $\Psi(a)$ is continuous map from T to $B(H)$ where the latter is endowed with norm topology. Moreover, for all $a \in C^*(A_f)$ the following equality holds $\Psi(a)(t_2) = U^*\Psi(a)(t_1)U$, where $\overline{T} = [t_1, t_2]$.

6. C^*-algebras $C^*(\pi_{\phi(t_1)})$ and $C^*(\pi_{\phi(t_2)})$ coincide for any $t_1, t_2 \in T$ as a subalgebras of $B(H)$. We have denoted this algebra by $\mathcal{A}$ for dynamical systems of type 1 and by $\mathcal{B}$ for type 2. Since $U \in \mathcal{A}$ and $U \in \mathcal{B}$ we denote by adU the inner automorphism $a \to U^*aU$, $a \in \mathcal{A}$ or $a \in \mathcal{B}$ as appropriate.

Proof. First two statements of the theorem are direct consequences of Theorems 2, 1. Let us show that for any $t_1, t_2 \in T$ algebras $C^*(\pi_{\phi(t_1)})$ and $C^*(\pi_{\phi(t_2)})$ coincide as a subalgebras of $B(H)$. $C^*(\pi_{\phi(t)})$ is generated by operators U and $C_{\phi(t)}$. Since $\phi(t)$ is not periodic there is point $x \in \phi(t)$ which occurs only finite number of times in the sequence $\phi(t)$, it is easy to see that x is isolated point in $\overline{\phi(t)}$. Hence if we put g to be equal to 1 at x and zero otherwise then g will be continuous function on spec $(C_{\phi(t)})$ and $g(C_{\phi(t)})$ will be a compact non-zero operator in $C^*(\pi_{\phi(t)})$. And since this algebra is prime it contains all compact operators. Hence by compact perturbation $C_{\phi(t)} + K$, where K is compact we can obtain any diagonal operator $C = \mathrm{diag}(c_k)_{k\in\mathbb{Z}}$ such that $\omega(\{c_k\}) = \omega(\phi(t))$ and $\alpha(\{c_k\}) = \alpha(\phi(t))$. Obvious equality $C^*(U, C_{\phi(t)}) = C^*(U, C_{\phi(t)} + K)$ completes the proof of our claim. It is easy to see that, up to isomorphism, $C^*(\pi_{\phi(t)})$ depends only on two integers $|\omega(\phi(t))|$ and $|\alpha(\phi(t))|$.

We proceed now to show that for every $a \in C^*(A_f)$ the map $\Psi(a)$ is continuous. Since X is a generator of $C^*(A_f)$ we need only to prove that $\Psi(X)(t) = U(C_{\phi(t)})^{1/2}$ is continuous in t

$$||\Psi(X)(t) - \Psi(X)(t')|| = \left\|C_{\phi(t)}^{1/2} - C_{\phi(t)}^{1/2}\right\| = \sup_{k\in\mathbb{Z}} \left|x_k^{1/2} - (x'_k)^{1/2}\right|.$$

Hence continuity at t' is equivalent to uniform convergence of $\phi(t)$ to $\phi(t')$ when $t \to t'$. Fix arbitrary $\epsilon > 0$. It can be inferred from the proof of Theorem 1 that if $\phi(t) = (y_s(t))_{s\in\mathbb{Z}}$ then $y_s(t) = g_s(t)$ for $s < 0$ where g_s is a composition of f_1^{-1} and f_2^{-1} and this composition is independent of $t \in T$. Let c_1 be $\alpha(\phi(t))$ and c_2 be $\omega(\phi(t))$ which are independent of $t \in T$. For $\epsilon > 0$ there is integer S such that $y_s(t) \in B_\epsilon(c_1) \cup B_\epsilon(c_2)$ for all $|s| > S$ and t in some neighborhood of t'. Thus we can find $\eta > 0$ such that $\sup\limits_{s:|s|>S} |y_s(t') - y_s(t)| < \epsilon$ for all t' : $|t - t'| < \eta$. Since functions g_s and $f^{(j)}$ are continuous we can choose η small enough for $|g_s(t') - g_s(t)| < \epsilon$ and $|f^{(s)}(t') - f^{(s)}(t)| < \epsilon$ to be true for all s: $|s| \le S$ and $|t - t'| < \eta$, i.e. $\sup\limits_{s:|s|\le S} |y_s(t') - y_s(t)| < \epsilon$. Hence, $\phi(t')$ uniformly converges to $\phi(t)$. Other statements of the theorem are straightforward. ∎

Remark 1. For any $a \in C^*(A_f)$ the map $\Psi(f)$ is a continuous map from $\overline{T}$ to $\mathcal{A}$ for type 1 dynamical systems (or $\mathcal{B}$ for type 2 dynamical systems) such that $\mathrm{ad}\, U(\Psi(a)(t_2)) = \Psi(a)(t_1)$.

Now we are ready to describe enveloping C^*-algebras. Define operators U_1 and U_2 on the basis as follows $U_1 e_k = e_{k+1}$ for $k < 0$ and $U_1 e_k = 0$ for $k \ge 0$ and $U_2 e_k = e_{k+1}$ for $k > 0$ and $U_1 e_k = 0$ for $k \le 0$. Consider two C^*-subalgebras $\mathcal{G}_1$ and $\mathcal{G}_2$ in $B(H)$ generated by operators U_1 and U_2 correspondingly. Then operator $U_1 + U_2$ generates C^*-subalgebra $\mathcal{G}_1 \oplus \mathcal{G}_2$ in $B(H)$ isomorphic to $\mathcal{T}(C(\mathbb{T})) \oplus \mathcal{T}(C(\mathbb{T}))$. Further on we will use notations of theorem 3 and will regard $\mathcal{T}(C(\mathbb{T})) \oplus \mathcal{T}(C(\mathbb{T}))$ as a concrete algebra in $B(H)$, namely $\mathcal{G}_1 \oplus \mathcal{G}_2$. Let (f, I) be of type 1 with attractive stable point. Let $\mathcal{C}$ denote C^*-algebra of all continuous maps ξ from $\overline{T} = [t_1, t_2]$ to $\mathcal{A}$ such that $\mathrm{ad}\, U(\xi(t_2)) = \xi(t_1)$.

Theorem 4. *Let (f, I) be of type 1 with attractive stable point. Then $\mathcal{T}(C(\mathbb{T})) \oplus \mathcal{T}(C(\mathbb{T}))$ is a C^*-subalgebra in $\mathcal{A}$. Let us denote by $\mathcal{M}_1$ the C^*-subalgebra in $\mathcal{C}$ comprised of those elements f such that $f(t_1) \in \mathcal{T}(C(\mathbb{T})) \oplus \mathcal{T}(C(\mathbb{T}))$ and $\pi(f(t)) = \pi(f(t'))$ for any one dimensional representation π of $\mathcal{A}$ from the first circle and $\rho(f(t)) = \rho(f(t'))$ for any one dimensional representation ρ of $\mathcal{A}$ from the second circle (see theorem 3) and for all $t, t' \in T$. Then $C^*(A_f)$ is isomorphic to $\mathcal{M}_1$.*

Proof. It is easy to verify that $\pi_{\phi(t_1)}$ is equivalent to the direct sum of Fock and anti-Fock representations. Hence representations $\pi_{\phi(t)}$ where $t \in \overline{T}$ comprise a residual family for $C^*(A_f)$. By Theorem 3 and the remark $C^*(A_f)$ is isomorphic under Gelfand transformation ($\Gamma(a)(\pi) = \pi(a)$, where $\pi \in \mathrm{Rep}(C^*(A_f))$) to a C^*-subalgebra in $\mathcal{C}$. Conditions $\pi(f(t)) = \pi(f(t'))$ and $\rho(f(t)) = \rho(f(t'))$ for all $t, t' \in T$ are easily verified on generator X. Since π and ρ are $*$-homomorphisms these conditions hold for every $a \in C^*(A_f)$. Hence $C^*(A_f)$ is a C^*-subalgebra in $\mathcal{M}_1$. Since it is a massive subalgebra in GCR C^*-algebra $\mathcal{M}_1$ we have $C^*(A_f) = \mathcal{M}_1$ by theorem 11.1.6 [5]. ■

Let $\mathcal{G}_3$ denote the C^*-subalgebra in $B(H)$ generated by operator X_f defined by $X_f e_k = x_k e_{k+1}$ for $k > 0$ and $X_f e_k = 0$ for $k \le 0$ (i.e. X_f is $\pi_f(X)$ if $l_2(\mathbb{N})$ is identified with subspace in $l_2(\mathbb{Z})$). Then operator $U_1 + X_f$ generates $\mathcal{G}_1 \oplus \mathcal{G}_3$ which is isomorphic to $\mathcal{T}(C(\mathbb{T})) \oplus M_2(\mathcal{T}(C(\mathbb{T})))$. Further on we will identify the latter with the concrete C^*-algebra $\mathcal{G}_1 \oplus \mathcal{G}_3$. Let $\mathcal{D}$ denote C^*-algebra of all continuous maps ξ from $\overline{T} = [t_1, t_2]$ to $\mathcal{B}$ such that $\mathrm{ad}\, U(\xi(t_1)) = \xi(t_2)$.

Theorem 5. *Let (f, I) be type two mapping and cycle of period two is attracting. Then $\mathcal{G}_1 \oplus \mathcal{G}_3$ is a subalgebra in $\mathcal{B}$. Let us denote by $\mathcal{M}_2$ the C^*-subalgebra of $\mathcal{D}$ comprised of those elements f such that $f(t_1) \in \mathcal{T}(C(\mathbb{T})) \oplus M_2(\mathcal{T}(C(\mathbb{T})))$ and $\eta(f(t)) = \eta(f(t'))$ for any one dimensional representation η of $\mathcal{B}$ and $\zeta(f(t)) = \zeta(f(t'))$ for any two dimensional representation ζ of $\mathcal{B}$ and for all $t, t' \in T$. Then $C^*(A_f)$ is isomorphic to $\mathcal{M}_2$.*

The proof is analogous to that of the previous theorem.

Corollary 1. *For $\mathcal{F}_{2^n}$ unimodal dynamical systems with zero Schwarzian and attractive cycle of length one or two isomorphism class of associated C^*-algebra depends only on the type of the system (whether it 1 or 2).*

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The Lipkin–Meshkov–Glick Model and its Deformations through Polynomial Algebras

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We search for solutions of the many-particle Hamiltonian of Lipkin, Meshkov and Glick in the context of the $sl(2,\mathbb{R})$ deformed polynomial algebra. The reducibility of the original model is proved according to the representations of this algebra. A new symmetry is uncovered, which further splits any matrix of a given j multiplet into two submatrices. In this way the diagonalization of the Hamiltonian matrix is simplified and the entire spectrum of the many-particle Hamiltonian is easily recovered. Supplementary eigenvalues stemming from the deformed algebra approach are also introduced. We indicate how they can lead to a new class of deformed-type models.

1 Introduction

Quantum mechanical equations with analytic solutions are rare. Only some interactions like e.g. the harmonic oscillator or the Coulomb potential give rise to a class of equations which are called exactly solvable. But sometimes one can weaken the condition of exact solvability by asking for an exact knowledge of a *finite* number of solutions only. This leads to what is referred to in [1] as quasi-exact solvability. Quasi-exactly solvable models have been essentially developed in a nonrelativistic context. They are characterized by the fact that, up to a change of variables as well as a transformation at the level of the wavefunctions, their Hamiltonians can be expressed as at most a quadratic function of the generators of a Lie algebra, namely $sl(2,\mathbb{R})$ for algebras of rank one. These generators stabilize a finite-dimensional space and so do the Hamiltonians which can be easily diagonalized within this space.

In physical examples at most a quadratic function of generators of $sl(2,\mathbb{R})$ is a consequence of the assumption of a two-body interaction between particles. One of the well-known quasi-exactly solvable models is that proposed by Lipkin, Meshkov and Glick [2], developed for treating many particle systems. Another one is the spin Van der Waals model used in statistical mechanics [3]. Interestingly enough, it has been shown that these two models are equivalent and represent particular cases of a more general Hamiltonian [4]. In these two cases the $sl(2,\mathbb{R})$ generators are called quasi-spin or pseudo-spin operators.

Here we refer to the work of Lipkin, Meshkov and Glick (LMG), who constructed a two N-fold degenerate level Hamiltonian where N is the number of fermions in the system. The two levels are separated by an energy ϵ. The simplified version of the LMG Hamiltonian, which we consider here, contains only terms which mix particle-hole configurations. The corresponding Hamiltonian reads

$$H_{\rm LMG} = \epsilon j_0 + \frac{\delta\epsilon}{2N}\left(j_+^2 + j_-^2\right), \tag{1}$$

where δ is the interaction strength, while the $sl(2,\mathbb{R})$ generators j_0, $j_\pm$ are realized as

$$j_0 = -\frac{N}{2} + \frac{1}{2}\sum_{m=1}^{N}\left(\alpha_m^\dagger\alpha_m + \beta_m^\dagger\beta_m\right), \tag{2}$$

$$j_+ = \sum_{m=1}^{N} \alpha_m^\dagger \beta_m^\dagger, \tag{3}$$

$$j_- = \sum_{m=1}^{N} \alpha_m \beta_m, \tag{4}$$

and satisfy

$$[j_0, j_\pm] = \pm j_\pm, \qquad [j_+, j_-] = 2j_0. \tag{5}$$

In the definitions (2)–(4) the fermion operators $\beta_m^\dagger$, β_m create and annihilate holes in the lower level, while $\alpha_m^\dagger$, α_m create and annihilate particles in the upper level. These operators are such that

$$\{\alpha_m, \alpha_n^\dagger\} = \{\beta_m, \beta_n^\dagger\} = \delta_{mn},$$
$$[\alpha_m, \beta_n] = [\alpha_m, \beta_n^\dagger] = [\beta_m, \alpha_n^\dagger] = [\alpha_m^\dagger, \beta_n^\dagger] = 0.$$

The Casimir operator of the $sl(2,\mathbb{R})$ algebra

$$C_1 = \frac{1}{2}\{j_+, j_-\} + j_0^2 \tag{6}$$

evidently commutes with the Hamiltonian (1). Hence the Hamiltonian matrix splits into submatrices each associated with a given value of j and of order $2j+1$. Each state in a j multiplet has a different number of excited particle-hole pairs. The interaction part of (1) mixes states within the same j multiplet but cannot mix states having different eigenvalues of C_1. It can only excite or de-excite two particle-hole pairs or in other words it can only change the eigenvalue of j_0 by two units. From the definition (2), it follows that the eigenvalues of j_0 are given by half the difference between the number of particles in the upper level and the number of particles in the lower level. Then the maximum eigenvalue of j_0 and of j is $\frac{N}{2}$. The largest matrix to be diagonalized in (1) is thus of dimension $N+1 = 2j+1$.

The main purpose of this paper is to revisit the LMG Hamiltonian (1) in the context of the $sl(2,\mathbb{R})$ deformed polynomial algebra. In such a context, we show that the largest matrix associated to a given N can be split into two submatrices of dimensions $\frac{N}{2}+1$ and $\frac{N}{2}$ for N even and two submatrices, both of dimensions $\frac{N+1}{2}$ for N odd. This is due to the presence, apart from (6), of an additional invariant, i.e. the Casimir operator of the deformed algebra. Moreover, the polynomial deformation technique leads to new representations corresponding to new eigenvalues appropriate to a deformed LMG model.

2 The deformed polynomial algebra approach

Instead of (1) in this section we propose to consider the following Hamiltonian [5]

$$H = \epsilon(2J_0 + \delta(J_+ + J_-)) \tag{7}$$

containing the operators J_0, $J_\pm$, which satisfy the following polynomial algebra (as compared with (5))

$$[J_0, J_\pm] = \pm J_\pm, \tag{8}$$

$$[J_+, J_-] = -\frac{16}{N^2} J_0^3 + \frac{2}{N^2}\left(2j^2 + 2j - 1\right) J_0, \tag{9}$$

where j is an eigenvalue of the operator C_1 as defined by (6). Such a choice is justified by the fact that the particular realization of the algebra (8)–(9)

$$J_0 = \frac{1}{2} j_0, \qquad J_\pm = \frac{1}{2N} j_\pm^2, \tag{10}$$

makes the Hamiltonian (7) to coincide with (1). However realizations other than (10) can be produced in general, leading to new eigenvalues, different from those of (1), as shown below.

Indeed the Casimir operator of the $sl(2,\mathbb{R})$ deformed polynomial algebra (8)–(9) is

$$C_2 = J_+ J_- - \frac{4}{N^2} J_0^4 + \frac{8}{N^2} J_0^3 + \frac{2j^2+2j-5}{N^2} J_0^2 - \frac{2j^2+2j-1}{N^2} J_0 \tag{11}$$

and two types of finite-dimensional representations arise. The first ones are defined according to

$$\begin{aligned} &J_0 |J,M\rangle = (M+c)|J,M\rangle, \\ &J_+ |J,M\rangle = f(M)|J,M+1\rangle, \qquad J_-|J,M\rangle = g(M)|J,M-1\rangle, \end{aligned} \tag{12}$$

with $M = -J, -J+1, \ldots, J-1, J,\ J = 0, \frac{1}{2}, 1, \ldots$ and

$$\begin{aligned} f(M-1)g(M) = {} & \frac{1}{N^2}(J-M+1)(J+M) \\ & \times \left(2j^2+2j-1-4J^2-4J-4M^2+4M+8(1-2M)c-24c^2\right). \end{aligned}$$

The real number c can take three distinct values [6] given by

$$c = 0 \qquad \text{and} \qquad c = \pm\sqrt{\frac{1}{4}j(j+1) - \frac{1}{8} - J(J+1)}.$$

The second type of representations are characterized by the following equations

$$\begin{aligned} &J_0|J',M'\rangle = \left(\frac{M'}{2}\right)|J',M'\rangle, \\ &J_+|J',M'\rangle = f'\left(M'\right)|J',M'+2\rangle, \qquad J_-|J',M'\rangle = g'\left(M'\right)|J',M'-2\rangle, \end{aligned} \tag{13}$$

where $J' = 0, 1, 2, \ldots$ and

$$\begin{aligned} f'\left(M'-2\right) g'\left(M'\right) = {} & \frac{1}{4N^2}\left(J'-M'+2\right)\left(J'+M'\right) \\ & \times \left(2j^2+2j-1-J'^2-2J'-M'^2+2M'\right) \end{aligned} \tag{14}$$

if $M' = -J', -J'+2, \ldots, J'-2, J'$ and

$$f'\left(M'-2\right) g'\left(M'\right) = \frac{1}{4N^2}\left(J'-M'+1\right)\left(J'+M'-1\right)\left(2j^2+2j-J'^2-M'^2+2M'\right) \tag{15}$$

if $M' = -J'+1, -J'+3, \ldots, J'-3, J'-1$. In the cases where $J' = \frac{1}{2}, \frac{3}{2}, \ldots$, J' must be equal to j (M' to m) and

$$f'(m-2)g'(m) = \frac{1}{4N^2}(j+m)(j+m-1)(j-m+1)(j-m+2).$$

It is important to note that the polynomial algebra provides a new "quantum number" c, as introduced above. It helps to distinguish between the eigenvalues of (7) corresponding to even and odd N. For N even one has $c = 0$ and for N odd $c = \pm 1/4$.

In the following we shall drop the representation (13) due to the fact that it is reducible. Indeed evaluating the eigenvalue of the Casimir operator C_2 of (11) within the invariant subspace of the representation (13) we obtain two distinct values which implies that the invariant subspace splits into the direct sum

$$\left(J' = n, c = 0\right)_{(13)} = \left(J = \frac{n}{2}, c = 0\right)_{(12)} \oplus \left(J = \frac{n-1}{2}, c = 0\right)_{(12)}, \tag{16}$$

where the left hand side refers to the representation space of (13) and each bracket in the right hand side designates an invariant subspace of (12). A similar decomposition holds for half-integer j

$$\left(J' = j = n + \frac{1}{2}, c = 0\right)_{(13)} = \left(J = \frac{n}{2}, c = \frac{1}{4}\right)_{(12)} \oplus \left(J = \frac{n}{2}, c = -\frac{1}{4}\right)_{(12)} \tag{17}$$

for any integer n. The original LMG model defined by (1) or equivalently by (7) with the realization (10) is clearly connected to the representations (13) with $J' = j$ (J' being an integer or a half integer). We can conclude that the LMG Hamiltonian matrix is reducible. More precisely, according to equations (16) and (17), a matrix Hamiltonian of dimension $2n+1$ can be split into a direct sum of two submatrices of dimensions $n+1$ and n for j even and a matrix of dimension $2n+2$ can be split into two matrices, each of dimension $n+1$, for j half integer. We thus obtain the result mentioned in the Introduction with $N = 2n$ and $N = 2n+1$ respectively, such result being significant for a large number of particles. Then searching for the spectrum of the Hamiltonian (7) amounts to the diagonalization of the matrix $\langle H \rangle$ given by

$$\begin{pmatrix} 2J+2c & \delta f(J-1) & 0 & 0 & \cdot & \cdot & 0 \\ \delta g(J) & 2J-2+2c & \delta f(J-2) & 0 & \cdot & \cdot & 0 \\ 0 & \delta g(J-1) & 2J-4+2c & \delta f(J-3) & \cdot & \cdot & 0 \\ 0 & 0 & \delta g(J-2) & 2J-6+2c & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -2J+2+2c & \delta f(-j) \\ 0 & 0 & 0 & 0 & \cdot & \delta g(-J+1) & -2J+2c \end{pmatrix}. \tag{18}$$

obtained in the invariant subspace defined by (12). In the following section we are going to illustrate these findings on specific examples.

3 Examples

3.1 The $N = 2$ case

We first consider the simplest $N = 2$ case in order to easily illustrate our results. The complete LMG matrix is of dimension 4, corresponding to the four possible states of two particles occupying two levels (the two particles can be on the lower level, or on the upper one, or one particle can be on the lower while the other can be on the upper level or vice-versa). Following the original LMG Hamiltonian (1), the matrix of dimension 4 splits into $3+1$ while the matrix of (7) in the invariant space of the representation (12) splits into $2+1+1$ (corresponding to $J = \frac{1}{2}$ and $J = 0$ twice). The eigenvalues E (in units of ϵ) are obtained from the diagonalization of three matrices of type (18) of dimensions 2, 1 and 1 respectively. The eigenvalues are summarized in the following table

j	J	E
0	0	0
1	0	0
	$\frac{1}{2}$	$\pm\sqrt{1+\frac{1}{4}\delta^2}$

3.2 The $N = 8$ case

For $N = 8$, there are $2^8 = 256$ states. The largest original LMG matrix corresponds to $j = \frac{N}{2} = 4$, the others being associated to $j = 3$ (7 times), $j = 2$ (20 times), $j = 1$ (28 times) and $j = 0$ (14 times). Following the decompositions (16)–(17) the polynomial algebra leads to other representations: $J = 2$ (1 time), $J = \frac{3}{2}$ (8 times), $J = 1$ (27 times), $J = \frac{1}{2}$ (48 times) and $J = 0$ (42 times). The corresponding eigenvalues come from the diagonalization of matrices of type (18) and are given in units of ϵ in the following table

j	J	E
0	0	0
1	0	0
	$\frac{1}{2}$	$\pm\sqrt{1+\frac{1}{64}\delta^2}$
2	$\frac{1}{2}$	$\pm\sqrt{1+\frac{9}{64}\delta^2}$
	1	$0, \pm\sqrt{4+\frac{3}{16}\delta^2}$
3	1	$0, \pm\sqrt{4+\frac{15}{16}\delta^2}$
	$\frac{3}{2}$	$\pm\sqrt{5+\frac{33}{64}\delta^2 \pm \sqrt{16+\frac{3}{2}\delta^2+\frac{27}{128}\delta^4}}$
4	$\frac{3}{2}$	$\pm\sqrt{5+\frac{113}{64}\delta^2 \pm \sqrt{16+\frac{19}{2}\delta^2+\frac{275}{128}\delta^4}}$
	2	$0, \pm\sqrt{10+\frac{59}{32}\delta^2 \pm \sqrt{36-\frac{9}{8}\delta^2+\frac{2025}{1024}\delta^4}}$

4 Supplementary eigenvalues

In the previous section the tables contain the eigenvalues of the Hamiltonian (1) only. They were obtained through the polynomial algebra technique. However the polynomial algebra is richer than the usual $sl(2,\mathbb{R})$ algebra, associated with the quasi-spin formalism in the LMG model. As seen above, its representations have three labels (J, c, j) instead of one (j) for $sl(2,\mathbb{R})$. Thus the number of representations is larger. This is particularly clear from the table corresponding to $N = 8$. Indeed when $j = 2$ for example, we can see that the eigenvalues of the LMG Hamiltonian are recovered when $J = \frac{1}{2}$ and $J = 1$ while the case $J = 0$ is missing and must correspond to another model. The same situation holds for $j = 3$, when $J = 0$ or $J = \frac{1}{2}$ and $j = 4$ when $J = 0$, $J = \frac{1}{2}$ and $J = 1$. These new possibilities are excluded by the Hamiltonian (1) but not by (7). They lead to supplementary eigenvalues as summarized in the following table

j	J	E
2	0	0
3	0	0
	$\frac{1}{2}$	$\pm\sqrt{1+\frac{21}{64}\delta^2}$
4	0	0
	$\frac{1}{2}$	$\pm\sqrt{1+\frac{37}{64}\delta^2}$
	1	$0, \pm\sqrt{4+\frac{31}{16}\delta^2}$

Taking for example the maximal value of j, i.e. $j = 4$ associated to $N = 8$ we can see that these supplementary eigenvalues are surprisingly close to some of the original LMG Hamiltonian. Indeed when $\delta = 1$, we have

j	J	E
4	0	0
	$\frac{1}{2}$	± 1.256
	1	$0, \pm 2.437$
	$\frac{3}{2}$	$\pm 1.228, \pm 3.467$
	2	$0, \pm 2.402, \pm 4.232$

i.e. very close to the numerical values shown in [5]. The same kind of results hold for any number of particles. In order to fix the ideas, for an even number $N = 2n$ of particles, the largest matrix corresponds to $j = n$, the values $J = \frac{n-1}{2}, \frac{n}{2}$ give rise to the LMG eigenvalues while the cases $J = 0, \frac{1}{2}, 1, \ldots, \frac{n}{2} - 1$ lead to supplementary solutions, close and larger than the LMG ones for a fixed j. Moreover the closeness is better realized for δ smaller, as it can be seen from the analytic expressions.

A natural question then arises: to what kind of model do correspond these supplementary eigenvalues? In order to answer this question, let us once again concentrate on the case of $N = 8$ particles and, this time, on the representations (13). We have five different values as far as J' is concerned, i.e. $J' = 0, 1, 2, 3, 4$. In fact, according to (13)–(15) we can generalize the realization (10) to

$$J_\pm = \frac{1}{16} M(J') j_\pm^2, \tag{19}$$

where $M(J')$ is a diagonal matrix depending on J' and of dimension $2J' + 1$. In principle this matrix should be different for each value of J'. It is interesting to note that this diagonal matrix reduces to the identity I for $J' = J'_{\max} = \frac{N}{2} = 4$ only, in agreement with (10). With this generalization the Hamiltonian (7) becomes

$$H = \epsilon j_0 + \frac{\delta\epsilon}{2N} \left(M(J') j_+^2 + j_-^2 M(J') \right) \tag{20}$$

with $J' = 0, 2, 4, \ldots, \frac{N}{2}$ and $M(\frac{N}{2}) = I$. In general the operators (19) can also be written as

$$J_+ = \frac{1}{2N} M(J') j_+^2 \equiv \frac{1}{2N} (j'_+)^2, \qquad J_- = \frac{1}{2N} j_-^2 M(J') \equiv \frac{1}{2N} (j'_-)^2$$

with

$$[j_0, j'_\pm] = \pm j'_\pm, \tag{21}$$

$$[j'_+, j'_-] = \sum_{k=0}^{J'-1} c_k j_0^{2k+1}, \tag{22}$$

where c_k are coefficients being fixed according to N and J'. The relations (21)–(22) are those of a polynomial deformation of $sl(2,\mathbb{R})$ except when $J' = 1$ and $J' = \frac{N}{2}$ where it is equivalent to $sl(2,\mathbb{R})$ ($J' = 0$ leading to trivial results). We can then conclude that our model (7) or equivalently (20) represents the usual LMG model (corresponding to $J' = \frac{N}{2}$) plus $\frac{N}{2}$ deformed LMG models (corresponding to $J' = 0, 1, \ldots, \frac{N}{2} - 1$ and $M(J') \neq I$), the deformed models giving rise to supplementary eigenvalues as discussed in this section. In all cases $M(J')$ is uniquely defined by the deformed algebra.

5 Summary

We have presented a derivation of the entire spectrum of the many-particle Hamiltonian of Lipkin, Meshkov and Glick in the context of the $sl(2,\mathbb{R})$ deformed polynomial algebra. For any given number N of particles the spectrum first splits into j multiplets of the $sl(2,\mathbb{R})$ algebra. The eigenvalues associated with the largest j are non-degenerate except for $E = 0$. We have shown that the Hamiltonian matrix of each j further splits into two submatrices corresponding to two distinct irreducible representations of the deformed polynomial algebra. In order to illustrate the method we have derived explicit analytic expressions for the eigenvalues of the LMG Hamiltonian for $N = 2$ and 8. Our method can evidently be extended to any N.

Furthermore we have shown that the deformed polynomial algebra related to the LMG model implies a larger spectrum than that of the model itself. Some of the new eigenvalues present characteristics similar to those of the LMG model and actually correspond to a superposition of specific deformed LMG models where, once again, the deformed polynomial algebra $sl(2,\mathbb{R})$ plays a prominent role.

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Ternary Hopf Algebras

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Properties of ternary semigroups, groups and algebras are briefly reviewed. It is shown that there exist three types of ternary units. A ternary analog of deformation is shortly discussed. Ternary coalgebras are defined in the most general manner, their classification with respect to the property "to be derived" is made. Three types of coassociativity and three kinds of counits are given. Ternary Hopf algebras with skew and strong antipods are defined. Concrete examples of ternary Hopf algebras, including the Sweedler example (which has two ternary generalizations), are presented. A ternary analog of quasitriangular Hopf algebras is constructed, and ternary abstract quantum Yang–Baxter equation (together with its classical counterpart) is obtained. A ternary "pairing" of three Hopf algebras is built.

I would like to report about the work done in part together with Andrzej Borowiec and Wieslaw Dudek, and I am grateful to them for fruitful collaboration.

Firstly ternary algebraic operations were introduced already in the XIX-th century by A. Cayley. As the development of Cayley's ideas it were considered n-ary generalization of matrices and their determinants [1] and general theory of n-ary algebras [2, 3] and ternary rings [4] (for physical applications in Nambu mechanics, supersymmetry, Yang–Baxter equation, etc. see [5] as surveys). The notion of an n-ary group was introduced in 1928 by W. Dörnte [6]. From another side, Hopf algebras [7] and their generalizations [8, 9, 10, 11] play a basic role in the quantum group theory (also see e.g. [12, 13]). We note that the derived ternary Hopf algebras are used as an intermediate tool in obtaining the Drinfeld's quantum double [14].

Here we first present necessary material on ternary semigroups, groups and algebras [15] in the abstract arrow language. Then using systematic reversing order of arrows [7], we define ternary bialgebras and Hopf algebras, investigate their properties and give some examples[1]. Most of the constructions introduced below are valid for n-ary case as well after obvious changes.

A non-empty set G with one *ternary* operation $[\;]: G\times G\times G\to G$ is called a *ternary groupoid* and is denoted by $(G,[\;])$ or $\left(G,m^{(3)}\right)$. If on G there exists a binary operation $\odot$ (or $m^{(2)}$) such that $[xyz]=(x\odot y)\odot z$ or

$$m^{(3)}=m^{(3)}_{\mathrm{der}}=m^{(2)}\circ\left(m^{(2)}\times\mathrm{id}\right) \tag{1}$$

for all $x,y,z\in G$, then we say that $[\;]$ or $m^{(3)}_{\mathrm{der}}$ is *derived* from $\odot$ or $m^{(2)}$ and denote this fact by $(G,[\;])=\mathrm{der}(G,\odot)$. If $[xyz]=((x\odot y)\odot z)\odot b$ holds for all $x,y,z\in G$ and some fixed $b\in G$, then a groupoid $(G,[\;]$ is b-*derived* from $(G,\odot)$. In this case we write $(G,[\;])=\mathrm{der}_b(G,\odot)$ [16, 17]. A *ternary isotopy* is a set of functions $f,g,h,w:G\to G$ such that $f\left([xyz]\right)=\left[g\left(x\right),h\left(y\right),w\left(z\right)\right]$ for all $x,y,z\in G$. If $g=h=w=f$, then f is *ternary isomorphism*.

A *ternary semigroup* is $(G,[\;])$ (or $\left(G,m^{(3)}\right)$) where the operation $[\;]$ ($m^{(3)}$) is *associative* $[[xyz]\,uv]=[x\,[yzu]\,v]=[xy\,[zuv]]$ (for all $x,y,z,u,v\in G$) or

$$m^{(3)}\circ\left(m^{(3)}\times\mathrm{id}\times\mathrm{id}\right)=m^{(3)}\circ\left(\mathrm{id}\times m^{(3)}\times\mathrm{id}\right)=m^{(3)}\circ\left(\mathrm{id}\times\mathrm{id}\times m^{(3)}\right) \tag{2}$$

[1]Due to the lack of place in the Proceedings we present only important results and constructions omitting most proofs and detailed derivations which will appear elsewhere.

A ternary operation $m^{(3)}_{\mathrm{der}}$ derived from a binary associative operation $m^{(2)}$ is also associative, but a ternary groupoid $(G, [\,])$ b-derived (b is a cancellative element) from a semigroup $(G, \odot)$ is a ternary semigroup if and only if b lies in the center of $(G, \odot)$. Fixing in a ternary operation $m^{(3)}$ one element a we obtain a binary operation $m^{(2)}_a$. A binary groupoid $(G, \odot)$ or $\left(G, m^{(2)}_a\right)$, where $x \odot y = [xay]$ or $m^{(2)}_a = m^{(3)} \circ (\mathrm{id} \times a \times \mathrm{id})$ for some fixed $a \in G$ is called a *retract* of $(G, [\,])$ and is denoted by $\mathrm{ret}_a(G, [\,])$ [16, 17]. It can be shown that if there exists an element e such that for all $y \in G$ we have $[eye] = y$, then this semigroup is derived from the binary semigroup $\left(G, m^{(2)}_e\right)$, where $m^{(2)}_e = m^{(3)} \circ (\mathrm{id} \times e \times \mathrm{id})$.

An element $e_m \in G$ is called a *middle identity* of $(G, [\,])$ if for all $x \in G$ we have $[e_m x e_m] = x$ or $m^{(3)} \circ (e_m \times \mathrm{id} \times e_m) = \mathrm{id}$. An element $e_l \in G$ satisfying the identity $[e_l e_l x] = x$ or $m^{(3)} \circ (e_l \times e_l \times \mathrm{id}) = \mathrm{id}$ is called a *left identity.* By analogy we define a *right identity*, satisfying $[x e_r e_r] = x$ or $m^{(3)} \circ (\mathrm{id} \times e_r \times e_r) = \mathrm{id}$ for all $x \in G$. An element which is a left, middle and right identity $e = e_m = e_l = e_r$ is called a *ternary identity* (briefly: *identity*), an element which is only left and right identity is a *semi-identity* $e_{\mathrm{semi}} = e_m = e_l$. There are ternary semigroups without left (middle, right) neutral elements, but there are also ternary semigroups in which all elements are identities [15, 18]. More general, a 2-sequence of elements $\alpha_2 = e_1 e_2$ is *neutral*, if $[e_1 e_2 x] = [x e_1 e_2] = x$ for all $x \in G$ and by analogy for n-sequence. Two sequences α and β are equivalent, if there are exist another two sequences γ and δ such that $[\gamma\alpha\delta] = [\gamma\beta\delta]$.

Lemma 1. *For any ternary semigroup $(G, [\,])$ with a left (right) identity there exists a binary semigroup $(G, \odot)$ and its endomorphism μ such that $[xyz] = x \odot \mu(y) \odot z$ for all $x, y, z \in G$.*

Proof. Let e_l be a left identity of $(G, [\ ])$. Then the operation $x \odot y = [x e_l y]$ is associative. Moreover, for $\mu(x) = [e_l x e_l]$, we have $\mu(x) \odot \mu(y) = [[e_l x e_l] e_l [e_l y e_l]] = [[e_l x e_l][e_l e_l y] e_l] = [e_l [x e_l y] e_l] = \mu(x \odot y)$ and $[xyz] = [x [e_l e_l y][e_l e_l z]] = [[x e_l [e_l y e_l]] e_l z] = x \odot \mu(y) \odot z$. In the case of right identity the proof is analogous. ■

A ternary groupoid $(G, [\ ])$ is a *left cancellative* if $[abx] = [aby] \Longrightarrow x = y$, a *middle cancellative* if $[axb] = [ayb] \Longrightarrow x = y$, a *right cancellative* if $[xab] = [yab] \Longrightarrow x = y$ hold for all $a, b \in G$. A ternary groupoid which is left, middle and right cancellative is called *cancellative.*

Definition 1. A ternary groupoid $(G, [\,])$ is *semicommutative* if $[xyz] = [zyx]$ for all $x, y, z \in G$. If the value of $[xyz]$ is independent on the permutation of elements x, y, z, viz.

$$[x_1 x_2 x_3] = \left[x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}\right] \tag{3}$$

or $m^{(3)} = m^{(3)} \circ \sigma$, then $(G, [\,])$ is a *commutative* ternary groupoid. If σ is fixed, then a ternary groupoid satisfying (3) is called σ-commutative.

The group S_3 is generated by two transpositions; (12) and (23). This means that $(G, [\,])$ is commutative if and only if $[xyz] = [yxz] = [xzy]$ holds for all $x, y, z \in G$. Further if in a ternary semigroup $(G, [\,])$ satisfying the identity $[xyz] = [yxz]$ there are a, b such that $[axb] = x$ for all $x \in G$, then $(G, [\,])$ is commutative.

Mediality in the binary case $(x \odot y) \odot (z \odot u) = (x \odot z) \odot (y \odot u)$ for groups coincides with commutativity. In the ternary case they do not coincide. A ternary groupoid $(G, [\,])$ is *medial* if it satisfies the identity

$$[[x_{11} x_{12} x_{13}][x_{21} x_{22} x_{23}][x_{31} x_{32} x_{33}]] = [[x_{11} x_{21} x_{31}][x_{12} x_{22} x_{32}][x_{13} x_{23} x_{33}]]$$

or

$$m^{(3)} \circ \left(m^{(3)} \times m^{(3)} \times m^{(3)}\right) = m^{(3)} \circ \left(m^{(3)} \times m^{(3)} \times m^{(3)}\right) \circ \sigma_{\mathrm{medial}}, \tag{4}$$

where $\sigma_{\mathrm{medial}} = \binom{123456789}{147258369} \in S_9$.

It is not difficult to see that a semicommutative ternary semigroup is medial. An element x such that $[xxx] = x$ is called an *idempotent*. A groupoid in which all elements are idempotents is called an *idempotent groupoid*. A left (right, middle) identity is an idempotent, also any neutral sequence e_1e_2 is an idempotent.

Definition 2. A ternary semigroup $(G, [\,])$ is a *ternary group* if for all $a, b, c \in G$ there are $x, y, z \in G$ such that $[xab] = [ayb] = [abz] = c$.

In a ternary group the equation $[xxz] = x$ has a unique solution which is denoted by $z = \overline{x}$ and called *skew element* [6], or equivalently

$$m^{(3)} \circ (\mathrm{id} \times \mathrm{id} \times \bar{\cdot}) \circ D^{(3)} = \mathrm{id},$$

where $D^{(3)}(x) = (x, x, x)$ is a ternary diagonal map.

Theorem 1. *In any ternary group $(G, [\,])$ for all $x, y, z \in G$ the following relations take place $[xx\,\overline{x}] = [x\,\overline{x}\,x] = [\,\overline{x}\,xx] = x$, $[yx\,\overline{x}] = [y\,\overline{x}\,x] = [x\,\overline{x}\,y] = [\,\overline{x}\,xy] = y$, $\overline{[xyz]} = [\,\overline{z}\,\overline{y}\,\overline{x}]$, $\overline{\overline{x}} = x$.*

Since in an idempotent ternary group $\overline{x} = x$ for all x, an idempotent ternary group is semicommutative. From [19, 20] it follows

Theorem 2. *A ternary semigroup $(G, [\,])$ with a unary operation $^{-} : x \to \overline{x}$ is a ternary group if and only if it satisfies identities $[yx\,\overline{x}\,] = [x\,\overline{x}\,y] = y$, or*

$$m^{(3)} \circ (\mathrm{id} \times \bar{\cdot} \times \mathrm{id}) \circ \left(D^{(2)} \times \mathrm{id}\right) = \mathrm{Pr}_2,$$
$$m^{(3)} \circ (\mathrm{id} \times \mathrm{id} \times \bar{\cdot}) \circ \left(\mathrm{id} \times D^{(2)}\right) = \mathrm{Pr}_1,$$

where $D^{(2)}(x) = (x, x)$ and $\mathrm{Pr}_1(x, y) = x$, $\mathrm{Pr}_2(x, y) = y$.

A ternary semigroup $(G, [\,])$ is an idempotent ternary group if and only if it satisfies identities $[yxx] = [xxy] = y$. Moreover, a ternary group with an identity is derived from a binary group.

Theorem 3 (Gluskin–Hosszú). *For a ternary group $(G, [\,])$ there exists a binary group $(G, \circledast)$, its automorphism φ and fixed element $b \in G$ such that $[xyz] = x \circledast \varphi(y) \circledast \varphi^2(z) \circledast b$.*

Proof. Let $a \in G$ be fixed. The binary operation $x \circledast y = [xay]$ ($a \in G$ fixed) is associative, because $(x \circledast y) \circledast z = [[xay]az] = [xa[yaz]] = x \circledast (y \circledast z)$ with identity $\overline{a}$ and $\varphi(x) = [\overline{a}xa]$, $b = [\overline{a}\,\overline{a}\,\overline{a}\,]$ (see [21]). ■

Theorem 4 (Post). *For any ternary group $(G, [\,])$ there exists a binary group $(G^*, \circledast)$ and $H \lhd G^*$, such that $G^*/H \simeq \mathbb{Z}_2$ and $[xyz] = x \circledast y \circledast z$ for all $x, y, z \in G$.*

Proof. Let c be a fixed element in G and let $G^* = G \times \mathbb{Z}_2$. In G^* we define binary operation $\circledast$ putting $(x, 0) \circledast (y, 0) = ([xy\overline{c}], 1)$, $(x, 0) \circledast (y, 1) = ([xyc], 0)$, $(x, 1) \circledast (y, 0) = ([xcy], 0)$, $(x, 1) \circledast (y, 1) = ([xcy], 1)$. This operation is associative and $(\overline{c}, 1)$ is its neutral element. The inverse element (in G^*) has the form $(x, 0)^{-1} = (\overline{x}, 0)$, $(x, 1)^{-1} = ([\overline{c}\,\overline{x}\,\overline{c}], 1)$. Thus G^* is a group such that $H = \{(x, 1) : x \in G\} \lhd G^*$. Obviously the set G can be identified with $G \times \{0\}$ and $[xyz] = ((x, 0) \circledast (y, 0)) \circledast (z, 0) = ([xy\overline{c}], 1) \circledast (z, 0) = ([[xy\overline{c}]cz], 0) = ([xy[\overline{c}cz]], 0) = ([xyz], 0)$, which completes the proof. ■

Let us consider ternary algebras. One can introduce autodistributivity property $[[xyz]\,ab] = [[xab]\,[yab]\,[zab]]$ (see [22]). If we take 2 ternary operations $\{\,,\;,\;\}$ and $[\,,\;,\;]$, then distributivity is $\{[xyz]\,ab\} = [\{xab\}\,\{yab\}\,\{zab\}]$. If $(+)$ is a binary operation (addition), then *left linearity* is $[(x + z)\,, a, b] = [xab] + [zab]$. By analogy one can define central (middle) and right linearity. *Linearity* is defined, when left, middle and right linearity hold valid simultaneously.

Definition 3. *Ternary algebra* is a triple $\left(A, m^{(3)}, \eta^{(3)}\right)$, where A is a linear space over a field $\mathbb{K}$, $m^{(3)}$ is a linear map $m^{(3)}: A\otimes A\otimes A \to A$ called *ternary multiplication* $m^{(3)}\left(a\otimes b\otimes c\right) = [abc]$ which is ternary associative $[[abc]\,de] = [a\,[bcd]\,e] = [ab\,[cde]]$ or

$$m^{(3)}\circ\left(m^{(3)}\otimes \mathrm{id}\otimes \mathrm{id}\right) = m^{(3)}\circ\left(\mathrm{id}\otimes m^{(3)}\otimes \mathrm{id}\right) = m^{(3)}\circ\left(\mathrm{id}\otimes \mathrm{id}\otimes m^{(3)}\right). \quad (5)$$

There are 3 types of ternary unit maps $\eta^{(3)}: \mathbb{K}\to A$: 1) One *strong* unit map $m^{(3)}\circ\left(\eta^{(3)}\otimes\eta^{(3)}\otimes \mathrm{id}\right) = m^{(3)}\circ\left(\eta^{(3)}\otimes \mathrm{id}\otimes\eta^{(3)}\right) = m^{(3)}\circ\left(\mathrm{id}\otimes\eta^{(3)}\otimes\eta^{(3)}\right) = \mathrm{id}$; 2) two *sequential* units $\eta_1^{(3)}$ and $\eta_2^{(3)}$ satisfying $m^{(3)}\circ\left(\eta_1^{(3)}\otimes\eta_2^{(3)}\otimes \mathrm{id}\right) = m^{(3)}\circ\left(\eta_1^{(3)}\otimes \mathrm{id}\otimes\eta_2^{(3)}\right) = m^{(3)}\circ\left(\mathrm{id}\otimes\eta_1^{(3)}\otimes\eta_2^{(3)}\right) = \mathrm{id}$; 3) Four *long* (left) ternary units

$$m^{(3)}\circ\left(\mathrm{id}\otimes\eta_1^{(3)}\otimes\eta_2^{(3)}\right)\circ\left(m^{(3)}\circ\left(\mathrm{id}\otimes\eta_3^{(3)}\otimes\eta_4^{(3)}\right)\right) = \mathrm{id}$$

which corresponds to $\left[\left[a\eta_1^{(3)}\eta_2^{(3)}\right], \eta_3^{(3)}, \eta_4^{(3)}\right] = a\in A$ (right and middle units are defined similarly). In first case the ternary analog of the binary relation $\eta^{(2)}\left(x\right) = x1$, where $x\in\mathbb{K}$, $1\in A$, is $\eta^{(3)}\left(x\right) = [x,x,1] = [x,1,x] = [1,x,x]$.

Let (A, m_A, η_A), (B, m_B, η_B) and (C, m_C, η_C) be ternary algebras, then the *ternary tensor product* space $A\otimes B\otimes C$ is naturally endowed with the structure of an algebra. The multiplication $m_{A\otimes B\otimes C}$ on $A\otimes B\otimes C$ reads $[(a_1\otimes b_1\otimes c_1)(a_2\otimes b_2\otimes c_2)(a_3\otimes b_3\otimes c_3)] = [a_1a_2a_3]\otimes[b_1b_2b_3]\otimes[c_1c_2c_3]$, and so the set of ternary algebras is closed under taking ternary tensor products. A ternary *algebra map* (homomorphism) is a linear map between ternary algebras $f: A\to B$ which respects the ternary algebra structure $f\left([xyz]\right) = [f\left(x\right), f\left(y\right), f\left(z\right)]$ and $f\left(1_A\right) = 1_B$.

A ternary (and n-ary) commutator can be obtained in different ways [23]. We will consider a simplest version called a Nambu bracket (see e.g. [24]). Let us introduce two maps $\omega_\pm^{(3)}: A\otimes A\otimes A\to A\otimes A\otimes A$ by

$$\omega_+^{(3)}\left(a\otimes b\otimes c\right) = a\otimes b\otimes c + b\otimes c\otimes a + c\otimes a\otimes b, \quad (6)$$

$$\omega_-^{(3)}\left(a\otimes b\otimes c\right) = b\otimes a\otimes c + c\otimes b\otimes a + a\otimes c\otimes b. \quad (7)$$

Thus obviously $m^{(3)}\circ\omega_\pm^{(3)} = \sigma_\pm^{(3)}\circ m^{(3)}$, where $\sigma_\pm^{(3)}\in S_3$ denotes sum of terms having even and odd permutations respectively. In the binary case $\omega_+^{(2)} = \mathrm{id}\otimes\mathrm{id}$ and $\omega_-^{(2)} = \tau$ is the twist operator $\tau: a\otimes b\to b\otimes a$, while $m^{(2)}\circ\omega_-^{(2)}$ is permutation $\sigma_-^{(2)}\left(ab\right) = ba$. So the Nambu product is $\omega_N^{(3)} = \omega_+^{(3)} - \omega_-^{(3)}$, and the ternary commutator is $[\,,\,,\,]_N = \sigma_N^{(3)} = \sigma_+^{(3)} - \sigma_-^{(3)}$, or simply $[a,b,c]_N = [abc] + [bca] + [cab] - [cba] - [acb] - [bac]$ (see [24] and refs. therein). An *abelian* ternary algebra is defined by vanishing of Nambu bracket $[a,b,c]_N = 0$ or ternary commutation relation $\sigma_+^{(3)} = \sigma_-^{(3)}$. By analogy with the binary case a *deformed* ternary algebra can be defined by

$$\sigma_+^{(3)} = q\sigma_-^{(3)} \text{ or } [abc] + [bca] + [cab] = q\left([cba] + [acb] + [bac]\right), \quad (8)$$

where multiplication by q is treated as an external operation. An opposite and more complicated possibility requires 2 deformation parameters and can be defined as $\sigma_+^{(3)}\left([a,b,c]\right) = \left[q, p, \sigma_-^{(3)}\left([a,b,c]\right)\right]$, which reminds the binary case $ab = qba$ in the following form $m^{(2)}\left(a,b\right) = m^{(2)}\left(q, \sigma_-^{(2)}\left(ab\right)\right)$. Here we will exploit (8).

Let C be a linear space over a field $\mathbb{K}$.

Definition 4. *Ternary comultiplication* $\Delta^{(3)}$ is a linear map over a field $\mathbb{K}$ such that

$$\Delta^{(3)}: C\to C\otimes C\otimes C. \quad (9)$$

In the standard Sweedler notations [7] $\Delta^{(3)}(a) = \sum_{i=1}^{n} a_i' \otimes a_i'' \otimes a_i''' = a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$. Consider different possible types of ternary coassociativity.

1. *Standard* ternary coassociativity

$$\left(\Delta^{(3)} \otimes \mathrm{id} \otimes \mathrm{id}\right) \circ \Delta^{(3)} = \left(\mathrm{id} \otimes \Delta^{(3)} \otimes \mathrm{id}\right) \circ \Delta^{(3)} = \left(\mathrm{id} \otimes \mathrm{id} \otimes \Delta^{(3)}\right) \circ \Delta^{(3)}. \tag{10}$$

2. *Nonstandard* ternary Σ-coassociativity (Gluskin-type — positional operatives)

$$\left(\Delta^{(3)} \otimes \mathrm{id} \otimes \mathrm{id}\right) \circ \Delta^{(3)} = \left(\mathrm{id} \otimes \left(\sigma \circ \Delta^{(3)}\right) \otimes \mathrm{id}\right) \circ \Delta^{(3)},$$

where $\sigma \circ \Delta^{(3)}(a) = \Delta_\sigma^{(3)}(a) = a_{(\sigma(1))} \otimes a_{(\sigma(2))} \otimes a_{(\sigma(3))}$ and $\sigma \in \Sigma \subset S_3$.

3. *Permutational* ternary coassociativity

$$\left(\Delta^{(3)} \otimes \mathrm{id} \otimes \mathrm{id}\right) \circ \Delta^{(3)} = \pi \circ \left(\mathrm{id} \otimes \Delta^{(3)} \otimes \mathrm{id}\right) \circ \Delta^{(3)},$$

where $\pi \in \Pi \subset S_5$.

Ternary comediality is

$$\left(\Delta^{(3)} \otimes \Delta^{(3)} \otimes \Delta^{(3)}\right) \circ \Delta^{(3)} = \sigma_{\mathrm{medial}} \circ \left(\Delta^{(3)} \otimes \Delta^{(3)} \otimes \Delta^{(3)}\right) \circ \Delta^{(3)},$$

where σ_{medial} is defined in (4). *Ternary counit* is defined as a map $\varepsilon^{(3)} : C \to \mathbb{K}$. In general, $\varepsilon^{(3)} \neq \varepsilon^{(2)}$ satisfying one of the conditions below. If $\Delta^{(3)}$ is derived, then maybe $\varepsilon^{(3)} = \varepsilon^{(2)}$, but another counits may exist. There are 3 types of ternary counits:

1. Standard (*strong*) ternary counit

$$\left(\varepsilon^{(3)} \otimes \varepsilon^{(3)} \otimes \mathrm{id}\right) \circ \Delta^{(3)} = \left(\varepsilon^{(3)} \otimes \mathrm{id} \otimes \varepsilon^{(3)}\right) \circ \Delta^{(3)} = \left(\mathrm{id} \otimes \varepsilon^{(3)} \otimes \varepsilon^{(3)}\right) \circ \Delta^{(3)} = \mathrm{id}\,. \tag{11}$$

2. Two *sequential* (polyadic) counits $\varepsilon_1^{(3)}$ and $\varepsilon_2^{(3)}$

$$\left(\varepsilon_1^{(3)} \otimes \varepsilon_2^{(3)} \otimes \mathrm{id}\right) \circ \Delta = \left(\varepsilon_1^{(3)} \otimes \mathrm{id} \otimes \varepsilon_2^{(3)}\right) \circ \Delta = \left(\mathrm{id} \otimes \varepsilon_1^{(3)} \otimes \varepsilon_2^{(3)}\right) \circ \Delta = \mathrm{id}\,. \tag{12}$$

3. Four long ternary counits $\varepsilon_1^{(3)}$–$\varepsilon_4^{(3)}$ satisfying

$$\left(\left(\mathrm{id} \otimes \varepsilon_3^{(3)} \otimes \varepsilon_4^{(3)}\right) \circ \Delta^{(3)} \circ \left(\left(\mathrm{id} \otimes \varepsilon_1^{(3)} \otimes \varepsilon_2^{(3)}\right) \circ \Delta^{(3)}\right)\right) = \mathrm{id}\,. \tag{13}$$

Below we will consider only the first standard type of associativity (10). By analogy with (3) *σ-cocommutativity* is defined as $\sigma \circ \Delta^{(3)} = \Delta^{(3)}$.

Definition 5. *Ternary coalgebra* is a triple $\left(C, \Delta^{(3)}, \varepsilon^{(3)}\right)$, where C is a linear space and $\Delta^{(3)}$ is a ternary comultiplication (9) which is coassociative in one of the above senses and $\varepsilon^{(3)}$ is one of the above counits.

Let $\left(A, m^{(3)}\right)$ be a ternary algebra and $\left(C, \Delta^{(3)}\right)$ be a ternary coalgebra and $f, g, h \in \mathrm{Hom}_{\mathbb{K}}(C, A)$. *Ternary convolution product* is

$$[f, g, h]_* = m^{(3)} \circ (f \otimes g \otimes h) \circ \Delta^{(3)} \tag{14}$$

or in the Sweedler notation $[f, g, h]_*(a) = \left[f\left(a_{(1)}\right) g\left(a_{(2)}\right) h\left(a_{(3)}\right)\right]$.

Definition 6. Ternary coalgebra is called *derived*, if there exists a binary (usual, see e.g. [7]) coalgebra $\Delta^{(2)} : C \to C \otimes C$ such that (cf. 1))

$$\Delta^{(3)}_{\text{der}} = \left(\mathrm{id} \otimes \Delta^{(2)}\right) \otimes \Delta^{(2)}. \tag{15}$$

Definition 7. Ternary bialgebra B is $\left(B, m^{(3)}, \eta^{(3)}, \Delta^{(3)}, \varepsilon^{(3)}\right)$ for which $\left(B, m^{(3)}, \eta^{(3)}\right)$ is a ternary algebra and $\left(B, \Delta^{(3)}, \varepsilon^{(3)}\right)$ is a ternary coalgebra and they are compatible

$$\Delta^{(3)} \circ m^{(3)} = m^{(3)} \circ \Delta^{(3)}. \tag{16}$$

One can distinguish four kinds of ternary bialgebras with respect to a "being derived" property:

1. Δ-*derived* ternary bialgebra

$$\Delta^{(3)} = \Delta^{(3)}_{\text{der}} = \left(\mathrm{id} \otimes \Delta^{(2)}\right) \circ \Delta^{(2)}. \tag{17}$$

2. m-*derived* ternary bialgebra

$$m^{(3)}_{\text{der}} = m^{(3)}_{\text{der}} = m^{(2)} \circ \left(m^{(2)} \otimes \mathrm{id}\right). \tag{18}$$

3. *Derived* ternary bialgebra is simultaneously m-derived and Δ-derived ternary bialgebra.
4. *Non-derived* ternary bialgebra which does not satisfy (17) and (18).

Let us consider a ternary analog of the Woronowicz example of a bialgebra construction, which in the binary case has two generators satisfying $xy = qyx$ (or $\sigma^{(2)}_{+}(xy) = q\sigma^{(2)}_{-}(xy)$), then the following coproducts $\Delta^{(2)}(x) = x \otimes x$, $\Delta^{(2)}(x) = y \otimes x + 1 \otimes y$ are algebra maps. In the derived ternary case using (8) we have $\sigma^{(3)}_{+}([xey]) = q\sigma^{(3)}_{-}([xey])$, where e is the ternary unit and ternary coproducts are $\Delta^{(3)}(e) = e \otimes e \otimes e$, $\Delta^{(3)}(x) = x \otimes x \otimes x$, $\Delta^{(3)}(x) = y \otimes x \otimes x + e \otimes y \otimes x + e \otimes e \otimes y$, which are ternary algebra maps, i.e. they satisfy $\sigma^{(3)}_{+}\left(\left[\Delta^{(3)}(x)\,\Delta^{(3)}(e)\,\Delta^{(3)}(y)\right]\right) = q\sigma^{(3)}_{-}\left(\left[\Delta^{(3)}(x)\,\Delta^{(3)}(e)\,\Delta^{(3)}(y)\right]\right)$.

Possible types of ternary antipodes can be defined using analogy with binary coalgebras.

Definition 8. *Skew ternary antipod* is

$$\begin{aligned} & m^{(3)} \circ \left(S^{(3)}_{\text{skew}} \otimes \mathrm{id} \otimes \mathrm{id}\right) \circ \Delta^{(3)} \\ & \quad = m^{(3)} \circ \left(\mathrm{id} \otimes S^{(3)}_{\text{skew}} \otimes \mathrm{id}\right) \circ \Delta^{(3)} = m^{(3)} \circ \left(\mathrm{id} \otimes \mathrm{id} \otimes S^{(3)}_{\text{skew}}\right) \circ \Delta^{(3)} = \mathrm{id}\,. \end{aligned} \tag{19}$$

If only one equality from (19) is satisfied, the corresponding skew antipod is called *left*, *middle* or *right*.

Definition 9. *Strong ternary antipod* is

$$\left(m^{(2)} \otimes \mathrm{id}\right) \circ \left(\mathrm{id} \otimes S^{(3)}_{\text{strong}} \otimes \mathrm{id}\right) \circ \Delta^{(3)} = 1 \otimes \mathrm{id},$$
$$\left(\mathrm{id} \otimes m^{(2)}\right) \circ \left(\mathrm{id} \otimes \mathrm{id} \otimes S^{(3)}_{\text{strong}}\right) \circ \Delta^{(3)} = \mathrm{id} \otimes 1,$$

where 1 is a unit of algebra.

If in a ternary coalgebra $\Delta^{(3)} \circ S = \tau_{13} \circ (S \otimes S \otimes S) \circ \Delta^{(3)}$, where $\tau_{13} = \binom{123}{321}$, then it is called *skew-involutive*.

Definition 10. Ternary Hopf algebra $\left(H, m^{(3)}, \eta^{(3)}, \Delta^{(3)}, \varepsilon^{(3)}, S^{(3)}\right)$ is a ternary bialgebra with a ternary antipod $S^{(3)}$ of the type corresponding to the above.

There are 8 types of associative ternary Hopf algebras and 4 types of medial Hopf algebras. Also it can happen that there are several ternary units $\eta_i^{(3)}$ and several ternary counits $\varepsilon_i^{(3)}$ (see (11)–(13)), as well as different skew antipodes (see (19) and below), which makes number of ternary Hopf algebras enormous.

Let us consider concrete constructions of ternary comultiplications, bialgebras and Hopf algebras. A *ternary group-like element* can be defined by $\Delta^{(3)}(g) = g \otimes g \otimes g$, and for 3 such elements we have $\Delta^{(3)}([g_1 g_2 g_3]) = \Delta^{(3)}(g_1)\,\Delta^{(3)}(g_2)\,\Delta^{(3)}(g_3)$. But an analog of the binary primitive element (satisfying $\Delta^{(2)}(x) = x \otimes 1 + 1 \otimes x$) cannot be chosen simply as $\Delta^{(3)}(x) = x \otimes e \otimes e + e \otimes x \otimes e + e \otimes e \otimes x$, since the algebra structure is not preserved. Nevertheless, if we introduce *two* idempotent units e_1, e_2 satisfying "semiorthogonality" $[e_1 e_1 e_2] = 0$, $[e_2 e_2 e_1] = 0$, then

$$\Delta^{(3)}(x) = x \otimes e_1 \otimes e_2 + e_2 \otimes x \otimes e_1 + e_1 \otimes e_2 \otimes x \tag{20}$$

and now $\Delta^{(3)}([x_1 x_2 x_3]) = \left[\Delta^{(3)}(x_1)\,\Delta^{(3)}(x_2)\,\Delta^{(3)}(x_3)\right]$. Using (20) $\varepsilon(x) = 0$, $\varepsilon(e_{1,2}) = 1$, and $S^{(3)}(x) = -x$, $S^{(3)}(e_{1,2}) = e_{1,2}$, one can construct a ternary universal enveloping algebra in full analogy with the binary case (see e.g. [12]).

One of the most important examples of noncommutative Hopf algebras is the well known Sweedler Hopf algebra [7] which in the binary case has two generators x and y satisfying (in the "arrow language") $m^{(2)}(x,x) = 1$, $m^{(2)}(y,y) = 0$, $\sigma_+^{(2)}(xy) = -\sigma_-^{(2)}(xy)$. It has the following comultiplication $\Delta^{(2)}(x) = x \otimes x$, $\Delta^{(2)}(y) = y \otimes x + 1 \otimes y$, unit $\varepsilon^{(2)}(x) = 1$, $\varepsilon^{(2)}(y) = 0$, and antipod $S^{(2)}(x) = x$, $S^{(2)}(y) = -y$, which respect to the algebra structure. In the derived case a *ternary Sweedler algebra* is generated also by two generators x and y obeying $m^{(3)}(x,e,x) = m^{(3)}(e,x,x) = m^{(3)}(x,x,e) = e$, $\sigma_+^{(3)}([yey]) = 0$, $\sigma_+^{(3)}([xey]) = -\sigma_-^{(3)}([xey])$. The derived Hopf algebra structure is given by

$$\Delta^{(3)}(x) = x \otimes x \otimes x, \qquad \Delta^{(3)}(y) = y \otimes x \otimes x + e \otimes y \otimes x + e \otimes e \otimes y, \tag{21}$$

$$\varepsilon^{(3)}(x) = \varepsilon^{(2)}(x) = 1, \qquad \varepsilon^{(3)}(y) = \varepsilon^{(2)}(y) = 0, \tag{22}$$

$$S^{(3)}(x) = S^{(2)}(x) = x, \qquad S^{(3)}(y) = S^{(2)}(y) = -y, \tag{23}$$

and it can be checked that (21)–(22) are algebra maps, while (23) is antialgebra maps. To obtain a non-derived ternary Sweedler example we have the possibilities: 1) one "even" generator x, two "odd" generators $y_{1,2}$ and one ternary unit e; 2) two "even" generators $x_{1,2}$, one "odd" generator y and two ternary units $e_{1,2}$. In the first case the ternary algebra structure is (no summation, $i = 1,2$)

$$\begin{gathered}
[xxx] = e, \qquad [y_i y_i y_i] = 0, \quad \sigma_+^{(3)}([y_i x y_i]) = 0, \qquad \sigma_+^{(3)}([x y_i x]) = 0,\\
[x e y_i] = -[x y_i e], \qquad [e x y_i] = -[y_i x e], \qquad [e y_i x] = -[y_i e x],\\
\sigma_+^{(3)}([y_1 x y_2]) = -\sigma_-^{(3)}([y_1 x y_2]).
\end{gathered} \tag{24}$$

The corresponding ternary Hopf algebra structure is

$$\begin{gathered}
\Delta^{(3)}(x) = x \otimes x \otimes x, \qquad \Delta^{(3)}(y_{1,2}) = y_{1,2} \otimes x \otimes x + e_{1,2} \otimes y_{2,1} \otimes x + e_{1,2} \otimes e_{2,1} \otimes y_{2,1},\\
\varepsilon^{(3)}(x) = 1, \qquad \varepsilon^{(3)}(y_i) = 0, \qquad S^{(3)}(x) = x, \qquad S^{(3)}(y_i) = -y_i.
\end{gathered} \tag{25}$$

In the second case we have for the algebra structure

$$\begin{gathered}
[x_i x_j x_k] = \delta_{ij}\delta_{ik}\delta_{jk} e_i, \qquad [yyy] = 0, \qquad \sigma_+^{(3)}([y x_i y]) = 0, \qquad \sigma_+^{(3)}([x_i y x_i]) = 0,\\
\sigma_+^{(3)}([y_1 x y_2]) = 0, \qquad \sigma_-^{(3)}([y_1 x y_2]) = 0,
\end{gathered} \tag{26}$$

and the ternary Hopf algebra structure is

$$\Delta^{(3)}(x_i) = x_i \otimes x_i \otimes x_i, \qquad \Delta^{(3)}(y) = y \otimes x_1 \otimes x_1 + e_1 \otimes y \otimes x_2 + e_1 \otimes e_2 \otimes y,$$
$$\varepsilon^{(3)}(x_i) = 1, \qquad \varepsilon^{(3)}(y) = 0, \qquad S^{(3)}(x_i) = x_i, \qquad S^{(3)}(y) = -y. \tag{27}$$

Let us consider the group $G = SL(n,\mathbb{K})$. Then the algebra generated by $a^i_j \in SL(n,\mathbb{K})$ can be endowed by the structure of ternary Hopf algebra (see e.g. [25] for binary case) by choosing the ternary coproduct, counit and antipod as (here summation is implied)

$$\Delta^{(3)}\left(a^i_j\right) = a^i_k \otimes a^k_l \otimes a^l_j, \qquad \varepsilon\left(a^i_j\right) = \delta^i_j, \qquad S^{(3)}\left(a^i_j\right) = \left(a^{-1}\right)^i_j. \tag{28}$$

This antipod is a skew one since from (19) it follows $m^{(3)} \circ (S^{(3)} \otimes \mathrm{id} \otimes \mathrm{id}) \circ \Delta^{(3)}\left(a^i_j\right) = S^{(3)}\left(a^i_k\right) a^k_l a^l_j = \left(a^{-1}\right)^i_k a^k_l a^l_j = \delta^i_l a^l_j = a^i_j$. This ternary Hopf algebra is derived since for $\Delta^{(2)} = a^i_j \otimes a^j_k$ we have $\Delta^{(3)} = \left(\mathrm{id} \otimes \Delta^{(2)}\right) \otimes \Delta^{(2)}\left(a^i_j\right) = \left(\mathrm{id} \otimes \Delta^{(2)}\right)\left(a^i_k \otimes a^k_j\right) = a^i_k \otimes \Delta^{(2)}\left(a^k_j\right) = a^i_k \otimes a^k_l \otimes a^l_j$. In the most important case $n = 2$ we can obtain the manifest action of the ternary coproduct $\Delta^{(3)}$ on components. Possible non-derived matrix representations of the ternary product can be done only by four-rank $n \times n \times n \times n$ twice covariant and twice contravariant tensors $\left\{a^{ij}_{kl}\right\}$. Among all products the non-derived ones are only the following $a^{oi}_{jk} b^{jl}_{oo} c^{ko}_{il}$ and $a^{ij}_{ok} b^{ol}_{io} c^{ko}_{il}$ (where o is any index). So using e.g. the first choice we can define the non-derived Hopf algebra structure by $\Delta^{(3)}\left(a^{ij}_{kl}\right) = a^{i\mu}_{v\rho} \otimes a^{v\sigma}_{kl} \otimes a^{\rho j}_{\mu\sigma}$, $\varepsilon\left(a^{ij}_{kl}\right) = \frac{1}{2}\left(\delta^i_k \delta^j_l + \delta^i_l \delta^j_k\right)$, and the skew antipod $s^{ij}_{kl} = S^{(3)}\left(a^{ij}_{kl}\right)$ which is a solution of the equation $s^{i\mu}_{v\rho} a^{v\sigma}_{kl} = \delta^i_\rho \delta^\mu_k \delta^\sigma_l$.

Next consider ternary dual pair $k(G)$ (push-forward) and $\mathcal{F}(G)$ (pull-back) which are related by $k^*(G) \cong \mathcal{F}(G)$ (see e.g. [26]). Here $k(G) = \mathrm{span}\,(G)$ is a ternary group algebra (G has a ternary product $[\,]_G$ or $m^{(3)}_G$) over a field k. If $u \in k(G)$ ($u = u^i x_i, x_i \in G$), then $[uvw]_k = u^i v^j w^l \left[x_i x_j x_l\right]_G$ is associative, and so $\left(k(G), [\,]_k\right)$ becomes a ternary algebra. Define a ternary coproduct $\Delta^{(3)}_k : k(G) \to k(G) \otimes k(G) \otimes k(G)$ by $\Delta^{(3)}_k(u) = u^i x_i \otimes x_i \otimes x_i$ (derived and associative), then $\Delta^{(3)}_k\left([uvw]_k\right) = \left[\Delta^{(3)}_k(u)\, \Delta^{(3)}_k(v)\, \Delta^{(3)}_k(w)\right]_k$, and $k(G)$ is a ternary bialgebra. If we define a ternary antipod by $S^{(3)}_k = u^i \bar{x}_i$, where $\bar{x}_i$ is a skew element of x_i, then $k(G)$ becomes a ternary Hopf algebra. In the dual case of functions $\mathcal{F}(G) : \{\varphi : G \to k\}$ a ternary product $[\,]_{\mathcal{F}}$ or $m^{(3)}_{\mathcal{F}}$ (derived and associative) acts on $\psi(x,y,z)$ as $\left(m^{(3)}_{\mathcal{F}} \psi\right)(x) = \psi(x,x,x)$, and so $\mathcal{F}(G)$ is a ternary algebra. Let $\mathcal{F}(G) \otimes \mathcal{F}(G) \otimes \mathcal{F}(G) \cong \mathcal{F}(G \times G \times G)$, then we define a ternary coproduct $\Delta^{(3)}_{\mathcal{F}} : \mathcal{F}(G) \to \mathcal{F}(G) \otimes \mathcal{F}(G) \otimes \mathcal{F}(G)$ as $\left(\Delta^{(3)}_{\mathcal{F}} \varphi\right)(x,y,z) = \varphi\left([xyz]_{\mathcal{F}}\right)$, which is derive and associative. Thus we can obtain $\Delta^{(3)}_{\mathcal{F}}\left([\varphi_1\varphi_2\varphi_3]_{\mathcal{F}}\right) = \left[\Delta^{(3)}_{\mathcal{F}}(\varphi_1)\, \Delta^{(3)}_{\mathcal{F}}(\varphi_2)\, \Delta^{(3)}_{\mathcal{F}}(\varphi_3)\right]_{\mathcal{F}}$, and therefore $\mathcal{F}(G)$ is a ternary bialgebra. If we define a ternary antipod by $S^{(3)}_{\mathcal{F}}(\varphi) = \varphi(\bar{x})$, where $\bar{x}$ is a skew element of x, then $\mathcal{F}(G)$ becomes a ternary Hopf algebra.

Let us introduce a ternary analog of R-matrix. For a ternary Hopf algebra H we consider a linear map $R^{(3)} : H \otimes H \otimes H \to H \otimes H \otimes H$.

Definition 11. A ternary Hopf algebra $\left(H, m^{(3)}, \eta^{(3)}, \Delta^{(3)}, \varepsilon^{(3)}, S^{(3)}\right)$ is called *quasifiveangular* (the reason of such notation is clear from (32)) if it satisfies

$$\left(\Delta^{(3)} \otimes \mathrm{id} \otimes \mathrm{id}\right) = R^{(3)}_{145} R^{(3)}_{245} R^{(3)}_{345}, \tag{29}$$

$$\left(\mathrm{id} \otimes \Delta^{(3)} \otimes \mathrm{id}\right) = R^{(3)}_{125} R^{(3)}_{145} R^{(3)}_{135}, \tag{30}$$

$$\left(\mathrm{id} \otimes \mathrm{id} \otimes \Delta^{(3)}\right) = R^{(3)}_{125} R^{(3)}_{124} R^{(3)}_{123}, \tag{31}$$

where as usual index of R denotes action component positions.

Using the standard procedure (see e.g. [12, 27, 13]), we obtain set of abstract *ternary quantum Yang–Baxter equations*, one of which has the form

$$R^{(3)}_{243}R^{(3)}_{342}R^{(3)}_{125}R^{(3)}_{145}R^{(3)}_{135} = R^{(3)}_{123}R^{(3)}_{132}R^{(3)}_{145}R^{(3)}_{245}R^{(3)}_{345}, \tag{32}$$

and others can be obtained by corresponding permutations. The classical ternary Yang–Baxter equations for one parameter family of solutions $R(t)$ can be obtained by the expansion $R^{(3)}(t) = e \otimes e \otimes e + rt + \mathcal{O}(t^2)$, where r is a ternary classical R-matrix, then e.g. for (32) we have

$$\begin{aligned}
&r_{342}r_{125}r_{145}r_{135} + r_{243}r_{125}r_{145}r_{135} + r_{243}r_{342}r_{145}r_{135} + r_{243}r_{342}r_{125}r_{135} + r_{243}r_{342}r_{125}r_{145} \\
&\quad = r_{132}r_{145}r_{245}r_{345} + r_{123}r_{145}r_{245}r_{345} + r_{123}r_{132}r_{245}r_{345} \\
&\quad + r_{123}r_{132}r_{145}r_{345} + r_{123}r_{132}r_{145}r_{245}.
\end{aligned}$$

For *three* ternary Hopf algebras

$$\left(H_A, m^{(3)}_A, \eta^{(3)}_A, \Delta^{(3)}_A, \varepsilon^{(3)}_A, S^{(3)}_A\right), \qquad \left(H_B, m^{(3)}_B, \eta^{(3)}_B, \Delta^{(3)}_B, \varepsilon^{(3)}_B, S^{(3)}_B\right) \quad \text{and}$$
$$\left(H_C, m^{(3)}_C, \eta^{(3)}_C, \Delta^{(3)}_C, \varepsilon^{(3)}_C, S^{(3)}_C\right)$$

we can introduce a non-degenerate ternary "pairing" (see e.g. [27] for binary case) $\langle\ ,\ ,\ \rangle^{(3)} : H_A \times H_B \times H_C \to \mathbb{K}$, trilinear over $\mathbb{K}$, satisfying

$$\begin{aligned}
&\left\langle \eta^{(3)}_A(a), b, c\right\rangle^{(3)} = \left\langle a, \varepsilon^{(3)}_B(b), c\right\rangle^{(3)}, \qquad \left\langle a, \eta^{(3)}_B(b), c\right\rangle^{(3)} = \left\langle \varepsilon^{(3)}_A(a), b, c\right\rangle^{(3)}, \\
&\left\langle b, \eta^{(3)}_B(b), c\right\rangle^{(3)} = \left\langle a, b, \varepsilon^{(3)}_C(c)\right\rangle^{(3)}, \qquad \left\langle a, b, \eta^{(3)}_C(c)\right\rangle^{(3)} = \left\langle a, \varepsilon^{(3)}_B(b), c\right\rangle^{(3)}, \\
&\left\langle a, b, \eta^{(3)}_C(c)\right\rangle^{(3)} = \left\langle \varepsilon^{(3)}_A(a), b, c\right\rangle^{(3)}, \qquad \left\langle \eta^{(3)}_A(a), b, c\right\rangle^{(3)} = \left\langle a, b, \varepsilon^{(3)}_C(c)\right\rangle^{(3)}, \\
&\left\langle m^{(3)}_A(a_1 \otimes a_2 \otimes a_3), b, c\right\rangle^{(3)} = \left\langle a_1 \otimes a_2 \otimes a_3, \Delta^{(3)}_B(b), c\right\rangle^{(3)}, \\
&\left\langle \Delta^{(3)}_A(a), b_1 \otimes b_2 \otimes b_3, c\right\rangle^{(3)} = \left\langle a, m^{(3)}_B(b_1 \otimes b_2 \otimes b_3), c\right\rangle^{(3)}, \\
&\left\langle a, m^{(3)}_B(b_1 \otimes b_2 \otimes b_3), c\right\rangle^{(3)} = \left\langle a, b_1 \otimes b_2 \otimes b_3, \Delta^{(3)}_C(c)\right\rangle^{(3)}, \\
&\left\langle a, \Delta^{(3)}_B(b), c_1 \otimes c_2 \otimes c_3\right\rangle^{(3)} = \left\langle a, b, m^{(3)}_C(c_1 \otimes c_2 \otimes c_3)\right\rangle^{(3)}, \\
&\left\langle a, b, m^{(3)}_C(c_1 \otimes c_2 \otimes c_3)\right\rangle^{(3)} = \left\langle \Delta^{(3)}_A(a), b, c_1 \otimes c_2 \otimes c_3\right\rangle^{(3)}, \\
&\left\langle a_1 \otimes a_2 \otimes a_3, b, \Delta^{(3)}_C(c)\right\rangle^{(3)} = \left\langle m^{(3)}_A(a_1 \otimes a_2 \otimes a_3), b, c\right\rangle^{(3)}, \\
&\left\langle S^{(3)}_A(a), b, c\right\rangle^{(3)} = \left\langle a, S^{(3)}_B(b), c\right\rangle^{(3)} = \left\langle a, b, S^{(3)}_C(c)\right\rangle^{(3)},
\end{aligned}$$

where $a, a_i \in H_A$, $b, b_i \in H_B$. The ternary "paring" between $H_A \otimes H_A \otimes H_A$ and $H_B \otimes H_B \otimes H_B$ is given by $\langle a_1 \otimes a_2 \otimes a_3, b_1 \otimes b_2 \otimes b_3\rangle^{(3)} = \langle a_1, b_1\rangle^{(3)} \langle a_2, b_2\rangle^{(3)} \langle a_3, b_3\rangle^{(3)}$. These constructions can naturally lead to ternary generalization of duality concept and quantum double [14, 12, 13].

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On the Center of q-Deformed Algebra $U'_q(\mathrm{so}_3)$ Related to Quantum Gravity at q a Root of 1

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It is known that Fairlie–Odesskii algebra $U'_q(\mathrm{so}_3)$ appears as algebra of observables in quantum gravity in $(2+1)$-dimensional de Sitter space with space being torus. In this paper, we study the center of this algebra at q a root of 1. It turns out that Casimir elements in this case are algebraically dependent. Using realization of the algebra $U'_q(\mathrm{so}_3)$ in terms of quantized lengths of geodesics on torus with one hole, we find this dependence in an explicit form. It is expressed in terms of Chebyshev polynomials of the first kind. The properties of Casimir elements in the cyclic type representations are studied.

1 Introduction

It is shown by Nelson, Regge and Zertuche [1] that the algebra of observables in quantum gravity in (2+1)-dimensional de Sitter space with space being torus is related to Fairlie–Odesskii algebra $U'_q(\mathrm{so}_3)$ [2, 3], where q is related to the Plank constant and the curvature of the de Sitter space. Thus it is important, from point of view of physics, to study the structure (in particular, the center) of this algebra. The center of the algebra $U'_q(\mathrm{so}_3)$ in the case of q being not a root of 1 is generated by the element C, which is deformation of the Casimir element of Lie algebra so_3. The center of this algebra at q a root of 1 contains three more elements C_1, C_2, C_3 [2, 4]. It turns out that all four Casimir elements are algebraically dependent. The main goal of this paper is to describe this dependence in an explicit form. To find it we use the realization of algebra $U'_q(\mathrm{so}_3)$ in terms of quantum geodesics on torus $\mathcal{T}$ with one hole proposed by Chekhov and Fock [5]. Namely, generators I_1 and I_2 (resp. Casimir element C) of algebra $U'_q(\mathrm{so}_3)$ are related to quantized lengths of geodesics corresponding to two basis cycles (resp. cycle around the hole) on $\mathcal{T}$. They are expressed in terms of z_1, z_2, z_3, which are “coordinates” on quantized Teichmüller space $\mathcal{A}_q$ of $\mathcal{T}$. In this realization, the fact that elements C, C_1, C_2, C_3 belong to the center of $U'_q(\mathrm{so}_3)$ is almost obvious. We note, that the same algebra $U'_q(\mathrm{so}_3)$ appeared also in the paper [6] as Kauffman bracket skein algebra of $\mathcal{T}$.

It is known that algebra $U'_q(\mathrm{so}_3)$, at q a root of 1, possesses cyclic type irreducible representations [7, 8]. The action formulas for Casimir operators on the spaces of these representations are presented in explicit form. It is shown that C_1, C_2, C_3 do not separate this type of representations. To separate them we also need to include C.

2 Fairlie–Odesskii algebra $U'_q(\mathrm{so}_3)$

The Fairlie–Odesskii algebra $U'_q(\mathrm{so}_3)$ [2, 3] is an associative unital algebra with generating elements I_1, I_2, I_3 and defining relations

$$q^{1/2}I_1I_2 - q^{-1/2}I_2I_1 = I_3, \qquad q^{1/2}I_2I_3 - q^{-1/2}I_3I_2 = I_1, \qquad q^{1/2}I_3I_1 - q^{-1/2}I_1I_3 = I_2,$$

where $q \neq 0, \pm 1$, is a complex number called deformation parameter. In the limit $q \to 1$, the algebra $U'_q(\mathrm{so}_3)$ reduces to the Lie algebra so_3. Algebra $U'_q(\mathrm{so}_3)$ has a linear basis $I_1^{k_1}I_2^{k_2}I_3^{k_3}$,

$k_1, k_2, k_3 \geq 0$ (Poincaré–Birkhoff–Witt basis) [7]. At arbitrary q, the algebra $U_q'(\mathrm{so}_3)$ has central element

$$C = -q^{1/2}\left(q - q^{-1}\right) I_1 I_2 I_3 + q I_1^2 + q^{-1} I_2^2 + q I_3^2. \tag{1}$$

It generates the center of $U_q'(\mathrm{so}_3)$ when q is not a root of 1 (see [6]).

Let us fix q to be a primitive root of 1 of order $p > 2$, that is $q^p = 1$, $q^{p'} \neq 1$, $1 \leq p' < p$. Then elements

$$C_k = 2\ T_p\left(I_k\left(q - q^{-1}\right)/2\right), \qquad k = 1, 2, 3, \tag{2}$$

where $T_p(x)$ is Chebyshev polynomial of the first kind, are also central in $U_q'(\mathrm{so}_3)$. The Chebyshev polynomial $T_p(x)$ is uniquely defined through $T_p(\cos\theta) = \cos(p\theta)$. It explicit form is

$$T_p(x) = \frac{p}{2}\sum_{k=0}^{\lfloor p/2\rfloor} \frac{(-1)^k (p-k-1)!}{k!(p-2k)!}(2x)^{p-2k}, \tag{3}$$

where $\lfloor p/2\rfloor$ is integral part of $p/2$. Some examples of Chebyshev polynomials of the first kind:

$$\begin{array}{lll} T_0(x) = 1, & T_1(x) = x, & T_2(x) = 2x^2 - 1, \\ T_3(x) = 4x^3 - 3x, & T_4(x) = 8x^4 - 8x^2 + 1, & T_5(x) = 16x^5 - 20x^3 + 5x, \quad \dots . \end{array}$$

The central elements C_1, C_2 and C_3 were first described in implicit form (and without proof) in [2]. In the paper [4], these elements were given in explicit form as sum of type (3). It was pointed out to me by V. Fock the coincidence of this sum with Chebyshev polynomial of the first kind. The elements C, C_1, C_2 and C_3 are algebraically dependent. Our main goal is to describe this dependence in an explicit form.

3 Algebra $U_q'(\mathrm{so}_3)$ as algebra of quantum geodesics on torus with one hole

Now we describe the algebra $\mathcal{A}_q$ of quantized Teichmüller space of torus with one hole [5]. It is an associative unital algebra with generating elements z_1, z_1^{-1}, z_2, z_2^{-1}, z_3, z_3^{-1} and defining relations

$$z_k z_k^{-1} = z_k^{-1} z_k = 1, \qquad z_1 z_2 = q z_2 z_1, \qquad z_2 z_3 = q z_3 z_2, \qquad z_3 z_1 = q z_1 z_3. \tag{4}$$

It is easy to realize that $z_1^{k_1} z_2^{k_2} z_3^{k_3}$, $k_1, k_2, k_3 \in \mathbb{Z}$, constitute a linear basis in $\mathcal{A}_q$. Geodesic functions G_1, G_2 and G_3, which are related to lengths L_1, L_2 and L_3 of geodesics $(1,0)$, $(0,1)$ (corresponding to two basis cycles) and $(1,1)$ (corresponding to sum of these cycles) on torus with one hole as $G_k = 2\cosh(L_k/2)$, after quantization take the form [5]:

$$G_1 = q^{-1/2} z_3^{-1} z_1^{-1} + q^{1/2} z_3^{-1} z_1 + q^{-1/2} z_3 z_1, \tag{5}$$

$$G_2 = q^{-1/2} z_2^{-1} z_3^{-1} + q^{1/2} z_2^{-1} z_3 + q^{-1/2} z_2 z_3, \tag{6}$$

$$G_3 = q^{-1/2} z_1^{-1} z_2^{-1} + q^{1/2} z_1^{-1} z_2 + q^{-1/2} z_1 z_2. \tag{7}$$

Proposition 1 ([5]). *The map ϕ given by*

$$\phi:\ I_k \mapsto G_k/\left(q - q^{-1}\right), \qquad k = 1, 2, 3,$$

defines an injective homomorphism $\phi : U_q'(\mathrm{so}_3) \to \mathcal{A}_q$.

Proof. It is easy to show by straightforward calculation that

$$q^{1/2}G_1G_2 - q^{-1/2}G_2G_1 = \left(q - q^{-1}\right)G_3,$$
$$q^{1/2}G_2G_3 - q^{-1/2}G_3G_2 = \left(q - q^{-1}\right)G_1,$$
$$q^{1/2}G_3G_1 - q^{-1/2}G_1G_3 = \left(q - q^{-1}\right)G_2.$$

It proves that ϕ defines a homomorphism. Let us show that $\ker(\phi) = 0$. We assume that there exists an element $a = \sum a_{k_1,k_2,k_3} I_1^{k_1} I_2^{k_2} I_3^{k_3}$, where only finite number of coefficients a_{k_1,k_2,k_3} are non-zero, such that $\phi(a) = 0$. Let $a_{l_1,l_2,l_3} \neq 0$ for some $l_1, l_2, l_3 \geq 0$ and $a_{k_1,k_2,k_3} = 0$ for all k_1, k_2, k_3 such that $k_1 + k_2 + k_3 > l_1 + l_2 + l_3$. Then $\phi(I_1^{l_1} I_2^{l_2} I_3^{l_3})$ contains summand $\alpha z_1^{l_1+l_3} z_2^{l_2+l_3} z_3^{l_1+l_2}$, $\alpha \neq 0$. It is unique summand with maximal sum of powers of z_1, z_2 and z_3. Only $\phi(I_1^{k_1} I_2^{k_2} I_3^{k_3})$ with $k_1 + k_2 + k_3 = l_1 + l_2 + l_3$ and $a_{k_1,k_2,k_3} \neq 0$ contain summands with the same sum of powers of z_1, z_2, z_3. But the very monomials in z_1, z_2, z_3 are not coinciding, because from $z_1^{k_1+k_3} z_2^{k_2+k_3} z_3^{k_1+k_2} = z_1^{l_1+l_3} z_2^{l_2+l_3} z_3^{l_1+l_2}$ it follows that $k_i = l_i$. Thus coefficient at $z_1^{l_1+l_3} z_2^{l_2+l_3} z_3^{l_1+l_2}$ in $\phi(a)$ is non-zero. It contradicts the assumption that $\phi(a) = 0$. ■

The injectivity of homomorphism ϕ follows from construction given in [5]. We proved the injectivity in purely algebraic way.

Now our strategy is following. We find the images of C, C_1, C_2, C_3 in $\mathcal{A}_q$. Then, due to Proposition 1, the relations between the obtained images will imply the relations between of C, C_1, C_2 and C_3. Instead of C, we will use

$$\partial = \left(q + q^{-1}\right)\mathbf{1} - \left(q - q^{-1}\right)^2 C. \tag{8}$$

Straightforward calculation shows that (see (1))

$$\phi(\partial) = q^{-2}\left(z_1^{-2} z_2^{-2} z_3^{-2} + z_1^2 z_2^2 z_3^2\right). \tag{9}$$

It is easy to see that $\phi(\partial)$ commutes with z_1, z_2, z_3 and, therefore, with $\phi(I_1)$, $\phi(I_2)$, $\phi(I_3)$. Hence, due to Proposition 1, ∂ is central in $U'_q(\mathrm{so}_3)$. The images of C_k, $k = 1, 2, 3$, in $\mathcal{A}_q$ are (see (2))

$$\phi(C_k) = 2T_p(G_k/2), \qquad k = 1, 2, 3. \tag{10}$$

Let us define an associative algebra L_q with generating elements Λ, Λ^{-1}, Λ_0 which satisfy the relations

$$\Lambda\Lambda^{-1} = \Lambda^{-1}\Lambda = 1, \qquad \Lambda\Lambda_0 = q^2\Lambda_0\Lambda,$$

where q is a non-zero complex number. In order to formulate an important lemma, we remind the standard notations for q-numbers:

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}, \tag{11}$$

q-factorials and q-binomial coefficients:

$$[m]! = [m][m-1]\cdots[1], \qquad \begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!} = \frac{[n][n-1]\cdots[n-m+1]}{[1][2]\cdots[m]}.$$

Lemma 1. *In algebra L_q at non-zero complex number q, we have*

$$2T_p\left(\frac{\Lambda + \Lambda_0 + \Lambda^{-1}}{2}\right) = \Lambda^p + \Lambda^{-p} + \sum R_{p,k,l}\Lambda^l\Lambda_0^k,$$

where sum runs over integral k and l such that $k > 0$, $k \pm l \leq p$, $k + l \equiv p \pmod 2$, and

$$R_{p,k,l} = q^{-kl}\frac{[p]}{[k]}\begin{bmatrix} \frac{p+k+l}{2} - 1 \\ k - 1 \end{bmatrix}\begin{bmatrix} \frac{p+k-l}{2} - 1 \\ k - 1 \end{bmatrix}.$$

Proof. We will prove the lemma by induction. It is easy to see that the lemma is correct at $p=0$ and $p=1$. For $p=2$, we have $2T_2(x/2)=x^2-2$, and validity of lemma follows from direct calculation. The left-hand sides of the relations given in lemma satisfy the recurrent relation which follows from recurrent relation for Chebyshev polynomials: $T_p(x)=2xT_{p-1}(x)-T_{p-2}(x)$. Hence, the right-hand sides also must satisfy the same relation. In terms of $R_{p,k,l}$ it looks like

$$R_{p,k,l}=R_{p-1,k,l-1}+R_{p-1,k,l+1}+R_{p-1,k,l+1}+q^{-2l}R_{p-1,k-1,l}-R_{p-2,k,l}.$$

Substituting explicit expressions for $R_{p,k,l}$ and cancelling common multiplier we obtain the relation

$$\begin{aligned}
&[p]\left[\frac{p+k+l}{2}-1\right]\left[\frac{p+k-l}{2}-1\right]=q^{-k}[p-1]\left[\frac{p-k-l}{2}\right]\left[\frac{p+k+l}{2}-1\right]\\
&\quad+q^{k}[p-1]\left[\frac{p-k+l}{2}\right]\left[\frac{p+k-l}{2}-1\right]+q^{-l}[p-1][k][k-1]\\
&\quad-[p-2]\left[\frac{p-k+l}{2}\right]\left[\frac{p-k-l}{2}\right],
\end{aligned}$$

which can be verified in direct way using definition (11) of q-numbers. ■

Corollary 1. *In algebra L_q, when q is a primitive root of 1 of order $p>2$, we have*

$$2T_p\left(\frac{\Lambda+\Lambda_0+\Lambda^{-1}}{2}\right)=\Lambda^p+\Lambda^{-p}+\Lambda_0^p$$

if p is odd, and

$$2T_p\left(\frac{\Lambda+\Lambda_0+\Lambda^{-1}}{2}\right)=\Lambda^p+\Lambda^{-p}+\Lambda_0^p+2\Lambda^{p/2}\Lambda_0^{p/2}+2\Lambda^{-p/2}\Lambda_0^{p/2}$$

if p is even.

Proof. Let us make some remarks on the values of q-numbers at q a root of 1. If p is odd, then $[p]=0$ and $[s]\neq 0$, if $s=1,2,\ldots,p-1$. If p is even, then $[p/2]=[p]=0$ and $[s]\neq 0$, if $s=1,2,\ldots,p/2-1,p/2+1,\ldots,p-1$. Let p be an odd number. Then numerators and denominators in the both q-binomial coefficients included in $R_{p,k,l}$ are non-zero. Thus all the $R_{p,k,l}=0$ (due to $[p]=0$), unless $k=p$. In the case $k=p$, we have $l=0$ and $R_{p,p,0}=1$. Now we consider the case of even p. Simple analysis shows that if the denominator of a q-binomial coefficient included in $R_{p,k,l}$ contains $[p/2]$ then the corresponding numerator also contains this q-number. Cancelling it, we obtain non-zero q-binomial coefficient. All the $R_{p,k,l}=0$ (due to $[p]=0$), unless $k=p$ or $k=p/2$. If $k=p$, we obtain $l=0$ and $R_{p,p,0}=1$ in full analogy with odd p case. Analyzing numerators and denominators in q-binomial coefficients in the case of $k=p/2$, we find that the q-binomial coefficients are non-zero only if $l=\pm p/2$. Since $q^{p/2}=-1$ (not $+1$ because q is a primitive root of 1), we have $[p]/[p/2]\equiv q^{p/2}+q^{-p/2}=-2$ and $(-1)^{\mp p^2/4}=(-1)^{p/2}$. Using the relation $[p-r]=-[r]$, we obtain $R_{p,p/2,\pm p/2}=2$. Thus we have found all the non-zero coefficients $R_{p,k,l}$. ■

Corollary 2. *The map ϕ on C_1, C_2 and C_3, when q is a primitive root of 1 of order $p>2$, is*

$$\left.\begin{aligned}
C_1&\mapsto q^{p/2}\left(z_3^{-p}z_1^{-p}+z_3^{-p}z_1^{p}+z_3^{p}z_1^{p}\right),\\
C_2&\mapsto q^{p/2}\left(z_2^{-p}z_3^{-p}+z_2^{-p}z_3^{p}+z_2^{p}z_3^{p}\right),\\
C_3&\mapsto q^{p/2}\left(z_1^{-p}z_2^{-p}+z_1^{-p}z_2^{p}+z_1^{p}z_2^{p}\right)
\end{aligned}\right\}\qquad \textit{at odd } p,$$

$$\left.\begin{aligned} &C_1 \mapsto z_3^{-p}z_1^{-p} + z_3^{-p}z_1^{p} + z_3^{p}z_1^{p} + (-1)^{p/2}2\left(z_1^{p} + z_3^{-p}\right),\\ &C_2 \mapsto z_2^{-p}z_3^{-p} + z_2^{-p}z_3^{p} + z_2^{p}z_3^{p} + (-1)^{p/2}2\left(z_3^{p} + z_2^{-p}\right),\\ &C_3 \mapsto z_1^{-p}z_2^{-p} + z_1^{-p}z_2^{p} + z_1^{p}z_2^{p} + (-1)^{p/2}2\left(z_2^{p} + z_1^{-p}\right) \end{aligned}\right\} \qquad \textit{at even } p.$$

Proof. Let us find $\phi(C_1)$ (see (10)). We denote three summands in $G_1 = q^{-1/2}z_3^{-1}z_1^{-1} + q^{1/2}z_3^{-1}z_1 + q^{-1/2}z_3z_1$ by Λ^{-1}, Λ_0 and Λ, respectively. It is easy to verify that these three objects give realization of algebra L_q in $\mathcal{A}_q$. Then this corollary can be obtained using Corollary 1 and commutation relations (4) for $\mathcal{A}_q$. The cases of the elements C_2 and C_3 can be analyzed in full analogy with the case of element C_1. ∎

It is obvious that images of C_1, C_2, C_3, at q a root of 1, commute with z_k, and therefore with $\phi(I_1)$, $\phi(I_2)$, $\phi(I_3)$. It gives one more proof of the fact that C_1, C_2, C_3 are central in $U'_q(\mathrm{so}_3)$.

Proposition 2. *The algebraic dependence of the central elements ∂, C_1, C_2, C_3 of $U'_q(\mathrm{so}_3)$ at q a primitive root of 1 of order $p > 2$ has the form*

$$\begin{aligned} p = 2k+1: \quad & -q^{p/2}C_1C_2C_3 + C_1^2 + C_2^2 + C_3^2 + 2T_p(\partial/2) - 2 = 0,\\ p = 4k: \quad & -C_1C_2C_3 + C_1^2 + C_2^2 + C_3^2 + 2T_p(\partial/2) + 16T_{p/2}(\partial/2) + 10\\ & \quad + 4(T_{p/2}(\partial/2)+1)(C_1+C_2+C_3) = 0,\\ p = 4k+2: \quad & -C_1C_2C_3 + C_1^2 + C_2^2 + C_3^2 + 2T_p(\partial/2) - 16T_{p/2}(\partial/2) + 10\\ & \quad - 4(T_{p/2}(\partial/2)-1)(C_1+C_2+C_3) = 0. \end{aligned} \tag{12}$$

(The relation between C and ∂ is given by (8)).

Proof. To prove this proposition we map by ϕ left-hand sides of these relations to $\mathcal{A}_q$. It is easy to verify (using (9) and $2T_k((t+t^{-1})/2) = t^k + t^{-k}$) the relations

$$2T_p(\phi(\partial)/2) = z_1^{-2p}z_2^{-2p}z_3^{-2p} + z_1^{2p}z_2^{2p}z_3^{2p}, \qquad 2T_{p/2}(\phi(\partial)/2) = z_1^{-p}z_2^{-p}z_3^{-p} + z_1^{p}z_2^{p}z_3^{p},$$

where second relation is given only for even p. Thus the images of left-hand sides of relations (12) can be rewritten in terms of commuting variables $x_k = z_k^p$, $k = 1,2,3$. We obtain three relations (with respect to cases $p = 2k+1$, $p = 4k$ and $p = 4k+2$) each of them not depending on p of commuting variables x_1, x_2 and x_3. They can be verified directly. ∎

In private communication, V. Fock informed me about the form of algebraic dependence of central elements, in the case of odd p. Independently, V. Levandovskyy found this dependence when $p = 3, 4$ by using Computer Algebra System PLURAL for Non-commuting Polynomial Computation. This information was very important for me to formulate Proposition 2. Note, that algebraic dependence of central elements of Drinfeld–Jimbo algebra $U_q(\mathrm{sl}_2)$ at q a root of unity is also expressed in terms of Chebyshev polynomials [9].

Conjecture 1. *The elements C (or, equivalently, ∂), C_1, C_2, C_3 of $U'_q(\mathrm{so}_3)$ at q a root of 1 generate the center of this algebra. All the algebraic relations among them follow from the relations described in Proposition 2.*

4 Cyclic type representations of $U'_q(\mathrm{so}_3)$ at q a root of 1

Let $q^p = 1$. Then all the irreducible representations of $U'_q(\mathrm{so}_3)$ are finite-dimensional [8]. We describe one class of such representations, namely, cyclic type representations $T \equiv T_{l,h,M}$, where h, l and M are complex numbers, $h, h+l, h-l \notin \frac{1}{2}\mathbb{Z}$. These representations are given on

p-dimensional vector space $\mathcal{V}_{l,h,M}$ with basis $|h\rangle$, $|h+1\rangle, \ldots, |h+p-1\rangle$. It useful to identify $|h+p\rangle \equiv |h\rangle$, $|h-1\rangle \equiv |h+p-1\rangle$. The action formulas are

$$
\begin{aligned}
T(I_1)|m\rangle &= \mathrm{i}[m]|m\rangle, \\
T(I_2)|m\rangle &= \frac{[m]}{[2m]} \left(M[l-m]|m+1\rangle - M^{-1}[l+m]|m-1\rangle \right), \\
T(I_1)|m\rangle &= \mathrm{i}q^{1/2} \frac{[m]}{[2m]} \left(Mq^m[l-m]|m+1\rangle + M^{-1}q^{-m}[l+m]|m-1\rangle \right),
\end{aligned}
$$

where $m = h, h+1, \ldots, h+p-1$, and definition of q-numbers (11) is used.

Proposition 3 ([8]). *Any of irreducible representations $T_{l',h',M'}$ has unique equivalent representation among $T_{l,h,M}$ with $|\mathrm{Re}\, h| < 1/4$, $0 < \mathrm{Re}\, l < p/4$, $0 \leq \arg M < 2\pi/p$.*

Proposition 4. *The action of $T(C)$, $T(C_1)$, $T(C_2)$ and $T(C_3)$ is given by the formulas:*

$$
T(C)|m\rangle = -[l][l+1]|m\rangle,
$$

if $p = 2k+1$:

$$
\begin{aligned}
T(C_1)|m\rangle &= \mathrm{i}^p \left(q^{ph} - q^{-ph} \right) |m\rangle, \\
T(C_2)|m\rangle &= \left(M^p A_+ - M^{-p} A_- \right) |m\rangle, \\
T(C_3)|m\rangle &= \mathrm{i}^p q^{p/2} \left(M^p q^{ph} A_+ + M^{-p} q^{-ph} A_- \right) |m\rangle, \\
A_\pm &= \frac{q^{p(l\mp h)} - q^{-p(l\mp h)}}{q^{ph} + q^{-ph}};
\end{aligned}
$$

if $p = 4k$:

$$
\begin{aligned}
T(C_1)|m\rangle &= \left(q^{ph} + q^{-ph} \right) |m\rangle, \\
T(C_2)|m\rangle &= \left(M^p \tilde{A}_+ + \tilde{A}_0 + M^{-p} \tilde{A}_- \right) |m\rangle, \\
T(C_3)|m\rangle &= \left(M^p q^{ph} \tilde{A}_+ + \tilde{A}_0 + M^{-p} q^{-ph} \tilde{A}_- \right) |m\rangle, \\
\tilde{A}_\pm &= \frac{\left(q^{\frac{p}{2}(l\mp h)} - q^{-\frac{p}{2}(l\mp h)} \right)^2}{\left(q^{\frac{p}{2}h} - q^{-\frac{p}{2}h} \right)^2}, \qquad \tilde{A}_0 = -2 \frac{\left(q^{\frac{p}{2}l} - q^{-\frac{p}{2}l} \right)^2}{\left(q^{\frac{p}{2}h} - q^{-\frac{p}{2}h} \right)^2};
\end{aligned}
$$

if $p = 4k+2$:

$$
\begin{aligned}
T(C_1)|m\rangle &= -\left(q^{ph} + q^{-ph} \right) |m\rangle, \\
T(C_2)|m\rangle &= \left(M^p \breve{A}_+ + \breve{A}_0 + M^{-p} \breve{A}_- \right) |m\rangle, \\
T(C_3)|m\rangle &= \left(-M^p q^{ph} \breve{A}_+ + \breve{A}_0 - M^{-p} q^{-ph} \breve{A}_- \right) |m\rangle, \\
\breve{A}_\pm &= \frac{\left(q^{\frac{p}{2}(l\mp h)} - q^{-\frac{p}{2}(l\mp h)} \right)^2}{\left(q^{\frac{p}{2}h} + q^{-\frac{p}{2}h} \right)^2}, \qquad \breve{A}_0 = -2 \frac{\left(q^{\frac{p}{2}l} + q^{-\frac{p}{2}l} \right)^2}{\left(q^{\frac{p}{2}h} + q^{-\frac{p}{2}h} \right)^2}.
\end{aligned}
$$

Proof. From Schur lemma, it follows that $T(C)$, $T(C_1)$, $T(C_2)$ and $T(C_3)$ are proportional to unit matrix. That is the vectors $|m\rangle$ are eigenvectors with eigenvalues not depending on m.

The action of $T(C)$ and $T(C_1)$ can be found directly using the definition of q-numbers (11). From the action formulas for $T(I_2)$ and $T(I_3)$, we can see that matrix elements of diagonal action of $T(C_2)$ and $T(C_3)$ may include only summands which are proportional to $M^{\pm p}$ or summands which have no dependence on M. To find the coefficients $\tilde{A}_\pm$, $\breve{A}_\pm$ at $M^{\pm p}$ in action formulas for $T(C_2)$ (resp. $T(C_3)$), we observe that only highest order summand in the expression of C_2 (resp. C_3) in terms of Chebyshev polynomial of I_2 (resp. I_3) gives contribution to these coefficients. It is easy to calculate them. Now we use the relations of Proposition 2 to find the coefficients $\tilde{A}_0$ and $\breve{A}_0$ in the case of even p. The coefficients at $M^{\pm 2p}$ after substitution of $\tilde{A}_\pm$ and $\breve{A}_\pm$ are zero. The condition on the coefficients at $M^{\pm p}$ to be zero gives $\tilde{A}_0$ and $\breve{A}_0$. Of course, the found matrix elements also identically satisfy relations of Proposition 2 constructed from terms not depending on M. ■

It follows from Proposition 3 that representations $T_{l,h,M}$ and $T_{l+1,h,M}$ with l, h, M as in that proposition are not equivalent. But, it is easy to see, $T_{l,h,M}(C_k) = T_{l+1,h,M}(C_k)$, $k = 1, 2, 3$. Thus central elements C_1, C_2 and C_3 do not separate non-equivalent cyclic type representations. In fact, they separate almost all of them, namely, there exists at least one of central elements C_k such that $T_{l,h,M}(C_k) \neq T_{l',h',M'}(C_k)$ if $(l - l') \notin \mathbb{Z}$. To separate all of them we also need to include C.

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Generalized Canonical Commutation Relations: Representations and Stability of Universal Enveloping C^*-Algebra

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We consider the deformation of canonical commutation relations in the class of Wick algebras. The irreducible representations of GCCR are classified. We study the universal bounded representation of GCCR and compute the K-theory for the twisted canonical commutation relations.

In this paper we study the deformation of CCR generalising both twisted CCR of W. Pusz and S.L. Woronowicz and some type of q_{ij}-CCR of M. Bozejko and R. Speicher (see [3, 1]). Namely, let us consider a $*$-algebra generated by elements $\{a_i, a_i^*,\ i = 1,\dots,d\}$ satisfying the following relations (GCCR)

$$\begin{aligned}
&a_i^* a_i = 1 + \alpha_i a_i a_i^* - \sum_{j<i:\ k_j \ge i} (1-\alpha_j) a_j a_j^*,\\
&a_i^* a_j = \lambda_{ij}\alpha_i a_j a_i^*, \qquad a_j a_i = \lambda_{ij}\alpha_i a_i a_j, \qquad i<j, \quad k_i \ge j,\\
&a_i^* a_j = \lambda_{ij} a_j a_i^*, \qquad a_j a_i = \lambda_{ij} a_i a_j, \qquad i<j, \quad k_i < j,\\
&0<\alpha_i<1, \qquad |\lambda_{ij}| = 1, \qquad i,j = 1,\dots,d, \quad i \ne j,
\end{aligned} \tag{1}$$

where the vector $\vec{k} = (k_1, k_2, \dots, k_{d-1})$ has the property that $d \ge k_i \ge i$ and if $j < i$ and $i \le k_j$ then $k_i \le k_j$.

Example 1. For $\vec{k} = (d,\dots,d)$, $\alpha_i = \mu^2$, $i = 1,\dots,d$ and $\lambda_{ij} = 1,\ i \ne j$ we have a well-known twisted CCR:

$$\begin{aligned}
&a_i^* a_i = 1 + \mu^2 a_i a_i^* - \left(1-\mu^2\right) \sum_{j<i} a_j a_j^*,\\
&a_i^* a_j = \mu a_j a_i^*, \qquad a_j a_i = \mu a_i a_j, \qquad i<j.
\end{aligned}$$

Example 2. If we put $\vec{k} = (1, 2, \dots, d-1)$ we get

$$a_i^* a_i = 1 + \alpha_i a_i a_i^*, \qquad a_i^* a_j = \lambda_{ij} a_j a_i^*, \qquad a_j a_i = \lambda a_j a_i, \qquad i<j,$$

i.e. the so-called generalised "quon" commutation relations, which form a special type of q_{ij}-CCR.

In the Section 1 we study the C^*-algebra A constructed by the bounded representations of these relations and show that it is isomorphic to the $C^*(s_i, s_i^*)$ where partial isometries s_i, s_i^* satisfy the relations

$$s_i^* s_i = 1 - \sum_{j<i:\ k_j \geq i} s_j s_j^*, \qquad s_i^* s_j = 0, \qquad s_j s_i = 0, \qquad i<j, \qquad k_i \geq j,$$
$$s_i^* s_j = \lambda_{ij} s_j s_i^*, \qquad s_j s_i = \lambda_{ij} s_i s_j, \qquad i<j, \qquad k_i < j.$$

As a corollary of this stability result we have that $K_0(A_\mu) = \mathbb{Z}$ and $K_1(A_\mu) = \{0\}$, where A_μ is the C^*-algebra associated with TCCR. We also prove that Fock representation of A is faithful.

In the Section 2 we study the unbounded representations of (GCCR) for several particular choices of parameters.

1 The universal bounded representations

The bounded representations of (1) were studied in [2]. Let π be irreducible bounded representation of GCCR. Denote $\pi(a_i)$ by A_i and consider the polar decomposition $A_i^* = U_i^* C_i$, $i = 1, \ldots, d$. Let us fix some subset $\Phi \subset \{1, \ldots, d\}$. Put $\Theta = \cup_{j\in\Phi}[j, k_j]$. Then the irreducible bounded representation of GCCR corresponding to Φ has the following form

$$C_i = U_i = 0, \qquad i \in \Theta,$$
$$C_i^2 = \bigotimes_{j<i, j\notin\Theta} d_{ij} \otimes D_i^2 \bigotimes_{j>i, j\notin\Theta} 1, \qquad i \notin \Phi,$$
$$U_i^* = \bigotimes_{j<i, j\notin\Theta} U_{ij} \otimes S \bigotimes_{j>i, j\notin\Theta} 1, \qquad i \notin \Phi,$$
$$C_i^2 = \frac{1}{1-\alpha_i} \bigotimes_{j<i, j\notin\Theta} d_{ij} \otimes \bigotimes_{j\geq i, j\notin\Theta} 1, \qquad i \in \Phi,$$
$$U_i^* = \bigotimes_{j<i, j\notin\Theta} U_{ij} \otimes \bigotimes_{j>i, j\notin\Theta} U_{ij} \otimes \widehat{U}_i, \qquad i \in \Phi,$$

where $d_{ij} \colon l_2(\mathbb{N}) \to l_2(\mathbb{N})$

$$d_{ij} e_n = \alpha_j^{n-1} e_n, \qquad k_j \geq i, \qquad d_{ij} = 1, \qquad k_j < i,$$

$D_i^2 \colon l_2(\mathbb{N}) \to l_2(\mathbb{N})$, $D_i^2 e_n = \frac{1-\alpha_i^{n-1}}{1-\alpha_i} e_n$, $U_{ij} e_n = \lambda_{ij}^{n-1} e_n$ and the family of unitary operators $\widehat{U}_i$, $i \in \Phi$ is irreducible and satisfies the relations

$$\widehat{U}_i \widehat{U}_j = \lambda_{ij} \widehat{U}_j \widehat{U}_i.$$

It can be easily seen that for any bounded representation we have the norm bound

$$\|\pi(a_i^* a_i)\| \leq \frac{1}{1-\alpha_i}.$$

Hence one can construct the universal bounded representation of GCCR, i.e. the C^*-algebra $A_{\alpha,\lambda}$ generated by a_i, a_i^* with norm:

$$\|X\| = \sup_\pi \|\pi(X)\|,$$

where X is any element of a $*$-algebra generated by GCCR, and sup is taken over all irreducible representations of GCCR.

Theorem 1. *The C^*-algebra $A_{\alpha,\lambda}$ is isomorphic to $A_{0,\lambda}$ for any choice of parameters α_i, $i = 1,\dots,d$, $0 < \alpha_i < 1$ where $A_{0,\lambda}$ is generated by partial isometries s_i, $i = 1,\dots,d$ satisfying the relations*

$$s_i^* s_i = 1 - \sum_{j<i:\ k_j \geq i} s_j s_j^*, \qquad s_i^* s_j = 0, \qquad s_j s_i = 0, \qquad i < j, \qquad k_i \geq j,$$

$$s_i^* s_j = \lambda_{ij} s_j s_i^*, \qquad s_j s_i = \lambda_{ij} s_i s_j, \qquad i < j, \qquad k_i < j.$$

For the particular case of TCCR we have the C^*-algebra A_0 generated by the relations

$$s_i^* s_j = \delta_{ij} \left(1 - \sum_{k<i} s_i s_i^* \right).$$

In the following theorem we suppose that the coefficients $\lambda_{ij} = e^{2\pi\theta_{ij}}$ have the additional property that the family $\{1, \theta_{ij}\}$ is linearly independent over $\mathbb{Q}$.

Theorem 2. *The Fock representation of $A_{0,\lambda}$ is faithful.*

For example for C^*-algebra generated by TCCR we have the following faithful realization

$$s_i = \bigotimes_{j<i} (1 - SS^*) \otimes S \otimes \bigotimes_{j>i} 1, \qquad i = 1,\dots,d.$$

Using this realization we compute the K-groups of A_0.

Theorem 3. *$K_0(A_\mu) \simeq \mathbb{Z}$ and $K_1(A_\mu) = \{0\}$.*

Proof. As it was noted above that $A_\mu \simeq C^*(s_i, s_i^*)$, where

$$s_i = \bigotimes_{j<i} (1 - ss^*) \otimes s \otimes \bigotimes_{j>i} 1, \qquad i = 1,\dots,d.$$

Let us consider the case $d = 2$. Let $\widetilde{\mathcal{T}}_0$ be the ideal generated by the element $(1 - ss^*) \otimes (1 - s)$. It is easy to see that $\widetilde{\mathcal{T}}_0 \simeq \mathcal{K} \otimes \mathcal{T}_0$, where $\mathcal{T}_0$ is an ideal in the Toeplitz algebra $\mathcal{T}$ generated by the element $1 - s$. It is known fact that $K_i(\mathcal{T}_0) = \{0\}$. Further, $A_\mu/\widetilde{\mathcal{T}}_0 \simeq \mathcal{T}$, i.e. we have the following short exact sequence

$$0 \longrightarrow \widetilde{\mathcal{T}}_0 \longrightarrow A_\mu \longrightarrow \mathcal{T} \longrightarrow 0.$$

Since $K_0(\mathcal{T}) \simeq \mathbb{Z}$ and $K_1(\mathcal{T}) = \{0\}$, the corresponding six-term exact sequence becomes

$$\begin{array}{ccccc} 0 & \longrightarrow & K_0(A_\mu) & \longrightarrow & \mathbb{Z} \\ \big\downarrow & & & & \big\downarrow \\ 0 & \longleftarrow & K_1(A_\mu) & \longleftarrow & 0 \end{array}$$

In the general case we consider the ideal $\widehat{\mathcal{T}}_0$ generated by the element $\bigotimes_{i=1}^{d-1} (1 - ss^*) \otimes (1 - s)$. Then

$$\widehat{\mathcal{T}}_0 \simeq \bigotimes_{i=1}^{d-1} \mathcal{K} \otimes \mathcal{T}_0 \simeq \mathcal{K} \otimes \mathcal{T}_0$$

and $A_0(d)/\widehat{\mathcal{T}}_0 \simeq A_0(d-1)$. Applying again the six-term sequence corresponding to the

$$0 \longrightarrow \mathcal{K} \otimes \mathcal{T}_0 \longrightarrow A_0(d) \longrightarrow A_0(d-1) \longrightarrow 0$$

and induction on d, we get $K_0(A_0(d)) \simeq \mathbb{Z}$ and $K_1(A_0(d)) = \{0\}$. ■

2 Representations of GCCR

In this section we restrict ourselves by the case $\alpha_i = \mu^2$ for any $i = 1, \dots, d$. To give the classification of irreducible representations of GCCR we need to introduce some notations. Let

$$\Phi = \{1 \le i_1 < i_2 < \cdots < i_m \le d \mid i_j > k_{i_{j-1}}\}.$$

Consider the function $l\colon \Phi \to \mathbb{N}$ such that for any $j \in \Phi$ we have $j \le l(j) \le k_j$. Construct the set Ψ:

$$\Theta = \cup_{j\in\Phi}([j+1, l(j)] \cap \mathbb{Z}),$$
$$\Psi := \{l(j)+1 \mid l(j)+1 \le k_j,\ j \in \Phi\}.$$

Let $F(j) := k_j + 1$, $j = 1, \dots, d-1$. For any $s \in \Psi$ denote by Ψ_s the following set:

$$\Psi_s = \{F^n(s),\ n \in \mathbb{Z}_+\} \cap [s, k_{m(s-1)}],$$

where $l(m(s-1)) = s-1$ for any $s \in \Psi$. Put also $M = \{1 \le j_1 < \cdots < j_t \le d\}$, such that $j_i \notin \cup_{l\in\Phi[l,k_l]}$ and $j_i > k_{j_{i-1}}$, $i = 1, \dots, t$. Finally, let

$$F = \{1, \dots, d\} \setminus \big(\cup_{i\in\Phi}[i, k_i] \cup \cup_{j\in M}[j, k_j]\big).$$

For any $j \in \Psi_s$, $s \in \Psi$ fix some $z_{js} > 0$ and put $\tau_{js} := \big(\mu^2 z_{js}, z_{js}\big]$. Fix any $x_{js} \in \tau_{js}$ and construct the function

$$g(x, x_{js}) = -\left(1-\mu^2\right)x_{js} + \mu^2 x.$$

For any $i \in M$ fix $y_i > \frac{1}{1-\mu^2}$ and set $\tau_i := (1+\mu^2 y_i, y_i]$. For any $x_i \in \tau_i$, $i \in M$ consider the function

$$f(x, x_i) = 1 - \left(1-\mu^2\right)x_i + \mu^2 x.$$

As in the bounded case we give the description of representations of GCCR using the polar decompositions $\pi(a_i^*) = U_i C_i$.

Theorem 4. *The irreducible representations of GCCR have (up to the unitary equivalence) the following form*

$$C_i^2 = \bigotimes_{j<i, j\notin\Theta} d_{ij} \otimes D_i^2 \otimes \bigotimes_{j>i, j\notin\Theta} 1, \qquad i \notin \Phi\cup\Theta,$$
$$U_i^* = \bigotimes_{j<i, j\notin\Theta} U_{ij} \otimes U_i \otimes \bigotimes_{j>i, j\notin\Theta} U_{ij}, \qquad i \notin \Phi\cup\Theta,$$
$$C_i^2 = \frac{1}{1-\mu^2} \bigotimes_{j<i, j\notin\Theta} d_{ij} \otimes D_i^2 \otimes \bigotimes_{j>i, j\notin\Theta} 1, \qquad i \in \Phi,$$
$$U_i^* = \bigotimes_{j<i, j\notin\Theta} U_{ij} \otimes U_i \otimes \bigotimes_{j>i, j\notin\Theta} U_{ij} \otimes \widehat{U}_i, \qquad i \in \Phi,$$
$$C_i^2 = 0, \qquad U_i = 0, \qquad i \in \Theta,$$

where $\{\widehat{U}_i,\ i \in \Phi\}$ form the irreducible representation of higher-dimensional non-commutative torus, i.e.

$$\widehat{U}_i \widehat{U}_j = \lambda_{ij} \widehat{U}_j \widehat{U}_i, \qquad i \ne j$$

and

$$
\begin{aligned}
&D_i^2 = D\left(\mu^2, x_{is}\right) : l_2(\mathbb{Z}) \to l_2(\mathbb{Z}), \qquad x_{is} \in \left(\mu^2 z_{is}, z_{is}\right], \qquad i \in \Psi_s, \qquad s \in \Psi,\\
&\qquad D\left(\mu^2, x_{is}\right) e_n = \mu^{2n} x_{is} e_n, \qquad n \in \mathbb{Z},\\
&D_i^2 = d\left(g^{-n}(0, x_{js})\right) : l_2(\mathbb{Z}_-) \to l_2(\mathbb{Z}_-), \qquad i \in [j+1, k(j)], \qquad j \in \Psi_s,\\
&\qquad d\left(g^{-n}(0, x_{js})\right) e_{-n} = g^{-n}(0, x_{js}) e_{-n}, \qquad n \in \mathbb{Z}_+,\\
&D_i^2 = D\left(f^n(x_i)\right) : l_2(\mathbb{Z}) \to l_2(\mathbb{Z}), \qquad x_i \in \left(1 + \mu^2 y_i, y_i\right], \qquad i \in M,\\
&\qquad D\left(f^n(x_i)\right) e_n = f^n(x_i) e_n, \qquad n \in \mathbb{Z},\\
&D_i^2 = d\left(f^{-n}(0, x_j)\right) : l_2(\mathbb{Z}_-) \to l_2(\mathbb{Z}_-), \qquad i \in [j+1, k_j], \qquad j \in M,\\
&\qquad d\left(f^{-n}(o, x_i)\right) e_{-n} = f^{-n}(0, x_i) e_{-n}, \qquad n \in \mathbb{Z}_+,\\
&D_i^2 = d\left(f^n(0)\right) : l_2(\mathbb{Z}_+) \to l_2(\mathbb{Z}_+), \qquad i \in F,\\
&\qquad d\left(f^n(0)\right) e_n = f^n(0) e_n, \qquad n \in \mathbb{Z}_+
\end{aligned}
$$

and

$$
\begin{aligned}
&U_i^* = U \colon l_2(\mathbb{Z}) \to l_2(\mathbb{Z}), \qquad i \in \Psi_s, \qquad s \in \Psi, \qquad U e_n = e_{n+1},\\
&U_i^* = \widehat{S} \colon l_2(\mathbb{Z}_-) \to l_2(\mathbb{Z}_-), \qquad i \in [j+1, k(j)], \qquad j \in \Psi_s,\\
&\qquad \widehat{S} e_{-n} = e_{-n+1}, \qquad n \in \mathbb{N}, \qquad \widehat{S} e_0 = 0,\\
&U_i^* = U, \qquad i \in M,\\
&U_i^* = \widehat{S}, \qquad i \in [j+1, k_j], \qquad j \in M,\\
&U_i^* = S \colon l_2(\mathbb{N}) \to l_2(\mathbb{N}), \qquad i \in F, \qquad S e_n = e_{n+1}.
\end{aligned}
$$

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On Four Orthogonal Projections that Satisfy the Linear Relation $\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = I$, $\alpha_i > 0$

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In the article we investigate the sets of orthogonal projections which satisfy the linear relation $\sum\limits_{i=1}^{n} \alpha_i P_i = I$, $\alpha_i > 0$, up to unitary equivalence. A problem of unitary classification of four projections that satisfy the linear relation $\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = I$, $\alpha_i > 0$ is considered in [1–4]. We present a new method for solving this problem that is based on functors of Coxeter, which are analogous to those introduced in [5].

Let $\mathfrak{P}_{n,\vec{\alpha}} = \mathbb{C}\langle p_1, p_2, \ldots, p_n \,|\, p_i^2 = p_i = p_i^*, \sum\limits_{i=1}^{n} \alpha_i p_i = e\rangle$ be a $*$-algebra, where the vector $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_i > 0$, $i = 1, \ldots, n$; $A = \sum\limits_{i=1}^{n} \alpha_i$. We study its representations, up to unitary equivalence, in the category of Hilbert spaces. Define Σ_n as a set of $\vec{\alpha}$ such that the category of representations $\operatorname{Rep} \mathfrak{P}_{n,\vec{\alpha}}$ is not empty.

1. Let us consider some properties of $\mathfrak{P}_{n,\vec{\alpha}}$.

Lemma 1. *If $\vec{\alpha} \in \Sigma_n$ then $A \geq 1$.*

Proof. Let π be a representation of the algebra $\mathfrak{P}_{n,\vec{\alpha}}$: $\sum\limits_{i=1}^{n} \alpha_i \pi(p_i) = I$ then $\sum\limits_{i=1}^{n} \alpha_i (I - \pi(p_i)) = (A-1)I$. Since the operator at the left hand-side is positive then $A \geq 1$. ■

Lemma 2. *If $A = 1$ then $\vec{\alpha} \in \Sigma_n$ and the algebra $\mathfrak{P}_{n,\vec{\alpha}}$ has (up to unitary equivalence) only one irreducible representation $\pi : \pi(p_i) = 1$.*

Proof. If $A = 1$ then $\sum\limits_{i=1}^{n} \alpha_i (I - \pi(p_i)) = 0$ and for all $i = 1, \ldots, n$: $\pi(p_i) = I$. ■

Definition 1. The algebra $\mathfrak{P}_{n,\vec{\alpha}}$ and the vector $\vec{\alpha}$ are called reduced if there exists such a number i_0 that for all representations π of the algebra we have $\pi(p_{i_0}) = 0$ or there exists a number j_0 that for all representations π of the algebra we have $\pi(p_{j_0}) = I$.

Remark 1. In the case of mapping of a reduced algebra to its enveloping C^*-algebra the elements p_{i_0} and $p_{j_0} - e$ belong to the $*$-radical, and the corresponding C^*-algebra will be generated by less than n linear connected projections.

Lemma 3. *If $\vec{\alpha} \in \Sigma_n : \exists\, \alpha_{i_0} > 1$ then for all representations π of the algebra $\mathfrak{P}_{n,\vec{\alpha}}$: $\pi(p_{i_0}) = 0$, e.g. the algebra $\mathfrak{P}_{n,\vec{\alpha}}$ is reduced.*

Proof. Take an arbitrary representation π of the algebra $\mathfrak{P}_{n,\vec{\alpha}}$ then $\sum\limits_{i \neq i_0} \alpha_i \pi(p_i) = I - \alpha_{i_0} \pi(p_{i_0})$. The operator at the left-hand side is positive. But the operator at the right-hand side is positive when $\pi(p_{i_0}) = 0$ only. ■

Lemma 4. *If $\vec{\alpha} \in \Sigma_n$ and the algebra $\mathfrak{P}_{n,\vec{\alpha}}$ is not reduced then $A \le n$.*

Proof. If $A > n$, then there exists a number $i_0 : \alpha_{i_0} > 1$ and according to the Lemma 3 the algebra $\mathfrak{P}_{n,\vec{\alpha}}$ will be reduced. ∎

Let $\Sigma_n^1 = \Sigma_n \bigcap (0,1)^n$ e.g. Σ_n^1 consists of such points $\vec{\alpha} \in \Sigma_n$ that $0 < \alpha_i < 1$.

Our aim is to describe the set Σ_n^1 $(1 \le A < n)$ and the set of representations of corresponding algebras. There are reduced and nonreduced ones among such class of algebras.

We define functors S and T (analogy with [5]), which act on the set of categories $\operatorname{Rep}\mathfrak{P}_{n,\vec{\alpha}}$. They are equivalences of categories (if $\operatorname{Rep}\mathfrak{P}_{n,\vec{\alpha}}$ is not empty, then $S(\operatorname{Rep}\mathfrak{P}_{n,\vec{\alpha}})$ (or $T(\operatorname{Rep}\mathfrak{P}_{n,\vec{\alpha}})$) is not empty and they are equivalent).

Let us define the functor T (*functor of hyperbolic reflection*).

Let $\alpha \in \Sigma_n$, $A > 1$, $\pi \in \operatorname{Rep}\mathfrak{P}_{n,\vec{\alpha}}$, then $\sum\limits_{i=1}^{n} \alpha_i \pi(p_i) = I$ and $\sum\limits_{i=1}^{n} \alpha_i (I - \pi(p_i)) = (A-1)I$ or $\sum\limits_{i=1}^{n} \frac{\alpha_i}{A-1}(I - \pi(p_i)) = I$. Define $T(\pi)(p_i) = I - \pi(p_i)$. Thus, we obtain the functor

$$T : \operatorname{Rep}\mathfrak{P}_{n,(\alpha_1,\alpha_2,\ldots,\alpha_n)} \to \operatorname{Rep}\mathfrak{P}_{n,\left(\frac{\alpha_1}{A-1},\frac{\alpha_2}{A-1},\ldots,\frac{\alpha_n}{A-1}\right)}$$

which is defined when $A > 1$.

It is easy to check that this functor is equivalence of categories (the corresponding algebras are isomorphic).

Let us define the functor S (*functor of linear reflection*).

Let $\vec{\alpha} \in \Sigma_n^1$, $\sum\limits_{i=1}^{n} \alpha_i \pi(p_i) = I$ and π be a representation of the algebra $\mathfrak{P}_{n,\vec{\alpha}}$ in the Hilbert space H_0. Since $\pi(p_i)$ is a projection then $\pi(p_i) = \Gamma_i \Gamma_i^*$, where Γ_i is the natural isometry of the space $H_i = \operatorname{Im}\pi(p_i)$ to H_0.

Let $H = H_1 \oplus H_2 \oplus \cdots \oplus H_n$. Define the linear operator $\Gamma : H \to H_0$ that is given by the matrix

$$\Gamma = \begin{pmatrix} \sqrt{\alpha_1}\,\Gamma_1 & \sqrt{\alpha_2}\,\Gamma_2 & \cdots & \sqrt{\alpha_n}\,\Gamma_n \end{pmatrix}.$$

Since $\Gamma\Gamma^* = \sum\limits_{i=1}^{n} \alpha_i \Gamma_i \Gamma_i^* = \sum\limits_{i=1}^{n} \alpha_i \pi(p_i) = I_{H_0}$, Γ^* is a partial isometry from H_0 to H. Let $\hat{H}_0 = (\operatorname{Im}\Gamma^*)^{\perp}$ and Δ^* is the natural isometry of $\hat{H}_0$ to H then $U^* = (\Gamma^*, \Delta^*)$ be a unitary operator from $\hat{H}_0 \oplus H_0$ to H. As $H = H_1 \oplus H_2 \oplus \cdots \oplus H_n$, the operators Δ and U have the Peirce decomposition

$$\Delta = \begin{pmatrix} \sqrt{1-\alpha_1}\Delta_1 & \sqrt{1-\alpha_2}\Delta_2 & \cdots & \sqrt{1-\alpha_n}\Delta_n \end{pmatrix},$$
$$U = \begin{pmatrix} \sqrt{\alpha_1}\,\Gamma_1 & \sqrt{\alpha_2}\,\Gamma_2 & \cdots & \sqrt{\alpha_n}\,\Gamma_n \\ \sqrt{1-\alpha_1}\Delta_1 & \sqrt{1-\alpha_2}\Delta_2 & \cdots & \sqrt{1-\alpha_n}\Delta_n \end{pmatrix}.$$

Since U is a unitary operator and $\Gamma_i^*\Gamma_i = I_{H_i}$, it is easy to obtain that $\Delta_i^*\Delta_i = I_{H_i}$ and $\Delta_i \Delta_i^* = Q_i$ are orthoprojections in the space $\hat{H}_0$. From $\Delta\Delta^* = I_{\hat{H}_0}$ (Δ is an isometry) it follows that $\sum\limits_{i=1}^{n}(1-\alpha_i)\Delta_i\Delta_i^* = I_{\hat{H}_0}$, $\sum\limits_{i=1}^{n}(1-\alpha_i)Q_i = I_{\hat{H}_0}$.

Define $S : \pi \to \hat{\pi}$, where $\hat{\pi}(p_i) = Q_i$. From the condition $\sum\limits_{i=1}^{n}(1-\alpha_i)Q_i = I$ we have $\hat{\pi} \in \operatorname{Ob}\operatorname{Rep}\mathfrak{P}_{n,(1-\alpha_1,1-\alpha_2,\ldots,1-\alpha_n)}$. One can see (in analogy with [5]), that the functor

$$S : \operatorname{Rep}\mathfrak{P}_{n,(\alpha_1,\alpha_2,\ldots,\alpha_n)} \to \operatorname{Rep}\mathfrak{P}_{n,(1-\alpha_1,1-\alpha_2,\ldots,1-\alpha_n)},$$

where $0 < \alpha_i < 1$ (therefore, $0 < A < n$), is an equivalence of categories.

Let π be a representation of the algebra $\mathfrak{P}_{n,\vec{\alpha}}$ in a finite-dimensional space H. We shall call the vector $(d; d_1, d_2, \ldots, d_n)$, where $d = \dim H$, $d_i = \dim \operatorname{Im} \pi(p_i)$, the generalized dimension of the representation π.

The functors T and S induce actions on the set of vectors $\vec{\alpha}$, on sums of their coordinates A and on generalized dimensions of representations of algebras $\mathfrak{P}_{n,\vec{\alpha}}$.

It it easy to check that

$$T(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\frac{\alpha_1}{A-1}, \frac{\alpha_2}{A-1}, \ldots, \frac{\alpha_n}{A-1}\right), \qquad T(A) = \frac{A}{A-1},$$
$$T(d; d_1, d_2, \ldots, d_n) = (d; d - d_1, d - d_2, \ldots, d - d_n),$$
$$S(\alpha_1, \alpha_2, \ldots, \alpha_n) = (1 - \alpha_1, 1 - \alpha_2, \ldots, 1 - \alpha_n), \qquad S(A) = n - A,$$
$$S(d; d_1, d_2, \ldots, d_n) = \left(\sum_{i=1}^{n} d_i - d; d_1, d_2, \ldots, d_n\right).$$

Define the functors of Coxeter as $\Phi^+ = TS$ and $\Phi^- = ST$. Φ^+ is defined when $A < n-1$, $\vec{\alpha} \in \Sigma_n^1$. Φ^- is defined when $A > 1$, $T(\vec{\alpha}) \in (0,1)^n$. Since $T^2 = Id$, $S^2 = Id$, then $\Phi^+\Phi^- = Id$ and $\Phi^-\Phi^+ = Id$.

Let $\Phi^{+(k)} = \Phi^+\Phi^{+(k-1)}$.

Lemma 5. $\lim\limits_{k\to\infty} \Phi^{+(k)}\left(1 + \frac{1}{n-2}\right) = \frac{n-\sqrt{n^2-4n}}{2}$ *and intervals*

$$\left[1, 1 + \tfrac{1}{n-2}\right), \ \left[1 + \tfrac{1}{n-2}, \Phi^+\left(1 + \tfrac{1}{n-2}\right)\right), \ \ldots, \ \left[\Phi^{+(k-1)}\left(1 + \tfrac{1}{n-2}\right), \Phi^{+(k)}\left(1 + \tfrac{1}{n-2}\right)\right), \ \ldots$$

do not intersect and cover the interval $\left[1, \frac{n-\sqrt{n^2-4n}}{2}\right)$.

Proof. It is easy to show that $\Phi^+(1) = 1 + \frac{1}{n-2}$ and the sequence $\Phi^{+(k)}\left(1 + \frac{1}{n-2}\right)$ is increasing. Since it is bounded by 2, the limit a of the sequence exists and it is a fixed point of the map $\Phi^+(A) = 1 + \frac{1}{n-A-1}$. From the equation $1 + \frac{1}{n-a-1} = a$ (taking into account that $a < 2$) we obtain $a = \frac{n-\sqrt{n^2-4n}}{2}$. ∎

Lemma 6. $\vec{\alpha} \in \Sigma_n^1, 0 < A \le \frac{n}{2}$, *if and only if* $T(\vec{\alpha}) \in \Sigma_n^1$ *and* $\frac{n}{2} \le T(A) < n$.

Proof. Obviously, the map S sets one-to-one correspondence between points of Σ_n^1 with the sum $A < n$ and points Σ_n^1 with the sum $n - A$. ∎

Lemma 7. *If* $n - 1 < A < n$ *then* $\vec{\alpha} \notin \Sigma_n^1$.

Proof. If $n-1 < A < n$ then $0 < S(A) < 1$, whence, by the Lemma 1, $S(\vec{\alpha}) \notin \Sigma_n$ and it means that $\vec{\alpha} \notin \Sigma_n^1$. ∎

Lemma 8. *If* $\vec{\alpha} \in \Sigma_n$, $A \ne 1$ *and* $\mathfrak{P}_{n,\vec{\alpha}}$ *is not reduced then* $\frac{\alpha_i}{A-1} \le 1$ *and* $A \ge \frac{n}{n-1}$.

Proof. If there exists a number i_0 that $\frac{\alpha_{i_0}}{A-1} > 1$, then the algebra $\mathfrak{P}_{n,T(\vec{\alpha})}$ will be reduced. Take any representation π of the algebra $\mathfrak{P}_{n,\vec{\alpha}}$. Denote $\hat{\pi}$ as the correspondent representation of the algebra $\mathfrak{P}_{n,T(\vec{\alpha})}$ then by the lemma 3 $\hat{\pi}(p_{i_0}) = 0$, so $\pi(p_{i_0}) = I$ and $\mathfrak{P}_{n,\vec{\alpha}}$ is reduced.

If for all $i : \frac{\alpha_i}{A-1} \le 1$ then $\frac{A}{A-1} \le n$ and from here $A \ge \frac{n}{n-1}$. ∎

2. Now we describe Σ_n^1, when $n = 3$ and $n = 4$.

Lemma 9. *Let* $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \Sigma_3$. *Then for some subset* $J \subseteq \{1,2,3\} : \sum\limits_{i\in J} \alpha_i = 1$ *or* $\alpha_1 + \alpha_2 + \alpha_3 = 2$. *To every pointed subset* J*, there corresponds a unique one-dimensional irreducible representation* π*:* $\pi(p_i) = 1$, $i \in J$*, and* $\pi(p_i) = 0$, $i \notin J$*. If* $\alpha_1 + \alpha_2 + \alpha_3 = 2$ *then, furthermore, the algebra has a unique, up to unitary equivalence, irreducible two-dimensional representation.*

Proof. The proof reduces to an easy computation, when taking into account that an irreducible pair of orthoprojections is a one-dimensionally or unitary equivalent to a pair

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} \tau & \sqrt{\tau-\tau^2} \\ \sqrt{\tau-\tau^2} & 1-\tau \end{pmatrix}, \qquad 0<\tau<1. \qquad \blacksquare$$

Lemma 10. *If $\vec{\alpha} \in \Sigma_4^1$, $0<A<2$, is reduced then the following condition, which we will call the R-condition, is satisfied: $\exists\, J \subset \{1,2,3,4\} : \Sigma_{i\in J}\alpha_i = 1$ or $\exists\, \alpha_{i_0} : 2-A = \alpha_{i_0}$.*

Proof. There are two possible cases.

1) Let $\pi(p_{i_0}) = 0$ then $\sum\limits_{i\neq i_0} \alpha_i \pi(p_i) = I$. Let $\vec{\alpha}'$ be obtained from $\vec{\alpha}$ by omitting the coordinate α_{i_0}. Obviously, $\vec{\alpha}' \in \Sigma_3$. So $\sum\limits_{i\in J}\alpha_i = 1$, for some subset $J \subset \{1,2,3,4\}\setminus\{i_0\}$, (if $\sum\limits_{i\neq i_0}\alpha_i = 2$, then $A>2$).

2) If for all $\pi : \pi(p_{i_0}) = I$ then $\sum\limits_{i\neq i_0}\alpha_i\pi(p_i) = (1-\alpha_{i_0})I$. The operator at the left hand-side is positive. From here $\alpha_{i_0} \le 1$. If $\alpha_{i_0} = 1$, then the R-condition is satisfied, else $\sum\limits_{i\neq i_0} \frac{\alpha_i}{1-\alpha_{i_0}}\pi(p_i) = I$. From the previous lemma we have either: $a)$ $\sum\limits_{i\in J}\frac{\alpha_i}{1-\alpha_4} = 1$, for some subset $J \subset \{1,2,3,4\}\setminus\{i_0\}$, hence $\sum\limits_{i\in J}\alpha_i + \alpha_4 = 1$ or $b)$ $\frac{\alpha_1}{1-\alpha_4} + \frac{\alpha_2}{1-\alpha_4} + \frac{\alpha_3}{1-\alpha_4} = 2$, $\alpha_1+\alpha_2+\alpha_3 = 2(1-\alpha_4)$ and $2-A = \alpha_4$. $\blacksquare$

Note, that if $\vec{\alpha}$ satisfies R-condition then $\vec{\alpha}$ is not necessary reduced.

Lemma 11. *If $\vec{\alpha} \in \Sigma_4 \setminus \Sigma_4^1$ then $T(\vec{\alpha})$ satisfies R-condition.*

Proof. From the condition $\vec{\alpha} \in \Sigma_4 \setminus \Sigma_4^1$, we obtain $\alpha_{i_0} \ge 1$ for some i_0. Suppose $\alpha_{i_0} > 1$, $\pi \in \operatorname{Rep}\mathfrak{P}_{4,T(\vec{\alpha})}$ then, by the Lemma 3, $T(\pi)(p_{i_0}) = 0$. From here $\pi(p_{i_0}) = I$, so $\vec{\alpha}$ is reduced.

Assume $\alpha_{i_0} = 1$. From $T(\vec{\alpha}) = \left(\frac{\alpha_1}{A-1}, \frac{\alpha_2}{A-1}, \frac{\alpha_3}{A-1}, \frac{\alpha_4}{A-1}\right) = \left(\frac{\alpha_1}{\sum\limits_{i\neq i_0}\alpha_i}, \frac{\alpha_2}{\sum\limits_{i\neq i_0}\alpha_i}, \frac{\alpha_3}{\sum\limits_{i\neq i_0}\alpha_i}, \frac{\alpha_4}{\sum\limits_{i\neq i_0}\alpha_i}\right)$, the sum $\sum\limits_{j\neq i_0}\left(\frac{\alpha_j}{\sum\limits_{i\neq i_0}\alpha_i}\right) = 1$, so $T(\vec{\alpha})$ satisfies R-condition. $\blacksquare$

From Lemmas 2, 3, 8, 10, it follows

Lemma 12. *If $1 \le A < 1 + \frac{1}{n-2}\Big|_{n=4} = \frac{3}{2}$ then $\vec{\alpha}$ satisfy R-condition.*

Using the lemmas proved above, we obtain:

Theorem 1. *Let $\vec{\alpha} = (\alpha_1,\alpha_2,\alpha_3,\alpha_4)$, $0<\alpha_i<1$, $A = \sum\limits_{i=1}^{4}\alpha_i$, Σ_4^1 be the set of such $\vec{\alpha}$ that the algebra $\mathfrak{P}_{4,\vec{\alpha}}$ has a nonzero representation.*

1) Dimensions of all irreducible representations of the algebra $\mathfrak{P}_{4,\vec{\alpha}}$ are finite.

2) If $A=1$ then $\vec{\alpha} \in \Sigma_4^1$ and the corresponding algebra $\mathfrak{P}_{4,\vec{\alpha}}$ has a unique irreducible representation π, which is a one-dimensional representation and $\pi(p_i) = 1$.

3) If $A=2$ then $\vec{\alpha} \in \Sigma_4^1$ and all irreducible representations has dimension one or two (their description see in [4]).

4) The functor S is equivalence of categories of representations of "symmetry" algebras $\mathfrak{P}_{4,(\alpha_1\alpha_2,\alpha_3,\alpha_4)}$ and $\mathfrak{P}_{4,(1-\alpha_1,1-\alpha_2,1-\alpha_3,1-\alpha_4)}$, $\vec{\alpha} \in \Sigma_4^1$, with the center of symmetry $A=2$.

5) Every point $\vec{\alpha} \in \Sigma_4^1, 1<A<2$, or satisfies R-condition or $\Phi^-(\alpha)$ belongs to Σ_4^1.

6) $\vec{\alpha} \in \Sigma_4^1$, $1<A<2$ if and only if $\Phi^{-(k)}(\vec{\alpha})$ satisfy R-condition for some k.The number k is bounded by $N : \Phi^{-(N)}(A) \in [1,\frac{3}{2})$. The functor $\Phi^{-(k)}$ is equivalence of categories of representations of algebra $\mathfrak{P}_{n,\vec{\alpha}}$ and reduced algebra $\mathfrak{P}_{n,\Phi^{-(k)}(\vec{\alpha})}$.

The theorem allows us to reduce the solution of the problem about belonging of a point $\vec{\alpha}$ to Σ_4^1 to verifying R-condition for some another point.

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Realizations of Real 4-Dimensional Solvable Decomposable Lie Algebras

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We construct a complete set of inequivalent realizations of real 4-dimensional solvable decomposable Lie algebras in vector fields on a space of an arbitrary (finite) number of variables.

Realizations of Lie algebras in vector fields are applied, in particular, for integrating of ordinary differential equations, group classification of partial differential equations, classification of gravity fields of a general form with respect to motion groups or groups of conformal transformations. In spite of importance for applications, the problem of complete description of realizations have not been solved even for cases when either the dimension of algebras or the dimension of realization space is a fixed small integer. An exception is Lie's classification of all possible Lie groups of point and contact transformations acting on a two-dimensional complex space without fixed points [1], which is equivalent to classification of all possible realizations of Lie algebras in vector fields on a two-dimensional complex space.

The necessary step to classify realizations of low-dimensional Lie algebras is classification of these algebras, i.e. classification of possible commutative relations between basis elements. All the possible complex Lie algebras of dimension less than 4 were already obtained by S. Lie [2]. L. Bianchi investigated three-dimensional real Lie algebras [3]. Complete and correct classification of four-dimensional real Lie algebras was firstly obtained by G.M. Mubarakzyanov [4].

C. Wafo Soh and F.M. Mahomed [5] used Mubarakzyanov's results to classify realizations of three- and four-dimensional real Lie algebras in the space of three variables and to describe systems of two second-order ODEs admitting real four-dimensional real symmetry Lie algebras, but unfortunately their paper contains a number of misprints and incorrect statements. Therefore, this classification cannot be considered as complete.

Preliminary classification of realizations of solvable three-dimensional Lie algebras in the space of any (finite) number of variables was given in [6]. In this paper we present a complete set of inequivalent realizations for real 4-dimensional solvable decomposable Lie algebras in vector fields on a space of an arbitrary (finite) number n of variables $x = (x_1, x_2, \ldots, x_n)$. Analogous results for indecomposable algebras in the case $n = 4$ have been obtained in [7].

The technics of classification is the following. For each algebra A from Mubarakzyanov's classification of abstract four-dimensional Lie algebras [4] we find the automorphism group $G(A)$ and the set of *megaideals* of A, i.e. the ideals invariant with respect to $G(A)$. Knowledge of the megaideals is important to construct realizations and to prove their inequivalence in a simpler way. Then, we take four linearly independent vector fields of the general form $e_i = \xi^{ia}(x)\partial_a$, where $\mathrm{rank}(\xi^{ia}) = 4$, $\partial_a = \partial/\partial x_a$, and demand from them to satisfy commutative relations of A. (Our notions of low-dimensional algebras, choice of their basis elements, and, consequently, the form of commutative relations coincide with Mubarakzyanov's ones.) As a result, we obtain a system of first-order PDEs for the coefficients ξ^{ia} and integrate it, considering all the possible cases. For each case we transform the found solution to the simplest form, using local diffeomorphisms of the space of x and automorphisms of A.

Consideration is essentially simplified if it is taken into account that any four-dimensional algebra contains a three-dimensional ideal. We can use classification of realizations of three-dimensional algebras with respect to local diffeomorphisms of the space of x, extending them to realizations of four-dimensional algebras by means of joining the fourth vector field. Then we obtain a system of first-order PDEs only for the coefficients ξ^{4a}.

Table 1. Realizations of real decomposable solvable four-dimensional Lie algebras.

Algebra	N	Realization
$A_{3.1}\oplus A_1$	1	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ \partial_2$
	2	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_2+x_5\partial_3,\ \partial_2$
$[e_2,e_3]=e_1$	3	$\partial_1,\ \partial_3,\ x_3\partial_1+\varphi(x_4)\partial_2+\psi(x_4)\partial_3,\ \partial_2$
	4	$\partial_1,\ \partial_3,\ x_3\partial_1+\partial_4,\ x_2\partial_1$
	5	$\partial_1,\ \partial_3,\ x_3\partial_1+x_4\partial_3,\ x_2\partial_1$
	6	$\partial_1,\ \partial_3,\ x_3\partial_1+\varphi(x_2)\partial_3,\ x_2\partial_1$
$A_{3.2}\oplus A_1$	1	$\partial_1,\ \partial_2,\ (x_1+x_2)\partial_1+x_2\partial_2+\partial_3,\ \partial_4$
	2	$\partial_1,\ \partial_2,\ (x_1+x_2)\partial_1+x_2\partial_2+\partial_3,\ x_4\partial_3$
$[e_1,e_3]=e_1,$	3	$\partial_1,\ \partial_2,\ (x_1+x_2)\partial_1+x_2\partial_2,\ \partial_3$
$[e_2,e_3]=e_1+e_2$	4	$\partial_1,\ \partial_2,\ (x_1+x_2)\partial_1+x_2\partial_2+\partial_3,\ x_4e^{x_3}(x_3\partial_1+\partial_2)$
	5	$\partial_1,\ \partial_2,\ (x_1+x_2)\partial_1+x_2\partial_2+\partial_3,\ e^{x_3}(x_3\partial_1+\partial_2)$
	6	$\partial_1,\ \partial_2,\ (x_1+x_2)\partial_1+x_2\partial_2+\partial_3,\ e^{x_3}\partial_1$
	7	$\partial_1,\ x_2\partial_1,\ x_1\partial_1-\partial_2,\ \partial_3$
	8	$\partial_1,\ x_2\partial_1,\ x_1\partial_1-\partial_2,\ x_3e^{-x_2}\partial_1$
	9	$\partial_1,\ x_2\partial_1,\ x_1\partial_1-\partial_2,\ e^{-x_2}\partial_1$
$A_{3.3}\oplus A_1$	1	$\partial_1,\ \partial_2,\ x_1\partial_1+x_2\partial_2+\partial_3,\ \partial_4$
	2	$\partial_1,\ \partial_2,\ x_1\partial_1+x_2\partial_2+\partial_3,\ x_4\partial_3$
$[e_1,e_3]=e_1,$	3	$\partial_1,\ \partial_2,\ x_1\partial_1+x_2\partial_2,\ \partial_3$
$[e_2,e_3]=e_2$	4	$\partial_1,\ \partial_2,\ x_1\partial_1+x_2\partial_2+\partial_3,\ e^{x_3}(\partial_1+x_4\partial_2)$
	5	$\partial_1,\ \partial_2,\ x_1\partial_1+x_2\partial_2+\partial_3,\ e^{x_3}\partial_1$
	6	$\partial_1,\ x_2\partial_1,\ x_1\partial_1+\partial_3,\ \partial_4$
	7	$\partial_1,\ x_2\partial_1,\ x_1\partial_1+\partial_3,\ x_4\partial_3$
	8	$\partial_1,\ x_2\partial_1,\ x_1\partial_1+\partial_3,\ \varphi(x_2)\partial_3$
	9	$\partial_1,\ x_2\partial_1,\ x_1\partial_1+\partial_3,\ e^{x_3}\partial_1$
$A_{3.4}\oplus A_1$	1	$\partial_1,\ \partial_2,\ x_1\partial_1+ax_2\partial_2+\partial_3,\ \partial_4$
	2	$\partial_1,\ \partial_2,\ x_1\partial_1+ax_2\partial_2+\partial_3,\ x_4\partial_3$
$[e_1,e_3]=e_1,$	3	$\partial_1,\ \partial_2,\ x_1\partial_1+ax_2\partial_2,\ \partial_3$
$[e_2,e_3]=ae_2,$	4	$\partial_1,\ \partial_2,\ x_1\partial_1+ax_2\partial_2+\partial_3,\ e^{x_3}\partial_1+x_4e^{ax_3}\partial_2$
$-1\le a<1,\ a\ne 0$	5	$\partial_1,\ \partial_2,\ x_1\partial_1+ax_2\partial_2+\partial_3,\ e^{x_3}\partial_1+e^{ax_3}\partial_2$
	6	$\partial_1,\ \partial_2,\ x_1\partial_1+ax_2\partial_2+\partial_3,\ e^{x_3}\partial_1$
	7	$\partial_1,\ \partial_2,\ x_1\partial_1+ax_2\partial_2+\partial_3,\ e^{ax_3}\partial_1,\quad 0<\lvert a\rvert<1$
	8	$\partial_1,\ x_2\partial_1,\ x_1\partial_1+(1-a)x_2\partial_2,\ \partial_3$
	9	$\partial_1,\ x_2\partial_1,\ x_1\partial_1+(1-a)x_2\partial_2,\ x_3\lvert x_2\rvert^{\frac{1}{1-a}}\partial_1$
	10	$\partial_1,\ x_2\partial_1,\ x_1\partial_1+(1-a)x_2\partial_2,\ \lvert x_2\rvert^{\frac{1}{1-a}}\partial_1$
$A_{3.5}\oplus A_1$	1	$\partial_1,\ \partial_2,\ (bx_1+x_2)\partial_1+(-x_1+bx_2)\partial_2+\partial_3,\ \partial_4$
	2	$\partial_1,\ \partial_2,\ (bx_1+x_2)\partial_1+(-x_1+bx_2)\partial_2+\partial_3,\ x_4\partial_3$
$[e_1,e_3]=be_1-e_2,$	3	$\partial_1,\ \partial_2,\ (bx_1+x_2)\partial_1+(-x_1+bx_2)\partial_2,\ \partial_3$
$[e_2,e_3]=e_1+be_2,$	4	$\partial_1,\ \partial_2,\ (bx_1+x_2)\partial_1+(-x_1+bx_2)\partial_2+\partial_3,\ x_4e^{bx_3}(\cos x_3\partial_1-\sin x_3\partial_2)$
$b\ge 0$	5	$\partial_1,\ \partial_2,\ (bx_1+x_2)\partial_1+(-x_1+bx_2)\partial_2+\partial_3,\ e^{bx_3}(\cos x_3\partial_1-\sin x_3\partial_2)$
	6	$\partial_1,\ x_2\partial_1,\ (b-x_2)x_1\partial_1-(1+x_2^2)\partial_2,\ \partial_3$
	7	$\partial_1,\ x_2\partial_1,\ (b-x_2)x_1\partial_1-(1+x_2^2)\partial_2,\ x_3\sqrt{1+x_2^2}\,e^{-b\arctan x_2}\partial_1$
	8	$\partial_1,\ x_2\partial_1,\ (b-x_2)x_1\partial_1-(1+x_2^2)\partial_2,\ \sqrt{1+x_2^2}\,e^{-b\arctan x_2}\partial_1$

Continuation of of Table 1.

Algebra	N	Realization
$A_{2.2} \oplus 2A_1$	1	$\partial_1,\ x_1\partial_1+\partial_4,\ \partial_2,\ \partial_3$
	2	$\partial_1,\ x_1\partial_1+x_4\partial_2+x_5\partial_3,\ \partial_2,\ \partial_3$
$[e_1,e_2]=e_1$	3	$\partial_1,\ x_1\partial_1+x_4\partial_2+\psi(x_4)\partial_3,\ \partial_2,\ \partial_3$
	4	$\partial_1,\ x_1\partial_1,\ \partial_2,\ \partial_3$
	5	$\partial_1,\ x_1\partial_1+x_3\partial_3,\ \partial_2,\ x_3\partial_1+x_4\partial_2$
	6	$\partial_1,\ x_1\partial_1+x_3\partial_3,\ \partial_2,\ x_3\partial_1$
	7	$\partial_1,\ x_1\partial_1+\partial_4,\ \partial_2,\ x_3\partial_2$
	8	$\partial_1,\ x_1\partial_1+x_4\partial_2,\ \partial_2,\ x_3\partial_2$
	9	$\partial_1,\ x_1\partial_1+\theta(x_3)\partial_2,\ \partial_2,\ x_3\partial_2$
	10	$\partial_1,\ x_1\partial_1+x_2\partial_2+x_3\partial_3,\ x_2\partial_1,\ x_3\partial_1$
$A_{2.1} \oplus A_{2.1}$	1	$\partial_1,\ x_1\partial_1+\partial_3,\ \partial_2,\ x_2\partial_2+\partial_4$
	2	$\partial_1,\ x_1\partial_1+\partial_3,\ \partial_2,\ x_2\partial_2+x_4\partial_3$
$[e_1,e_2]=e_1,$	3	$\partial_1,\ x_1\partial_1+\partial_3,\ \partial_2,\ x_2\partial_2+C\partial_3,\ \lvert C\rvert\le 1$
$[e_3,e_4]=e_3$	4	$\partial_1,\ x_1\partial_1+x_3\partial_2,\ \partial_2,\ x_2\partial_2+x_3\partial_3$
	5	$\partial_1,\ x_1\partial_1,\ \partial_2,\ x_2\partial_2$
	6	$\partial_1,\ x_1\partial_1+x_2\partial_2,\ x_2\partial_1,\ -x_2\partial_2+\partial_3$
	7	$\partial_1,\ x_1\partial_1+x_2\partial_2,\ x_2\partial_1,\ -x_2\partial_2$
$4A_1$	1	$\partial_1,\ \partial_2,\ \partial_3,\ \partial_4$
	2	$\partial_1,\ \partial_2,\ \partial_3,\ x_4\partial_1+x_5\partial_2+x_6\partial_3$
	3	$\partial_1,\ \partial_2,\ \partial_3,\ x_4\partial_1+x_5\partial_2+\lambda(x_4,x_5)\partial_3$
	4	$\partial_1,\ \partial_2,\ \partial_3,\ x_4\partial_1+\varphi(x_4)\partial_2+\psi(x_4)\partial_3$
	5	$\partial_1,\ \partial_2,\ x_3\partial_1+x_4\partial_2,\ x_5\partial_1+x_6\partial_2$
	6	$\partial_1,\ \partial_2,\ x_3\partial_1+x_4\partial_2,\ x_5\partial_1+\theta(x_3,x_4,x_5)\partial_2$
	7	$\partial_1,\ \partial_2,\ x_3\partial_1+\varphi(x_3,x_4)\partial_2,\ x_4\partial_1+\psi(x_3,x_4)\partial_2$
	8	$\partial_1,\ \partial_2,\ x_3\partial_1+\varphi(x_3)\partial_2,\ \lambda(x_3)\partial_1+\psi(x_3)\partial_2$
	9	$\partial_1,\ x_2\partial_1,\ x_3\partial_1,\ x_4\partial_1$
	10	$\partial_1,\ x_2\partial_1,\ x_3\partial_1,\ \lambda(x_2,x_3)\partial_1$
	11	$\partial_1,\ x_2\partial_1,\ \varphi(x_2)\partial_1,\ \psi(x_2)\partial_1$

We plan to publish our results on complete classification of realizations for real Lie algebras of dimensions less than 5 in vector fields on a space of an arbitrary (finite) number of variables in the near future, giving detailed description of the technics of classification and a number of applications of obtained realizations to theory of differential invariants, integrating of ODEs and group classification of PDEs.

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Positive Conjugacy for Simple Dynamical Systems

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In the article the question of topological conjugacy is considered, which provides sufficient condition for the algebras isomorphism. The concept of positive conjugacy for some classes of simple dynamical systems is presented.

1 Introduction

It is well-known that dynamical systems play important role in the representation theory of C^*-algebras. In the book by Yu.S. Samoilenko and V.L. Ostrovskii (see [1]) some results with respect to connection between the theory of representations of $*$-algebras given by generators and relations with, generally speaking, non-bijective dynamical systems were presented. In recent paper (see [2]) the issue of description of isomorfism classes of C^*-algebras assosiated with $SU \cap \mathcal{F}_{2^n}$-mappings has been considered. The question of classification of C^*-algebras connected with dynamical systems up to isomorphism leads to studying conjugacy of dynamical systems on the set of positive orbits. In the present paper this question is considered for simple dynamical systems.

2 Topological conjugacy for simple unimodal dynamical systems

First we recall the necessary material from [4, 5, 6] and the results about conjugacy for $SU \cap \mathcal{F}_{2^n}$-mappings (from [3]).

Mapping $f \in C(I, I)$, $I = [0, 1]$ is called unimodal if there is unique extreme point $c \in (0, 1)$ and f is homeomorphism on the intervals $J_1 = [0, c]$, $J_2 = [c, 1]$. We consider unimodal mappings $f \in C(I, I)$ such that $f(0) = f(1) = 0$ and extreme point c is point of maximum. Let $f \in C^3(I, I)$. Schwarzian derivative is defined by the formula

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2, \quad \text{for } x \text{ such that } f'(x) \neq 0.$$

By SU we mean a class of unimodal mappings on interval I with $Sf(x) < 0$ for all x different from c.

Dynamical system is called simple if every its trajectory is periodic or asymptoticly periodic. A simple dynamical system can have only cycles of periods 2^k, $k = 0, 1, 2, \ldots$ [4, p. 64]. By necessary and sufficient condition of a simplicity of dynamical system is $\mathrm{Per}\, f = \bigcup\limits_{n=1}^{\infty} \{x : f^n(x) = x\}$ is closed. Denote $\mathcal{F}_n = \{f \in C(I, I) : \mathrm{Per}\, f = \mathrm{Fix}\, f^n\}$. The class $\mathcal{F}_n$ consists of mappings for which the period of each cycle is not greater than n and is a divisor of n. In fact n can be only the power of 2. It is known that for $f \in C^{(1)}(I, I)$: $f \in \mathcal{F}_{2^k} \Leftrightarrow (f, I)$ is a simple dynamical system.

Cycle $B = \{\beta_1, \ldots, \beta_m\}$ is called attractive if there is a neighborhood U of B such that $f(U) \subseteq U$ and $\bigcap\limits_{i>0} f^i(U) = B$. Cycle B is called repellent if there is a neighborhood U such that for every $x \in U \setminus B$ there is integer $k > 0$ and $f^{(k)}(x) \notin U$.

By the positive orbit of dynamical system (f, I) we will mean a sequence $\delta = (x_k)_{k\in\mathbb{Z}}$ such that $f(x_k) = x_{k+1}$ and $x_k > 0$ for all integer k. Unilateral positive orbit is a sequence $\delta = (x_k)_{k\in\mathbb{N}}$ (Fock orbit) such that $x_1 = 0$ and $f(x_k) = x_{k+1}$, $x_k > 0$ for $k > 1$ or $\delta = (x_{-k})_{k\in\mathbb{N}}$ (anti-Fock orbit) such that $x_{-1} = 0$ and $f(x_k) = x_{k+1}$, $x_k > 0$ for $k < -1$. Define $\mathrm{Orb}_+(f, I)$ be the set of all positive orbits on interval I.

We say that two maps $f, g : X \to X$ are topologically conjugate if there exists a homeomorphism $h : X \to X$ such that $h \circ f = g \circ h$. This implies that $h \circ f^n = g^n \circ h$ for every integer n.

Further we will need some concepts of symbolic dynamics. By the address of a point $x \in I$ we mean the value $A(x) = \begin{cases} J_s, & \text{if } x \in J_s \text{ and } x \neq c, \ s = 1, 2; \\ c, & \text{if } x = c. \end{cases}$

The itinerary of a point x is the sequence of addresses

$$A_f(x) = \big(A(x), A(f(x)), A(f^2(x)), \ldots\big) = (A_0, A_1, A_2, \ldots).$$

Define the sign of an interval J_s as $\varepsilon(J_s) = (-1)^{s+1}$, $\varepsilon(c) = 0$ and put $\theta_f(x) = (\theta_0, \theta_1, \theta_2, \ldots)$ where $\theta_0 = \varepsilon_0$, $\theta_1 = \varepsilon_0\varepsilon_1$, …, $\theta_n = \varepsilon_0\varepsilon_1 \cdots \varepsilon_n$, …, $\varepsilon_i = \varepsilon(A_i)$. By the dynamical coordinate of a point $x \in I$ we mean a formal power series $\theta(x) = \sum_{i=0}^{\infty} \theta_i(x)t^i$. Series $\nu_f = \theta(c^-) = \lim_{x\uparrow c} \theta(x)$ is called kneading invariant of f.

We will need the following theorem from [7]:

Theorem 1. *Let $f, g \in SU$, $\nu_f = \nu_g$. Then*

a) if the series ν_f is nonperiodic then f and g are topologically conjugate;

b) if the series ν_f is periodic of period n then f and g have an attractive or neutral trajectory of period n or $n/2$; moreover f and g are topologically conjugate when these trajectories are of the same type (i.e. simultaneously either attractive or neutral) and corresponding points of these trajectories have the same dynamical coordinates.

Theorem 2. *Let (f_1, I), (f_2, I) be dynamical systems such that $(f_i, I) \in SU \cap \mathcal{F}_{2^n}$, $i = \{1, 2\}$, c_i is the greatest point of local maximum of function $f_i^{2^n}$. Then* $\operatorname{sign}(f_1^{2^n}(c_1) - c_1) = \operatorname{sign}(f_2^{2^n}(c_2) - c_2)$ *if and only if f and g are topologically conjugate.*

Proof. We need only to verify that conditions the previous theorem are satisfied. As we can uniquely define a series ν_f when $A_f(c^-)$ is given we consider $A_f(c^-)$ instead of ν_f.

At first let us demonstrate the statement in case $n = 0$, i.e. $f \in SU \cap \mathcal{F}_1$. In this case the dynamical system has one or two fixed points and does not have cycles of period greater or equal 2. It is easy to see that if f has only one fixed point ($s_0 = 0$) then $A_f(c^-) = (+, +, \ldots)$, $A_f(0) = (+, +, \ldots)$ and $\operatorname{sign}(f(c) - c) = -1$. Assume that f has two fixed points $s_0 = 0$ and $0 < s_1 < 1$. In this case s_0 is repellent, s_1 is attractive and there are three alternatives:

1) $s_1 < c$ (i.e. $f(c) - c < 0$);

2) $s_1 = c$ (i.e. $f(c) - c = 0$);

3) $s_1 > c$ (i.e. $f(c) - c > 0$).

In the case 1) and 2): $A_f(c^-) = (+, +, \ldots)$, hence for every $f_1, f_2 \in SU \cap \mathcal{F}_1$ fulfilling condition 1) or 2): $\nu_{f_1} = \nu_{f_2} = (+, +, \ldots)$. If $s_1 > c$ then $A_f(c^-) = (+, -, -, \ldots)$. Hence for every $f_1, f_2 \in SU \cap \mathcal{F}_1$ satisfying condition 3): $\nu_{f_1} = \nu_{f_2} = (+, -, +, -, \ldots)$. And also if f_1 and f_2 simultaneously satisfy one of the conditions 1), 2), 3) then dynamical coordinates of their attractive periodic trajectories coinside:

1) $A_f(s_1) = (+, +, \ldots)$, $\theta_f(s_1) = (+, +, \ldots)$;

2) $A_f(s_1) = (0, 0, \ldots)$, $\theta_f(s_1) = (0, 0, \ldots)$;

3) $A_f(s_1) = (-, -, \ldots)$, $\theta_f(s_1) = (-, +, -, +, \ldots)$.

Thus by previous theorem if f_1 and f_2 simultaneously satisfy one of the conditions 1), 2), 3) then they are topologically conjugate.

We now turn to the case $n \geq 1$. Let us consider the mapping $g(x) = f^2(x)$ and let a and d be preimages of s_1 under g such that $a < s_1 < d$ and $[a, d]$ contains no other preimages of s_1 (in other words a and d are the closest preimages to s_1). Let $I_1 = [a, s_1]$, $I_2 = [s_1, d]$. By simple calculations we have $f(I_1) \subseteq I_2$, $f(I_2) \subseteq I_1$, i.e. intervals I_1, I_2 are invariant under function g.

Let us note that dynamical system (g, I_2) is simple with unimodal g and $Sf < 0$ when $x \neq c'$ (c' be the point of local maximum of function g on I_2). Moreover (g, I_2) is conjugate to dynamical system $(\tilde{f}, [0, 1])$ where $\tilde{f} = f^2((d - s_1)x + s_1) - s_1$ and satisfying all the conditions of the theorem.

Let us consider the behavior of the trajectory $f^m(c^-)$, $m \geq 0$. $f^0(c^-) = c^- \in J_1$ hence $\varepsilon\left(f^0(c^-)\right) = +1$. Since $f(I_1) \subseteq I_2$, $f(I_2) \subseteq I_1$ we have

$$f^m(c^-) \subset I_2, \quad \text{if} \quad m = 2k + 1, \qquad f^m(c^-) \subset I_1, \quad \text{if} \quad m = 2k,$$

hence $\varepsilon\left(f^{2k+1}(c^-)\right) = -1$. In the case $m = 2k$ we have $f^{2k}(c^-) \subset I_1$ and since $c = f(c')$ then

$$f\left(f^{2k+1}(c^-)\right) \subset I_1 \cap J_1, \quad \text{if} \quad f^{2k+1}(c^-) > c', \quad f\left(f^{2k+1}(c^-)\right) \subset I_1 \cap J_2, \quad \text{if} \quad f^{2k+1}(c^-) < c'.$$

Thus the sign of $f^{2k}(c^-)$ is calculated by the recursive formula $\varepsilon\left(f^{2k}(c^-)\right) = -\varepsilon\left(g^k(c'^-)\right)$. And if c_{2^n} is the greatest point of local maximum of function f^{2^n} then the value $A_f(c^-)$ is uniquely defined by $A_{f^{2^n}}(c_{2^n}^-)$, i.e. by value sign $\left(f^{2^n}(c_{2^n}) - c_{2^n}\right)$.

And also it is follows from above that the dynamical coordinate of attractive periodical trajectory also uniquely defined by this value. ■

Corollary 1. *For every n there is no more than three isomorphism classes of enveloping C^*-algebras (see [2]).*

Corollary 2. *Let $(f_1, I), (f_2, I) \in SU \cap \mathcal{F}_{2^n}$ and* $\operatorname{sign}\left(\mu(B^1_{2^n})\right) = \operatorname{sign}\left(\mu(B^2_{2^n})\right)$. *Then $f_1 \sim f_2$.*

3 Positive conjugacy

Define the support of dynamical system (f, I) to be the union of positive orbits $X = X(f, I) = \{x \in \delta \mid \delta \in \mathrm{Orb}_+(f, I)\}$.

Definition 1. We will say that two maps $f_1 : [0, a_1] \to [0, a_1]$ and $f_2 : [0, a_2] \to [0, a_2]$ are positively conjugate if they are topologically conjugate on their supports, i.e. $X_1 = X(f_1, [0, a_1])$, $X_2 = X(f_2, [0, a_2])$ and there exist a homeomorphism $\varphi : X_1 \to X_2$ such that $\varphi \circ f_1 = f_2 \circ \varphi$.

Proposition 1. *If $f_1 : [0, a_1] \to [0, a_1]$ and $f_2 : [0, a_2] \to [0, a_2]$ are topologically conjugate then they are positively conjugate.*

Proof. Since f_1 and f_2 are conjugate then there exist a homeomorphism $\varphi : [0, a_1] \to [0, a_2]$ and $\varphi \circ f_1(x) = f_2 \circ \varphi(x)$ for all $x \in [0, a_1]$. Let us show that $\varphi(X_1) = X_2$. Indeed, if $x \in X_1$ then there exist $\delta \in \mathrm{Orb}_+(f_1, [0, a_1])$ such that $x \in \delta$. Since $\varphi(\delta) \in \mathrm{Orb}_+(f_2, [0, a_2])$ (where $\varphi(\delta) = (\varphi(x_k))_{k \in \mathbb{Z}}$) we have $\varphi(x) \in X_2$, i.e. $\varphi(X_1) \subseteq X_2$. Considering φ^{-1} analogiously we get $\varphi^{-1}(X_2) \subseteq X_1$ and consequently $X_2 \subseteq \varphi(X_1)$. Obviously $\varphi|_{X_1}$ is homeomorphism on X_2. ■

The converse statement to proposition 1 is not true in general. Let us consider the notion of positive conjugacy for the class $SU \cap \mathcal{F}_{2^n}$-maps. In the case $n = 0$ two maps can be positively conjugate but not topologically conjugate.

Proposition 2. *Let $f_1, f_2 \in SU \cap \mathcal{F}_1$, c_i is the point of maximum of function f_i. Then f_1 and f_2 are positively conjugate iff one of the following statements holds:*
1) $\operatorname{sign}(f_1(c_1) - c_1) = \operatorname{sign}(f_2(c_2) - c_2) \neq 0$;
2) $\operatorname{sign}(f_1(c_1) - c_1) \leq 0$, $\operatorname{sign}(f_2(c_2) - c_2) = 0$.

Proof. If $f_i \in SU \cap \mathcal{F}_1$ then f_i has one or two fixed points and does not have any cycles of period more than one (see theorem 1.2 in [3]). Theorem 2 implies 1. In the case 2) f_1 has support $X_1 = [0, s_1]$, where s_1 is fixed point of f_1 that is nonequal to 0. Mapping f_2 has support $X_2 = [0, c_2]$. Since the functions f_1 on X_1 and f_2 on X_2 are monotonely increasing and fixed points are the ends of the intervals X_1 and X_2 correspondingly hence f_1 and f_2 are topologically conjugate on their supports. Thus f_1 and f_2 are positively conjugate. ■

Theorem 3. *Let $f_1, f_2 \in SU \cap \mathcal{F}_{2^n}$, $n \geq 1$. If maps f_1 and f_2 are positively conjugate then they are topologically conjugate.*

Proof. Let us prove that if f_1 and f_2 are not topologically conjugate then they are not positively conjugate. There are two cases when f_1 and f_2 are not topologically conjugate: 1) $\operatorname{sign}\left(f_1^{2^n}(c_1) - c_1\right) = 0$, $\operatorname{sign}\left(f_2^{2^n}(c_2) - c_2\right) \neq 0$ (c_i is the greatest point of local maximum of function $f_i^{2^n}$). The support X_i of the dynamical system (f_i, I) is interval $[0, M_i]$, where $M_i = \max\limits_{x \in I} f_i(x)$. Let us note that M_1 is periodic point of period 2^n of function f_1 but M_2 is not periodic. Therefore f_1 and f_2 are not conjugate on their supports. 2) $\operatorname{sign}\left(f_1^{2^n}(c_1) - c_1\right) < 0$, $\operatorname{sign}\left(f_2^{2^n}(c_2) - c_2\right) > 0$. Let us consider the function $f_i^{2^n}$ on the interval $I_i^{2^n}$ that is bounded by the two greatest preimages of fixed point s_i under $f_i^{2^n}$). Mapping $f_1^{2^n}$ is monotone on the support of $\left(f_1^{2^n}, I_1^{2^n}\right)$ but $f_2^{2^n}$ is not monotone on the support of $\left(f_2^{2^n}, I_2^{2^n}\right)$. Hence f_1 and f_2 are not positively conjugate. ■

Let us consider the notion of positive conjugacy for mappings $f(x) = 1 + ax - bx^2$, $a > 0$, $b > 0$. Define F_+^{-1}, F_-^{-1} to be two branches of inverse to F mapping such that $F_-^{-1}(0) = 0$.

Definition 2. Let $(F, I) \in SU$. By p-truncated mapping (or truncated by p) of F we mean the mapping $f_p(x) = F(x + p) - p$, $p \in (0, c)$, $x \in I_p$, where $I_p = [0, F_+^{-1}(p) - p]$. F truncated by interval $P \subset (0, c)$ is the family of mappings $f_p(x)$, where $p \in P$.

Remark 1. In general $f_p(I_p) \nsubseteq I_p$, i.e. (f_p, I_p) is not always a dynamical system.

Obviously, we can consider the mapping $f(x) = 1 + ax - bx^2$ ($a > 0$, $b > 0$) as truncated mapping of some $F \in SU$.

Proposition 3. *Let $(f_1, [0, a_1])$, $(f_2, [0, a_2])$ are dynamical systems such that f_i is p_i-truncated mapping of $F_i \in SU \cap \mathcal{F}_1$. Then f_1 and f_2 are positively conjugate.*

Proof. Since dynamical system $(f_i, [0, a_i])$ does not have repellent points then the set $\operatorname{Orb}_+(f_i)$ consists of Fock-orbit only. Hence the support of such dynamical system is a sequence of points of the interval $[0, a_i]$ which converge to attractive fixed point s_i. Let $\delta_1 = \{x_k \mid x_0 = 0,\ x_{k+1} = f_1(x_k),\ k \geq 0\}$ is Fock orbit of f_1, $\delta_2 = \{y_k \mid y_0 = 0,\ y_{k+1} = f_2(y_k),\ k \geq 0\}$ is Fock orbit of f_2.

The mapping $\varphi : \delta_1 \to \delta_2$ defined by the formula $\varphi(x_k) = y_k$ is homeomorphism since it preserves convergence in both directions and satisfies the condition $f_2 \circ \varphi(x_k) = \varphi \circ f_1(x_k)$ for all $k \geq 0$. Thus f_1 and f_2 are positively conjugate. ■

Further we will consider only dynamical systems truncated by interval $[0, F^2(M)]$. This condition guarantees an absence of anti-Fock orbits.

Theorem 4. *Let $(F, I) \in SU \cap \mathcal{F}_2$, $F^2(c') - c' < 0$ (c' is the greatest point of local maximum of the fuction F^2). Then there exist a countable number of positive conjugacy classes of truncated mappings of F.*

Proof. Define $B = \{\beta_1, \beta_2\}$ to be the cycle of period 2 of function F, $\beta_1 < \beta_2$. Let us consider the sequence of intervals T_k, $k \geq 0$:

$$T_{2k} = F_-^{-k}([\beta_1, \beta_2]), \qquad T_{2k+1} = F_-^{-k}\left(\left(F_-^{-1}(\beta_2), \beta_1\right)\right).$$

Interval $T = [\beta_1, \beta_2]$ has the property:

$$\text{if} \quad x \in T \quad \text{then} \quad F^n(x) \in T \quad \text{for all} \quad n \geq 1 \tag{1}$$

(since $F^2(c') - c' < 0$ then every trajectory of F^2 is attracted to the fixed point β_2 monotonely) and

$$\text{for every} \quad x \in T \quad \text{there is} \quad x' \in T \quad \text{such that} \quad F(x') = x \tag{2}$$

since $\max\limits_{x \in I} F(x) > \beta_2$.

It is evidently that for all $x \in T_{2k}$ there is $m \geq 0$ such that $f^m(x) \in T$. If $x \in T_{2k+1}$ then for every $m \geq 0$: $f^m(x) \notin T$. Any truncated mapping f of F has only one repellent cycle (fixed point s) and attractive cycle of period 2. Consequently, taking into account properties (1), (2) we can see that for every $x \in T$ there is $\delta \in \mathrm{Orb}_+(f)$ such that $x \in \delta$. Thus $T \subset X$.

Further let us consider Fock-orbits. If $f(0) \in T_{2k+1}$, $k \geq 0$ then support X of f is the set $T \cup \{x_n \mid x_0 = 0,\ x_{n+1} = f(x_n),\ n \geq 0\}$. Since F is monotone on T then all mappings truncated by T_{2k+1}, $k \geq 0$ are positively conjugate. If f is truncated by T_{2k}, $k \geq 1$ then the support of f be the set $T \cup \{x_n \mid x_0 = 0,\ x_{n+1} = f(x_n),\ x_n \notin T\}$. Thus k defines a class of positive conjugacy of mappings truncated by interval T_{2k}. ■

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Realizations of Indecomposable Solvable 4-Dimensional Real Lie Algebras

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Inequivalent classes of realizations for indecomposable four-dimensional solvable real Lie algebras in the space of four variables are obtained.

Inequivalent two- and three-dimensional Lie algebras were classified in XIX century by Lie [1]. In 1963 Mubaraksyanov classified three- and four-dimensional real Lie algebras [2] (see also those results in Patera and Winternitz [3]). In 1989 Mahomed and Leach obtained realizations of three-dimensional Lie algebras in terms of vector fields defined on the plane [4]. Mahomed and Soh tried to obtain realizations of three- and four-dimensional Lie algebras in the space of variables (t, x, y) [5], but their attempt can not be considered successful. Their article contains some misprints and a number of realizations are omitted. The results of [5] are used in [6] to solve the problem of linearization of systems of second-order ordinary differential equations, so some results from [6] are incorrect too. Realizations of solvable third-dimensional Lie algebras in the space of any number of variables are considered in [7].

In the present paper we give a complete set of inequivalent realizations of real indecomposable four-dimensional Lie algebras in the space of four variables, realizations of real decomposable four-dimensional Lie algebras are considered in [8]. Obtained results can be applied to integration of ordinary differential equations (or systems of ordinary differential equations) (see, for example, [9]), and to the problems of group classification (see, for example, [10]). We look for the realizations in the class of vector fields:

$$Q = \sum_{i=1}^{4} \xi^i \left(x_1, x_2, x_3, x_4\right) \partial_{x_i}.$$

After using Mubaraksyanov classification [2] we consider ten indecomposable solvable four-dimensional Lie algebras:

$$
\begin{array}{lllll}
A_{4.1} & [Q_2, Q_4] = Q_1, & [Q_3, Q_4] = Q_2; & & \\
A_{4.2} & [Q_2, Q_4] = Q_2, & [Q_1, Q_4] = qQ_1, & [Q_3, Q_4] = Q_2 + Q_3, & q \neq 0; \\
A_{4.3} & [Q_1, Q_4] = Q_1, & [Q_3, Q_4] = Q_2; & & \\
A_{4.4} & [Q_1, Q_4] = Q_1, & [Q_2, Q_4] = Q_1 + Q_2, & [Q_3, Q_4] = Q_2 + Q_3; & \\
A_{4.5} & [Q_1, Q_4] = Q_1, & [Q_2, Q_4] = qQ_2, & [Q_3, Q_4] = pQ_3, & \\
 & & -1 \leq p \leq q \leq 1, & pq \neq 0; & \\
A_{4.6} & [Q_1, Q_4] = qQ_1, & [Q_2, Q_4] = pQ_2 - Q_3, & [Q_3, Q_4] = Q_2 + pQ_3, & \\
 & & q \neq 0,\ p \geq 0; & & \\
A_{4.7} & [Q_2, Q_3] = Q_1, & [Q_1, Q_4] = 2Q_1, & [Q_2, Q_4] = Q_2, & \\
 & [Q_3, Q_4] = Q_2 + Q_3; & & &
\end{array}
$$

$A_{4.8}$ $[Q_2, Q_3] = Q_1,$ $[Q_1, Q_4] = (1 + q)\, Q_1,$ $[Q_2, Q_4] = Q_2,$
$[Q_3, Q_4] = qQ_3,$ $|q| \leq 1;$

$A_{4.9}$ $[Q_2, Q_3] = Q_1,$ $[Q_1, Q_4] = 2qQ_1,$ $[Q_2, Q_4] = qQ_2 - Q_3,$
$[Q_3, Q_4] = Q_2 + qQ_3,$ $q \geq 0;$

$A_{4.10}$ $[Q_1, Q_3] = Q_1,$ $[Q_1, Q_4] = -Q_2,$ $[Q_2, Q_4] = Q_1,$
$[Q_2, Q_3] = Q_2.$

For each algebra we write down only the non-zero commutation relations. We start from a given Lie algebra with a set of structure constants and look which vector fields in at most four variables satisfy the given set of commutator relations with none of the operators vanishing. We thus look for possible realizations or representations of our Lie algebra. Two realizations of the same Lie algebra will be considered equivalent or similar if there exists an invertible transformation mapping one of the realizations to the other.

We arrange all results in the next Table 1 (below $\partial_i = \partial_{x_i}$, $i = 1, \ldots, 4$; $A^k_{n.m}$ denotes the k-th realizations of algebra $A_{n.m}$).

Table 1. Realizations of real indecomposable solvable four-dimensional Lie algebras.

$A^1_{4.1}$	$\partial_1,\ \partial_2,\ \partial_3,\ x_2\partial_1 + x_3\partial_2 + \partial_4;$
$A^2_{4.1}$	$\partial_1,\ \partial_2,\ \partial_3,\ x_2\partial_1 + x_3\partial_2 + x_4\partial_3;$
$A^3_{4.1}$	$\partial_1,\ \partial_2,\ \partial_3,\ x_2\partial_1 + x_3\partial_2;$
$A^4_{4.1}$	$\partial_1,\ \partial_2,\ x_3\partial_1 + x_4\partial_2,\ x_2\partial_1 + x_4\partial_3 - \partial_4;$
$A^5_{4.1}$	$\partial_1,\ \partial_2,\ -\frac{x_3^2}{2}\partial_1 + x_3\partial_2,\ x_2\partial_1 - \partial_3;$
$A^6_{4.1}$	$\partial_1,\ x_3\partial_1,\ \partial_2,\ x_2x_3\partial_1 - \partial_3$
$A^7_{4.1}$	$\partial_1,\ x_2\partial_1,\ x_3\partial_1,\ -\partial_2 - x_2\partial_3;$
$A^8_{4.1}$	$\partial_1,\ x_2\partial_1,\ \frac{x_2^2}{2}\partial_1,\ -\partial_2$
$A^1_{4.2}$	$\partial_1,\ \partial_2,\ \partial_3,\ qx_1\partial_1 + (x_2 + x_3)\,\partial_2 + x_3\partial_3 + \partial_4;$
$A^2_{4.2}$	$\partial_1,\ \partial_2,\ \partial_3,\ qx_1\partial_1 + (x_2 + x_3)\,\partial_2 + x_3\partial_3;$
$A^3_{4.2}$	$\partial_1,\ \partial_2,\ x_3\partial_1 + x_4\partial_2,\ qx_1\partial_1 + x_2\partial_2 + (q-1)\,x_3\partial_3 - \partial_4;$
$A^4_{4.2}$	$\partial_1,\ \partial_2,\ e^{(1-q)x_3}\partial_1 + x_3\partial_2,\ qx_1\partial_1 + x_2\partial_2 - \partial_3,\ q \neq 1$
$A^5_{4.2}$	$\partial_1,\ x_3\partial_1,\ \partial_2,\ (qx_1 + x_2x_3)\,\partial_1 + x_2\partial_2 + (q-1)\,x_3\partial_3;$
$A^6_{4.2}$	$\partial_1,\ x_3\partial_1,\ \partial_2,\ (x_1 + x_2x_3)\,\partial_1 + x_2\partial_2 + \partial_4,\ q = 1;$
$A^7_{4.2}$	$\partial_1,\ x_2\partial_1,\ x_3\partial_1,\ qx_1\partial_1 + (q-1)\,x_2\partial_2 + ((q-1)\,x_3 - x_2)\,\partial_3;$
$A^8_{4.2}$	$\partial_1,\ x_2\partial_1,\ \frac{x_2}{1-q}\ln\lvert x_2\rvert\partial_1,\ qx_1\partial_1 + (q-1)\,x_2\partial_2,\ q \neq 1$
$A^1_{4.3}$	$\partial_1,\ \partial_2,\ \partial_3,\ x_1\partial_1 + x_3\partial_2 + \partial_4;$
$A^2_{4.3}$	$\partial_1,\ \partial_2,\ \partial_3,\ x_1\partial_1 + x_3\partial_2 + x_4\partial_3;$
$A^3_{4.3}$	$\partial_1,\ \partial_2,\ \partial_3,\ x_1\partial_1 + x_3\partial_2;$
$A^4_{4.3}$	$\partial_1,\partial_2,\ x_3\partial_1 + x_4\partial_2,\ x_1\partial_1 + x_3\partial_3 - \partial_4;$
$A^5_{4.3}$	$\partial_1,\ \partial_2,\ x_3\partial_2 + ce^{-x_3}\partial_1,\ x_1\partial_1 - \partial_3,\ c \in \{0; 1\};$
$A^6_{4.3}$	$\partial_1,\ x_3\partial_1,\ \partial_2,\ (x_1 + x_2x_3)\,\partial_1 + x_3\partial_3$
$A^7_{4.3}$	$\partial_1,\ x_2\partial_1,\ x_3\partial_1,\ x_1\partial_1 + x_2\partial_2 + (x_3 - x_2)\,\partial_3;$
$A^8_{4.3}$	$\partial_1,\ x_2\partial_1,\ -x_2\ln\lvert x_2\rvert\partial_1,\ x_1\partial_1 + x_2\partial_2$

Continuation of Table 1.

$A^1_{4.4}$	$\partial_1,\ \partial_2,\ \partial_3,\ (x_1+x_2)\,\partial_1+(x_2+x_3)\,\partial_2+x_3\partial_3+\partial_4;$
$A^2_{4.4}$	$\partial_1,\ \partial_2,\ \partial_3,\ (x_1+x_2)\,\partial_1+(x_2+x_3)\,\partial_2+x_3\partial_3;$
$A^3_{4.4}$	$\partial_1,\ \partial_2,\ x_3\partial_1+x_4\partial_2,\ (x_1+x_2)\,\partial_1+x_2\partial_2+x_4\partial_3-\partial_4;$
$A^4_{4.4}$	$\partial_1,\ \partial_2,\ -\frac{{x_3}^2}{2}\partial_1+x_3\partial_2,\ (x_1+x_2)\,\partial_1+x_2\partial_2-\partial_3;$
$A^5_{4.4}$	$\partial_1,\ x_3\partial_1,\ \partial_2,\ (x_1+x_2x_3)\,\partial_1+x_2\partial_2-\partial_3$
$A^6_{4.4}$	$\partial_1,\ x_2\partial_1,\ x_3\partial_1,\ x_1\partial_1-\partial_2-x_2\partial_3;$
$A^7_{4.4}$	$\partial_1,\ x_2\partial_1,\ \frac{x_2^2}{2}\partial_1,\ x_1\partial_1-\partial_2$
$A^1_{4.5}$	$\partial_1,\ \partial_2,\ \partial_3,\ x_1\partial_1+qx_2\partial_2+px_3\partial_3+\partial_4$
$A^2_{4.5}$	$\partial_1,\ \partial_2,\ \partial_3,\ x_1\partial_1+qx_2\partial_2+px_3\partial_3;$
$A^3_{4.5}$	$\partial_1,\ \partial_2,\ x_3\partial_1+x_4\partial_2,\ x_1\partial_1+qx_2\partial_2+(1-p)\,x_3\partial_3+(q-p)\,x_4\partial_4;$
$A^4_{4.5}$	$\partial_1,\ \partial_2,\ x_3\partial_1,\ x_1\partial_1+qx_2\partial_2+\partial_4;$
$A^5_{4.5}$	$\partial_1,\ \partial_2,\ x_3\partial_1,\ x_1\partial_1+qx_2\partial_2;$
$A^6_{4.5}$	$\partial_1,\ \partial_2,\ x_3\partial_2,\ x_1\partial_1+qx_2\partial_2+\partial_4;$
$A^7_{4.5}$	$\partial_1,\ \partial_2,\ x_3\partial_2,\ x_1\partial_1+qx_2\partial_2;$
$A^8_{4.5}$	$\partial_1,\ \partial_2,\ x_3\partial_1+f\,(x_3)\,\partial_2,\ x_1\partial_1+qx_2\partial_2+\partial_4;$
$A^9_{4.5}$	$\partial_1,\ \partial_2,\ x_3\partial_1+f\,(x_3)\,\partial_2,\ x_1\partial_1+qx_2\partial_2;$
$A^{10}_{4.5}$	$\partial_1,\ \partial_2,\ c_1e^{(1-p)x_3}\partial_1+c_2e^{(q-p)x_3}\partial_2,\ x_1\partial_1+qx_2\partial_2+\partial_3,$ $c_i\in\{0;1\}$, $c_1=0$ when $p=1$, $c_2=0$ when $q=p$
$A^{11}_{4.5}$	$\partial_1,\ x_3\partial_1,\ \partial_2,\ x_1\partial_1+px_2\partial_2+(1-q)\,x_3\partial_3;$
$A^{12}_{4.5}$	$\partial_1,\ x_2\partial_1,\ x_3\partial_1,\ x_1\partial_1+(1-q)\,x_2\partial_2+(1-p)\,x_3\partial_3;$
$A^{13}_{4.5}$	$\partial_1,\ x_2\partial_1,\ x_3\partial_1, x_1\partial_1+\partial_4,\ p=q=1;$
$A^{14}_{4.5}$	$\partial_1,\ x_2\partial_2,\ f\,(x_2)\,\partial_1,\ x_1\partial_1+\partial_3,\ f\neq c_1x_2+c_2,\ p=q=1;$
$A^{15}_{4.5}$	$\partial_1,\ x_2\partial_2,\ f\,(x_2)\,\partial_1,\ x_1\partial_1,\ f\neq c_1x_2+c_2,\ p=q=1;$
$A^{16}_{4.5}$	$\partial_1,\ e^{(1-q)x_2}\partial_1,\ e^{(1-p)x_2}\partial_1,\ x_1\partial_1+\partial_2,\ q\neq 1,\ p\neq 1,\ q\neq p$
$A^1_{4.6}$	$\partial_1,\ \partial_2,\ \partial_3,\ qx_1\partial_1+(px_2+x_3)\,\partial_2+(-x_2+px_3)\,\partial_3+\partial_4;$
$A^2_{4.6}$	$\partial_1,\ \partial_2,\ \partial_3,\ qx_1\partial_1+(px_2+x_3)\,\partial_2+(-x_2+px_3)\,\partial_3;$
$A^3_{4.6}$	$\partial_1,\ \partial_2,\ x_3\partial_1+x_4\partial_2,\ (qx_1-x_2x_3)\,\partial_1+(p-x_4)\,x_2\partial_2+(q-p-x_4)\,x_3\partial_3-\left(1+x_4^2\right)\partial_4;$
$A^4_{4.6}$	$\partial_1,\ \partial_2,\ c\left(\sqrt{x_3^2+1}e^{(p-q)\arctan x_3}\right)\partial_1+x_3\partial_2,$ $\left(qx_1-cx_2\left(\sqrt{x_3^2+1}e^{(p-q)\arctan x_3}\right)\right)\partial_1+(p-x_3)\,x_2\partial_2-\left(x_3^2+1\right)\partial_3,\ c\in\{0;1\}$
$A^5_{4.6}$	$\partial_1,\ x_2\partial_1,\ x_3\partial_1,\ qx_1\partial_1+((q-p)\,x_2+x_3)\,\partial_2+((q-p)\,x_3-x_2)\,\partial_3$
$A^6_{4.6}$	$\partial_1,\ e^{(q-p)x_2}\cos x_2\partial_1,\ -e^{(q-p)x_2}\sin x_2\partial_1,\ qx_1\partial_1+\partial_2$
$A^1_{4.7}$	$\partial_1,\ \partial_2,\ x_2\partial_1+\partial_3,\ \left(2x_1+\frac{{x_3}^2}{2}\right)\partial_1+(x_2+x_3)\,\partial_2+x_3\partial_3+\partial_4$
$A^2_{4.7}$	$\partial_1,\ \partial_2,\ x_2\partial_1+\partial_3,\ \left(2x_1+\frac{{x_3}^2}{2}\right)\partial_1+(x_2+x_3)\,\partial_2+x_3\partial_3;$
$A^3_{4.7}$	$\partial_1,\ \partial_2,\ x_2\partial_1+x_3\partial_2,\ 2x_1\partial_1+x_2\partial_2-\partial_3;$
$A^4_{4.7}$	$\partial_1,\ x_2\partial_1,\ -\partial_2,\ \left(2x_1-\frac{{x_2}^2}{2}\right)\partial_1+x_2\partial_2+\partial_3;$
$A^5_{4.7}$	$\partial_1,\ x_2\partial_1,\ -\partial_2,\ \left(2x_1-\frac{{x_2}^2}{2}\right)\partial_1+x_2\partial_2$
$A^1_{4.8}$	$\partial_1,\ \partial_2,\ x_2\partial_1+\partial_3,\ (1+q)\,x_1\partial_1+x_2\partial_2+qx_3\partial_3+\partial x_4;$
$A^2_{4.8}$	$\partial_1,\ \partial_2,\ x_2\partial_1+\partial_3,\ (1+q)\,x_1\partial_1+x_2\partial_2+qx_3\partial_3,\ q\neq 0;$
$A^3_{4.8}$	$\partial_1,\ \partial_2,\ x_2\partial_1+\partial_3,\ x_1\partial_1+x_2\partial_2+x_4\partial_3,\ q=0;$
$A^4_{4.8}$	$\partial_1,\ \partial_2,\ x_2\partial_1+\partial_3,\ x_1\partial_1+x_2\partial_2+c\partial_3,\ q=0,\ c\in\mathbb{R};$
$A^5_{4.8}$	$\partial_1,\ \partial_2,\ x_2\partial_1+x_3\partial_2,\ 2x_1\partial_1+x_2\partial_2+\partial_4,\ q=1;$

Continuation of Table 1.

$A^6_{4.8}$	$\partial_1,\ \partial_2,\ x_2\partial_1 + x_3\partial_2,\ (1+q)\,x_1\partial_1 + x_2\partial_2 + (1-q)\,x_3\partial_3;$
$A^7_{4.8}$	$\partial_1,\ \partial_2,\ x_2\partial_1,\ (1+q)\,x_1\partial_1 + x_2\partial_2 + \partial_3,\ q\neq 1;$
$A^8_{4.8}$	$\partial_1,\ \partial_2,\ x_2\partial_1,\ (1+q)\,x_1\partial_1 + x_2\partial_2,\ q\neq 1;$
$A^9_{4.8}$	$\partial_1,\ \partial_2,\ x_2\partial_1,\ x_3\partial_1 + x_2\partial_2,\ q=-1;$
$A^{10}_{4.8}$	$\partial_1,\ -x_2\partial_1,\ \partial_2,\ (1+q)\,x_1\partial_1 + qx_2\partial_2 + \partial_3;$
$A^{11}_{4.8}$	$\partial_1,\ -x_2\partial_1,\ \partial_2,\ (1+q)\,x_1\partial_1 + qx_2\partial_2$
$A^1_{4.9}$	$\partial_1,\ \partial_2,\ x_2\partial_1 + \partial_3,\ \left(2qx_1 + \frac{{x_3}^2-{x_2}^2}{2}\right)\partial_1 + (qx_2 + x_3)\,\partial_2 + (qx_3 - x_2)\,\partial_3 + \partial_4;$
$A^2_{4.9}$	$\partial_1,\ \partial_2,\ x_2\partial_1 + \partial_3,\ \left(2qx_1 + \frac{{x_3}^2-{x_2}^2}{2}\right)\partial_1 + (qx_2 + x_3)\,\partial_2 + (qx_3 - x_2)\,\partial_3;$
$A^3_{4.9}$	$\partial_1,\ \partial_2,\ x_2\partial_1 + \partial_3,\ \left(\frac{{x_3}^2-{x_2}^2}{2} + x_4\right)\partial_1 + x_3\partial_2 - x_2\partial_3,\ q=0;$
$A^4_{4.9}$	$\partial_1,\ \partial_2,\ x_2\partial_1 + x_3\partial_2,\ \left(2qx_1 - \frac{{x_2}^2}{2}\right)\partial_1 + (q - x_3)\,x_2\partial_2 - \left(1 + {x_3}^2\right)\partial_3$
$A^1_{4.10}$	$\partial_1,\ \partial_2,\ x_1\partial_1 + x_2\partial_2 + \partial_3,\ x_2\partial_1 - x_1\partial_2 + \partial_4;$
$A^2_{4.10}$	$\partial_1,\ \partial_2,\ x_1\partial_1 + x_2\partial_2 + \partial_3,\ x_2\partial_1 - x_1\partial_2 + x_4\partial_3;$
$A^3_{4.10}$	$\partial_1,\ \partial_2,\ x_1\partial_1 + x_2\partial_2 + \partial_3,\ x_2\partial_1 - x_1\partial_2 + c\partial_3,\ c\in\mathbb{R}$
$A^4_{4.10}$	$\partial_1,\ x_2\partial_1,\ x_1\partial_1 + \partial_3,\ -x_1x_2\partial_1 - \left(1 + {x_2}^2\right)\partial_2;$
$A^5_{4.10}$	$\partial_1,\ \partial_2,\ x_1\partial_1 + x_2\partial_2,\ x_2\partial_1 - x_1\partial_2 + \partial_3;$
$A^6_{4.10}$	$\partial_1,\ \partial_2,\ x_1\partial_1 + x_2\partial_2,\ x_2\partial_1 - x_1\partial_2;$
$A^7_{4.10}$	$\partial_1,\ x_2\partial_1,\ x_1\partial_1,\ -x_1x_2\partial_1 - \left(1 + {x_2}^2\right)\partial_2$

This can be applied to the integrating of fourth-order differential equations of or some system classes of four first-order differential equations and their classification.

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Jacobson Generators of (Quantum) $sl(n+1|m)$. Related Statistics

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A description of the quantum superalgebra $U_q[sl(n+1|m)]$ and hence (at $q=1$) of the special linear superalgebra $sl(n+1|m)$ via a new set of generators, called Jacobson generators, is given. It provides an alternative to the canonical description of $U_q[sl(n+1|m)]$ in terms of Chevalley generators. The Jacobson generators satisfy three linear supercommutation relations and define $U_q[sl(n+1|m)]$ as a deformed Lie supertriple system. Fock representations are constructed and the action of the Jacobson generators on the Fock basis is written down. The Jacobson generators and the Fock representations allow for an interpretation in terms of quantum statistics, and the properties of the underlying statistics are shortly discussed.

1 Introduction

The Lie superalgebra $sl(n+1|m)$ is one of the basic classical simple Lie superalgebras in Kac's classification [1]. It can be considered as the superanalogue of the special linear Lie algebra $sl(n+1)$. The quantum superalgebra $U_q[sl(n+1|m)]$ is a Hopf superalgebra deformation of the universal enveloping superalgebra $U[sl(n+1|m)]$ of $sl(n+1|m)$.

Usually, $U_q[sl(n+1|m)]$ is defined by its Chevalley generators e_i, f_i, h_i, $i=1,\ldots,n+m$, subject to the Cartan–Kac relations and the Serre relations [2, 3, 4]. Beside these defining relations, also the other Hopf superalgebra maps (comultiplication, co-unit and antipode) are part of the definition. In the present talk, however, we do not use these other Hopf superalgebra maps; so we shall concentrate on $U_q[sl(n+1|m)]$ as an associative superalgebra.

The definition in terms of Chevalley generators has the advantage that the comultiplication, co-unit and antipode are easy to give. Furthermore, certain representations can be constructed explicitly (e.g. for the essentially typical representations a Gelfand–Zetlin basis exist for which the action of the Chevalley generators is known [5]). Having certain physical applications in mind, however, it is sometimes more useful to work with a different set of generators for $U_q[sl(n+1|m)]$.

The different set of generators for $U_q[sl(n+1|m)]$ given here are the Jacobson generators (JGs) (denoted by a_i^+, a_i^- and H_i, with $i=1,\ldots,n+m$). For the case of $sl(n+1)$, such generators were originally introduced by Jacobson [6, 7]. The use of Jacobson generators has a number of advantages.

First of all, in certain applications it is necessary to have a complete basis of $U_q[sl(n+1|m)]$ (following from the Poincaré–Birkhoff–Witt theorem). Such a basis is given in terms of the

Cartan–Weyl elements. Although it is possible to express all Cartan–Weyl elements in terms of the Chevalley generators, such expressions soon become rather unmanageable. In terms of the Jacobson generators, the description of all Cartan–Weyl elements is very easy.

Secondly, the Jacobson generators allow for the definition of a simple class of representations, the Fock representations of $U_q[sl(n+1|m)]$. In these representations, the Jacobson generators a_i^+ and a_i^- share certain properties with ordinary creation and annihilation operators.

A disadvantage of the Jacobson generators compared to the Chevalley generators is that the explicit expressions for the other Hopf (super)algebra maps (comultiplication, co-unit and antipode) become very lengthy and complicated.

In Section 2 we define the Jacobson generators of $U_q[sl(n+1|m)]$ as a special subset of the Cartan–Weyl elements. The description of all Cartan–Weyl elements in terms of the Jacobson generators becomes very simple. In order to apply these results (e.g. in representations) one must have a list of all (super)commutation relations between these Cartan–Weyl elements; in terms of Jacobson generators, this means one has to determine certain triple relations. These are given in Theorem 2. In Section 3 we define Fock representations for $U_q[sl(n+1|m)]$, related to the Jacobson generators. The Fock representations are labeled by a number p; when p is a nonnegative integer, the Fock representation is finite-dimensional. These representations are further analyzed. Following conditions required in a physical context, it is determined when these Fock representations are unitary, see Theorem 4. In that case, an orthonormal basis of the Fock space is given, together with the action of the Jacobson generators on these basis elements. Finally, in Section 4 the Jacobson generators are interpreted as operators creating or annihilating a “particle”, and the underlying quantum statistics is discussed.

2 Jacobson generators of $U_q[sl(n+1|m)]$

The Hopf superalgebra $U_q[sl(n+1|m)]$ is defined in the sense of Drinfeld [8], as a topologically free $\mathbb{C}[[h]]$ module. As a superalgebra, $U_q[sl(n+1|m)]$ is usually defined by means of its Chevalley generators, subject to the Cartan–Kac relations and the Serre relations [2, 3, 4]. Here, we present an alternative description of $U_q[sl(n+1|m)]$ in terms of the so-called Jacobson generators. The definition of JGs can be best presented in the framework of a set of Cartan–Weyl elements e_{ij}, $i,j = 0,\ldots,n+m$ of $U_q[gl(n+1|m)]$ [9]. The elements e_{ij} are the q-analogues of the defining basis of $gl(n+1|m)$; their grading is given by $\deg(e_{ij}) = \theta_{ij} = \theta_i + \theta_j$, where

$$\theta_i = \begin{cases} \bar{0} & \text{if} \quad i = 0,\ldots,n, \\ \bar{1} & \text{if} \quad i = n+1,\ldots,n+m. \end{cases}$$

We shall refer to e_{ij} as a positive root vector (resp. negative root vector) if $i<j$ (resp. $i>j$). For the formulation of the Poincaré–Birkhoff–Witt theorem, it is necessary to fix a total order for the set of elements e_{ij}. Among the positive root vectors, this order is given by

$$e_{ij} < e_{kl}, \quad \text{if} \quad i<k \quad \text{or} \quad i=k \quad \text{and} \quad j<l; \tag{1}$$

for the negative root vectors e_{ij} one takes the same rule (1), and total order is fixed by choosing

$$\text{positive root vectors} < \text{negative root vectors} < e_{ii}.$$

The difference between $U_q[sl(n+1|m)]$ and $U_q[gl(n+1|m)]$ is in the elements of the Cartan subalgebra. For $U_q[gl(n+1|m)]$ the Cartan subalgebra is generated by e_{ii} ($i=0,\ldots,n+m$). For $U_q[sl(n+1|m)]$ the Cartan subalgebra is generated by the elements H_i, with

$$H_i = e_{00} - (-1)^{\theta_i} e_{ii}, \qquad i = 1,\ldots,n+m. \tag{2}$$

We will use also the elements L_i and $\bar{L}_i$, where

$$L_i = q^{H_i}, \qquad \bar{L}_i = q^{-H_i}, \qquad i = 1,\ldots,n+m. \tag{3}$$

The Cartan–Weyl elements of $U_q[sl(n+1|m)]$ are now given by $\{H_i; i = 1,\ldots,n+m\} \cup \{e_{ij}; i \neq j = 0,\ldots,n+m\}$. The complete set of supercommutation relations between these Cartan–Weyl elements is given by

$$[H_i, H_j] = 0; \tag{4}$$

$$[H_i, e_{jk}] = (\delta_{0j} - \delta_{0k} - (-1)^{\theta_i}(\delta_{ij} - \delta_{ik}))e_{jk}; \tag{5}$$

for two positive root vectors $e_{ij} < e_{kl}$:

$$[\![e_{ij}, e_{kl}]\!]_{q^{(-1)^{\theta_j}\delta_{jl}-(-1)^{\theta_j}\delta_{jk}+(-1)^{\theta_i}\delta_{ik}}} = \delta_{jk}e_{il} + \left(q - q^{-1}\right)(-1)^{\theta_k}\theta(l > j > k > i)e_{kj}e_{il}; \tag{6}$$

for two negative root vectors $e_{ij} > e_{kl}$:

$$[\![e_{ij}, e_{kl}]\!]_{q^{-(-1)^{\theta_j}\delta_{jl}+(-1)^{\theta_j}\delta_{jk}-(-1)^{\theta_i}\delta_{ik}}} = \delta_{jk}e_{il} - \left(q - q^{-1}\right)(-1)^{\theta_k}\theta(i > k > j > l)e_{kj}e_{il}; \tag{7}$$

and finally for a positive root vector e_{ij} and a negative root vector e_{kl}:

$$\begin{aligned}
[\![e_{ij}, e_{kl}]\!] &= \frac{\delta_{il}\delta_{jk}}{q - q^{-1}}\left(L_j^{(-1)^{\theta_i}}\bar{L}_i^{(-1)^{\theta_i}} - \bar{L}_j^{(-1)^{\theta_i}}L_i^{(-1)^{\theta_i}}\right) \\
&+ \Big(\left(q - q^{-1}\right)\theta(j > k > i > l)(-1)^{\theta_k}e_{kj}e_{il} - \delta_{il}\theta(j > k)(-1)^{\theta_{kl}}e_{kj} + \delta_{jk}\theta(i > l)e_{il}\Big)L_i\bar{L}_k \\
&+ L_j\bar{L}_l\Big(-\left(q - q^{-1}\right)\theta(k > j > l > i)(-1)^{\theta_j}e_{il}e_{kj} - \delta_{il}\theta(k > j)(-1)^{\theta_{ij}}e_{kj} + \delta_{jk}\theta(l > i)e_{il}\Big),
\end{aligned} \tag{8}$$

where

$$[a, b]_x = ab - xba, \qquad \{a, b\}_x = ab + xba, \qquad [\![a, b]\!]_x = ab - (-1)^{\deg(a)\deg(b)}xba,$$

$$\theta(i_1 > i_2 > \ldots > i_r) = \begin{cases} 1, & \text{if } i_1 > i_2 > \ldots > i_r, \\ 0, & \text{otherwise.} \end{cases}$$

Define the Jacobson generators of $U_q[sl(n+1|m)]$ to be the following Cartan–Weyl vectors:

$$a_i^- = e_{0i}, \qquad a_i^+ = e_{i0}, \qquad H_i, \qquad i = 1,\ldots,n+m. \tag{9}$$

Then from (8) one obtains:

$$[\![a_i^-, a_j^+]\!] = -(-1)^{\theta_i}L_i e_{ji}, \quad (i < j); \qquad [\![a_i^-, a_j^+]\!] = -(-1)^{\theta_j}e_{ji}\bar{L}_j, \quad (i > j). \tag{10}$$

In terms of the JGs the definition of $U_q[sl(n+1|m)]$ reads

Theorem 1. *$U_q[sl(n+1|m)]$ is a unital associative algebra with generators $\{H_i,\ a_i^\pm\}_{i=1,\ldots,n+m}$ and relations*

$$\begin{aligned}
&[H_i, H_j] = 0, \qquad [H_i, a_j^\pm] = \mp(1 + (-1)^{\theta_i}\delta_{ij})a_j^\pm, \\
&[\![a_i^-, a_i^+]\!] = \frac{L_i - \bar{L}_i}{q - \bar{q}}, \qquad L_i = q^{H_i}, \qquad \bar{L}_i \equiv L_i^{-1} = q^{-H_i}, \qquad \bar{q} \equiv q^{-1}, \\
&[\![[\![a_i^\eta, a_{i+\xi}^{-\eta}]\!], a_k^\eta]\!]_{q^{\xi(1+(-1)^{\theta_i}\delta_{ik})}} = \eta^{\theta_k}\delta_{k,i+\xi}L_k^{-\xi\eta}a_i^\eta, \\
&[\![a_1^\xi, a_2^\xi]\!]_q = 0, \qquad [\![a_1^\xi, a_1^\xi]\!] = 0, \qquad \xi, \eta = \pm \quad \text{or} \quad \pm 1.
\end{aligned} \tag{11}$$

The set of relations (11) is the minimal one defining the algebra $U_q[sl(n+1|m)]$. This description of $U_q[sl(n+1|m)]$ (resp. $sl(n+1|m)$) is somewhat similar to the Lie triple system description of Lie algebras, initiated by Jacobson [6, 7] and generalized to Lie superalgebras by Okubo [10]. Therefore we have defined $U_q[sl(n+1|m)]$ (resp. $sl(n+1|m)$) as a (deformed) Lie supertriple system.

In order to be able to reorder the Cartan–Weyl elements, which appear when computing the transformations of the Fock spaces, it is convenient to write down all triple relations between the JGs (which certainly follow from the relations (11)).

Theorem 2. *A set of Cartan–Weyl elements of $U_q[sl(n+1|m)]$ is given by H_i, $a_i^{\pm}$, $[\![a_i^+, a_j^-]\!]$ ($i \neq j = 1, \ldots, n+m$). A complete set of supercommutation relations between these elements is given by:*

$$[H_i, H_j] = 0; \qquad [H_i, a_j^{\pm}] = \mp(1 + (-1)^{\theta_i}\delta_{ij})a_j^{\pm}; \tag{12}$$

$$[\![a_i^-, a_i^+]\!] = \frac{L_i - \bar{L}_i}{q - q^{-1}}; \tag{13}$$

$$[\![a_i^\eta, a_j^\eta]\!]_q = 0 \quad (i < j); \qquad (a_i^{\pm})^2 = 0 \quad (i = n+1, \ldots, n+m); \tag{14}$$

$$\begin{aligned}[\![[\![a_i^\eta, a_j^{-\eta}]\!], a_k^\eta]\!]_{q^{\xi(1+(-1)^{\theta_i}\delta_{ik})}} &= \eta^{\theta_j}\delta_{jk}L_k^{-\xi\eta}a_i^\eta + (-1)^{\theta_k}\epsilon(j,k,i)(q - \bar{q})[\![a_k^\eta, a_j^{-\eta}]\!]a_i^\eta \\ &= \eta^{\theta_j}\delta_{jk}L_k^{-\xi\eta}a_i^\eta + (-1)^{\theta_k\theta_j}\epsilon(j,k,i)q^{\xi}(q - \bar{q})a_i^\eta[\![a_k^\eta, a_j^{-\eta}]\!],\end{aligned} \tag{15}$$

where $(j - i)\xi > 0$, ξ, $\eta = \pm$ and

$$\epsilon(j,k,i) = \begin{cases} 1, & \text{if } j > k > i; \\ -1, & \text{if } j < k < i; \\ 0, & \text{otherwise,} \end{cases}$$

and we have used the notation $\bar{q} = q^{-1}$.

3 Fock representations

We construct the Fock modules using the induced module procedure. $G = U_q[sl(n+1|m)]$, with Cartan–Weyl elements H_i, $a_i^{\pm}$ and $[\![a_i^+, a_j^-]\!]$ ($i \neq j = 1, \ldots, n+m$), has a subalgebra $A = U_q[gl(n|m)]$ with Cartan–Weyl elements H_i and $[\![a_i^+, a_j^-]\!]$ ($i \neq j = 1, \ldots, n+m$). Define a trivial one-dimensional A module as follows:

$$[\![a_i^-, a_j^+]\!]|0\rangle = 0, \qquad (i \neq j = 1, \ldots, n+m), \tag{16}$$

$$H_i|0\rangle = p|0\rangle, \tag{17}$$

where p is any complex number. Let P be the (associative) subalgebra of $G = U_q[sl(n+1|m)]$ generated by the elements of A and $\{a_i^-; i = 1, \ldots, n+m\}$. The one-dimensional module $\mathbb{C}|0\rangle$ can be extended to a one-dimensional P module by requiring:

$$a_i^-|0\rangle = 0, \qquad i = 1, \ldots, n+m. \tag{18}$$

Now the G module $\bar{W}_p$ is defined as

$$\bar{W}_p = \mathrm{Ind}_P^G\, \mathbb{C}|0\rangle.$$

Clearly $\bar{W}_p$ is freely generated by the generators a_i^+ ($i = 1, \ldots, n+m$) acting on $|0\rangle$. Therefore a basis for $\bar{W}_p$ is given by

$$|p; r_1, r_2, \ldots, r_{n+m}\rangle \equiv (a_1^+)^{r_1}(a_2^+)^{r_2}\cdots(a_n^+)^{r_n}(a_{n+1}^+)^{r_{n+1}}(a_{n+2}^+)^{r_{n+2}}\cdots(a_{n+m}^+)^{r_{n+m}}|0\rangle, \tag{19}$$

where $r_i \in \mathbb{Z}_+$ for $i = 1, \ldots, n$ and $r_i \in \{0, 1\}$ for $i = n+1, \ldots, n+m$.

Theorem 3. *The transformation of the basis (19) of $\bar{W}_p$ under the action of the JGs reads:*

$$H_i|p;r_1,r_2,\ldots,r_{n+m}\rangle = \left(p-(-1)^{\theta_i}r_i-\sum_{j=1}^{n+m}r_j\right)|p;r_1,r_2,\ldots,r_{n+m}\rangle, \tag{20}$$

$$\begin{aligned} a_i^-|p;r_1,r_2,\ldots,r_{n+m}\rangle &= (-1)^{\theta_1 r_1+\theta_2 r_2+\cdots+\theta_{i-1}r_{i-1}}q^{r_1+\cdots+r_{i-1}}[r_i]\left[p-\sum_{j=1}^{n+m}r_j+1\right] \\ &\quad\times|p;r_1,r_2,\ldots,r_{i-1},r_i-1,r_{i+1},\ldots,r_{n+m}\rangle, \end{aligned} \tag{21}$$

$$\begin{aligned} a_i^+|p;r_1,r_2,\ldots,r_{n+m}\rangle &= (-1)^{\theta_1 r_1+\theta_2 r_2+\cdots+\theta_{i-1}r_{i-1}}\bar{q}^{r_1+\cdots+r_{i-1}}(1-\theta_i r_i) \\ &\quad\times|p;r_1,r_2,\ldots,r_{i-1},r_i+1,r_{i+1},\ldots,r_{n+m}\rangle, \end{aligned} \tag{22}$$

where $i=1,\ldots,n+m$.

Proof. We sketch the proof. Equation (20) is an immediate consequence of $[H_i,a_j^+]=-(1+(-1)^{\theta_i}\delta_{ij})a_j^+$, which is one of the last relations in (12). Also the action of a_i^+ on the basis vectors is easy: (22) follows directly from (14). The proof of (21) follows from the following relations [11]:

- $$\begin{aligned} &[\![A,B_1B_2\cdots B_{i-1}B_iB_{i+1}\cdots B_j]\!]_{q^{b_1+b_2+\cdots+b_j}} \\ &= \sum_{i=1}^{j}q^{b_1+b_2+\cdots+b_{i-1}}(-1)^{\alpha(\beta_1+\cdots+\beta_{i-1})}B_1B_2\cdots B_{i-1}[\![A,B_i]\!]_{q^{b_i}}B_{i+1}\cdots B_j, \\ &\qquad \text{where}\quad \alpha=\deg(A)\quad\text{and}\quad \beta_i=\deg(B_i); \end{aligned} \tag{23}$$
- $$[\![a_i^-,(a_j^+)^r]\!] = \begin{cases} \dfrac{\bar{q}^{2r}-1}{\bar{q}^2-1}(a_j^+)^{r-1}[\![a_i^-,a_j^+]\!] & \text{when } i<j, \\ \dfrac{q^{2r}-1}{q^2-1}(a_j^+)^{r-1}[\![a_i^-,a_j^+]\!] & \text{when } i>j; \end{cases} \tag{24}$$
- $$[\![a_i^-,(a_i^+)^r]\!] = \frac{(a_i^+)^{r-1}}{q-\bar{q}}\left(\frac{\bar{q}^{2r}-1}{\bar{q}^2-1}L_i-\frac{q^{2r}-1}{q^2-1}\bar{L}_i\right); \tag{25}$$
- $$[\![[\![a_i^-,a_j^+]\!],(a_i^+)^r]\!]_{q^r} = -(-1)^{\theta_j}\frac{\bar{q}^{2r}-1}{\bar{q}^2-1}\bar{L}_ia_j^+(a_i^+)^{r-1},\quad i>j, \tag{26}$$
- $$[\![[\![a_i^-,a_j^+]\!],(a_k^+)^r]\!]_{q^r} = (-1)^{\theta_j}(q^{2r}-1)a_j^+(a_k^+)^{r-1}[\![a_i^-,a_k^+]\!],\quad i>k>j, \tag{27}$$
- $$\begin{aligned} &[\![a_i^-,a_1^+]\!](a_2^+)^{r_2}\cdots(a_{n+m}^+)^{r_{n+m}}|0\rangle \\ &= -(-1)^{\theta_1+\theta_2r_2+\theta_3r_3+\cdots+\theta_{i-1}r_{i-1}}q^{2r_2+\cdots+2r_{i-1}+r_i+\cdots+r_{n+m}-p}[r_i] \\ &\times a_1^+(a_2^+)^{r_2}\ldots(a_{i-1}^+)^{r_{i-1}}(a_i^+)^{r_i-1}(a_{i+1}^+)^{r_{i+1}}\cdots(a_{n+m}^+)^{r_{n+m}}|0\rangle,\quad i>1. \end{aligned} \tag{28}$$

■

The action of the elements H_i and $a_i^\pm$ ($i=1,\ldots,n+m$) on the basis vectors of $\bar{W}_p$, determined in Theorem 3, imply that $\bar{W}_p$ has an invariant submodule when p is a nonnegative integer. From now on we shall assume that $p\in\mathbb{Z}_+$. Then we have

Corollary 1. *The $U_q[sl(n+1|m)]$ module $\bar{W}_p$ has an invariant submodule V_p with basis vectors*

$$|p;r_1,r_2,\ldots,r_{n+m}\rangle,\qquad \textit{with}\quad \sum_{i=1}^{n+m}r_i>p.$$

The quotient module $W_p = \bar{W}_p/V_p$ is an irreducible representation for $U_q[sl(n+1|m)]$. The basis vectors of W_p are given by (the representatives of)

$$|p; r_1, r_2, \ldots, r_{n+m}\rangle, \qquad \textit{with} \quad \sum_{i=1}^{n+m} r_i \le p. \tag{29}$$

Now we select a class of Fock modules important for physical applications. These are the ones for which the standard Fock metric is positive definite, and for which the representatives of $a_i^{\pm}$ and H_i $(i = 1, \ldots, n+m)$ satisfy the Hermiticity conditions:

$$(a_i^+)^\dagger = a_i^-, \qquad (a_i^-)^\dagger = a_i^+, \qquad (H_i)^\dagger = H_i. \tag{30}$$

For the Fock representation W_p, we can define a Hermitian form $(\,,\,)$ by requiring

$$(|0\rangle\,,\,|0\rangle) = \langle 0|0\rangle = 1, \tag{31}$$

and by postulating that the Hermiticity conditions (30) should be satisfied, i.e.

$$(a_i^{\pm} v, w) = (v, a_i^{\mp} w), \qquad \forall\, v, w \in W_p. \tag{32}$$

Then any two vectors $|p; r_1, r_2, \ldots, r_{n+m}\rangle$ and $|p; r_1', r_2', \ldots, r_{n+m}'\rangle$ with $(r_1, r_2, \ldots, r_{n+m}) \neq (r_1', r_2', \ldots, r_{n+m}')$ are orthogonal and

$$(|p; r_1, r_2, \ldots, r_{n+m}\rangle, |p; r_1, r_2, \ldots, r_{n+m}\rangle) = \frac{[p]!}{[p-R]!} \prod_{i=1}^{n+m} [r_i]! = \frac{[p]!}{[p-R]!} \prod_{i=1}^{n} [r_i]!, \tag{33}$$

where $R = r_1 + r_2 + \cdots + r_{n+m}$. The straightforward computations show that Hermiticity conditions hold if q is a phase, i.e.

$$q = e^{i\phi} \qquad (-\pi < \phi < \pi). \tag{34}$$

Let us now further investigate when the Hermitian form $(\,,\,)$ is an inner product. This means that for every $(r_1, \ldots, r_{n+m})$ with $0 \le R \le p$, the value in (33) should be positive. In particular, this implies that all the numbers

$$[p],\ [p-1],\ [p-2],\ \ldots,\ [2],\ [1]$$

should be positive. However, since $q = e^{i\phi}$ is a phase, we have

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}} = \frac{\sin(k\phi)}{\sin(\phi)}.$$

The common domain where all functions

$$\frac{\sin(2\phi)}{\sin(\phi)},\ \frac{\sin(3\phi)}{\sin(\phi)},\ \ldots,\ \frac{\sin(p\phi)}{\sin(\phi)}$$

are positive is

$$-\frac{\pi}{p} < \phi < \frac{\pi}{p}.$$

Thus we have

Theorem 4. *The irreducible Fock module W_p $(p \ge 2)$ is unitary if and only if q is a phase, i.e. $q = e^{i\phi}$, with $-\frac{\pi}{p} < \phi < \frac{\pi}{p}$.*

Observe that whether q is a root of unity or not does not have any effect on the irreducibility or unitarity of the Fock module W_p, as long as the conditions of Theorem 4 are satisfied. Indeed, suppose that $q = e^{i\phi}$ is a root of unity with ϕ a rational multiple of π and $-\frac{\pi}{p} < \phi < \frac{\pi}{p}$. Then the smallest integer N for which $q^N = -1$ is greater than p. As a consequence, the rhs in (33) is never zero. This implies that there are no singular vectors among the weight vectors $|p; r_1, \ldots, r_{n+m}\rangle$, and thus irreducibility holds.

Under the conditions of Theorem 4, we can define an orthonormal basis of W_p :

$$|p; r_1, r_2, \ldots, r_{n+m}) = \sqrt{\frac{[p - \sum_{l=1}^{n+m} r_l]!}{[p]![r_1]! \cdots [r_{n+m}]!}} |p; r_1, r_2, \ldots, r_{n+m}\rangle, \tag{35}$$

where $0 \le \sum_{l=1}^{n+m} r_l \le p$. In the new basis (35) the transformation formulas (20)–(22) read $(i = 1, \ldots, n+m)$:

$$H_i|p; r_1, r_2, \ldots, r_{n+m}) = \left(p - (-1)^{\theta_i} r_i - \sum_{j=1}^{n+m} r_j\right) |p; r_1, r_2, \ldots, r_{n+m}), \tag{36}$$

$$\begin{aligned} a_i^-|p; r_1, \ldots, r_{n+m}) &= (-1)^{\theta_1 r_1 + \cdots + \theta_{i-1} r_{i-1}} \\ &\times q^{r_1 + \cdots + r_{i-1}} \sqrt{[r_i]\left[p - \sum_{l=1}^{n+m} r_l + 1\right]} |p; r_1, \ldots, r_{i-1}, r_i - 1, r_{i+1}, \ldots, r_{n+m}), \end{aligned} \tag{37}$$

$$\begin{aligned} a_i^+|p; r_1, \ldots, r_{n+m}) &= (-1)^{\theta_1 r_1 + \cdots + \theta_{i-1} r_{i-1}} \bar{q}^{r_1 + \cdots + r_{i-1}} (1 - \theta_i r_i) \\ &\times \sqrt{[r_i + 1]\left[p - \sum_{l=1}^{n+m} r_l\right]} |p; r_1, \ldots, r_{i-1}, r_i + 1, r_{i+1}, \ldots, r_{n+m}). \end{aligned} \tag{38}$$

4 Properties of the underlying statistics

In the present section we indicate that each $U_q[sl(n+1|m)]$ module W_p can be considered as a state space, where a_i^+ (resp. a_i^-) can be interpreted as operators creating (resp. annihilating) "particles" with, say, energy ε_i. To this end consider a "free" Hamiltonian

$$H = \sum_{i=1}^{n+m} \varepsilon_i e_{ii}. \tag{39}$$

Then

$$[H, a_i^\pm] = \pm \varepsilon_i a_i^\pm. \tag{40}$$

This result together with equations (37)–(38) allows one to interpret a_i^+ as an operator creating a particle with energy ε_i, or more precisely, creating a particle on the i-th orbital. The operator a_i^- annihilates a particle with energy ε_i, or equivalently annihilates a particle on the i-th orbital. On every orbital i with $i = n+1, \ldots, n+m$ there cannot be more than one particle since $(a_i^+)^2 = 0$ for $i = n+1, \ldots, n+m$, whereas such a restriction does not hold for the first n orbitals. These are Fermi like (resp. Bose like) properties. There is however one essential difference. If the corresponding Fock module is characterized by p then no more than p "particles" can be accommodated in the system, $\sum_{i=1}^{n+m} r_i \le p$. Hence the number of particles that can be accommodated on a given orbital, keeping the number of particles on all other orbitals fixed, depends on how many particles have already been accommodated in the system. If $\sum\limits_{i=1}^{n+m} r_i < p$ the particles behave similar to bosons and fermions, but are neither bosons nor fermions since the maximum number of the particles in the system cannot exceed p. This condition together with the restrictions for the orbitals with $i = n+1, \ldots, n+m$ is the analogue of the Pauli principle for this statistics.

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On One Algebra of Temperley–Lieb Type

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An algebra generated by projections with relations of Temperley–Lieb type is considered. Knowledge of Gröbner basis of the ideal allows to find a linear basis of the algebra. Some questions of representation theory for this algebra were studied in [13]. Obtained in the present paper are the additional relations, which hold in all finite-dimensional irreducible $*$-representations, although they do not hold in the algebra.

1 Introduction

Temperley–Lieb algebras generated by n projections $p_1, \dots, p_n$ with relations

$$p_i p_j = p_j p_i, \quad |i-j| > 1, \quad p_i p_{i\pm 1} p_i = \tau p_i, \quad \tau \in \mathbb{R},$$

appeared in [1, 2] in the context of ice-type models. On the other hand, they were applied to studying of von Neumann algebras and problems of knots theory by V. Jones (see [3, 4]). Representations of Temperley–Lieb algebras were studied and used by H. Wenzl, F.M. Goodman, P.P. Martin (see, e.g., [5, 6, 7, 8, 9]) and other authors. Values of parameter τ such that the representations exist were found, a description of irreducible representations was given, their dimensions were calculated and other questions were considered.

In [13] we considered the analogous questions of representation theory for modification of Temperley–Lieb algebra: algebra generated by projections $p_1, \dots, p_n$ with relations

$$p_i p_j = 0, \quad |i-j| > 1, \quad (i,j) \neq (1,n)\,; \quad p_i p_{i\pm 1} p_i = \tau p_i, \quad p_1 p_n p_1 = \tau p_1, \quad p_n p_1 p_n = \tau p_n.$$

In the present paper we find the linear basis of this algebra and consider its properties. Furthermore, some properties of the representations of the algebra are studied by using the results of [13]. New relations in the finite-dimensional irreducible $*$-representations of the algebra allow to prove that the representations obtained by the action of group $\mathbb{Z}_n$ on the operators $P_1, \dots, P_n$ are equivalent.

The paper is arranged as follows. In Section 2 we give main definitions and designations. A set of values of parameter τ when the finite-dimensional $*$-representations exist and a description of irreducible $*$-representations up to a unitary equivalence are presented (see [13]). In Section 3 we find the linear basis of algebra in question using the Diamond Lemma (see, e.g., [10, 11, 12]) and discovery additional relations in the finite-dimensional irreducible $*$-representations of the algebra.

2 Description of all finite-dimensional irreducible $*$-representations of algebra $TL_{\tau,n,\Gamma}$

We are going to study $*$-algebra generated by n ($n \geq 3$) projections with relations depending on real parameter τ:

$$TL_{\tau,n,\Gamma} = \mathbb{C}\Big\langle e, p_1, \dots, p_n \mid p_i = p_i^2 = p_i^*, p_i p_j p_i = \gamma_{ij} p_i,$$

$$(\gamma_{ij}) = \Gamma = \begin{pmatrix} 1 & \tau & 0 & \cdots & 0 & \tau \\ \tau & 1 & \tau & 0 & \cdots & 0 \\ 0 & \tau & 1 & \tau & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \tau & 1 & \tau \\ \tau & 0 & \cdots & 0 & \tau & 1 \end{pmatrix} \Big\rangle.$$

The theorems giving information about all finite-dimensional $*$-representations of $TL_{\tau,n,\Gamma}$ can be found in [13], but we need some results about these representations here. First of all we give the theorem about the set of the values of parameter τ when the $*$-representations exist and the description of construction of operators of these representations. In the following we consider only nontrivial finite-dimensional irreducible $*$-representations and name them simply 'representations'. If π is a representation of algebra $TL_{\tau,n,\Gamma}$ then P_i will denote $\pi(p_i)$.

Theorem 1. *Representations of algebra $TL_{\tau,n,\Gamma}$ exist in finite-dimensional space H iff*

$$\tau \in \left[0, \frac{1}{4\cos^2 \frac{\pi}{n}}\right] =: \Sigma_n.$$

Then, if $\tau = 0$ all p_i are orthogonal and if $\tau \neq 0$ then a basis of H exists such that operators of the representation are as follows:

$$P_1 = \operatorname{diag}(1, 0, \ldots, 0),$$

$$P_i = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ 0 & \cdots & 0 & \tau_{i-2} & \sqrt{\tau_{i-2} - \tau_{i-2}^2} & 0 & \cdots \\ 0 & \cdots & 0 & \sqrt{\tau_{i-2} - \tau_{i-2}^2} & 1 - \tau_{i-2} & 0 & \cdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \quad i = 2, \ldots, n-1,$$

where $\tau_i = \frac{\tau}{1-\tau_{i-1}}$, $i = 1, \ldots, n-3$, $\tau_0 = \tau$ and the number of zeroes on the top of diagonal is equal to $i-2$.

$$P_n = \begin{pmatrix} \tau & l_1 & l_2 & \cdots & l_{n-3} & \lambda & \mu \\ l_1 & \frac{l_1^2}{\tau} & \frac{l_1 l_2}{\tau} & \cdots & \frac{l_1 l_{n-3}}{\tau} & \frac{l_1 \lambda}{\tau} & \frac{l_1 \mu}{\tau} \\ l_2 & \frac{l_1 l_2}{\tau} & \frac{l_2^2}{\tau} & \cdots & \frac{l_2 l_{n-3}}{\tau} & \frac{l_2 \lambda}{\tau} & \frac{l_2 \mu}{\tau} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ l_{n-3} & \frac{l_1 l_{n-3}}{\tau} & \frac{l_2 l_{n-3}}{\tau} & \cdots & \frac{l_{n-3}^2}{\tau} & \frac{l_{n-3}\lambda}{\tau} & \frac{l_{n-3}\mu}{\tau} \\ \bar{\lambda} & \frac{l_1 \bar{\lambda}}{\tau} & \frac{l_2 \bar{\lambda}}{\tau} & \cdots & \frac{l_{n-3}\bar{\lambda}}{\tau} & \frac{|\lambda|^2}{\tau} & \frac{\bar{\lambda}\mu}{\tau} \\ \mu & \frac{l_1 \mu}{\tau} & \frac{l_2 \mu}{\tau} & \cdots & \frac{l_{n-3}\mu}{\tau} & \frac{\mu\lambda}{\tau} & \frac{\mu^2}{\tau} \end{pmatrix},$$

where $l_i = (-1)^i \tau \prod_{j=0}^{i-1} \frac{\tau_j}{\sqrt{\tau_j - \tau_j^2}}$. λ is such that

$$\left(l_{n-3} + \lambda \frac{\sqrt{\tau_{n-3} - \tau_{n-3}^2}}{\tau_{n-3}} \right) \left(l_{n-3} + \bar{\lambda} \frac{\sqrt{\tau_{n-3} - \tau_{n-3}^2}}{\tau_{n-3}} \right) = \frac{\tau^2}{\tau_{n-3}},$$

and $\mu^2 = \tau - \tau^2 - \sum_{j=1}^{n-3} l_j^2 - |\lambda|^2$, $\mu \geq 0$.

Proof. The proof of this theorem can be found in [13]. ■

Remark 1. If $\tau \in \Sigma_n \setminus \{0\}$ then dimension of H is equal to n if $\lambda, \mu \neq 0$, to $n-1$ if $\lambda \neq 0, \mu = 0$ and to $n-2$ if $\lambda = \mu = 0$ (i.e. $\tau_{n-3} = 1$).

In the following we assume that $\tau \neq 0$.

Remark 2. Theorem 1 gives explicit construction of operators of representations. One can easily check that different λ's define inequivalent representations. So, we say that each irreducible representation of $*$-algebra $TL_{\tau,n,\Gamma}$ is given by the number λ.

3 Linear basis in the algebra $TL_{\tau,n,\Gamma}$

To found a linear basis in the algebra $TL_{\tau,n,\Gamma}$ we use the Diamond Lemma (see, e.g., [10, 11, 12]).

Let $F_n = \mathbb{C}\langle e, p_1, \dots, p_n\rangle$ be a free associative algebra and W be a set of words on the alphabet $\{e, p_1, \dots, p_n\}$ with homogeneous lexicographic order and minimal element e.

Let I be the ideal generated by

$$R = \{p_i^2 - p_i,\ p_i p_{i\pm1} p_i - \tau p_i,\ p_1 p_n p_1 - \tau p_1,\ p_n p_1 p_n - \tau p_n,\ p_i p_j \mid \\ |i-j| > 1,\ (i,j) \neq (1,n), (n,1)\}.$$

It is not difficult to prove that R is the reduced Gröbner basis of the ideal I. This implies that the next theorem holds:

Theorem 2. *A linear basis of the algebra $TL_{\tau,n,\Gamma}$ is:*

$$e, p_1,\ p_1p_2, \dots, (p_1p_2\dots p_n)^k,\ (p_1p_2\dots p_n)^k p_1\dots p_j,\ k\in\mathbb{N},\ j = 1,\dots,n-1;$$
$$p_2,\ p_2p_3, \dots, (p_2p_3\dots p_np_1)^k,\ (p_2p_3\dots p_np_1)^k p_2\dots p_j,\ k\in\mathbb{N},\ j = 2,\dots,n;$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$p_n,\ p_np_1, \dots, (p_np_1p_2\dots p_{n-1})^k,\ (p_np_1\dots p_{n-1})^k p_n\dots p_j,\ k\in\mathbb{N},\ j = n, 1,\dots,n-2$$

and adjoint elements of these words.

Direct calculations imply that the basis of modification of Temperley–Lieb algebra has the analogous property to the basis of Temperley–Lieb algebra:

Proposition 1. *Product of any two basis elements of algebra $TL_{\tau,n,\Gamma}$ is either zero or a power of τ times another basis element.*

Proposition 2. *For any representation π the following relations hold:*

$$P_1P_2\cdots P_nP_1 = f(\lambda)\,P_1,\quad P_iP_{i+1}\cdots P_nP_1\cdots P_{i-1}P_i = f(\lambda)\,P_i,\quad i = 2,\dots,n,$$

where

$$f(\lambda) = \left(\tau_{n-3}l_{n-3} + \sqrt{\tau_{n-3} - \tau_{n-3}^2}\,\bar{\lambda}\right)\prod_{j=0}^{n-4}\sqrt{\tau_j - \tau_j^2}.$$

Note that these relations are not valid in the algebra $TL_{\tau,n,\Gamma}$ because left and right parts of the equations are the elements of the linear basis of the algebra $TL_{\tau,n,\Gamma}$.

Corollary 1. *The algebra $TL_{\tau,n,\Gamma}$ is infinite algebra. But for any finite-dimensional irreducible $*$-representation π the algebra $\pi(TL_{\tau,n,\Gamma})$ is infinite algebra.*

Corollary 2. *(Action on the set $\{P_1, \dots, P_n\}$ of the group $\mathbb{Z}_n$.) Let π, $\tilde{\pi}$ be the representations of the algebra $TL_{\tau,n,\Gamma}$ such that $\pi(p_i) = P_i$, $\tilde{\pi}(p_1) = P_i$, $\tilde{\pi}(p_2) = P_{i+1}$, $\dots$, $\tilde{\pi}(p_{n-i+2}) = P_1$, $\dots$, $\tilde{\pi}(p_n) = P_{i-1}$ $(i = 1, \dots, n)$. Then π and $\tilde{\pi}$ are equivalent.*

Proof. Theorem 1 implies that $\tilde{\pi}$ is equivalent to the representation $\hat{\pi}$ such that $\hat{\pi}(p_i) = P_i$ (but a parameter $\hat{\lambda}$ which defines this representation is possible different from the parameter λ that defines the representation π), i.e., there exists an unitary operator C that

$$CP_iC^{-1} = P_1, \quad CP_{i+1}C^{-1} = P_2, \quad \dots, \quad CP_1C^{-1} = P_{n-i+2}, \quad \dots, \quad CP_{i-1}C^{-1} = P_n.$$

From Proposition 2 it follows that

$$P_1P_2\cdots P_nP_1 = f\left(\hat{\lambda}\right)P_1$$

that implies

$$P_iP_{i+1}\cdots P_nP_1\cdots P_{i-1}P_i = f\left(\hat{\lambda}\right)P_i.$$

But

$$P_iP_{i+1}\cdots P_nP_1\cdots P_{i-1}P_i = f(\lambda)P_i$$

that implies $f(\lambda) = f\left(\hat{\lambda}\right)$ or $\lambda = \hat{\lambda}$ what proves the statement of Corollary 2. ■

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On Involutions which Preserve Natural Filtration

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In this work we study involutions in finitely presented $*$-algebras which preserve the natural filtration.

1 Introduction

Introducing additional structures is often useful in a study of algebraic objects, in particular finitely presented algebras and their representations, – introducing topology in algebras gives a comprehensive theory of Banach algebras or, more generally, a theory of locally convex algebras; introducing an involution, which we can consider as some inner symmetry, calls into being the theory of $*$-algebras; considering an involution together with the corresponding norm gives the theory of C^*-algebras. Moreover, on the one hand, studying not all representations but only those which "conserve" this additional structure (for example, $*$-representations) is simple (for example, $*$-representations are indecomposable if and only if they is irreducible, see [1]) on the other hand, this is often sufficient for applications.

In [1] the theory of $*$-representations of finitely presented $*$-algebras is studied, and the involution in the considered $*$-algebras often preserves filtration (see Definition 1). In this article we consider the following question. Let $\mathbb{F}_n$ be a free algebra with n generators $x_1, \ldots, x_n$ and an identity e, and lets us also have a unital finitely presented algebra

$$\mathbf{A} = \mathbb{C}\langle\, x_1, \ldots, x_n \,|\, q_1 = 0, \ldots, q_m = 0 \,\rangle,$$

where $q_k \in \mathbb{F}$, $k = 1, \ldots, m$. We can assume, without loss of generality, that all relations q_k are nonlinear, for otherwise, the algebra $\mathbf{A}$ is isomorphic to an algebra with a smaller number of generators (roughly speaking, we can exclude generators that are linear combinations of the others). We will denote by $V(\mathbf{A})$ the linear subspace of $\mathbf{A}$ generated by the elements $x_0 = e, x_1, \ldots, x_n$. Then the question is how many involutions which map $V(\mathbf{A})$ into itself exist in the algebra $\mathbf{A}$ such that the corresponding $*$-algebras are not $*$-isomorphic.

The answer is that such an involution is unique and so we can always suppose that the generators are self-adjoint (see Theorem 1 and Proposition 1). Moreover, in some cases there is a $*$-isomorphism between the corresponding $*$-algebras such as it "conserves" the relations (see Theorem 1 and examples).

2 Main result

We will denote the free $*$-algebra with n self-adjoint generators z_k by $\mathbb{F}_n^*$. Some other involution will be denoted by $\star$. It is given by defining its values on generators. We will denote the free $*$-algebra with such an involution by

$$\mathbb{F}_n^\star = \mathbb{C}\langle\, x_1, \ldots, x_n \,|\, x_k^\star = p_k, k = 1, \ldots, n \,\rangle,$$

where $p_k \in \mathbb{F}_n$.

Definition 1. We say that an involution $\star$ of a $*$-algebra $\mathbf{A}^\star$ preserves the natural filtration iff the involution maps $\mathrm{V}(\mathbf{A}^\star)$ into itself.

Theorem 1. *Let an involution $\star$ of the $*$-algebra $\mathbb{F}_n^\star$ preserve the natural filtration. Then there is a $*$-isomorphism $\varphi : \mathbb{F}_n^* \to \mathbb{F}_n^\star$. Moreover, $\varphi(V(\mathbb{F}_n^*)) = V(\mathbb{F}_n^\star)$.*

Proof. We can assume that the first $n-l$ generators are self-adjoint and the others are not, such otherwise, we can renumber the generators. We will prove the theorem by induction on the number l of the generators that are not self-adjoint.

If $l = 0$ then there is nothing to prove, since all the generators are self-adjoint.

Let $1 \leqslant l \leqslant n$. Put

$$y_k = \frac{x_k + x_k^\star}{2}, \qquad k = 0, \dots, n.$$

It is evident that $y_k^\star = y_k$. Because the involution preserves the filtration, $x_k^\star \in V(\mathbb{F}_n^\star)$ and so $y_k \in V(\mathbb{F}_n^\star)$.

If $y_0 = e, y_1, \dots, y_n$ are linearly independent then we define $\varphi : \mathbb{F}_n^* \to \mathbb{F}_n^\star$ on the generators by $\varphi(z_k) = y_k$, $k = 0, \dots, n$, $z_0 = e$. Since $\dim V(\mathbb{F}_n^\star) = n+1$ and $y_0, y_1, \dots, y_n$ are linearly independent and lie in $V(\mathbb{F}_n^\star)$, y_k, $k = 0, \dots, n$, is a linear basis of $V(\mathbb{F}_n^\star)$ and so

$$x_k = \sum_{j=0}^{n} \alpha_k^j y_j, \qquad \alpha_k^j \in \mathbb{C}.$$

Then the homomorphism inverse to φ is defined on the generators by

$$\varphi^{-1}(x_k) = \sum_{j=0}^{n} \alpha_k^j z_j.$$

So φ is an isomorphism of the algebras $\mathbb{F}_n^\star$ and $\mathbb{F}_n^*$. It is evident that φ is also a $*$-homomorphism and $\varphi(V(\mathbb{F}_n^*)) = V(\mathbb{F}_n^\star)$.

Let now $y_0 = e, y_1, \dots, y_n$ be linearly dependent. Then, since the first $n-l$ generators are self-adjoint, $y_j = x_j$ for $j = 0, \dots, n-l$ and, consequently, y_j are linearly independent. Then there exists k $(n-l < k \leqslant n)$ such that

$$y_k = \sum_{j \neq k} \lambda_j y_j, \qquad \lambda_j \in \mathbb{C}.$$

And since y_j are self-adjoint,

$$y_k = \sum_{j \neq k} \overline{\lambda}_j y_j, \qquad \lambda_j \in \mathbb{C}.$$

If we put $a_j = (\lambda_j + \overline{\lambda}_j)/2$ then we get

$$y_k = \sum_{j \neq k} a_j y_j, \qquad a_j \in \mathbb{R}.$$

Renumbering the generators we can suppose that $k = n-l+1$.

Put

$$\mathbb{F}_n^{\star_1} = \mathbb{C}\langle\, v_1, \dots, v_n \,|\, v_j^{\star_1} = v_j, j = 1, \dots, k, v_j^{\star_1} = q_j, j > k \,\rangle,$$

where

$$q_j = p_j\left(v_1,\dots,v_{k-1},-2iv_k+\sum_{j\neq k}a_jv_j,v_{k+1},\dots,v_n\right),\qquad j>k.$$

Define $\psi:\mathbb{F}_n^{\star_1}\to\mathbb{F}_n^{\star}$ on generators by the formula $\psi(v_j)=x_j$, if $j\neq k$, and

$$\psi(v_k)=\frac{i}{2}\left(x_k-\sum_{j\neq k}a_jx_j\right).$$

It is evident that ψ is an isomorphism of the algebras $\mathbb{F}_n^{\star_1}$ and $\mathbb{F}_n^{\star}$. Let us show that ψ is a $*$-homomorphism.

If $j<k$, then $\psi(v_j)^\star = x_j^\star = x_j = \psi(v_j) = \psi(v_j^{\star_1})$.

If $j>k$, then $\psi(v_j)^\star = x_j^\star = p_j$ and again

$$\begin{aligned}\psi(v_j^{\star_1}) = \psi(q_j) &= \psi\left(p_j\left(v_1,\dots,v_{k-1},-2iv_k+\sum_{j\neq k}a_jv_j,v_{k+1},\dots,v_n\right)\right)\\ &= p_j\left(x_1,\dots,x_{k-1},x_k-\sum_{j\neq k}a_jx_j+\sum_{j\neq k}a_jx_j,x_{k+1},\dots,x_n\right) = p_j = \psi(v_j^{\star_1}).\end{aligned}$$

Finally,

$$\psi(v_k)^\star = -\frac{i}{2}\left(x_k^\star-\sum_{j\neq k}a_jx_j^\star\right)\quad\text{and}\quad \psi(v_k^{\star_1})=\psi(v_k)=\frac{i}{2}\left(x_k-\sum_{j\neq k}a_jx_j\right).$$

So

$$\psi(v_k^{\star_1})-\psi(v_k)^\star = i\left(y_k-\sum_{j\neq k}a_jy_j\right)=0,$$

i.e., $\psi(v_k^*)=\psi(v_k)^*$.

We have proved that $\mathbb{F}_n^\star$ and $\mathbb{F}_n^{\star_1}$ are $*$-isomorphic. Further, by the definition of ψ we again have $\psi(V(\mathbb{F}_n^{\star_1}))=V(\mathbb{F}_n^\star)$. And now we have $l-1$ generators in $\mathbb{F}_n^{\star_1}$ that are not self-adjoint and so, by the inductive assumption, $\mathbb{F}_n^{\star_1}$ is $*$-isomorphic to $\mathbb{F}_n^*$ and, consequently, $\mathbb{F}_n^\star$ is $*$-isomorphic to $\mathbb{F}_n^*$. ■

3 Corollary and examples

In this section we will obtain a corollary of Theorem 1 and consider some examples.

Consider the $*$-algebra

$$\mathbf{A}=\mathbb{C}\langle\, x_1,\dots,x_n \,|\, x_k^\star = p_k, k=1,\dots,n,\ r_1=0,\dots,r_m=0\,\rangle,$$

where $r_k\in\mathbb{F}_n$, $k=0,\dots,m$. Let I be a $*$-ideal generated by $r_1,\dots,r_m$, i.e., $\mathbf{A}$ is a $*$-isomorphic to the factor $\mathbb{F}_n^\star/\mathrm{I}$.

By increasing the number of generators (not more than two times) and adding new relations we always can construct a $*$-algebra which is $*$-isomorphic to $\mathbf{A}$ such that its generators are self-adjoint. The corollary of Theorem 1 claims that if the involution is “good” then we can leave the number of the generators and relations the same as in $\mathbf{A}$ and the length of words in the relations does not grow.

Proposition 1. *Let the involution $\star$ preserves the filtration. Then the $*$-algebra* $\mathbf{A}$ *is $*$-isomorphic to the $*$-algebra*

$$\mathbf{B} = \mathbb{C}\langle\, z_1, \dots, z_n \,|\, z_k^* = z_k, k = 1, \dots, n,\ s_1 = 0, \dots, s_m = 0 \,\rangle,$$

where s_k have the same degrees as r_k, $k = 1, \dots, m$.

Proof. Since the involution $\star$ preserves the filtration then, there exists a $*$-isomorphism $\varphi : \mathbb{F}_n^\star \to \mathbb{F}_n^*$. Denote by $\mathrm{J} = \varphi(\mathrm{I})$ the $*$-ideal generated by the relations $s_1 = \varphi(r_1), \dots, s_m = \varphi(r_m)$. It is evident that so defined s_k have the same degrees as r_k. Then we can put $\mathbf{B} = \mathbb{F}_n^*/\mathrm{J}$.

Let i be an injection of I into $\mathbb{F}_n^\star$ and π a projection of the latter into $\mathbf{A}$. Similarly, let i_0 be an injection of J into $\mathbb{F}_n^*$ and π_0 a projection into $\mathbf{B}$. The restriction of φ to I will be denoted by φ_0. Then we get a commutative diagram of $*$-homomorphisms,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{I} & \xrightarrow{i} & \mathbb{F}_n^\star & \xrightarrow{\pi} & \mathbf{A} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle \varphi_0} & & \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle \psi} & & \\
0 & \longrightarrow & \mathrm{J} & \xrightarrow{i_0} & \mathbb{F}_n^* & \xrightarrow{\pi_0} & \mathbf{B} & \longrightarrow & 0
\end{array}$$

where ψ is defined by the formula $\psi(\pi(a)) = \pi_0(\varphi(a))$, for any $a \in \mathbb{F}_n^\star$.

Now we show that ψ is well-defined. Indeed, since π is surjective, ψ is defined for all elements of $\mathbf{A}$. If $\pi(a) = 0$ then $a \in \mathrm{I}$ and so $\varphi(a) \in \mathrm{J}$, consequently, $\psi(\pi(a)) = \pi_0(\varphi(a)) = 0$.

It is evident that ψ is surjective. Now we show that it is injective.Indeed, if $\psi(\pi(a)) = 0$, then it means that $\pi_0(\varphi(a)) = 0$ and so $\varphi(a) \in \mathrm{J}$, consequently, $a \in \mathrm{I}$, from where we get $\pi(a) = 0$. It is also evident that ψ is a $*$-homomorphism.

So we have constructed a $*$-isomorphism of the $*$-algebras $\mathbf{A}$ and $\mathbf{B}$. ■

Actually we have "changed" the generators in $\mathbf{A}$ so that the new generators are self-adjoint. But the next example shows that, generally speaking, the relations are changed too.

Example 1. Consider the $*$-algebra

$$\mathbf{Q_2} = \mathbb{C}\langle\, q_1, q_2 \,|\, q_1^\star = q_2, q_2^\star = q_1, q_1^2 = q_1, q_2^2 = q_2 \,\rangle.$$

A $*$-isomorphism $\varphi : \mathbb{F}_2^\star \to \mathbb{F}_2^*$ is defined by the formulas

$$\varphi(q_1) = z_1 + iz_2, \qquad \varphi(q_2) = z_1 - iz_2.$$

Then

$$\varphi(q_1^2 - q_1) = (z_1 + iz_2)^2 - z_1 - iz_2 = z_1^2 - z_2^2 + i\{z_1, z_2\} - z_1 - iz_2,$$

similarly

$$\varphi(q_2^2 - q_2) = z_1^2 - z_2^2 - i\{z_1, z_2\} - z_1 + iz_2,$$

where $\{\,,\}$ is the anticommutator.

It is evident that the ideal generated by these relations is also generated by the relations

$$z_1^2 - z_2^2 = z_1 \quad \text{and} \quad \{z_1, z_2\} = z_2.$$

So $\mathbf{Q_2}$ is $*$-isomorphic to the $*$-algebra

$$\mathbb{C}\langle\, z_1, z_2 \,|\, z_1^* = z_1, z_2^* = z_2, z_1^2 - z_2^2 = z_1, \{z_1, z_2\} = z_1 \,\rangle.$$

On the other hand, it is not difficult to show that there is no $*$-isomorphisms between $\mathbf{Q_2}$ and the $*$-algebra

$$\mathbb{C}\langle\, x_1, x_2 \,|\, x_1^* = x_1, x_2^* = x_2, x_1^2 = x_1, x_2^2 = x_2 \,\rangle.$$

The next two examples show that there are algebras that are not free for which an analogue of Theorem 1 is also true.

Example 2. Consider the $*$-algebra of polynomials in n variables, P_n. It is a factor of the free algebra by the ideal I generated by the relations

$$[x_j, x_k] = 0, \qquad j, k = 1, \dots n,$$

where $[\,,]$ is the commutator. All elements of the ideal I can be written as $[p_1, p_2]$, where $p_1, p_2 \in \mathbb{F}_n$. Then, for any involution in $\mathbb{F}_n$, $[p_1, p_2]^\star = [p_2^\star, p_1^\star] \in \mathrm{I}$ so I is a $*$-ideal. Let $\star$ preserves the filtration. Then the $*$-ideal $\varphi(\mathrm{I})$ consists of all elements which can be written as $[\varphi(p_1), \varphi(p_2)]$. So it is generated by the relations

$$[z_j, z_k] = 0, \qquad j, k = 1, \dots n,$$

And we have the $*$-isomorphism of P_n^* and $P_n^\star$.

Example 3. Consider one more algebra for which a theorem analogous to Theorem 1 holds. Let

$$\mathbf{A} = \mathbb{C}\langle\, p, q \mid [[p, q], p] = 0, [[p, q], q] = 0 \,\rangle.$$

Let I be an ideal generated by the corresponding relations. Then it is evident that for any $a, b, c \in V(\mathbb{F}_n^\star)$ we have $[[a, b], c] \in \mathrm{I}$.

Now, let us introduce in $\mathbf{A}$ an involution $\star$ which preserves the filtration. Let us show that the ideal I is a $*$-ideal,

$$-[[p, q], p]^\star = [p, [p, q]]^\star = [[p, q]^\star, p^\star] = [[q^\star, p^\star], p^\star],$$

but $p^\star, q^\star \in V(\mathbb{F}_n^\star)$ so $[[p, q], p]^\star \in \mathrm{I}$. Similarly, $[[p, q], q]^\star \in \mathrm{I}$.

Since $\star$ preserves the filtration, by Theorem 1 there is a $*$-isomorphism $\varphi : \mathbb{F}_2^\star \to \mathbb{F}_2^*$ and there exist elements $a_1, a_2 \in V(\mathbb{F}_2^\star)$ such that $\varphi(a_1) = z_1$ and $\varphi(a_2) = z_2$, where z_1 and z_2 are generators of $\mathbb{F}_2^*$. Then the $*$-ideal $\varphi(\mathrm{I})$ is generated by the relations

$$[[z_1, z_2], z_1] = 0, \qquad [[z_1, z_2], z_2] = 0.$$

So we have a $*$-isomorphism of $\mathbf{A}^\star$ and the $*$-algebra

$$\mathbb{C}\langle\, z_1, z_2 \mid z_1^* = z_1, z_2^* = z_2, [[z_1, z_2], z_1] = 0, [[z_1, z_2], z_2] = 0 \,\rangle.$$

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Supersymmetry

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Extended SUSY with Central Charges in Quantum Mechanics

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We present a new form of supersymmetric quantum mechanics which is characterized by presence of non-trivial central charges. We show that the corresponding extended SUSY appears in a number of popular quantum mechanical models.

1 Introduction

Supersymmetric quantum mechanics (SUSY QM) [1] appeared first as a toy model for better understanding of SUSY itself. However, it turns out that SUSY QM systems themselves are so rich in structure and deal with new properties which became recently subjects per se studied by many mathematicians and theoretical physicists. The basic problems and various applications of SUSY quantum mechanics are discussed in a number of papers, refer, e.g., to surway [2].

Soon it appears that SUSY QM systems can admit more than two supercharges and so to have richer symmetry called extended SUSY. Such extended SUSY has good physical grounds, since there exists a number of realistic physical systems which admit more than two supercharges, see e.g., [3, 4, 6, 7]. Moreover, it was demonstrated in [3, 6, 7] that the Schrödinger–Pauli and the Dirac equations admit not only extended SUSY but also rather large algebras of discrete involutive symmetries isomorphic to $gl(4,\mathbb{C})$ and $gl(8,\mathbb{R})$ respectively. Thus it seams that extended SUSY is closely connected to discrete symmetries.

In the present paper we continue in our investigations [3, 6, 7] and study QM systems which admit extended SUSY. Moreover, we consider generalized extensions of symmetry superalgebras generated by additional supercharges and even operators as well. We prove that the Coulomb, Aharonov–Bohm–Colomb (ABC) and Aharonov–Casher systems admit extended SUSY with six supercharges and central charge and, besides, they admit extended algebras of discrete symmetries isomorphic to $gl(8,\mathbb{R})$. All mentioned symmetries are responsible for degeneracy of the corresponding energy spectra.

We introduce the concept of general SUSY QM systems with central charges, and prove that many popular quantum mechanical models are perfect examples of them.

2 Quantum mechanics with extended SUSY

We say that the Schrödinger type equation

$$H\psi = E\psi \tag{1}$$

is supersymmetric and has $N = 2n$ SUSY, if it admits a set of integrals of motion $Q_1, Q_2, \dots, Q_n$ which commute with Hamiltonian H and satisfy the following relations

$$\{Q_a, \bar{Q}_b\} = Q_a\bar{Q}_b + \bar{Q}_bQ_a = 2\delta_{ab}H, \qquad a, b = 1, 2, \dots, n,$$
$$\{Q_a, Q_b\} = \{\bar{Q}_a, \bar{Q}_b\} = 0 \tag{2}$$

with δ_{ab} being the Kronecker symbol and $\bar{Q} = Q^\dagger$.

For $n = 1$ we recognize in (2) the Witten superalgebra which is characteristic algebra appearing in SUSY QM models. This algebra contains two odd elements (supercharges) Q_1 and $\bar{Q}_1$ and the only even element H, thus in this case we have $N = 2$ SUSY. For $n > 1$ one has a QM model with the so-called extended SUSY. Realistic QM models admitting extended SUSY are discussed in [3, 4, 6, 7].

Of course, relations (2) admit a formal generalization to the case when the number of even elements is larger than 1. Then the corresponding defining relations can be transformed to the following ones:

$$\{Q_a, \bar{Q}_b,\} = 2\delta_{ab}H + Z_{ab}, \qquad a, b = 1, 2, \dots, n,$$
$$\{Q_a, Q_b\} = \{\bar{Q}_a, \bar{Q}_b\} = 0, \tag{3}$$

where Z_{ab} are the so called *central charges* which commute with all elements Q_a, $\bar{Q}_a$, H of the superalgebra.

We shall show in Sections 3, 4 that such a generalization appears naturally for some popular QM problems.

3 Extended SUSY for the Coulomb problem

First we shall consider the free Dirac equation

$$(\gamma_\mu p^\mu - m)\psi(x) = 0, \tag{4}$$

where $p_\mu = i\frac{\partial}{\partial x^\mu}$, $\mu = 0, 1, 2, 3$, $x = (x_0, x_1, x_2, x_3)$, γ_μ are the Dirac matrices.

It was shown in [3, 6] that equation (4) admits a 64-dimensional algebra of involutive discrete symmetries. Basis elements of this algebra can be chosen in the following form

$$\Gamma_m, \quad \Gamma_m\Gamma_n, \quad \Gamma_m\Gamma_n\Gamma_p, \quad I_4, \tag{5}$$

were $m, n, p = 0, 1, \dots, 6$, I_4 is the 4×4 unit matrix,

$$\Gamma_\mu = i\gamma_4\gamma_\mu\hat{\theta}_\mu \quad (\text{no sum over } \mu), \qquad \Gamma_4 = i\gamma_4\hat{\theta}, \qquad \gamma_4 = \gamma_0\gamma_1\gamma_2\gamma_3,$$
$$\Gamma_5 = \gamma_4\gamma_2 c\theta, \qquad \Gamma_6 = \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\Gamma_5, \tag{6}$$

$\hat{\theta}$ and c are reflection and complex conjugation operators defined by the following relations:

$$\hat{\theta}_\mu\psi(x) = \psi(\theta_\mu x), \qquad \hat{\theta}\psi(x) = \psi(-x), \qquad c\psi(x) = \psi^*(x),$$
$$\theta_0 x = (-x_0, x_1, x_2, x_3), \qquad \theta_1 x = (x_0, -x_1, x_2, x_3), \qquad \theta_2 x = (x_0, x_1, -x_2, x_3),$$
$$\theta_3 x = (x_0, x_1, x_2, -x_3).$$

Operators (5) transform solutions of the Dirac equation into solutions and form a Lie algebra isomorphic to $gl(8, \mathbb{R})$. We notice that the set of operators (5) include reflections Γ_μ, Γ_4 and pure rotations $\Gamma_\mu\Gamma_\nu$ as well.

The Dirac equation with non-trivial potentials

$$L\psi \equiv \left(\gamma^\mu \pi_\mu - m\right)\psi = 0, \qquad \pi_\mu = p_\mu - eA_\mu \tag{7}$$

does not admit all symmetry operators (5) but only a part of them instead. Nevertheless, we shall show that for some vector-potentials A_μ equation (7) admits extra symmetries which form basses of extended algebras isomorphic to (5).

As an example consider the relativistic Coulomb system described by the Dirac equation (7) with

$$A_1 = A_2 = A_3 = 0, \qquad eA_0 = \frac{\alpha}{|x|} \tag{8}$$

and $|x| = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}$.

Equation (7), (8) admits a specific integral of motion discovered by Johnson and Lippman [8]. We present this constant of motion in the following form

$$\hat{Q} = m\alpha\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{|x|} + iD\left(\boldsymbol{\sigma}\cdot\boldsymbol{p} + i\gamma_4\frac{\alpha}{|x|}\right). \tag{9}$$

Here $D = \gamma_0\left(\boldsymbol{\sigma}\cdot\boldsymbol{J} - \frac{1}{2}\right)$ with $\boldsymbol{J} = \boldsymbol{x}\times\boldsymbol{p} + \boldsymbol{\sigma}/2$ is the Dirac constant of motion, $\boldsymbol{\sigma} = i\boldsymbol{\gamma}\times\boldsymbol{\gamma}/2$.

Operators $\hat{Q}$ and D commute with the Dirac Hamiltonian $H = \gamma_0\boldsymbol{\gamma}\cdot\boldsymbol{p} + \gamma_0 m + \frac{\alpha}{|x|}$ and anticommute among themselves. They are odd elements of the five dimensional superalgebra which contains also three even elements, namely H, $\hat{Q}^2$ and D^2. The commutation relations for odd-even and even-even elements have the form $[A, B] = 0$.

We notice that eigenvalues of Hamiltonian H can be expressed via eigenvalues of D and $\hat{Q}$. Indeed, using the relations

$$D^2 = J^2 + \frac{1}{4}, \qquad Q^2 = D^2\left(H^2 + m^2\right) - \alpha^2 m^2$$

and denoting eigenvalues of mutually commuting operators D^2, Q^2 and H by κ^2, q^2 and E respectively, we obtain the following relation

$$E^2 = \frac{q^2}{\kappa^2} + m^2\left(1 - \frac{\alpha^2}{\kappa^2}\right), \qquad \kappa = 0, 1, 2, \ldots.$$

Using this expression we shall demonstrate that the Coulomb system defined in (7)and (8) admits extended superalgebra which include six supercharges Q_a, $\bar{Q}_a$, $a = 1, 2, 3$ and one central charge $Z_{ab} = \delta_{ab}Z$ where

$$\begin{aligned}
&Q_1 = (1 + i\Gamma_5\Gamma_1\Gamma_2)\hat{Q}, && \bar{Q}_1 = (1 - i\Gamma_5\Gamma_1\Gamma_2)\hat{Q},\\
&Q_2 = i(\Gamma_1 + \Gamma_5)\Gamma_2\Gamma_3)\hat{Q}, && \bar{Q}_2 = i(\Gamma_1 - G_5)\Gamma_2\Gamma_3)\hat{Q},\\
&Q_3 = \Gamma_5(1 + i\Gamma_1\Gamma_3)\hat{Q}, && \bar{Q}_3 = \Gamma_5(1 - i\Gamma_1\Gamma_3)\hat{Q},\\
&Z = \left(\alpha^2 - D^2\right)m^2/\kappa^2, && \hat{H} = H^2.
\end{aligned} \tag{10}$$

Using the relations

$$[\Gamma_k, H] = [\Gamma_k, D] = 0, \qquad \{\Gamma_k, Q\} = \{\Gamma_5, \Gamma_a\} = \{\Gamma_5, i\} = 0, \qquad \{\Gamma_a, \Gamma_b\} = 2\delta_{ab},$$

where $k = 1, 2, 3, 5$, $a, b = 1, 2, 3$, we find that operators (10) commute with H and satisfy superalgebra (3). Thus, *the Coulomb system admits $N = 6$ extended SUSY with non-trivial central charge.* This symmetry algebra is closely related to the 64-dimensional algebra of involutive symmetries described in [3, 6]. Indeed, for any $q \neq 0$ we can define the following symmetry operators of the stationary Dirac equation

$$\hat{\Gamma}_0 = i\Gamma_1\Gamma_2\Gamma_3, \qquad \hat{\Gamma}_a = (Q_a + \bar{Q}_a)/2q, \qquad \hat{\Gamma}_{3+a} = (Q_a - \bar{Q}_a)/2iq, \qquad a = 1,2,3 \tag{11}$$

which satisfy

$$\{\hat{\Gamma}_K, \hat{\Gamma}_N\} = 2g_{KN}, \qquad K, N = 0, 1, \ldots, 6.$$

The only nonzero elements of tensor g_{KN} are $g_{00} = g_{11} = g_{22} = g_{33} = -g_{44} = -g_{55} = -g_{66} = 1$.

All linearly independent products of $\hat{\Gamma}_K$ have the same form (5) as for Γ_μ and form again a basis of algebra $gl(8, \mathbb{R})$.

4 Extended SUSY for Aharonov–Bohm–Coulomb and Aharonov–Casher systems

Let us search for extended SUSY of the system defined by the Dirac equation (7) with an external field being a superposition of the Coulomb potential and the potential generated by a solenoid directed along the third co-ordinate axis. Such configuration corresponds to the so-called Aharonov–Bohm–Coulomb (ABC) system which has been studied by a number of investigators (see, e.g., [9, 10]). The related vector-potential has the form

$$eA_0 = \frac{\alpha}{|x|}, \qquad eA_1 = \xi\frac{x_2}{r^2}, \qquad eA_2 = -\xi\frac{x_1}{r^2}, \qquad A_3 = 0, \tag{12}$$

where $r^2 = x_1^2 + x_2^2$.

Using the fact that A_1 and A_2 are locally pure gauges we can prove that there exist constants of motion for the ABC system which are analogues of Johnson–Lippman and Dirac constants of motion for the Coulomb system. They have the following form

$$\begin{aligned} \hat{Q}' &= m\alpha\frac{\boldsymbol{\sigma}\cdot\boldsymbol{x}}{\boldsymbol{x}^2} + iD'\left(\boldsymbol{\sigma}\cdot\boldsymbol{p} + i\gamma_4\frac{\alpha}{|x|}\right), \\ D' &= \gamma_0\left(\boldsymbol{\sigma}\cdot\boldsymbol{J} + \frac{1}{2} + \frac{\xi}{x^2}\left(\sigma_3\boldsymbol{x}^2 - x_3\boldsymbol{\sigma}\cdot\boldsymbol{x}\right)\right) \end{aligned} \tag{13}$$

and commute with the corresponding Hamiltonian

$$H' = \gamma_0\boldsymbol{\gamma}\cdot\boldsymbol{\pi} + \gamma_0 m + \frac{\alpha}{|x|}.$$

Commutation and anticommutation relations for operators $\hat{Q}'$, D' and H' are the same as for unprimed operators considered in the previous section. Thus we can construct two supercharges $Q = \frac{1}{\kappa\sqrt{2}}(1+P)\hat{Q}'$, $\bar{Q} = \frac{1}{\kappa\sqrt{2}}(1-P)\hat{Q}'$ and central charge $Z_{ab} = 2\delta_{ab}(\alpha^2 - D'^2)m^2/\kappa^2$ which satisfy relations (3) together with $\hat{H}' = H'^2$. Thus *the ABC system admits extended SUSY with one non-trivial central charge.*

Additional involutive symmetries for this system can be found in the form

$$\begin{aligned} &R_{12} = i\gamma_1\gamma_2\hat{\theta}_1\hat{\theta}_2, \qquad R_{31} = i\gamma_3\gamma_1\hat{\theta}_3\hat{\theta}_1, \qquad R = \gamma_4\gamma_0\hat{\theta}, \\ &R_{23} = i\exp(i\varphi)\gamma_2\gamma_3\hat{\theta}_2\hat{\theta}_3, \qquad \hat{C} = i\exp(i\varphi)\gamma_2 c. \end{aligned} \tag{14}$$

Here $\varphi = 2\arctan\frac{x_1}{x_2}$, $\hat{\theta}$, $\hat{\theta}_a$ and c are reflection and complex conjugation operators defined in the previous section.

Operators (14) commute with the Dirac operator L of equation (7), with potentials (12) and satisfy the following relations

$$\{R, \hat{Q}'\} = \{\hat{C}, Q'\} = \{R, \hat{C}\} = \{\hat{C}, R_{ab}\} = \{R_{ab}, R_{cd}\} = 0,$$
$$[R_{ab}, R] = [R_{ab}, Q'] = 0. \tag{15}$$

Using (15) we can construct six supercharges for the ABC system, namely

$$Q'_1 = (1 + \hat{C}R_{12})Q', \qquad Q'_2 = (\hat{C}R_{23} + R)Q', \qquad Q'_3 = \hat{C}(1 + R_{31}),$$
$$\bar{Q}'_1 = (1 - \hat{C}R_{12})Q', \qquad \bar{Q}'_2 = (\hat{C}R_{23} - R)Q', \qquad \hat{Q}'_3 = \hat{C}(1 + R_{31})Q'. \tag{16}$$

Operators (16) and $\hat{H}' = Q'^2$ satisfy relations (3) and form a basis of $N = 6$ extended superalgebra for ABC system. This system admits also the 64-dimensional algebra $gl(8,\mathbb{R})$ of involutive symmetries. Basis elements of this algebra can be obtained using formulae (11) with Q'_a, $\bar{Q}'_a$ (16) instead of operators (10).

Let us consider now the relativistic Aharonov–Casher (AC) system [9, 12]. This system includes chargeless particle with non-trivial electric quadrupole momentum, interacting with an infinite homogeneously charged cylinder. It is described by the Dirac equation with anomalous interaction instead of a minimal one:

$$\left(\gamma_\mu p^\mu - m + \frac{ik}{m}\gamma_\mu\gamma_\nu F^{\mu\nu}\right)\psi = 0, \tag{17}$$

where $F^{\mu\nu}$ is the strength tensor of the external electromagnetic field generated by infinite homogeneously charged cylinder which we suppose be directed along the third co-ordinate axis.

We shall consider more general system (17) with an external field of the following form

$$F_{ab} = 0, \qquad F_{0a} = \frac{\partial\varphi}{\partial x_a}, \qquad a, b = 1, 2, 3,$$

where $\varphi = \varphi(\boldsymbol{x})$ is a potential of the electric field which is an even function of spatial variables. In the case $\varphi = \sqrt{x_1^2 + x_2^2}$ equation (17) reduces to the AC system.

The considered system admits $N = 6$ extended SUSY generated by the following supercharges

$$Q_1 = (\Gamma_1 + \Gamma_0)H, \qquad \bar{Q}_1 = (\Gamma_1 - \Gamma_0)H,$$
$$Q_2 = (\Gamma_2 + \Gamma_5)H, \qquad \bar{Q}_2 = (\Gamma_2 - \Gamma_5)H,$$
$$Q_3 = (\Gamma_3 + \Gamma_6)H, \qquad \bar{Q}_3 = (\Gamma_3 - \Gamma_6)H, \tag{18}$$

where $\Gamma_1, \ldots, \Gamma_6$ are discrete symmetries (6) and

$$H = \gamma_0\gamma_\alpha p^\alpha + \frac{ik}{m}\gamma_\alpha E^\alpha + \gamma_0 m.$$

Operators (18) and $\hat{H} = H^2$ satisfy relations (3) with $Z_{ab} \equiv 0$ and so generate $N = 6$ SUSY algebra for the AC system.

The AC system admits also the algebra $gl(8,\mathbb{R})$ whose basis elements are given by relations (6), and so has the same involutive symmetry algebra as the free Dirac equation.

5 Stueckelberg systems

Relativistic Stueckelberg equation [13] describes quantum mechanical systems which have two spin states corresponding to values of spin $s = 1$ and $s = 0$. It is a system of equations for an antisymmetric tensor field $\psi^{\mu\nu}$, a four-vector field ψ^{μ} and a scalar field ψ of the following form

$$\begin{aligned}
&p^{\mu}\psi^{\nu} - p^{\nu}\psi^{\mu} = m\psi^{\mu\nu},\\
&p_{\nu}\psi^{\mu\nu} = p^{\mu}\psi + m\psi^{\mu},\\
&p_{\nu}\psi^{\nu} = m\psi.
\end{aligned} \tag{19}$$

Introducing the minimal and anomalous interaction with an external e.m. field into (19) we obtain the following system

$$\begin{aligned}
&\pi^{\mu}\psi^{\nu} - \pi^{\nu}\psi^{\mu} = m\psi^{\mu\nu},\\
&\pi_{\nu}\psi^{\mu\nu} = \pi^{\mu}\psi + m\psi^{\mu} + \frac{e}{m}F_{\mu\nu}\psi^{\nu},\\
&\pi_{\nu}\psi^{\nu} = m\psi.
\end{aligned} \tag{20}$$

Here $\pi_{\mu} = p_{\mu} - eA_{\mu}$ and $F_{\mu\nu} = -\frac{i}{e}[\pi_{\mu}, \pi_{\nu}]$ is the strength tensor of the electromagnetic field.

A special form of anomalous interaction chosen in (20) yields to extended SUSY for this equation. Other interactions for the Stueckelberg equation are discussed in [14].

Expressing $\psi^{\mu\nu}$, and ψ in (20) via ψ^{μ} we come to the second-order equation

$$\left(\pi_{\nu}\pi^{\nu} - m^2\right)\psi^{\mu} + 2eF^{\mu\nu}\psi_{\nu} = 0. \tag{21}$$

Its symmetries will be investigated in few steps.

We begin with the constant and homogeneous external magnetic field directed along the third co-ordinate axis. The corresponding vector-potential and tensor $F^{\mu\nu}$ have the form

$$A_0 = A_2 = A_3 = 0, \qquad A_1 = -Hx_2, \qquad F_{0a} = F_{23} = F_{31} = 0, \qquad F_{12} = H. \tag{22}$$

Substituting (22) into (21) and representing ψ_{ν} as

$$\psi_{\nu} = \exp\left(iEt + ip_1x_1 + ip_3x_3\right)\varphi_{\nu}(x_2), \qquad x_2 = \left(\frac{p_1}{\sqrt{eH}} + y\right)$$

equation (21) can be reduced to the form

$$E^2\varphi = \left(m^2 + p_3^2 - \frac{\partial^2}{\partial y^2} + \omega^2y^2 + 2S_3\omega\right)\varphi, \tag{23}$$

where $\omega = eH$, $\varphi = \mathrm{column}(\varphi_0\varphi_1\varphi_2\varphi_3)$ and

$$S_3 = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & -i & 0\\ 0 & i & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Equation (23) admits a large extended supersymmetry. First, we indicate two sets of constants of motion which generate extended SUSY with non-trivial central charges. The basis elements of the corresponding superalgebras have the following forms:

$$\begin{aligned}
&\tilde{Q}_1 = \frac{1}{2}(\sigma_1 + i\sigma_2)(p + i\omega y), \qquad \bar{\tilde{Q}}_1 = \frac{1}{2}(\sigma_1 - i\sigma_2)(p - i\omega y),\\
&\hat{H} = -\frac{\partial^2}{\partial y^2} + \omega^2y^2 + 2S_3\omega + p_3^2 + m^2, \qquad \tilde{Z}_{ab} = 2\delta_{ab}\left(p_3^2 + m^2 - \frac{1}{2}\tau_3\omega\right)
\end{aligned} \tag{24}$$

and

$$\tilde{Q}'_1 = \frac{1}{2}(\tau_1 + i\tau_2)(p + i\omega y), \qquad \bar{\tilde{Q}}'_1 = \frac{1}{2}(\tau_1 - i\tau_2)(p - i\omega y),$$
$$\hat{H}' = \hat{H}, \qquad \tilde{Z}'_{ab} = 2\delta_{ab}\left(p_3^2 + m^2 - \frac{1}{2}\sigma_3\omega\right). \tag{25}$$

Here

$$\sigma_1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$
$$\tau_1 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}. \tag{26}$$

It can be verified by a direct calculation that operators (24) and (25) commute with $\hat{H}$ and are, therefore, constants of motion for equation (23). In addition, operators (24) and (25) satisfy relations (3), so *the Stueckelberg equation with the constant and homogeneous external magnetic field admits extended SUSY with non-trivial central charges.*

The superalgebras (24) and (25) can be jointed in frames of a more extended superalgebra including ten elements. Four of them are odd elements (supercharges), namely, Q_α, $\bar{Q}_\alpha$, $\alpha = 1, 2$:

$$Q_1 = \frac{1}{2}\left(\sigma_1 p + \sigma_2\omega y + i\sigma_3(\tau_1 p + \tau_2\omega y)\right),$$
$$\bar{Q}_1 = Q_1^\dagger = \frac{1}{2}\left(\sigma_1 p + \sigma_2\omega y - i\sigma_3(\tau_1 p + \tau_2\omega y)\right),$$
$$Q_2 = \frac{1}{2}(\sigma_2 p - \sigma_1\omega y + i\sigma_3(\tau_2 p - \tau_1\omega y)),$$
$$\bar{Q}_2 = Q_2^\dagger = \frac{1}{2}(\sigma_2 p - \sigma_1\omega y - i\sigma_3(\tau_2 p - \tau_1\omega y)) \tag{27}$$

and six of them are even. They include the central charge $Z_{ab} = 2\delta_{ab}(p_3^2 + m^2)$, and five additional elements of the form

$$\hat{H} = -\frac{\partial^2}{\partial y^2} + \omega^2 y^2 + 2S_3\omega + p_3^2 + m^2, \qquad I_0 = \omega(\sigma_3 + \tau_3)/2,$$
$$I_1 = \omega(\sigma_2\tau_1 - \sigma_1\tau_2)/2, \qquad I_\pm = \omega\left(\sigma_3 - \tau_3 \pm (\sigma_1\tau_1 + \sigma_2\tau_2)\right)/4. \tag{28}$$

Anticommutation relations for odd elements are given by the following formulae

$$\{Q_a, Q_b^\dagger\} = \delta_{ab}(H - Z - I_0) - i\varepsilon_{ab}I_1, \qquad \{Q_a, Q_b\} = \delta_{ab}I_-, \tag{29}$$

where $\varepsilon_{12} = -\varepsilon_{21} = 1$, $\varepsilon_{11} = \varepsilon_{22} = 0$. The remaining (commutation) relations of odd elements with even and even elements with even ones are of the form

$$[Q_a, \hat{H}] = [I_0, \hat{H}] = [I_\pm, \hat{H}] = 0, \qquad [Q_a, I_0] = -i\varepsilon_{ab}Q_b, \qquad [Q_a, I_1] = Q_a,$$
$$[Q_a, I_-] = 0, \qquad [Q_a, I_+] = -i\varepsilon_{ab}Q_b^\dagger, \qquad [I_0, I_1] = [I_0, I_\pm] = 0,$$
$$[I_1, I_\pm] = \pm I_\pm, \qquad [I_+, I_-] = I_1. \tag{30}$$

In addition Z commutes with all operators enumerated in (27), (28).

Thus Stueckelberg particle interacting with a constant homogeneous external magnetic field forms a system admitting extended superalgebra characterized by relations (29), (5). We will further denote this algebra as $\mathcal{A}$.

Consider now Stueckelberg equation for the case when external field is generated by a point charge. The related vector-potential can be chosen in the form (8) and equation (21) reads

$$\left(p_0 + \frac{\alpha^2}{|x|}\right)^2 \Psi = \left(\boldsymbol{p}^2 + m^2 + i\alpha\frac{(\sigma_a - \tau_a)x_a}{|x|^3}\right)\Psi, \tag{31}$$

where $\Psi = \text{column}\,(\psi_0, \psi_1, \psi_2, \psi_3)$.

Rather surprisingly, equation (31) also admits extended invariance superalgebra, isomorphic to $\mathcal{A}$. This can be shown by writing Ψ in the form $\Psi = \exp(iEt)\varphi(\boldsymbol{x})$ (i.e., considering the related eigenvalue problem) and introducing new space variables $\boldsymbol{r} = E\boldsymbol{x}$. Equation (31) then takes the form

$$\mu\varphi = \hat{H}\varphi,$$

where

$$\hat{H} = p'^2 + i\alpha\frac{(\sigma_a - \tau_a)r_a}{|r|^3} - \left(\frac{\alpha}{|r|} - 1\right)^2, \qquad \text{and} \qquad \mu = -\frac{m^2}{E^2}.$$

The corresponding radial equation can be written as [15]:

$$\mu\varphi(r) = \hat{H}\varphi \equiv \left(-\frac{d^2}{dr^2} + \begin{pmatrix} V_1 & 0 & 0 & 0 \\ 0 & V_2 & 0 & 0 \\ 0 & 0 & V_3 & 0 \\ 0 & 0 & 0 & V_4 \end{pmatrix}\right)\varphi(r), \tag{32}$$

where

$$V_1 = \frac{b^2 - \frac14}{r^2} - \frac{2\alpha}{r}, \qquad V_2 = V_3 = \frac{(b+1)^2 - \frac14}{r^2} - \frac{2\alpha}{x}, \qquad V_4 = \frac{(b-1)^2 - \frac14}{r^2} - \frac{2\alpha}{r}. \tag{33}$$

It can be proven by a direct verification that equation (32) admits nine constants of motion, namely

$$Q_1 = \begin{pmatrix} 0 & a_-^1 & ia_-^1 & 0 \\ a_+^1 & 0 & 0 & -ia_-^2 \\ ia_+^1 & 0 & 0 & a_-^2 \\ 0 & -ia_+^2 & a_+^2 & 0 \end{pmatrix}, \qquad Q_2 = \begin{pmatrix} 0 & -ia_-^1 & a_-^1 & 0 \\ ia_+^1 & 0 & 0 & -a_-^2 \\ -a_+^1 & 0 & 0 & -ia_-^2 \\ 0 & a_+^2 & ia_+^2 & 0 \end{pmatrix},$$

$$\bar{Q}_1 = \begin{pmatrix} 0 & a_+^1 & -ia_+^1 & 0 \\ a_-^1 & 0 & 0 & +ia_+^2 \\ -ia_-^1 & 0 & 0 & a_+^2 \\ 0 & ia_+^2 & a_-^2 & 0 \end{pmatrix}, \qquad \bar{Q}_2 = \begin{pmatrix} 0 & +ia_+^1 & -a_+^1 & 0 \\ -ia_-^1 & 0 & 0 & a_+^2 \\ a_-^1 & 0 & 0 & ia_+^2 \\ 0 & -a_-^2 & -ia_+^2 & 0 \end{pmatrix},$$

$$I_0 = C\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad I_1 = iC\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$I_\pm = C\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm1 & 0 \\ 0 & \pm1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad Z = m^2 + p_3^2 \tag{34}$$

which satisfy relations (29) and (5). Here

$$a^1_\pm = p' \pm i\left(\frac{b+\frac{1}{2}}{x} + \frac{\alpha}{b+\frac{1}{2}}\right), \qquad a^2_\pm = p' \pm i\left(\frac{b-\frac{1}{2}}{x} + \frac{\alpha}{b-\frac{1}{2}}\right),$$

$$C = \frac{\alpha b}{a^2 - \frac{1}{4}}, \qquad b^2 = \left(j + \frac{1}{2}\right)^2$$

and j is the quantum number defining the spectrum of total angular momentum $\boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{p}'+\boldsymbol{\sigma}+\boldsymbol{\tau}$ of a system in state Ψ, i.e., $\boldsymbol{J}^2\Psi = j(j+1)\Psi$.

Thus the Stueckelberg equation with Coulomb potential is invariant with respect to extended superalgebra whose generators are given in (34). As a consequence it admits two symmetry superalgebras (24) and (25), and hence is characterized by extended SUSY with non-trivial central charges. Using discrete involutive symmetries of the Stueckelberg equation it is possible to construct extra supercharges which enlarge superalgebras (24) and (25) to $N = 6$ extended SUSY.

6 Representations of superalgebra $\mathcal{A}$

In order to describe other QM systems invariant with respect to superalgebra $\mathcal{A}$ which admit extended SUSY we construct representations of this algebra realized by differential operators defined on four-component vector-functions. The related supercharges and even elements of the superalgebra can then chosen in the form (34) where

$$a_1^\pm = p \pm iW_1, \qquad a_2^\pm = p \pm iW_2, \qquad p = -i\frac{d}{dx}. \tag{35}$$

Here W_1 and W_2 are functions of x satisfying the following relation

$$W_1^2 - W_2^2 + W_1' + W_2' = C \tag{36}$$

with C being a constant and prime denoting derivative of W_α with respect to x.

Operators (34), (35) satisfy relations (29), (5) for the case when W_1, W_2 are arbitrary functions satisfying condition (36). Choosing $W_1 = W_2 = \omega x$ in (35) we obtain supercharges for the Stueckelberg system with constant, homogeneous external magnetic field. The choice $W_1 = \frac{b+\frac{1}{2}}{x} + \frac{\alpha}{b+\frac{1}{2}}$, $W_2 = \frac{b-\frac{1}{2}}{x} + \frac{\alpha}{b-\frac{1}{2}}$ corresponds to the Stueckelberg–Coulomb system. Two other choices, namely, $W_1 = -W_2 = \omega x$ and $W_1 = -W_2 = \frac{b}{x} + \frac{\alpha}{b}$ correspond to Dirac particle in the constant, homogeneous external magnetic field and Coulomb field respectively, where all states have additional two fold degeneracy.

7 Discussion

We have shown that extended SUSY with non-trivial central charges appears as internal symmetry of many quantum mechanical systems. In particular we have proven that symmetry of the relativistic Coulomb system as well as of the Aharonov–Bohm–Coulomb and Aharonov–Casher systems can be described by the superalgebra including six supercharges. The Stueckelberg systems systems are characterized even by more extended SUSY described by ten-dimensional superalgebra with non-trivial central charges.

One more goal of our analysis was searching for realistic quantum mechanical systems which are invariant with respect to algebra $gl(8,\mathbb{R})$ of involutive discrete symmetries. This invariance algebra for the free Dirac equation was found in papers [3, 6]. In the present paper we prove that this symmetry is valid also for the Coulomb, ABC and AC systems.

A natural question arises what are the practical consequences of the found symmetries. Using the technique developed in [3, 7] it is possible to use $gl(8,\mathbb{R})$ symmetry to decouple the related Dirac equation and construct complete sets of solutions.

A standard application of SUSY consists in prediction and interpretation of degeneration of energy spectra of the related QM systems. Energy levels for the exactly solvable Coulomb–Dirac problem are degenerated with respect to quantum numbers sign j_3 and sign κ where j_3 and κ are eigenvalues of mutually commuting operators of the third component of the total angular momentum J_3 and D respectively. One more degeneration which is non-observable is connected with the change of sign of the phase multiplier of the Dirac-Coulomb wave function. Extended SUSY presents a specific interpretation of these degenerations. A particular importance of such interpretation consists in the fact that such a degeneration appears for all the systems which admit extended SUSY, e.g., for the AC system.

The other application of (extended) SUSY is to construct exact solutions of QM systems with sharp invariant potentials using purely algebraic methods [2] which admit a straitforward generalization to the case of a more general superalgebra (29), (5). The potentials (33) of the Stueckelberg–Coulomb system are shape invariant which enables us to find easily its energy eigenvalues. They can be written in the following form

$$E_{n\kappa\lambda} = m\left[1 + \frac{\alpha^2}{\left(n+\frac{1}{2}+b+\lambda\right)^2}\right]^{\frac{1}{2}},$$

where $n = 0, 1, 2, \ldots$, $\lambda = 0, \pm 1$, $b = \sqrt{\kappa^2 - \alpha^2}$, $|\kappa| = 1, 2, \ldots$

Finally we notice that representations of superalgebra $\mathcal{A}$ considered in the previous section can be used in non-relativistic quantum mechanics. It seems to us that such generalized SUSY quantum mechanics has better physical grounds than parasupersymmetric quantum mechanics [16, 17] and $n = N$ ($N > 1$) SUSY quantum mechanics [18], since it is realized in a number of quite realistic QM systems. We plane to study possible applications of superalgebra in quantum mechanics $\mathcal{A}$ elsewhere.

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Nonlinear Supersymmetry

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After a short discussion of the intimate relation between the generalized statistics and supersymmetry, we review the recent results on the nonlinear supersymmetry obtained in the context of the quantum anomaly problem and of the universal algebraic construction associated with the holomorphic nonlinear supersymmetry.

Introduction

Nonlinear supersymmetry is a natural generalization of the usual linear supersymmetry [1, 2]. It is realized variously in such different systems as the parabosonic [3] and parafermionic [4] oscillator models, and the P,T-invariant models of planar fermions [5] and Chern–Simons fields [6]. It is also the symmetry of the fermion-monopole system [7, 8]. The algebraic structure of the nonlinear supersymmetry resembles the structure of the finite W-algebras [9] for which the commutator of generating elements is proportional to a finite order polynomial in them. In the simplest case the nonlinear supersymmetry is characterized by the superalgebra of the form

$$[Q^{\pm}, H] = 0, \qquad (Q^{\pm})^2 = 0, \qquad \{Q^+, Q^-\} = P_n(H), \tag{1}$$

where $P_n(\cdot)$ is a polynomial of the n-th degree. The nonlinear supersymmetry with such a superalgebra was investigated for the first time by Andrianov, Ioffe and Spiridonov [10].

The pseudoclassical construction underlies the supersymmetric quantum mechanics of Witten [1, 2] corresponding to the linear ($n = 1$) case of the superalgebra (1). Though the nonlinear supersymmetry can also be realized classically, there is an essential difference from the linear case: the attempt to quantize the nonlinear supersymmetry immediately faces the problem of the quantum anomaly [3, 11]. It was shown [12] that the universal algebraic structure with associated "integrability conditions" in the form of the Dolan–Grady relations [13] underlies the so called holomorphic nonlinear supersymmetry [11]. This structure allows one to find a broad class of anomaly-free quantum mechanical systems related to the exactly and quasi-exactly solvable systems [14, 15, 16, 17, 18, 19], and gives a nontrivial centrally extended generalization of the superalgebra (1) [12].

In this talk, after a short discussion of the intimate relation between the generalized statistics and supersymmetry [3], we shall review the recent results on the nonlinear supersymmetry obtained in the context of the quantum anomaly problem and of the universal algebraic construction associated with the holomorphic nonlinear supersymmetry [11, 12].

Nonlinear supersymmetry in purely parabosonic systems

Some time ago it was shown that the linear supersymmetry can be realized without fermions [20, 21, 22]. The nonlinear supersymmetry admits a similar realization revealing the close relationship between the generalized statistics and supersymmetry [3]. The relationship can be observed in the following way. Let us consider a single-mode paraboson system defined by the relations

$$[\{a^+, a^-\}, a^\pm] = \pm a^\pm, \qquad a^-a^+|0\rangle = p|0\rangle, \qquad a^-|0\rangle = 0,$$

where $p \in \mathbb{N}$ is the order of a paraboson [23]. Then the direct calculation shows that the pure parabosonic system of the even order $p = 2(k+1)$, $k \in \mathbb{Z}_+$, with the Hamiltonian of the simplest quadratic form $H = a^+a^-$ reveals a spectrum typical for the nonlinear supersymmetry: all its states are paired in doublets except the $k+1$ singlet states $|2l\rangle \propto (a^+)^{2l}|0\rangle$, $l = 0, \ldots, k$. In correspondence with this property, the system has two integrals of motion

$$Q^+ = (a^+)^{2k+1} \sin^2 \frac{\pi}{4}\{a^+, a^-\}, \qquad Q^- = (a^-)^{2k+1} \cos^2 \frac{\pi}{4}\{a^+, a^-\}, \tag{2}$$

which together with the Hamiltonian form the nonlinear superalgebra (1) of the order $n = 2k+1$ with $P_{2k+1}(H) = H \cdot \prod_{m=1}^{k} \left(H^2 - 4m^2\right)$. This simplest system reflects the peculiar feature of the parabosonic realization of supersymmetry: the supercharges are realized in the form of the infinite series in $a^\pm$, and the role of the grading operator is played here by $R = (-1)^N = \cos \pi N$, where $N = \frac{1}{2}\{a^+, a^-\} - \frac{1}{2}p$ is the parabosonic number operator.

It is known that the deformed Heisenberg algebra with reflection

$$[a^-, a^+] = 1 + \nu R, \qquad \{R, a^\pm\} = 0, \qquad R^2 = 1, \tag{3}$$

underlies the parabosons [24, 25]. This algebra possesses unitary infinite-dimensional representations for $\nu > -1$, and at the integer values of the deformation parameter, $\nu = p - 1$, $p \in \mathbb{N}$, is directly related to parabosons of order p [23, 24, 25]. On the other hand, at $\nu = -(2p+1)$ the R-deformed Heisenberg algebra has finite-dimensional representations corresponding to the deformed parafermions of order $2p$ [25]. In the coordinate representation the operator R is the parity operator and the operators $a^\pm$ can be realized in the form $a^\pm = \frac{1}{\sqrt{2}}(x \mp iD_\nu)$ with $D_\nu = -i\left(\frac{d}{dx} - \frac{\nu}{2x}R\right)$. In the context of the Calogero-like models, the operator D_ν is known as the Yang–Dunkl operator [26, 27], where R is treated as the exchange operator. In the coordinate representation the Hamiltonian $H = a^+a^-$ and supercharges (2) read as [3]

$$H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 + \frac{\nu^2}{4x^2} - 1 + \nu\left(\frac{1}{2x^2} - 1\right)R\right), \tag{4}$$

$$Q^+ = (Q^-)^\dagger = \frac{1}{2^{3\left(k+\frac{1}{2}\right)}}\left(\left(-\frac{d}{dx} + x + \frac{\nu}{2x}\right)(1 - R)\right)^{2k+1} \tag{5}$$

with $\nu = 2k + 1$. The system given by the Hamiltonian (4) can be treated as a 2-particle Calogero-like model with exchange interaction, where x has a sense of a relative coordinate and R has to be understood as the exchange operator [28, 29]. Therefore, at odd values of the parameter ν, the class of Calogero-like systems (4) possesses a hidden supersymmetry, which at $\nu = 1$ is the linear ($n = 1$) supersymmetry in the unbroken phase, whereas at $\nu = 2k+1$, the supersymmetry is characterized by the supercharges being differential operators of order $2k + 1$ satisfying the nonlinear superalgebra (1). Recently the realization of the nonlinear supersymmetry was extended within the standard approach with fermion degrees of freedom to the case of multi-particle Calogero and related models [30].

Classical supersymmetry

Let us turn now to the classical formulation of the supersymmetry (1). For the purpose, we consider a non-relativistic particle in one dimension described by the Lagrangian

$$L = \frac{1}{2}\dot{x}^2 - V(x) - L(x)N + i\theta^+\dot{\theta}^-, \tag{6}$$

where $\theta^{\pm}$ are the Grassman variables, $(\theta^+)^* = \theta^-$, $N = \theta^+\theta^-$, and $V(x)$ and $L(x)$ are two real functions. The nontrivial Poisson–Dirac brackets for the system are $\{x, p\}_* = 1$ and $\{\theta^+, \theta^-\}_* = -i$, and the Hamiltonian is

$$H = \frac{1}{2}p^2 + V(x) + L(x)N. \tag{7}$$

The Hamiltonian H and the nilpotent quantity N are the even integrals of motion for any choice of the functions $V(x)$ and $L(x)$, and one can put the question: when the system (6) has also local in time odd integrals of motion of the form $Q^{\pm} = B^{\mp}(x, p)\theta^{\pm}$, where $(B^+)^* = B^-$? It is obvious that such odd integrals can exist only for a special choice of the functions $V(x)$ and $L(x)$. Restricting ourselves to the physically interesting class of the systems given by the potential $V(x)$ bounded from below, we can generally represent it in terms of a superpotential $W(x)$: $V(x) = \frac{1}{2}W^2(x) + v$, $v \in \mathbb{R}$. Then all the supersymmetric systems are separated into the three classes defined by the behaviour of the superpotential and the results can be summarized as follows [11].

i) When the physical domain given by $z = W(x) + ip$ includes the origin $z = 0$ ($a < W(x) < b$, $a < 0$, $b > 0$), the corresponding supersymmetric system is characterized by the Hamiltonian and the supercharges of the form

$$\begin{aligned} H &= \frac{p^2}{2} + \frac{1}{2}W^2(x) + v + W'(x)\left[n + W(x)M\left(W^2(x)\right)\right]N, \\ Q^+ &= (Q^-)^* = z^n e^{i\int_0^p M\left(p^2 - y^2 + W^2(x)\right)dy}\theta^+, \qquad n \in \mathbb{Z}, \end{aligned} \tag{8}$$

where $M(W^2)$ is an arbitrary regular function, $|M(0)| < \infty$. The appearance of the integer parameter illustrates in this case the known classical "quantization phenomenon" [31]. The appropriate canonical transformation reduces the system with these Hamiltonian and supercharges to the form of the supersymmetric system with the holomorphic supercharges [11]:

$$H = \frac{1}{2}p^2 + \frac{1}{2}W^2(x) + v + nW'(x)\theta^+\theta^-, \qquad Q^+ = (Q^-)^* = z^n\theta^+, \qquad n \in \mathbb{Z}_+. \tag{9}$$

The integrals (9) obey the classical nonlinear superalgebra:

$$\left\{Q^-, Q^+\right\}_* = -iH^n, \qquad \{Q^{\pm}, H\}_* = 0. \tag{10}$$

The presence of the integer number n in the Hamiltonian means that the instant frequencies of the oscillator-like odd, $\theta^{\pm}$, and even, z, $\bar{z}$, variables are commensurable. Only in this case the regular odd integrals of motion can be constructed, and the factor z^n in the supercharges $Q^{\pm}$ corresponds to the n-fold conformal mapping of the complex plane (or the strip $a < \mathsf{Re}\, z < b$) on itself (or on the corresponding region in $\mathbb{C}$).

ii) The physical domain is defined by the condition $\mathsf{Re}\, z \geq 0$ (or $\mathsf{Re}\, z \leq 0$) and also includes the origin of the complex plane. But unlike the previous case, there are no closed contours around $z = 0$. In this case the most general form of the Hamiltonian and the supercharge is

$$H = \frac{p^2}{2} + \frac{1}{2}W^2(x) + v + W'(x)\left[\alpha + R(W(x))\right]N, \qquad Q^+ = z^{\alpha} e^{i\int_{\varphi_0}^{\varphi} R(\rho\cos\lambda)\,d\lambda}\theta^+, \tag{11}$$

where $\alpha \in \mathbb{R}$, and we assume that the function $R(W)$ is analytical at $W = 0$ and $R(0) = 0$. By the canonical transformation [11], the Hamiltonian and the supercharges can be reduced to

$$H = \frac{1}{2}p^2 + \frac{1}{2}W^2(x) + v + \alpha W'(x)\theta^+\theta^-, \qquad Q^+ = (Q^-)^* = z^{\alpha}\theta^+, \qquad \alpha \in \mathbb{R}_+.$$

iii) The physical domain is defined by the condition $\mathsf{Re}\, z > 0$ (or $\mathsf{Re}\, z < 0$), i.e. the origin of the complex plane is not included. Though in this case the Hamiltonian and the supercharges have a general form

$$H = \frac{p^2}{2} + \frac{1}{2}W^2 + v + W'(x)\phi(W(x))N, \qquad Q^+ = (Q^-)^* = f(H)e^{i\int_{\varphi_0}^{\varphi}\phi(\rho\cos\lambda)\,d\lambda}\theta^+, \quad (12)$$

where ϕ is some function, the appropriate canonical transformation reduces it to [11]

$$H = \frac{1}{2}p^2 + \frac{1}{2}W^2(x) + v, \qquad Q^\pm = \theta^\pm.$$

This means that *classically* the supersymmetry of any system with bounded non-vanishing superpotential has a "fictive" nature.

In what follows we will refer to the nonlinear supersymmetry generated by the holomorphic supercharges with the Poisson bracket (anticommutator) being proportional to the n-th order polynomial in the Hamiltonian as to the *holomorphic n-supersymmetry.*

Though the form of the Hamiltonians (8), (11), and (12) can be simplified by applying in every case the appropriate canonical transformation reducing the associated supercharges to the holomorphic or antiholomorphic form, the quantization breaks the equivalence between the corresponding classical systems (even in the linear case $n = 1$) [11]. Moreover, alternative classical forms for the Hamiltonians and associated supercharges are important because of the quantum anomaly problem to be discussed below. Having in mind the importance of alternative classical formulations of the nonlinear supersymmetry from the viewpoint of subsequent quantization, one can look for the classical formulation characterized by the supercharges of the n-th degree polynomial form in p [11]. The problem of finding such a formulation can be solved completely in the simplest case $n = 2$, for which the supercharges are given by

$$Q^\pm = \frac{1}{2}\left[(\pm ip + W(x))^2 + \frac{c}{W^2(x)}\right]\theta^\pm, \qquad c \in \mathbb{R}, \quad (13)$$

while the Hamiltonian is

$$H = \frac{1}{2}\left[p^2 + W^2(x) - \frac{c}{W^2(x)}\right] + 2W'(x)N + v. \quad (14)$$

Note that the Hamiltonian (14) has the Calogero-like form: at $W(x) = x$ its projection to the unit of Grassman algebra takes the form of the Hamiltonian of the two-particle Calogero system. Depending on the value of the parameter c, classically the Calogero-like $n = 2$ supersymmetric system (14) is symplectomorphic to the holomorphic n-supersymmetry with $n = 0$ $(c > 0)$, $n = 1$ $(c < 0)$ or $n = 2$ $(c = 0)$ [11].

Quantum anomaly and quasi-exactly solvable (QES) systems

According to the results on the supersymmetry in pure parabosonic systems, a priori one cannot exclude the situation characterized by the supercharges to be the nonlocal operators represented in the form of some infinite series in the operator $\frac{d}{dx}$. Since such nonlocal supercharges have to anticommute for some function of the Hamiltonian being a usual local differential operator of the second order, they have to possess a very peculiar structure. We restrict ourselves by the discussion of the supersymmetric systems with the supercharges being the differential operators of order n. Classically this corresponds to the system (9) with the holomorphic supercharges or to the system (14).

In the simplest case of the superoscillator possessing the nonlinear n-supersymmetry and characterized by the holomorphic supercharges of the form (9) with $W(x) = x$, the form of the classical superalgebra $\{Q_n^+, Q_n^-\} = H^n$ is changed for $\{Q_n^+, Q_n^-\} = H(H-\hbar)(H-2\hbar)\cdots(H-\hbar(n-1))$. Moreover, it was pointed out in [3] that for $W(x) \neq ax+b$ a global quantum anomaly arises in a generic case: the direct quantum analogues of the superoscillators and the Hamiltonian do not commute, $[Q_n^\pm, H_n] \neq 0$. Therefore, we arrive at the problem of looking for the classes of superpotentials and corresponding quantization prescriptions leading to the anomaly-free quantum n-supersymmetric systems.

Let us begin with the quantum supercharges in the holomorphic form corresponding to the classical n-supersymmetry,

$$Q^\pm = (A^\mp)^n \theta^\pm, \qquad \text{where} \qquad A^\pm = \mp\hbar\frac{d}{dx} + W(x). \tag{15}$$

Choosing the quantum Hamiltonian in the form (7), from the requirement of conservation of the supercharges, $[Q^\pm,\, H] = 0$, one arrives at the supersymmetric quantum system given by the Hamiltonian [11]

$$H = \frac{1}{2}\left(-\hbar^2\frac{d^2}{dx^2} + W^2(x) + 2v + n\hbar\sigma_3 W'\right), \qquad W(x) = w_2x^2 + w_1x + w_0. \tag{16}$$

For any other form of the superpotential, the nilpotent operators (15) are not conserved. The family of supersymmetric systems (16) is reduced to the superoscillator at $w_2 = 0$ with the associated exact n-supersymmetry [3]. For $w_2 \neq 0$, the n-supersymmetry is realized always in the spontaneously broken phase since in this case the supercharges (15) have no zero modes (normalized eigenfunctions of zero eigenvalue).

One can also look for the supercharges in the form of polynomial of order n in the oscillator-like operators $A^\pm$ defined in (15):

$$Q^\pm = \left(A^\mp\right)^n\theta^\pm + \sum_{k=0}^{n-1} q_{n-k}\left(A^\mp\right)^k\theta^\pm, \tag{17}$$

where q_k are real parameters which have to be fixed. As in the case of the supercharges (15), the requirement of conservation of (17) results in the Hamiltonian (16) but with the exponential superpotential [11]:

$$W(x) = w_+e^{\omega x} + w_-e^{-\omega x} + w_0, \qquad \omega^2 = -\frac{24q}{n\left(n^2-1\right)}, \tag{18}$$

where all the parameters $w_{\pm,0}$ are real, while the parameter ω is real or pure imaginary depending on the sign of the real parameter q, and for the sake of simplicity we put $\hbar = 1$. In the limit $\omega \to 0$ this superpotential is reduced to the quadratic form (16) via the appropriate rescaling of the parameters $w_{\pm,0}$.

The family of n-supersymmetric systems given by the superpotential (18) is tightly related to the so called quasi-exactly solvable problems [15, 16, 17, 18, 19]. Indeed, both of the Hamiltonians constituting the supersymmetric Hamiltonian of the form (16) with the exponential superpotential belong to the $sl(2,\mathbb{R})$ scheme of one-dimensional QES systems [15, 16, 17]. Besides, the QES family given by the superpotential (18) is related to the exactly solvable Morse potential for some choice of the parameters [11].

The $n = 2$ non-holomorphic supersymmetry corresponding to equations (13), (14) occupies an especial position. Like the linear supersymmetry, it admits the anomaly-free quantum formulation in terms of an arbitrary superpotential. Indeed, the quantization of the supersymmetric system (14) leads to [11]

$$H = \frac{1}{2}\left[-\hbar^2\frac{d^2}{dx^2} + W^2 - \frac{c}{W^2} + v + 2\hbar W'\sigma_3 + \Delta(W)\right], \tag{19}$$

$$Q^+ = (Q^-)^\dagger = \frac{1}{2}\left[\left(\hbar\frac{d}{dx} + W\right)^2 + \frac{c}{W^2} - \Delta(W)\right]\theta^+, \tag{20}$$

$$\Delta = \frac{\hbar^2}{4W^2}\left(2W''W - W'^2\right). \tag{21}$$

Looking at the quantum Hamiltonian (19) and supercharges (20), we see that the presence of the quadratic in $\hbar^2$ term (21) in the operators H and Q^+ is crucial for preserving the supersymmetry at the quantum level. Therefore, one can say that the quantum correction (21) cures the problem of the quantum anomaly since without it the operators $Q^\pm$ would not be the integrals of motion. The supercharges (20) satisfy the relation $\{Q^+, Q^-\} = (H - v)^2 + c$, and the structure of the lowest bounded states in the cases $c > 0$, $c < 0$ and $c = 0$ for $v = 0$ is reflected in the table and on the figure (for the details see Ref. [11]).

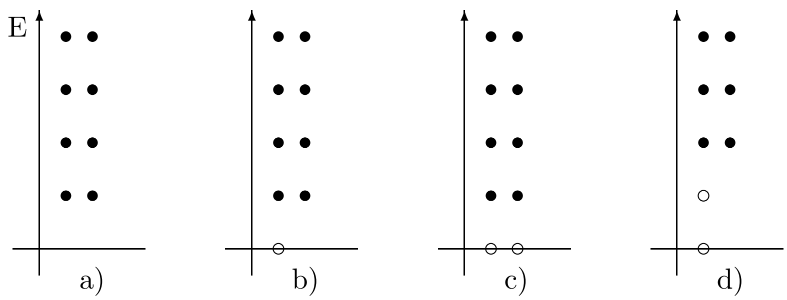

Figure. The four types of the spectra for the $n = 2$ supersymmetry for bounded states.

Table. The structure of the lowest states for the $n = 2$ supersymmetry.

		$c > 0$	$c = 0$	$c < 0$
a)	Completely broken phase, there are no singlet states	+	+	+
b)	One singlet state in either bosonic or fermionic sector		+	+
c)	Two singlet states with $E = 0$, one is in fermionic sector, another is in bosonic sector		+	+
d)	Two singlet states in one of two sectors			+

From this structure one can see, in particular, that the quantum theory "remembers" its classical origin: the case $c > 0$ corresponding classically to the holomorphic $n = 0$ supersymmetry gives the systems in the completely broken phase for any superpotential providing the existence of bounded states.

In conclusion of the discussion of the nonlinear supersymmetry for the 1D quantum systems, we note that for the first time the close relationship between the nonlinear supersymmetry and QES systems was observed in Ref. [11]. Recently, it was demonstrated [32] that the so called type A $\mathcal{N}$-fold supersymmetry [33] being a generalization of the one-dimensional holomorphic supersymmetry is, in essence, equivalent to the one-dimensional QES systems associated with the $sl(2,\mathbb{R})$ algebra.

Nonlinear supersymmetry on plane in magnetic field

The nonlinear holomorphic supersymmetry we have discussed has a universal nature due to the algebraic construction underlying it and revealed in Ref. [12]. This universality allows us, in particular, to generalize the above analysis to the case of the two-dimensional systems.

The classical Hamiltonian of a charged spin-1/2 particle ($-e = m = 1$) with gyromagnetic ratio g moving on a plane and subjected to a magnetic field $B(\boldsymbol{x})$ is given by

$$H = \frac{1}{2}\boldsymbol{\mathcal{P}}^2 + gB(\boldsymbol{x})\theta^+\theta^-, \tag{22}$$

where $\boldsymbol{\mathcal{P}} = \boldsymbol{p} + \boldsymbol{A}(\boldsymbol{x})$, $\boldsymbol{A}(\boldsymbol{x})$ is a 2D gauge potential, $B(\boldsymbol{x}) = \partial_1 A_2 - \partial_2 A_1$. The variables x_i, p_i, $i = 1, 2$, and complex Grassman variables $\theta^\pm$, $(\theta^+)^* = \theta^-$, are canonically conjugate with respect to the Poisson–Dirac brackets, $\{x_i, p_j\}_* = \delta_{ij}$, $\{\theta^-, \theta^+\}_* = -i$. For even values of the gyromagnetic ratio $g = 2n$, $n \in \mathbb{N}$, the system (22) is endowed with the nonlinear n-supersymmetry. In this case the Hamiltonian (22) takes the form

$$H_n = \frac{1}{2}Z^+Z^- + \frac{i}{2}n\left\{Z^-, Z^+\right\}_* \theta^+\theta^-, \qquad Z^\pm = \mathcal{P}_2 \mp i\mathcal{P}_1, \tag{23}$$

which admits the existence of the odd integrals of motion

$$Q^\pm = 2^{-\frac{n}{2}}\left(Z^\mp\right)^n\theta^\pm \tag{24}$$

generating the nonlinear n-superalgebra (10). The n-superalgebra does not depend on the explicit form of the even complex conjugate variables $Z^\pm$. Therefore, in principle, $Z^\pm$ can be arbitrary functions of the bosonic dynamical variables of the system.

The nilpotent quantity $N = \theta^+\theta^-$ is, as in the 1D case, the even integral of motion. When the gauge potential $\boldsymbol{A}(\boldsymbol{x})$ is a 2D vector, the system (23) possesses the additional even integral of motion $L = \varepsilon_{ij}x_i p_j$. The integrals N and L generate the $U(1)$ rotations of the odd, $\theta^\pm$, and even, $Z^\pm$, variables, respectively. Their linear combination $J = L + nN$ is in involution with the supercharges, $\{J, Q^\pm\}_*{=}0$, and plays the role of the central charge of the classical n-superalgebra. As we shall see, at the quantum level the form of the nonlinear n-superalgebra (10) is modified generically by the appearance of the nontrivial central charge in the anticommutator of the supercharges.

A spin-1/2 particle moving on a plane in a constant magnetic field represents the simplest case of a quantum 2D system admitting the nonlinear supersymmetry. Such a system corresponds to the n-supersymmetric quantum oscillator [12]. As in the case of the one-dimensional theory, the attempt to generalize the n-supersymmetry of the system to the case of the magnetic field of general form faces the problem of quantum anomaly. The generalization is nevertheless possible for the magnetic field of special form [12].

To analyse the nonlinear n-supersymmetry for arbitrary $n \in \mathbb{N}$, it is convenient to introduce the complex oscillator-like operators

$$Z = \partial + W(z, \bar{z}), \qquad \bar{Z} = -\bar{\partial} + \bar{W}(z, \bar{z}), \tag{25}$$

where the complex superpotential is defined by $\mathsf{Re}\, W = A_2(\boldsymbol{x})$, $\mathsf{Im}\, W = A_1(\boldsymbol{x})$, and the notations $z = \frac{1}{2}(x_1 + ix_2)$, $\bar{z} = \frac{1}{2}(x_1 - ix_2)$, $\partial = \partial_z$, $\bar{\partial} = \partial_{\bar{z}}$ are introduced.

The magnetic field is defined by the relation $[Z, \bar{Z}] = 2B(z, \bar{z})$. The n-supersymmetric Hamiltonian has the form

$$H_n = \frac{1}{4}\left\{\bar{Z}, Z\right\} + \frac{n}{4}\left[Z, \bar{Z}\right]\sigma_3. \tag{26}$$

For $n = 1$ we reproduce the usual supersymmetric Hamiltonian. Unlike the linear supersymmetry, the nonlinear holomorphic supersymmetry exists only when the operators (25) obey the relations

$$\left[Z, \left[Z, \left[Z, \bar{Z}\right]\right]\right] = \omega^2\left[Z, \bar{Z}\right], \qquad \left[\bar{Z}, \left[\bar{Z}, \left[Z, \bar{Z}\right]\right]\right] = \bar{\omega}^2\left[Z, \bar{Z}\right]. \tag{27}$$

Here $\omega \in \mathbb{C}$ and $\bar{\omega} = \omega^*$. Using equation (27), one can prove algebraically by the mathematical induction that for the system (26) the odd operators defined by the recurrent relations

$$Q^+_{n+2} = \frac{1}{2}\left(Z^2 - \left(\frac{n+1}{2}\right)^2 \omega^2\right) Q^+_n, \qquad Q^+_0 = \theta^+, \qquad Q^+_1 = 2^{-\frac{1}{2}} Z\theta^+, \tag{28}$$

are the integrals of motions, i.e. they are supercharges. One can make sure [11] that in the 1D case these operators generate the nonlinear supersymmetry with the polynomial superalgebra (1).

In the representation (25) the conditions (27) acquire the form of the differential equations for magnetic field:

$$\left(\partial^2 - \omega^2\right) B(z,\bar{z}) = 0, \qquad \left(\bar{\partial}^2 - \bar{\omega}^2\right) B(z,\bar{z}) = 0. \tag{29}$$

The general solution to these equations is

$$B(z,\bar{z}) = w_+ e^{\omega z + \bar{\omega}\bar{z}} + w_- e^{-(\omega z + \bar{\omega}\bar{z})} + w e^{\omega z - \bar{\omega}\bar{z}} + \bar{w} e^{-(\omega z - \bar{\omega}\bar{z})}, \tag{30}$$

where $w_\pm \in \mathbb{R}$, $w \in \mathbb{C}$, $\bar{w} = w^*$. On the other hand, for $\omega = 0$ the solution to equation (29) is the polynomial,

$$B(\boldsymbol{x}) = c\left((x_1 - x_{10})^2 + (x_2 - x_{20})^2\right) + c_0, \tag{31}$$

with c, c_0, x_{10}, x_{20} being some real constants. Though the latter solution can be obtained formally from (30) in the limit $\omega \to 0$ by rescaling appropriately the parameters $w_\pm$, w, the corresponding limit procedure is singular and the cases (30) and (31) have to be treated separately.

Since the conservation of the supercharges is proved algebraically, the operators Z, $\bar{Z}$ can have any nature (the action of Z, $\bar{Z}$ is supposed to be associative). For example, they can have a matrix structure. With this observation the nonlinear supersymmetry can be applied to the case of matrix Hamiltonians [34, 35, 36].

Thus, the introduction of the operators Z, $\bar{Z}$ allows us to reduce the two-dimensional holomorphic n-supersymmetry to the pure algebraic construction. It is worth noting that in the literature the algebraic relations (27) are known as Dolan–Grady relations. The relations of such a form appeared for the first time in the context of integrable models [13].

The essential difference of the n-supersymmetric 2D system (26) from the corresponding 1D supersymmetric system is the appearance of the central charge

$$J_n = -\frac{1}{4}\left(\omega^2 \bar{Z}^2 + \bar{\omega}^2 Z^2\right) + \partial B \bar{Z} + \bar{\partial} B Z - B^2 + \frac{n}{2}\bar{\partial}\partial B \sigma_3, \tag{32}$$

$[H_n, J_n] = [Q^\pm_n, J_n] = 0$. The anticommutator of the supercharges contains it for any $n > 1$. For example, the $n = 2$ nonlinear superalgebra is

$$\left\{Q^-_2, Q^+_2\right\} = H_2^2 + \frac{1}{4} J_2 + \frac{|\omega|^4}{64}. \tag{33}$$

The systems (26) with the magnetic field (30) of the pure hyperbolic ($w = 0$) or pure trigonometric ($w_\pm = 0$) form can be reduced to the one-dimensional problems with the nonlinear holomorphic supersymmetry [12].

Let us turn now to the polynomial magnetic field (31). One can see that this case reveals a nontrivial relation of the holomorphic n-supersymmetry of the 2D system to the nonholomorphic 1D $\mathcal{N}$-fold supersymmetry of Aoyama et al [33].

In the system (26) with the polynomial magnetic field (31) the central charge has the form

$$J_n = \frac{1}{4c}\left(\partial B(z,\bar{z})\bar{Z} + \bar{\partial} B(z,\bar{z}) Z - B^2(z,\bar{z}) + \frac{n}{2}\bar{\partial}\partial B(z,\bar{z})\sigma_3\right). \tag{34}$$

It can be obtained from the operator (32) in the limit $\omega \to 0$ via the same rescaling of the parameters of the exponential magnetic field which transforms (30) into (31). The essential feature of this integral is its linearity in derivatives.

The polynomial magnetic field (31) is invariant under rotations about the point (x_{10}, x_{20}). Therefore, one can expect that the operator (34) should be related to a generator of the axial symmetry. To use the benefit of this symmetry, one can pass over to the polar coordinate system with the origin at the point (x_{10}, x_{20}). Then the magnetic field is radial, $B(r) = cr^2 + c_0$. The supercharges have the simple structure: $Q_n^+ = 2^{-\frac{n}{2}} Z^n \theta^+ = (Q_n^-)^\dagger$. As in the case $\omega \neq 0$, the anticommutator of the supercharges is a polynomial of the n-th degree in H_n, $\{Q_n^-, Q_n^+\} = H_n^n + P(H_n, J_n)$, where $P(H_n, J_n)$ denotes a polynomial of the $(n-1)$-th degree. For example, for $n = 2$ one has

$$\{Q_2^-, Q_2^+\} = H_2^2 + cJ_2.$$

For the radial magnetic field it is convenient to use the gauge

$$A_\varphi = \frac{1}{4}cr^4 + \frac{1}{2}c_0 r^2, \qquad A_r = 0. \tag{35}$$

In this gauge the Hamiltonian (26) reads

$$H_n = -\frac{1}{2}\left(\partial_r^2 + r^{-1}\partial_r - r^{-2}\left(A_\varphi^2(r) - 2iA_\varphi(r)\partial_\varphi - \partial_\varphi^2\right)\right) + \frac{n}{2}B(r)\sigma_3, \tag{36}$$

while the central charge (34) takes the form $J_n = -i\partial_\varphi - \frac{c_0^2}{4c} + \frac{n}{2}\sigma_3$. Thus, the integral J_n is associated with the axial symmetry of the system under consideration. The simultaneous eigenstates of the operators H_n and J_n have the structure

$$\Psi_m(r, \varphi) = \begin{pmatrix} e^{i(m-n)\varphi}\chi_m(r) \\ e^{im\varphi}\psi_m(r) \end{pmatrix}. \tag{37}$$

Since the angular variable φ is cyclic, the 2D Hamiltonian (36) can be reduced to the 1D Hamiltonian. The kinetic term of the Hamiltonian (36) is Hermitian with respect to the measure $d\mu = rdrd\varphi$. In order to obtain a one-dimensional system with the usual scalar product defined by the measure $d\mu = dr$, one has to perform the similarity transformation $H_n \to UH_nU^{-1}$, $\Psi \to U\Psi$ with $U = \sqrt{r}$. Since the system obtained after such a transformation is originated from the two-dimensional system, one should keep in mind that the variable r belongs to the half-line, $r \in [0, \infty)$. After the transformation, the reduced one-dimensional Hamiltonian acting on the lower (Bose) component of the state (37) reads as

$$\mathcal{H}_n^{(2)} = -\frac{1}{2}\frac{d^2}{dr^2} + \frac{c^2}{32}r^6 + \frac{c_0 c}{8}r^4 + \frac{1}{8}\left(c_0^2 - 2c(2n - m)\right)r^2 + \frac{m^2 - \frac{1}{4}}{2r^2} - \frac{1}{2}(n - m)c_0. \tag{38}$$

This Hamiltonian gives the well-known family of the quasi-exactly solvable systems [15, 16, 34, 19]. The superpartner $\mathcal{H}_n^{(1)}$ acting on the upper (Fermi) component of the state (37) can be obtained from $\mathcal{H}_n^{(2)}$ by the substitution $n \to -n$, $m \to m - n$.

The reduced supercharges have the form

$$\mathcal{Q}_n^+ = 2^{-\frac{n}{2}}\mathcal{Z}_n\theta^+ = (\mathcal{Q}_n^-)^\dagger, \qquad \mathcal{Z}_n = \left(A - \frac{n-1}{r}\right)\left(A - \frac{n-2}{r}\right)\cdots A,$$

where $A = \frac{d}{dr} + W(r)$ and the superpotential is

$$W(r) = \frac{1}{4}cr^3 + \frac{1}{2}c_0 r + \frac{m - \frac{1}{2}}{r}.$$

The operators $\mathcal{Q}_n^{\pm}$, $\mathcal{H}_n^{(i)}$, $i = 1, 2$, generate the non-holomorphic type A $\mathcal{N}$-fold supersymmetry discussed in [33]. The supersymmetry is exact for $c > 0$ ($c < 0$) and corresponding zero modes of the supercharge $\mathcal{Q}_n^{+}$ ($\mathcal{Q}_n^{-}$) can be found. The relation of the $\mathcal{N}$-fold supersymmetry with the cubic superpotential to the family of QES system (38) with the sextic potential was also noted in Ref. [37].

Resume

To conclude, let us summarize the main results of our consideration of the nonlinear supersymmetry.

- Generalized statistics and supersymmetry are intimately related.
- Linear supersymmetry at the classical level is a particular case of a classical supersymmetry characterized by the Poisson algebra being nonlinear in Hamiltonian.
- Any classical 1D supersymmetric system is symplectomorphic to the supersymmetric system of the canonical form characterized by the holomorphic supercharges. There are three different classes of the classical canonical supersymmetric systems defined by the behaviour of the superpotential.
- The anomaly-free quantization of the classical 1D holomorphic n-supersymmetry is possible for the quadratic and exponential superpotentials.
- The nonlinear supersymmetry is closely related to the quasy-exactly solvable systems.
- The $n = 2$ supersymmetric Calogero-like systems (14) admit the anomaly-free quantization for any superpotential; the specific quantum term ($\sim \hbar^2$) "cures" the quantum anomaly problem.
- The anomaly-free quantization of the classical 2D holomorphic n-supersymmetry fixes the form of the magnetic field to be the quadratic or exponential one.
- Realization of the holomorphic n-supersymmetry in 2D systems leads to the appearance of the central charge entering nontrivially into the superalgebra.
- The holomorphic nonlinear supersymmetry can be related to other known forms of nonlinear supersymmetry via the dimensional reduction procedure.
- There is the universal algebraic foundation associated with the Dolan–Grady relations which underlies the holomorphic n-supersymmetry.

The universal algebraic structure underlying the holomorphic nonlinear supersymmetry opens the possibility to apply the latter for investigation of the wide class of the quantum mechanical systems including the models described by the matrix Hamiltonians, the models on the noncommutative space, and integrable models [38].

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Time-Dependent Supersymmetry and Parasupersymmetry in Quantum Mechanics

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Concepts of supersymmetry and parasupersymmetry known for the one-dimensional stationary Schrödinger equation are generalized to the time-dependent equation. Our approach is based on differential transformation operators for the non-stationary Schrödinger equation called Darboux transformation operators and on chains of such operators. As an illustration new exactly solvable time-dependent potentials are derived.

1 Introduction

Supersymmetry has been introduced in quantum mechanics by Nicolai [1] and later by Witten [2]. It was realized afterwards that this approach is really a particular case of transformation operators method well known in mathematics (see e.g. [3]) when it is applied to the stationary Schrödinger equation and when the transformation operator has a differential form [4]. In particular, when the transformation operator is a first order differential operator this approach is equivalent to the one studied by Darboux in 1882 [5]. When the same method is applied to the transformed equation one gets a chain of transformations. We shall see further that the algebraic structure underlying such a chain is parasupersymmetry.

In this lecture I am planning to show how this approach may be generalized to the time-dependent case, i.e. to the time-dependent Schrödinger equation. This generalization is straightforward. Therefore I will develop the time-dependent constructions in parallel lines with the time-independent ones. The left-hand lines of the most formulae will be devoted to the stationary (known) results and the right-hand lines will show their time-dependent generalization.

2 Time-dependent Darboux transformations and time-dependent supersymmetry

The main idea of the transformation operators method is so called *intertwining relation* (see e.g. [4]). Let us suppose that one knows the solutions of the Schrödinger equation (stationary or non-stationary)

$$h_0\psi_E = E\psi_E, \qquad (i\partial_t - h_0)\psi = 0, \tag{1}$$
$$h_0 = -\partial_x^2 + V_0(x), \qquad x \in [a,b].$$

For the stationary case they are supposed to be known for all real and if necessary complex values of the parameter E.

To solve another Schrödinger equation

$$h_1\varphi_E = E\varphi_E, \qquad (i\partial_t - h_1)\psi = 0, \tag{2}$$
$$h_1 = -\partial_x^2 + V_1(x), \qquad x \in [a,b]$$

one may introduce so called *transformation operator* which I will denote by L. The defining relation for this operator is the *intertwining relation*

$$Lh_0 = h_1 L, \qquad L(i\partial_t - h_0) = (i\partial_t - h_1)L. \tag{3}$$

Therefore it is also called *intertwiner*. It is clear from (3) that $\varphi = L\psi$ is a solution to (2) provided ψ is a solution to (1). The equation (1) is called the initial equation, the Hamiltonian h_0 is the initial Hamiltonian and the potential V_0 is the initial potential. The equation (2), Hamiltonian h_1 and the potential V_1 are transformed entities.

In the simplest case one can try to find the operator L as a first order differential operator

$$L = L_0(x) + L_1(x)\partial_x, \qquad L = L_0(x,t) + L_1(x,t)\partial_x.$$

Note that for the time-dependent case I do not include to L the derivative with respect to time. If it would be included to it, L should become a second order operator since it follows from (1) that $i\partial_t = -\partial_x^2 + V_0$ but we want to have the operator L only as a first order differential operator.

If one introduces the potential difference $A = h_1 - h_0 = V_1(x) - V_0(x)$ then the intertwining relation reduces to the system of differential equations for A and for the coefficients of the operator L. Note that this system can be integrated both in stationary and in non-stationary cases. I give here only the final result

$$L = -\frac{u_x(x)}{u(x)} + \partial_x, \qquad L = L_1(t)\left(-\frac{u_x(x,t)}{u(x,t)} + \partial_x\right). \tag{4}$$

Here u is a solution to the initial equation

$$h_0 u(x) = \alpha u(x), \qquad (i\partial_t - h_0)u(x,t) = 0.$$

The main difference between time-dependent and time-independent cases is that for the time-independent case the coefficient L_1 is an arbitrary constant which always may be put equal to equal to 1, but for the time-dependent case it is an arbitrary function of time.

The potential difference depends on the function u but for the time-dependent case it depends also on the function $L_1(t)$. For the time-independent case the function u can always be chosen real for all real values of the parameter E whereas for the time-dependent case this function takes essentially complex values. Our main idea for the time-dependent case is to dispose of the arbitrary function $L_1(t)$ for satisfying the reality condition for the potential difference. As it happens this is possible only if the function u is subject to an additional condition which we call *the reality condition of the new potential* or simply *the reality condition*

$$[\log u/\overline{u}]_{xxx} = 0.$$

The bar means the complex conjugation. Under this condition the function $L_1(t)$ becomes real

$$L_1(t) = \exp\left[2\int dt\ \mathrm{Im}\,(\log u)_{xx}\right] \tag{5}$$

and for the potential difference one gets

$$A(x) = -\left[\log u^2(x)\right]_{xx}, \qquad A(x,t) = -\left[\log |u(x,t)|^2\right]_{xx}. \tag{6}$$

We see from (4), (5), (6) that the potential difference and the transformation operator are defined only by the function u. Therefore we call it the *transformation function*. As it follows from (6) a sole condition which should be imposed on u is the absence of zeros for x belonging to the interval (a,b) where the initial Schrödinger equation is solved. No boundary or asymptotic condition should be imposed on it.

Note, that when the potential V_0 is independent of time, one can take $u(x,t)$ in the form

$$u(x,t) = u(x)e^{-i\alpha t}.$$

In this case the reality condition is satisfied and the function $L_1 = \text{const}$. The time-dependent transformation reduces just to the known time-independent one.

Once one knows the operator L one can introduce so called Laplace adjoint to it which is defined by the formal relations

$$(c\partial_x)^+ = -\overline{c}\partial_x, \qquad c \in \mathbb{C}, \qquad (AB)^+ = B^+A^+.$$

Then

$$L^+ = -\frac{u_x(x)}{u(x)} - \partial_x, \qquad L^+ = -L_1(t)\left(\frac{\overline{u}_x(x,t)}{\overline{u}(x,t)} + \partial_x\right)$$

and

$$(i\partial_t - h_0)^+ = i\partial_t - h_0.$$

The conjugation of the intertwining relations gives us corresponding relations for L^+

$$h_0L^+ = L^+h_1, \qquad (i\partial_t - h_0)L^+ = L^+(i\partial_t - h_1).$$

These relations mean that the operator L^+ realizes the transformation in the inverse direction, i.e. from the solutions of the transformed equation to the solutions of the initial one. It is clear now that the superposition L^+L transforms solutions of the initial equation into solutions of the same equation and hence this is a symmetry operator for the initial Schrödinger equation. By the same reason the operator LL^+ is a symmetry operator for the transformed equation. For the stationary case there exists only one second order differential symmetry operator, this is the Hamiltonian (may be displaced by a constant). For the non-stationary case the Hamiltonian in general is not an integral of motion. So, our transformation is possible only for such systems which have symmetry operators either of the second order in ∂_x or of the first order in ∂_x and ∂_t. In other words

$$L^+L = h_0 - \alpha, \qquad L^+L = g_0 - \alpha \tag{7}$$

and

$$LL^+ = h_1 - \alpha, \qquad LL^+ = g_1 - \alpha. \tag{8}$$

We denote by g_0 and g_1 corresponding symmetry operators for the nonstationary Schrödinger equation. These relations may be treated as factorizations of the operators g_0 (h_0) and g_1 (h_1).

It follows from (7) and (8) that for the non-stationary case the following intertwining relations take place

$$Lg_0 = g_1L, \qquad g_0L^+ = L^+g_1. \tag{9}$$

Moreover, when a Hilbert space is introduced and formally adjoint operator coincides with the adjoint with respect to an inner product, the operators L^+L and LL^+ are nonnegative. Hence, the symmetry operators g_0 (h_0) and g_1 (h_1) are bounded from below. Furthermore, by constructions one has $Lu = 0$. It follows from here and (7) that $g_0u = \alpha u$.

The intertwining relations and factorization properties may be rewritten in another form. Let us introduce the following matrices

$$\mathcal{H} = \begin{pmatrix} h_0 & 0 \\ 0 & h_1 \end{pmatrix}, \qquad \mathcal{G} = \begin{pmatrix} g_0 & 0 \\ 0 & g_1 \end{pmatrix}$$

and

$$\mathcal{Q}^+ = \begin{pmatrix} 0 & L^+ \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{Q} = \begin{pmatrix} 0 & 0 \\ L & 0 \end{pmatrix}.$$

It is easy to see now that the factorization property may be rewritten in the form

$$\mathcal{Q}^+\mathcal{Q} + \mathcal{Q}\mathcal{Q}^+ = \mathcal{H} - \alpha\mathcal{I}, \qquad \mathcal{Q}^+\mathcal{Q} + \mathcal{Q}\mathcal{Q}^+ = \mathcal{G} - \alpha\mathcal{I},$$

where $\mathcal{I}$ is the unity 2×2 matrix, and the intertwining relations result in

$$\mathcal{Q}\mathcal{H} - \mathcal{H}\mathcal{Q} = 0, \qquad \mathcal{Q}\mathcal{G} - \mathcal{G}\mathcal{Q} = 0.$$

We see from here that the operators $\mathcal{H}$, Q, Q^+ or $\mathcal{G}$, Q, Q^+ form a simplest superalgebra. In the time-dependent case the operators $\mathcal{G}$, Q, Q^+ depend on time. Therefore we have a *time-dependent superalgebra*.

The operators L and L^+ have non-trivial kernels. Nevertheless if one introduces the space of the solutions of the initial equation, T_0, and the space of the solutions of the transformed equation, T_1, one can establish a one-to one correspondence between these spaces. Let us decompose the spaces $T_{0,1}$ into a direct sums

$$T_{0,1} = T_{0,1}^0 \oplus T_{0,1}^1, \qquad T_0^1 = \ker L^+L, \qquad T_1^1 = \ker LL^+.$$

The spaces $T_{0,1}^1$ are two-dimensional. It is clear by constructions that $u \in T_0^1$. The equation $L^+L = 0$ except for u has another solution linearly independent with u which has the form

$$\tilde{u} = uL_1^{-2} \int \frac{dx}{u\overline{u}}\,.$$

It is easy to show that the function $v = L\tilde{u} = 1/(L_1\overline{u})$ is such that $L^+v = 0$. This means that $v \in \ker LL^+$. Another solution of the equation $LL^+ = 0$ linearly independent with v has the form $\tilde{v} = vL_1^{-2}\int 1/(v\overline{v})\,dx$ and $L^+\tilde{v} = u$. Once we know the basis functions $u, \tilde{u} \in T_0^1$ and $v, \tilde{v} \in T_1^1$ we can define a linear one-to-one correspondence between T_0^1 and T_1^1 by defining the correspondence between the bases: $u \longleftrightarrow \tilde{v}$ and $\tilde{u} \longleftrightarrow v$. The equations $L^+L\psi = 0$ and $LL^+\varphi = 0$ has no solutions when solving on T_0^0 and T_0^1 respectively. These operators are hence invertible on these spaces. This means that they establish the one-to-one correspondence between T_0^0 and T_0^1. So, we have established the one-to one correspondence between T_0 and T_1.

This correspondence is very useful for finding all square integrable solutions of the transformed equation. It is easy to see that the function $\varphi = L\psi$ is square integrable provided so is $\psi \in T_0^0$ and when ψ is not square integrable φ is not either. Hence, to find all square integrable solutions of the transformed equation it remains to analyze the functions v and $\tilde{v}$.

As I have mentioned, u should be a nodeless solution of the initial Schrödinger equation and the operator g_0 is bounded from below. Let E_0 be its lower bound. Then, according to the oscillator theorem u may be nodeless only if $\alpha \le E_0$. When g_0 has a discrete spectrum, E_0 may be associated with the ground state level. If we take $\alpha = E_0$ then neither v nor $\tilde{v}$ are square integrable and this level will be absent in the spectrum of g_1. All other levels of g_0 are unchanged in the course of the Darboux transformation. When $\alpha < E_0$ there are two possibilities. The first one corresponds to the case when the function $v = 1/(L_1\overline{u})$ is square integrable. In this case the operator g_1 has an additional discrete spectrum level with respect to g_0. In the second case neither v nor $\tilde{v}$ are square integrable and g_1 has exactly the same spectrum as g_0, i.e. they are strictly isospectral.

3 Chains of transformations and parasupersymmetry

Once we know the potential V_1 we can take it as V_0 and realize the Darboux transformation once again, etc. In such a way we obtain a chain of exactly solvable symmetry operators

$$h_0 \ \to h_1 \ \to \cdots \ \to h_N, \qquad g_0 \ \to g_1 \ \to \cdots \ \to g_N \tag{10}$$

and a chain of first order transformation operators

$$L_{0,1} \ \to L_{1,2} \ \to \cdots \ \to L_{N-1,N}.$$

If one is not interested in the intermediate operators, one can expunge all the intermediate transformation functions from the final result and express it only in terms of solutions of the initial equation. Moreover, in this case one does not have to impose the reality condition on the intermediate potentials. This leads to the following formulae for solutions of the transformed equation

$$\varphi = L_{0N}(t)W(u_1, u_2, \ldots, u_N) \begin{vmatrix} u_1 & u_2 & \cdots & \psi \\ u_{1x} & u_{2x} & \cdots & \psi_x \\ \cdots & \cdots & \cdots & \cdots \\ u_{1x}^{(N)} & u_{2x}^{(N)} & \cdots & \psi_x^{(N)} \end{vmatrix}. \tag{11}$$

Here $g_0 u_k = \alpha_k u_k$. For the stationary case $L_{0N}(t) = 1$ and this formula reduces to the known Krum–Krein formula [6, 7]. The formula (11) defines an N-order transformation operator $\varphi = L_{0,N}\psi$, $L_{0,N} = L_{N-1,N}L_{N-2,N-1}\cdots L_{0,1}$. This operator is an intertwiner for the symmetry operators g_0 and g_N. The operators $L_{0,N}$ and its adjoint $L^+_{0,N}$ factorize now a polynomial of the operators g_0 and g_N

$$L^+_{0,N}L_{0,N} = \prod_{k=1}^{N}(h_0 - \alpha_k), \qquad L^+_{0,N}L_{0,N} = \prod_{k=1}^{N}(g_0 - \alpha_k), \tag{12}$$

$$L_{0,N}L^+_{0,N} = \prod_{k=1}^{N}(h_N - \alpha_k), \qquad L_{0,N}L^+_{0,N} = \prod_{k=1}^{N}(g_N - \alpha_k). \tag{13}$$

Let us consider a chain in which all elements are good. Such chains are known as *completely reducible* ones. For this chain one can consider nth order transformation operators

$$L_{p,p+n} = L_{p+n-1,p+n}L_{p+n-2,p+n-1}\cdots L_{p,p+1}, \qquad n \le N$$

and their adjoint. They factorize polynomials of the symmetry operators g_p and g_{p+n}

$$L^+_{p,p+n}L_{p,p+n} = \prod_{k=1}^{n}(g_p - \alpha_{p+k}), \qquad L_{p,p+n}L^+_{p,p+n} = \prod_{k=1}^{n}(g_{p+n} - \alpha_{p+k})$$

and they are intertwiners for g_p and g_{p+n} and for the Schrödinger equations with the Hamiltonians h_p and h_{p+n}.

Let us introduce now the diagonal matrix operators

$$\mathcal{H} = \mathrm{diag}\,(h_0, h_1, \ldots, h_N), \qquad \mathcal{G} = \mathrm{diag}\,(g_0, g_1, \ldots, g_N)$$

and nilpotent supercharges

$$\mathcal{Q}^+_{p,q} = L_{p,q}e_{p,q}, \qquad \mathcal{Q}_{p,q} = L^+_{p,q}e_{q,p},$$

where $e_{p,q}$ is $(N+1)\times(N+1)$ matrix with a single non-zero entry which is equal to one and stands at the intersection of pth column and qth row.

Instead of the chain of the Schrödinger equations one can write now the single equation (supersymmetric Schrödinger equation)

$$(i\mathcal{I}\partial_t - \mathcal{H})\Psi(x,t) = 0.$$

Intertwining relations between transformation operators and $i\partial_t - h_p$ are equivalent to the commutation of the supercharges $\mathcal{Q}_{p,q}$ with $i\mathcal{I}\partial_t - \mathcal{H}$. This means that all $\mathcal{Q}_{p,q}$ are integrals of motion for the system with the superhamiltonian $\mathcal{H}$. The condition of the complete reducibility leads to the following non-linear algebra

$$\begin{aligned}
&\mathcal{Q}_{s,p}\mathcal{Q}_{p,q} = \mathcal{Q}_{s,q}, \qquad N+1 \geq q > p > s,\\
&\mathcal{Q}^+_{p,p+n}\mathcal{Q}_{p,p+n+m} = \prod_{i=1}^{n}(\mathcal{G}_0 - \alpha_{p+i})\mathcal{Q}_{p+n,p+n+m}, \qquad p+n+m \leq N+1,\\
&\mathcal{Q}_{p-n-m,p}\mathcal{Q}^+_{p-n,p} = \prod_{i=1}^{n}(\mathcal{G}_0 - \alpha_{p+i-1})\mathcal{Q}_{p-n-m,p-n}, \qquad p-n-m \geq 0, \quad p \leq N+1,\\
&\mathcal{Q}_{p,p+n}\mathcal{Q}^+_{p,p+n}\mathcal{Q}_{p,p+n} = \prod_{i=1}^{n}(\mathcal{G}_0 - \alpha_{p+i})\mathcal{Q}_{p,p+n}, \qquad p+n \leq N+1, \qquad n,m = 1,2,\ldots
\end{aligned}$$

Similar non-linear algebras are known for the stationary Schrödinger equation as parasuperalgebras (see e.g. [8, 9]). The operators involved in this algebra depend on time. Hence one has here a time-dependent parasuperalgebra.

4 Time-dependent exactly solvable potentials

4.1 Harmonic oscillator with a time varying frequency

Consider first a time-dependent generalization of the harmonic oscillator

$$h_0 = -\partial_x^2 + \omega^2(t)x^2. \tag{14}$$

Some solutions of the Schrödinger equation with such a Hamiltonian are well-known but we will need other solutions for using as transformation functions. To get them we will use the method of separation of the variables in its general formulation as R-separation of variables well-described in the book by Miller [10]. This method is based on classification of orbits in adjoint representation of a symmetry group for a given equation.

Symmetry algebra of the Schrödinger equation with the Hamiltonian (14) is the well-known Schrödinger algebra. Consider first representation of this algebra suitable for our purpose.

Operators $a = \epsilon\partial_x - \frac{i}{2}\dot{\epsilon}x$, $a^+ = \bar{\epsilon}\partial_x + \frac{i}{2}\dot{\bar{\epsilon}}x$, $aa^+ - a^+a = \frac{1}{4}$, where $\epsilon = \epsilon(t)$ is a solution of a classical equation of motion for the Harmonic oscillator with a time-varying frequency $\ddot{\epsilon}(t) + 4\omega^2(t)\epsilon(t) = 0$ are creation and annihilation operators and together with the identity operator close the Heisenberg–Weil algebra. All operators of the Schrödinger algebra are constructed in terms of a and a^+

$$\begin{aligned}
&K_1 = a - a^+, \qquad K_{-1} = -i(a+a^+), \qquad K_0 = i,\\
&K_{-2} = -i(a+a^+)^2, \qquad K_2 = -i(a-a^+)^2, \qquad K^0 = -2\left[a^2 - (a^+)^2\right].
\end{aligned}$$

Symmetry operators are classified by the orbits of adjoint representation of the symmetry group. It is well-known that in the case under consideration there exist five different orbits. We shall consider every orbit successively.

Two orbits with representatives $J_1 = K_1$ and $J_1 = K_2$ give the same solution of the Schrödinger equation

$$\psi = \gamma^{-1/2} \exp\left[\frac{i\lambda x}{8\gamma} + \frac{ix^2\dot{\gamma}}{4\gamma} - \frac{i\lambda^2\delta}{64\gamma}\right], \tag{15}$$

$$2\gamma = \epsilon + \bar{\epsilon}, \qquad 2i\delta = \epsilon - \bar{\epsilon}, \qquad \dot{\epsilon}\bar{\epsilon} - \epsilon\dot{\bar{\epsilon}} = \frac{i}{2}.$$

Using the function (15) we construct the transformation function

$$u = \gamma^{-1/2} \cosh\frac{\lambda x}{8\gamma} \exp\left[\frac{ix^2\dot{\gamma}}{4\gamma} - \frac{i\lambda^2\delta}{64\gamma}\right], \qquad L_1(t) = \gamma = (\epsilon + \bar{\epsilon})/2,$$

which gives us the following potential

$$V_1 = \omega^2(t)x^2 - \frac{\lambda^2}{32\gamma^2}\cosh^{-2}\frac{\lambda x}{8\gamma}.$$

When $\omega = 0$ it reduces to the well-known one soliton potential. Therefore it may be considered as a non-stationary generalization of the one soliton potential. The Fig. 1 shows the behavior of this potential for $\omega = 1/2$ (stationary case) and $\gamma = \frac{1}{2}\cos t$. At the bottom of the harmonic oscillator parabola one can see an additional minimum of the varying depth.

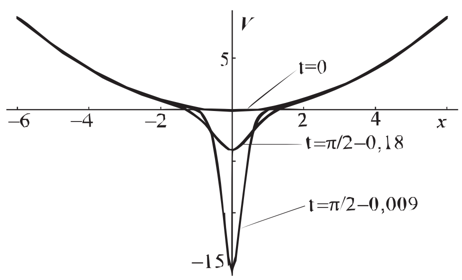

Figure 1. Potential with a time-dependent anharmonic member.

Using the same function (15) one can construct the transformation function of a more general form

$$u = u_\lambda + u_{\bar{\lambda}} = \gamma^{-1/2}\cosh\left(\frac{\nu x}{8\gamma} + \mu\nu\frac{\delta}{32\gamma}\right)\exp\left[\frac{ix^2\dot{\gamma}}{4\gamma} - \frac{i\mu x}{8\gamma} + i\left(\nu^2 - \mu^2\right)\frac{\delta}{64\gamma}\right],$$

$$\lambda = -\mu - i\nu, \qquad L_1(t) = \gamma$$

which gives the following potential

$$V_1 = \omega^2(t)x^2 - \frac{\nu^2}{32\gamma^2}\cosh^{-2}\left(\frac{\nu x}{8\gamma} + \mu\nu\frac{\delta}{32\gamma}\right). \tag{16}$$

When $\omega = 0$ this potential reduces to the known non-stationary soliton potential which gives rise to a one soliton solution to the Kadomtsev–Petviashili (two-dimensional KdV) nonlinear equation. The next figure shows the plot of this potential at different time-moments. Here an additional minimum of a varying depth oscillates between the parabola walls.

The next orbit is presented by the operator $J_2 = K_2 - K_1$. Corresponding solution of the Schrödinger equation has the form

$$\psi = \delta^{-1/2}\exp\left(ix^2\frac{\dot{\delta}}{4\delta} - ix\frac{\gamma}{2\delta^1} + i\frac{\gamma^3}{6\delta^3} + i\lambda\frac{\gamma}{\delta}\right) Q\left(2^{-1/2}\left(\frac{x}{\delta} - \frac{\gamma^2}{2\delta^2}\right) - 2^2/3\lambda\right),$$

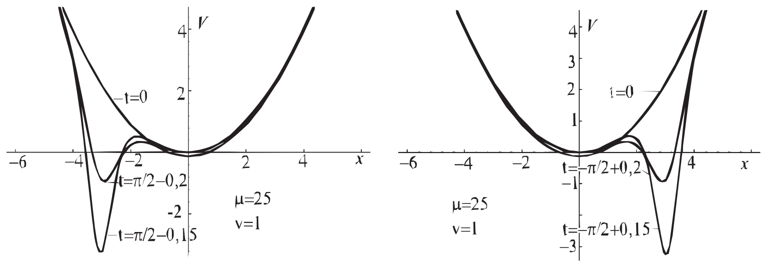

Figure 2. Potential with a time-dependent anharmonic member at different time moments.

where $\gamma = \epsilon + \bar{\epsilon}$, $i\delta = \epsilon - \bar{\epsilon}$, and $Q(z)$ is the Airy function satisfying the equation $Q_{zz}(z) = zQ(z)$. Exactly solvable potential is expressed in this case in terms of the Airy function. To obtain a real and regular on the whole real line potential one can realize a second order transformation with the mutually conjugated transformation functions u_λ and $u_{\overline{\lambda}}$. For $\omega = 0$ the plot of one of these potentials is shown by the Fig. 3.

The fourth orbit has the representative $J_3 = K_2 - K_{-2}$ and creates the following solution of the Schrödinger equation

$$\psi = \gamma^{-1/4}\left(\frac{\epsilon}{\bar{\epsilon}}\right)^{\lambda/2}\exp\left(i\frac{\dot{\gamma}x^2}{8\gamma}\right)Q\left(\frac{x}{2\sqrt{\gamma}}\right), \qquad \gamma = \epsilon\bar{\epsilon},$$

where $Q(z)$ is the parabolic cylinder function satisfying the equation $Q_{zz}(z) - \left(z^2/4 + \lambda\right)Q(z) = 0$. At $\lambda = -n - 1/2$ one gets the discrete basis functions of corresponding Hilbert space

$$\psi_n = N_n\gamma^{-1/4}\left(\frac{\bar{\epsilon}}{\epsilon}\right)^{n/2+1/4}\exp\left(\frac{2i\dot{\gamma} - 1}{16\gamma}x^2\right)He_n\left(\frac{x}{2\sqrt{\gamma}}\right),$$

where $He_n(z) = 2^{-n/2}H_n(z/\sqrt{2})$ are Hermite polynomials.

The same functions with $\lambda = n + 1/2$ are suitable for the Darboux transformations and they generate the following potential differences

$$A_2^{m,l} = \frac{1}{2\gamma}\left[1 + \frac{f''_{ml}(z)}{f_{ml}(z)} - \left(\frac{f'_m l(z)}{f_{ml}(z)}\right)^2\right],$$
$$f_{ml}(z) = q_m(z)q_{l+1}(z) - q_l(z)q_{m+1}(z), \qquad z = x/(2\sqrt{\gamma})$$

which are well-defined for $m = 0, 2, 4, \ldots$, $l = m+1, m+3, \ldots$. For $m = 2$ and $l = 5$ the behavior of the transformed potential is shown by the Fig. 4.

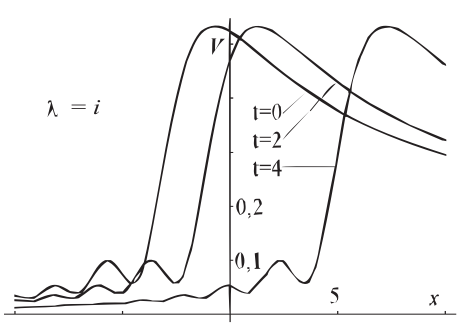

Figure 3. Potential generated with the help of the Airy function.

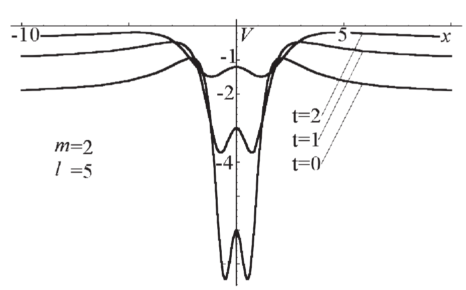

Figure 4. Potentials $V_2^{m,l}(x,t)$ at $m = 2$ and $l = 5$.

One can take a general solution of the equation for the parabolic cylinder functions as transformation function. For example, when $\lambda = 1/2$ one has

$$u = \gamma^{-1/4} \left(\frac{\epsilon}{\bar{\epsilon}}\right) \exp\left(\frac{2i\dot{\gamma}+1}{16\gamma}x^2\right) \left[C + \operatorname{erf}\left(\frac{x}{2\sqrt{2\gamma}}\right)\right],$$

which gives the following potential

$$V_1 = \omega^2(t)x^2 - \frac{1}{4\gamma}\left[1 - 2zQ^{-1}(z)e^{-z^2/2} - 2Q^{-2}(z)e^{-z^2}\right],$$

$$Q(z) = \sqrt{\frac{\pi}{2}}\left[C + \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right)\right], \qquad z = \frac{x}{2\sqrt{\gamma}}, \qquad |C| > 1.$$

For $\omega = \text{const} \neq 0$ these potentials reduce to the known isospectral potentials with an equidistant spectrum. For $\omega = 0$ their behavior is shown by the Fig. 5. The cases a) and b) differ by the values of parameters the potential depends on.

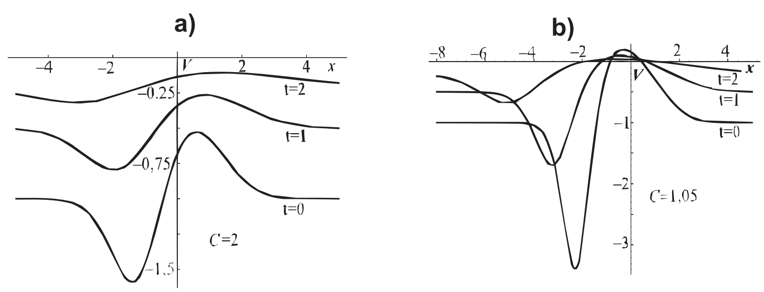

Figure 5. Time-dependent generalization of isospectral potentials.

4.2 Singular oscillator with a time dependent frequency

Consider now the following Hamiltonian:

$$h_0 = -\partial_x^2 + \omega^2(t)\,x^2 + gx^{-2}.$$

Symmetry algebra of the Schrödinger equation with this Hamiltonian is $su(1.1) \sim sl(2,\mathbb{R})$. We use the following representation for this algebra:

$$[K_+ = 2\left[\left(a^+\right)^2 - \bar{\varepsilon}^2 g x^{-2}\right], \quad K_- = 2\left[a^2 - \varepsilon^2 g x^{-2}\right],$$

$$K_0 = \frac{1}{2}\left(K_-K_+ - K_+K_-\right) = \frac{1}{2}\left[K_-, K_+\right].$$

Consider solutions of the Schrödinger equation which are eigenstates of K_0: $K_0\varphi_\lambda(x,t) = \lambda\varphi_\lambda(x,t)$. When $\lambda = n + k$, $n = 0, 1, 2, \ldots$ we have a discrete basis of the Hilbert space

$$\varphi_n(x,t) = 2^{1/2-3k}\sqrt{\frac{n!}{\Gamma(n+2k)}}\,\gamma^{-k}\left(\frac{\bar{\varepsilon}}{\varepsilon}\right)^{n+k} x^{2k-1/2}$$

$$\times \exp\left[i\frac{x^2\dot{\gamma}}{8\gamma} - \frac{x^2}{16\gamma}\right] L_n^{2k-1}\left(\frac{x^2}{8\gamma}\right), \qquad k = \frac{1}{2} + \frac{1}{4}\sqrt{1+4g}, \qquad \gamma = \varepsilon\bar{\varepsilon}.$$

To construct spontaneously broken supersymmetric model we need transformation functions $u(x,t)$ such that neither $u(x,t)$ nor $u^{-1}(x,t)$ are from the Hilbert space and $u(x,t)$ is nodeless for all real values of t and $x > 0$. These conditions are fulfilled for the functions

$$u_p(x,t) = \gamma^{-k}\left(\frac{\overline{\varepsilon}}{\varepsilon}\right)^{-p-k} x^{2k-1/2}\exp\left[i\frac{x^2\dot{\gamma}}{8\gamma}+\frac{x^2}{16\gamma}\right] L_p^{2k-1}\left(\frac{-x^2}{8\gamma}\right),$$
$$K_0 u_p(x,t) = -(p+k)u_p(x,t).$$

These transformation functions create the following exactly solvable family of potential differences $A(x,t) = \omega^2(t)x^2 + gx^{-2} - V_1(x,t)$:

$$A(x,t) = A_p(x,t) = \frac{1}{4\gamma} - \frac{4k-1}{x^2} - \frac{1}{8}\left(\frac{xL_{p-1}^{2k}(z)}{\gamma L_p^{2k-1}(z)}\right)^2 + \frac{x^2 L_{p-2}^{2k+1}(z) + 4\gamma L_{p-1}^{2k}(z)}{8\gamma^2 L_p^{2k-1}(z)}, \qquad z = -\frac{x^2}{8\gamma}.$$

To construct a model with exact supersymmetry we need transformation functions $u(x,t)$ such that $u^{-1}(x,t)$ is square integrable on semiaxis $x \geq 0$ and satisfies the zero boundary condition at the origin for all values of t. The following solution of the Schrödinger equation may be chosen in this case:

$$u_p(x,t) = \gamma^{k-1}\left(\frac{\overline{\varepsilon}}{\varepsilon}\right)^{k-p-1} x^{3/2-2k}\exp\left[i\frac{x^2\dot{\gamma}}{8\gamma}+\frac{x^2}{16\gamma}\right] L_p^{1-2k}\left(\frac{-x^2}{8\gamma}\right),$$
$$K_0 u_p(x,t) = (k-p-1)\, u_p(x,t).$$

It is not difficult to establish the possible values of p. If p is even it may takes the values $p < 2k-1$ and $p = [2k]+1, [2k]+3, \ldots$. For odd p values we may use only $p = [2k], [2k]+2, \ldots$, where $[2k] \equiv \mathrm{entire}\,(2k)$. For regular potential differences we obtain

$$A_p(x,t) = \frac{1}{4\gamma} + \frac{4k-3}{x^2} - \frac{1}{2}\left(\frac{xL_{p-1}^{2-2k}(z)}{2\gamma L_p^{1-2k}(z)}\right)^2 + \frac{x^2 L_{p-2}^{3-2k}(z) + 4\gamma L_{p-1}^{2-2k}(z)}{8\gamma^2 L_p^{1-2k}(z)}.$$

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Geometry of Nonlinear Supersymmetry in Curved Spacetime and Unity of Nature

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A new Einstein–Hilbert type action of superon-graviton model (SGM) for space-time and matter is obtained based upon the geometrical arguments of the higher symmetric (SGM) space-time. SGM action is invariant under [global NL SUSY] $\otimes$ [local $GL(4,\mathbb{R})$] $\otimes$ [local Lorentz] $\otimes$ [global $SO(N)$]. The explicit form of SGM action is given in terms of the fields of the graviton and superons by using the affine connection formalism. Some characteristic structures of the gravitational coupling of superons are manifested (in two dimensional space-time) with some details of the calculations. SGM cosmology is discussed briefly.

1 Introduction

To explore the new physics and the new framework for the unification of space-time and matter beyond the standard mode (SM), new (gauge) symmetries and new particles yet to be observed are introduced in the model building. Supersymmetry [1, 2] may be the most promising notion beyond SM, especially for the unification of space-time and matter.

In the previous paper [3] we have introduced a new fundamental constituent with spin 1/2 superon and proposed superon-graviton model (SGM) as a model for unity of space-time and matter. In SGM, the fundamental entities of nature are the graviton with spin-2 and a quintet of superons with spin-1/2. They are the elementary gauge fields corresponding to the local $GL(4,\mathbb{R})$ and the global nonlinear supersymmetry (NL SUSY) with a global $SO(10)$, respectively. All observed elementary particles including gravity are assigned to a single massless irreducible representation of $SO(10)$ super-Poincaré (SP) symmetry and reveal a remarkable potential for the phenomenology, e.g. the three-generations structure of quarks and leptons, stability of proton, mixings, etc. [3]. And except graviton they are supposed to be the (massless) composite-eigenstates of superons of $SO(10)$ SP symmetry [4] of space-time and matter. The uniqueness of $N = 10$ among all $SO(N)$ SP is pointed out. The arguments are group theoretical so far.

In order to obtain the fundamental action of SGM which is invariant at least under local $GL(4,\mathbb{R})$, local Lorentz, global NL SUSY transformations and global $SO(10)$, we have performed the similar arguments to Einstein general relativity theory (EGRT) in the SGM space-time, where the tangent (Riemann-flat) Minkowski space-time is specified by the coset space $SL(2,\mathbb{C})$ coordinates (corresponding to Nambu–Goldstone (N–G) fermion) of NL SUSY of Volkov–Akulov (V–A) [2] in addition to the ordinary Lorentz $SO(3,1)$ coordinates [3], which are locally homomorphic groups. As shown in Ref. [5] the SGM action obtained by the geometrical arguments of SGM space-time is naturally the analogue of Einstein–Hilbert (E–H) action of GR and has the similar concise expression. (The similar systematic arguments are applicable to spin 3/2 N–G case [6].)

In this article, after a brief review of SGM for the self contained arguments we expand SGM action in terms of the fields of graviton and superons in order to see some characteristic structures of our model and to show some details of the calculations. For the sake of simplicity the expansion is performed by the affine connection formalism.

Finally some hidden symmetries and a potential cosmology, especially the birth of the universe are mentioned briefly.

2 Fundamental action of superon-graviton model (SGM)

SGM space-time is defined as the space-time whose tangent(flat) space-time is specified by $SO(1,3)$ Lorentz coordinates x^a and the coset space $SL(2,\mathbb{C})$ coordinates ψ of NL SUSY of Volkov–Akulov (V–A) [2]. The unified vierbein ${w_a}^\mu$ and the unified metric $s^{\mu\nu}(x) \equiv {w_a}^\mu(x) w^{a\nu}(x)$ of SGM space-time are defined by generalizing the NL SUSY invariant differential forms of V–A to the curved space-time [5]. SGM action is given as follows [5]

$$L_{\rm SGM} = -\frac{c^3}{16\pi G}|w|(\Omega+\Lambda), \tag{1}$$

$$|w| = \det {w_a}^\mu = \det({e_a}^\mu + {t_a}^\mu), \qquad {t_a}^\mu = \frac{\kappa}{2i}\sum_{j=1}^{10}(\bar\psi^j\gamma_a\partial^\mu\psi^j - \partial^\mu\bar\psi^j\gamma_a\psi^j), \tag{2}$$

where κ is an arbitrary constant of V–A up now with the dimension of the fourth power of length, ${e_a}^\mu$ and ψ^j $(j = 1,2,\ldots,10)$ are the fundamental elementary fields of SGM, i.e. the vierbein of (EGRT) and the superons of N–G fermion of NL SUSY of Volkov–Akulov [2], respectively. Λ is a cosmological constant which is necessary for SGM action to reduce to V–A model with the first order derivative terms of the superon in the Riemann-flat space-time. Ω is a unified scalar curvature of SGM space-time analogous to the Ricci scalar curvature R of EGRT. SGM action (1) is invariant under the following new SUSY transformations

$$\delta\psi^i(x) = \zeta^i + i\kappa(\bar\zeta^j\gamma^\rho\psi^j(x))\partial_\rho\psi^i(x), \tag{3}$$

$$\delta{e^a}_\mu(x) = i\kappa(\bar\zeta^j\gamma^\rho\psi^j(x))D_{[\rho}{e^a}_{\mu]}(x), \tag{4}$$

where ζ^i, $(i = 1,\ldots,10)$ is a constant spinor, $D_{[\rho}{e^a}_{\mu]}(x) = D_\rho{e^a}_\mu - D_\mu{e^a}_\rho$ and D_μ is a covariant derivative containing a symmetric affine connection. The explicit expression of Ω is obtained by just replacing ${e_a}^\mu(x)$ in Ricci scalar R of EGRT by the vierbein ${w_a}^\mu(x) = {e_a}^\mu + {t_a}^\mu$ of the SGM curved space-time, which gives the gravitational interaction of $\psi(x)$ invariant under (3) and (4). The overall factor of our model is fixed to $\frac{-c^3}{16\pi G}$, which reproduces E–H action of GR in the absence of superons(matter). Also in the Riemann-flat space-time, i.e. ${e_a}^\mu(x) \to {\delta_a}^\mu$, it reproduces V–A action of NL SUSY[2] with $\kappa_{\rm V\text{–}A}^{-1} = \frac{c^3}{16\pi G}\Lambda$ in the first order derivative terms of the superon. Therefore our model (SGM) predicts a (small) non-zero cosmological constant, provided $\kappa_{\rm V\text{–}A} \sim O(1)$, and posesses two mass scales. Furthermore it fixes the coupling constant of superon (N–G fermion) with the vacuum to $\left(\frac{c^3}{16\pi G}\Lambda\right)^{\frac{1}{2}}$ (from the low energy theorem viewpoint), which may be relevant to the birth of the universe.

It is interesting that our action is the vacuum (matter free) action in SGM space-time as read off from (1) but gives in ordinary Riemann space-time the E–H action with matter (superons) accompanying the spontaneous supersymmetry breaking.

The commutators of new SUSY transformations induce the generalized general coordinate transformations

$$[\delta_{\zeta_1}, \delta_{\zeta_2}]\psi = \Xi^\mu\partial_\mu\psi, \tag{5}$$

$$[\delta_{\zeta_1}, \delta_{\zeta_2}]{e^a}_\mu = \Xi^\rho\partial_\rho{e^a}_\mu + {e^a}_\rho\partial_\mu\Xi^\rho, \tag{6}$$

where Ξ^μ is defined by

$$\Xi^\mu = 2ia(\bar\zeta_2\gamma^\mu\zeta_1) - \xi_1^\rho\xi_2^\sigma{e_a}^\mu(D_{[\rho}{e^a}_{\sigma]}). \tag{7}$$

We have shown that our action is invariant at least under [7]

$$[\text{global NL SUSY}] \otimes [\text{local } GL(4,\mathbb{R})] \otimes [\text{local Lorentz}] \otimes [\text{global } SO(N)], \tag{8}$$

which is isomorphic to $N = 10$ extended (global $SO(10)$) SP symmetry through which SGM reveals the spectrum of all observed particles in the low energy [4]. In contrast with the ordinary SP SUSY, SGM SUSY may be regarded as a square root of a generalized $GL(4,\mathbb{R})$. The usual local $GL(4,\mathbb{R})$ invariance is obvious by the construction.

The simple expression (1) invariant under the above symmetry may be universal for the gravitational coupling of Nambu–Goldstone (N–G) fermion, for by performing the parallel arguments we obtain the same expression for the gravitational interaction of the spin-3/2 N–G fermion [6].

Now to clarify the characteristic features of SGM we focus on $N = 1$ SGM for simplicity without loss of generality and write down the action explicitly in terms of ${t^a}_\mu$ (or ψ) and $g^{\mu\nu}$ (or ${e^a}_\mu$). We will see that the graviton and superons (matter) are complementary in SGM and contribute equally to the curvature of SGM space-time. Contrary to its simple expression (1), it has rather complicated and rich structures.

We use the Minkowski tangent space metric $\frac{1}{2}\{\gamma^a, \gamma^b\} = \eta^{ab} = (+,-,-,-)$ and $\sigma^{ab} = \frac{i}{4}[\gamma^a, \gamma^b]$. (Latin $(a, b, \ldots)$ and Greek $(\mu, \nu, \ldots)$ are the indices for local Lorentz and general coordinates, respectively.) By requiring that the unified action of SGM space-time should reduce to V–A in the flat space-time which is specified by x^a and $\psi(x)$ and that the graviton and superons contribute equally to the unified curvature of SGM space-time, it is natural to consider that the unified vierbein ${w^a}_\mu(x)$ and the unified metric $s^{\mu\nu}(x)$ of unified SGM space-time are defined through the NL SUSY invariant differential forms ω^a of V–A [2] as follows:

$$\omega^a = {w^a}_\mu dx^\mu, \tag{9}$$

$${w^a}_\mu(x) = {e^a}_\mu(x) + {t^a}_\mu(x), \tag{10}$$

where ${e^a}_\mu(x)$ is the vierbein of EGRT and ${t^a}_\mu(x)$ is defined by

$${t^a}_\mu(x) = i\kappa\bar{\psi}\gamma^a\partial_\mu\psi, \tag{11}$$

where the first and the second indices of ${t^a}_\mu$ represent those of the γ matrices and the general covariant derivatives, respectively. We can easily obtain the inverse ${w_a}^\mu$ of the vierbein ${w^a}_\mu$ in the power series of ${t^a}_\mu$ as follows, which terminates with t^4 (for 4 dimensional space-time):

$${w_a}^\mu = {e_a}^\mu - {t^\mu}_a + {t^\rho}_a {t^\mu}_\rho - {t^\rho}_a {t^\sigma}_\rho {t^\mu}_\sigma + {t^\rho}_a {t^\sigma}_\rho {t^\kappa}_\sigma {t^\mu}_\kappa. \tag{12}$$

Similarly a new metric tensor $s_{\mu\nu}(x)$ and its inverse $s^{\mu\nu}(x)$ are introduced in SGM curved space-time as follows:

$$s_{\mu\nu}(x) \equiv {w^a}_\mu(x) w_{a\nu}(x) = {w^a}_\mu(x)\eta_{ab}{w^b}_\nu(x) = g_{\mu\nu} + t_{\mu\nu} + t_{\nu\mu} + {t^\rho}_\mu t_{\rho\nu}, \tag{13}$$

$$\begin{aligned} s^{\mu\nu}(x) \equiv {w_a}^\mu(x) w^{a\nu}(x) &= g^{\mu\nu} - t^{\mu\nu} - t^{\nu\mu} + t^{\rho\mu}{t^\nu}_\rho + t^{\rho\nu}{t^\mu}_\rho + t^{\mu\rho}{t^\nu}_\rho - t^{\rho\mu}{t^\sigma}_\rho {t^\nu}_\sigma \\ &\quad - t^{\rho\nu}{t^\sigma}_\rho {t^\mu}_\rho - t^{\mu\sigma}{t^\rho}_\sigma {t^\nu}_\rho - t^{\nu\rho}{t^\sigma}_\rho {t^\mu}_\sigma + t^{\rho\mu}{t^\sigma}_\rho {t^\kappa}_\sigma {t^\nu}_\kappa + t^{\rho\nu}{t^\sigma}_\rho {t^\kappa}_\sigma {t^\mu}_\kappa \\ &\quad + t^{\mu\sigma}{t^\rho}_\sigma {t^\sigma}_\rho {t^\nu}_\sigma + t^{\nu\sigma}{t^\rho}_\sigma {t^\sigma}_\rho {t^\mu}_\sigma + t^{\rho\kappa}{t^\sigma}_\kappa {t^\mu}_\rho {t^\nu}_\sigma. \end{aligned} \tag{14}$$

We can easily show

$${w_a}^\mu w_{b\mu} = \eta_{ab}, \qquad s_{\mu\nu}{w_a}^\mu {w_b}^\nu = \eta_{ab}. \tag{15}$$

Furthermore they have generalized $GL(4,\mathbb{R})$ transformations under (3) and (4) [5, 7]. It is obvious from the above general covariant arguments that (1) is invariant under the ordinaly $GL(4,\mathbb{R})$ and under (3) and (4).

By using (10), (12), (13) and (14) we can express SGM action (1) in terms of $e^a{}_\mu(x)$ and $\psi^j(x)$, which describes explicitly the fundamental interaction of graviton with superons. The expansion of the action in terms of the power series of κ (or $t^a{}_\mu$) can be carried out straightforwardly. After the lengthy calculations concerning the complicated structures of the indices we obtain

$$\begin{aligned} L_{\rm SGM} = &-\frac{c^3\Lambda}{16\pi G}e|w_{\rm V\text{-}A}| - \frac{c^3}{16\pi G}eR + \frac{c^3}{16\pi G}e\Bigg[2t^{(\mu\nu)}R_{\mu\nu} + \frac{1}{2}\Big\{g^{\mu\nu}\partial^\rho\partial_\rho t_{(\mu\nu)} - t_{(\mu\nu)}\partial^\rho\partial_\rho g^{\mu\nu} \\ &+ g^{\mu\nu}\partial^\rho t_{(\mu\sigma)}\partial^\sigma g_{\rho\nu} - 2g^{\mu\nu}\partial^\rho t_{(\mu\nu)}\partial^\sigma g_{\rho\sigma} - g^{\mu\nu}g^{\rho\sigma}\partial^\kappa t_{(\rho\sigma)}\partial^\kappa g_{\mu\nu}\Big\} \\ &+ \left(t^\mu{}_\rho t^{\rho\nu} + t^\nu{}_\rho t^{\rho\mu} + t^{\mu\rho}t^\nu{}_\rho\right)R_{\beta\mu} - \Big\{2t^{(\mu\rho)}t^{(\nu}{}_{\rho)}R_{\mu\nu} + t^{(\mu\rho)}t^{(\nu\sigma)}R_{\mu\nu\rho\sigma} \\ &+ \frac{1}{2}t^{(\mu\nu)}\left(g^{\rho\sigma}\partial^\mu\partial_\nu t_{(\rho\sigma)} - g^{\rho\sigma}\partial^\rho\partial_\mu t_{(\sigma\nu)} + \cdots\right)\Big\} \\ &+ \left\{O\left(t^3\right)\right\} + \left\{O\left(t^4\right)\right\} + \cdots + \left\{O\left(t^{10}\right)\right\}\Bigg], \end{aligned} \tag{16}$$

where $e = \det e^a{}_\mu$, $t^{(\mu\nu)} = t^{\mu\nu} + t^{\nu\mu}$, $t_{(\mu\nu)} = t_{\mu\nu} + t_{\nu\mu}$, and $|w_{\rm V-A}| = \det w^a{}_b$ is the flat space V–A action [2] containing up to $O\left(t^4\right)$ and R and $R_{\mu\nu}$ are the Ricci curvature tensors of GR.

Remarkably the first term can be regarded as a space-time dependent cosmological term and reduces to V–A action [2] with $\kappa_{\rm V-A}^{-1} = \frac{c^3}{16\pi G}\Lambda$ in the Riemann-flat $e_a{}^\mu(x) \to \delta_a{}^\mu$ space-time. The second term is the familiar E–H action of GR. These expansions show the complementary relation of graviton and (the stress-energy tensor of) superons. The existence of (in the Riemann-flat space-time) NL SUSY invariant terms with the (second order) derivatives of the superons beyond V–A model is manifested. For example, such terms of the lowest order appear in $O\left(t^2\right)$ and have the following expressions (up to the total derivative terms)

$$+\epsilon^{abcd}\epsilon_a{}^{efg}\partial_c t_{(be)}\partial_f t_{(dg)}. \tag{17}$$

Existence of such derivative terms in addition to the original V–A model are already pointed out and exemplified in part in [8]. Note that (17) vanishes in 2 dimensional space-time.

Here we just mention that we can consider two types of the flat space in SGM, which are not equivalent. One is SGM-flat, i.e. $w_a{}^\mu(x) \to \delta_a{}^\mu$, space-time and the other is Riemann-flat, i.e. $e_a{}^\mu(x) \to \delta_a{}^\mu$, space-time, where SGM action reduces to $-\frac{c^3\Lambda}{16\pi G}$ and $-\frac{c^3\Lambda}{16\pi G}|w_{\rm V-A}| - \frac{c^3}{16\pi G}$ (*derivative terms*), respectively. Note that SGM-flat space-time may allow Riemann space-time, e.g. $t_a{}^\mu(x) \to -e_a{}^\mu + \delta_a{}^\mu$ realizes Riemann space-time *and* SGM-flat space-time. The cosmological implications are mentioned in the discussions.

3 SGM in two dimensional space-time

Now we go to two dimensional SGM space-time to simplify the arguments without loss of generality and demonstrate some details of the computations. It is well known that two dimensional GR has no physical degrees of freedom (due to the local $GL(2,\mathbb{R})$). SGM in SGM space-time is also the case. However the general covariant arguments shed light on the universal characteristic features of the theory in any space-time dimensions. Especialy for SGM, it is also useful to see explicitly the superon-graviton coupling in (two dimensional) Riemann space-time which is realized spontaneously from SGM space-time. We adopt the affine connection formalism. Knowledge of the complete structure of SGM action including the surface terms is useful to linearize SGM into the equivalent linear theory and to find the symmetry breaking of the model.

Following EGRT the scalar curvature tensor Ω of SGM space-time is given as follows

$$\Omega = s^{\beta\mu}\Omega^\alpha{}_{\beta\mu\alpha} = s^{\beta\mu}\left[\left\{\partial_\mu\Gamma^\lambda{}_{\beta\alpha} + \Gamma^\alpha{}_{\lambda\mu}\Gamma^\lambda{}_{\beta\alpha}\right\} - \{\text{ lower indices } (\mu \leftrightarrow \alpha)\}\right], \tag{18}$$

where the Christoffel symbol of the second kind of SGM space-time is

$$\Gamma^{\alpha}{}_{\beta\mu} = \frac{1}{2} s^{\alpha\rho} \left\{ \partial_\beta s_{\rho\mu} + \partial_\mu s_{\beta\rho} - \partial_\rho s_{\mu\beta} \right\}. \tag{19}$$

The straightforward expression of SGM action (1) in two dimensional space-time (which is 3^6 times more complicated than the 2 dimensional GR) is given as follows

$$\begin{aligned}
L_{2d\mathrm{SGM}} = & -\frac{c^3}{16\pi G} e \left\{ 1 + t^a{}_a + \frac{1}{2} \left(t^a{}_a t^b{}_b - t^a{}_b t^b{}_a \right) \right\} \left(g^{\beta\mu} - \tilde{t}^{(\beta\mu)} + \tilde{t}^{2(\beta\mu)} \right) \\
& \times \left[\left\{ \frac{1}{2} \partial_\mu \left(g^{\alpha\sigma} - \tilde{t}^{(\alpha\sigma)} + \tilde{t}^{2(\alpha\sigma)} \right) \partial_{\dot\beta} \left(g_{\dot\sigma\dot\alpha} + \underline{t}_{\dot\sigma\dot\alpha} + \underline{t}^2{}_{\dot\sigma\dot\alpha} \right) \right.\right. \\
& \left. + \frac{1}{2} \left(g^{\alpha\sigma} - \tilde{t}^{(\alpha\sigma)} + \tilde{t}^{2(\alpha\sigma)} \right) \partial_\mu \partial_{\dot\beta} \left(g_{\dot\sigma\dot\alpha} + \underline{t}_{\dot\sigma\dot\alpha} + \underline{t}^2{}_{\dot\sigma\dot\alpha} \right) \right\} - \{ \text{lower indices } (\mu \leftrightarrow \alpha) \} \\
& + \left\{ \frac{1}{4} \left(g^{\alpha\sigma} - \tilde{t}^{(\alpha\sigma)} + \tilde{t}^{2(\alpha\sigma)} \right) \partial_{\dot\lambda} \left(g_{\dot\sigma\dot\mu} + \underline{t}_{\dot\sigma\dot\mu} + \underline{t}^2{}_{\dot\sigma\dot\mu} \right) \right. \\
& \left.\left. \times \left(g^{\lambda\rho} - \tilde{t}^{(\lambda\rho)} + \tilde{t}^{2(\lambda\rho)} \right) \partial_{\dot\beta} \left(g_{\dot\rho\dot\alpha} + \underline{t}_{\dot\rho\dot\alpha} + \underline{t}^2{}_{\dot\rho\dot\alpha} \right) \right\} - \{ \text{lower indices } (\mu \leftrightarrow \alpha) \} \right] \\
& - \frac{c^3 \Lambda}{16\pi G} e |w_{\mathrm{V\text{-}A}}|,
\end{aligned} \tag{20}$$

where we have put

$$\begin{aligned}
& s_{\alpha\beta} = g_{\alpha\beta} + \underline{t}_{(\alpha\beta)} + \underline{t}^2{}_{(\alpha\beta)}, \qquad s^{\alpha\beta} = g^{\alpha\beta} - \tilde{t}^{(\alpha\beta)} + \tilde{t}^{2(\alpha\beta)}, \\
& \underline{t}_{(\mu\nu)} = t_{\mu\nu} + t_{\nu\mu}, \qquad \underline{t}^2{}_{(\mu\nu)} = t^\rho{}_\mu t_{\rho\nu}, \\
& \tilde{t}^{(\mu\nu)} = t^{\mu\nu} + t^{\nu\mu}, \qquad \tilde{t}^{2(\mu\nu)} = t^\mu{}_\rho t^{\rho\nu} + t^\nu{}_\rho t^{\rho\mu} + t^{\mu\rho} t^\nu{}_\rho,
\end{aligned} \tag{21}$$

and the Christoffel symbols of the first kind of SGM space-time contained in (19) are abbreviated as

$$\begin{aligned}
& \partial_{\dot\mu} g_{\dot\sigma\dot\nu} = \partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\nu\mu}, \\
& \partial_{\dot\mu} \underline{t}_{\dot\sigma\dot\nu} = \partial_\mu \underline{t}_{(\sigma\nu)} + \partial_\nu \underline{t}_{(\mu\sigma)} - \partial_\sigma \underline{t}_{(\nu\mu)}, \\
& \partial_{\dot\mu} \underline{t}^2{}_{\dot\sigma\dot\nu} = \partial_\mu \underline{t}^2{}_{(\sigma\nu)} + \partial_\nu \underline{t}^2{}_{(\mu\sigma)} - \partial_\sigma \underline{t}^2{}_{(\nu\mu)}.
\end{aligned} \tag{22}$$

By expanding the scalar curvature Ω in the power series of t which terminates with t^4, we have the following complete expression of two dimensional SGM,

$$\begin{aligned}
L_{2d\mathrm{SGM}} = & -\frac{c^3 \Lambda}{16\pi G} e |w_{\mathrm{V\text{-}A}}| - \frac{c^3}{16\pi G} e |w_{\mathrm{V\text{-}A}}| \left[R - 2\tilde{t}^{(\mu\nu)} R_{\mu\nu} + \frac{1}{2} \Big\{ g^{\mu\nu} \partial^\rho \partial_\rho \underline{t}_{(\mu\nu)} \right. \\
& - \underline{t}^{(\mu\nu)} \partial^\rho \partial_\rho g_{\mu\nu} + g^{\mu\nu} \partial^\rho \underline{t}_{(\mu\sigma)} \partial^\sigma g_{\rho\nu} - 2 g^{\mu\nu} \partial^\rho \underline{t}_{(\mu\nu)} \partial^\sigma g_{\rho\sigma} - g^{\mu\nu} g^{\rho\sigma} \partial^\kappa \underline{t}_{(\rho\sigma)} \partial_\kappa g_{\mu\nu} \Big\} \\
& + \tilde{t}^{2(\beta\mu)} R_{\beta\mu} + \tilde{t}^{(\beta\mu)} \tilde{t}^{(\alpha\sigma)} R_{\mu\alpha\sigma\beta} - \frac{1}{2} \tilde{t}^{(\beta\mu)} \Big\{ g^{\alpha\sigma} \partial_\mu \partial_\beta \underline{t}_{(\alpha\sigma)} - \partial^\sigma \partial_\beta \underline{t}_{(\sigma\mu)} \\
& + \partial_\mu \tilde{t}^{(\alpha\sigma)} \partial_\beta g_{\sigma\alpha} - \partial_\mu g^{\alpha\sigma} \partial_\beta \underline{t}_{(\sigma\alpha)} + \partial_\alpha g^{\alpha\sigma} \partial_\beta \underline{t}_{(\sigma\mu)} - \partial_\alpha \tilde{t}^{(\alpha\sigma)} \partial_\beta g_{\sigma\mu} + 2 \partial^\rho \underline{t}_{(\sigma\mu)} \partial^\sigma g_{\beta\rho} \\
& - 2 g^{\alpha\sigma} \partial_\lambda \underline{t}_{(\sigma\mu)} \partial^\lambda g_{\alpha\beta} + g^{\alpha\sigma} g^{\lambda\rho} \partial_\mu \underline{t}_{(\lambda\sigma)} \partial^\beta g_{\rho\alpha} - 2 g^{\alpha\sigma} \partial^\rho \underline{t}_{(\sigma\alpha)} \partial_\beta g_{\rho\mu} \\
& + g^{\alpha\sigma} \partial_\lambda \underline{t}_{(\sigma\alpha)} \partial^\lambda g_{\mu\beta} \Big\} - g^{\beta\mu} \partial_\mu \left(g^{\alpha\sigma} \partial_\beta \underline{t}^2{}_{(\sigma\alpha)} + \tilde{t}^{2(\alpha\sigma)} \partial_\beta g_{\sigma\alpha} - \tilde{t}^{(\alpha\sigma)} \partial_\beta \tilde{t}_{(\sigma\alpha)} \right)
\end{aligned}$$

$$
\begin{aligned}
&- \tilde t^{(\alpha\sigma)} \left(2\partial_\beta \underline{t}_{(\sigma\mu)} - \partial_\sigma \underline{t}_{(\mu\beta)}\right) + g^{\beta\mu}\partial_\alpha \Big\{ g^{\alpha\sigma}\left(2\partial_\beta \underline{t}^2{}_{(\sigma\mu)} - \partial_\sigma \underline{t}^2{}_{(\mu\beta)}\right) \\
&+ \tilde t^{2(\alpha\sigma)} \left(2\partial_\beta g_{\sigma\mu} - \partial_\sigma g_{\mu\beta}\right) \Big\} + 2\partial^\alpha g_{\lambda\mu} g^{\beta\lambda}\left(2\partial^\mu \underline{t}^2{}_{(\alpha\beta)} - \partial_\alpha \underline{t}^2{}_{(\beta\rho)} g^{\mu\rho}\right) \\
&+ 2\tilde t^{2(\lambda\rho)}\partial_\lambda g_{\sigma\mu} g^{\beta\mu}\left(2\partial^\sigma g_{\beta\rho} - \partial_\rho g_{\alpha\beta}\right) - 2\tilde t^{(\lambda\rho)}\partial_\lambda g_{\sigma\mu} g^{\alpha\sigma}\left(2\partial^\mu \underline{t}_{(\rho\alpha)} - \partial_\rho \underline{t}_{(\alpha\beta)} g^{\beta\mu}\right) \\
&+ \partial^\rho \underline{t}_{(\sigma\mu)} g^{\beta\mu}\left(2\partial^\sigma \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)} g^{\alpha\sigma}\right) + \tilde t^{(\alpha\sigma)}\tilde t^{(\lambda\rho)}\Big\{\partial^\beta g_{\lambda\sigma}\partial_\beta g_{\rho\alpha} \\
&+ 2\partial_\lambda g_{\sigma\mu} g^{\mu\beta}\left(\partial_\alpha g_{\beta\rho} - \partial_\rho g_{\alpha\beta}\right)\Big\} - 2\tilde t^{(\lambda\rho)}\partial_\lambda \underline{t}_{(\sigma\mu)} g^{\beta\mu}\left(2\partial^\sigma g_{\beta\rho} - \partial_\rho g_{\alpha\beta} g^{\alpha\sigma}\right) \\
&- \partial^\rho g_{\sigma\alpha} g^{\sigma\alpha}\left(2\partial^\mu \underline{t}^2{}_{(\rho\mu)} - \partial_\rho \underline{t}^2{}_{(\mu\beta)} g^{\beta\mu}\right) - \partial^\rho \underline{t}^2{}_{(\sigma\alpha)}\left(2\partial^\mu g_{\rho\mu} g^{\sigma\alpha} - \partial^\lambda g_{\mu\beta} g^{\mu\beta}\right) \\
&- \tilde t^{2(\lambda\rho)}\partial_\lambda g_{\sigma\alpha} g^{\sigma\alpha}\left(2\partial^\mu g_{\rho\mu} - \partial_\rho g_{\mu\beta} g^{\mu\beta}\right) - \tilde t^{2(\alpha\sigma)}\partial^\rho g_{\sigma\alpha}\left(2\partial^\mu g_{\rho\mu} - \partial^\rho g_{\mu\beta} g^{\mu\beta}\right) \\
&- \tilde t^{(\lambda\rho)}\partial_\lambda g_{\sigma\alpha} g^{\alpha\sigma}\left(2\partial^\mu \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)} g^{\beta\mu}\right) - \tilde t^{(\alpha\sigma)}\partial^\rho t_{(\sigma\alpha)}\left(2\partial^\mu g_{\rho\mu} - \partial_\rho g_{\mu\beta} g^{\beta\mu}\right) \\
&+ g^{\alpha\sigma}\partial^\rho \tilde t_{(\sigma\alpha)}\left(2\partial^\mu \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)} g^{\beta\mu}\right) + \tilde t^{(\alpha\sigma)}\tilde t^{(\lambda\rho)}\partial_\lambda g_{\sigma\alpha} g^{\alpha\sigma}\left(2\partial^\mu g_{\rho\mu} - \partial_\rho g_{\mu\beta} g^{\beta\mu}\right) \\
&- \tilde t^{(\lambda\rho)}\partial_\lambda \underline{t}_{(\sigma\alpha)} g^{\alpha\sigma}\left(2\partial^\mu g_{\rho\mu} - \partial_\rho g_{\mu\beta} g^{\beta\mu}\right) - \tilde t^{(\alpha\sigma)}\partial^\rho g_{\sigma\alpha}\left(2\partial^\mu \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)} g^{\beta\mu}\right) \\
&+ \frac{1}{2}\tilde t^{2(\beta\mu)}\Big\{ g^{\alpha\sigma}\partial_\mu\partial_\beta \underline{t}_{(\alpha\sigma)} - \partial^\sigma\partial_\beta \underline{t}_{(\sigma\mu)} + \partial_\mu \tilde t^{(\alpha\sigma)}\partial_\beta g_{\sigma\alpha} - \partial_\mu g^{\alpha\sigma}\partial_\beta \underline{t}_{(\sigma\alpha)} + \partial_\alpha g^{\alpha\sigma}\partial_\beta \underline{t}_{(\sigma\mu)} \\
&- \partial_\alpha \tilde t^{(\alpha\sigma)}\partial_\beta g_{\sigma\mu} + 2\partial^\rho \underline{t}_{(\sigma\mu)}\partial^\sigma g_{\beta\rho} - 2g^{\alpha\sigma}\partial_\lambda \underline{t}_{(\sigma\mu)}\partial^\lambda g_{\alpha\beta} + g^{\alpha\sigma} g^{\lambda\rho}\partial_\mu \underline{t}_{(\lambda\sigma)}\partial^\beta g_{\rho\alpha} \\
&- 2g^{\alpha\sigma}\partial^\rho \underline{t}_{(\sigma\alpha)}\partial_\beta g_{\rho\mu} + g^{\alpha\sigma}\partial_\lambda \underline{t}_{(\sigma\alpha)}\partial^\lambda g_{\mu\beta}\Big\} - \frac{1}{2}\tilde t^{(\beta\mu)}\Big\{\partial_\mu\left(g^{\alpha\sigma}\partial_\beta \underline{t}^2{}_{(\sigma\alpha)} - \tilde t^{(\alpha\sigma)}\partial_\beta \tilde t_{(\sigma\alpha)}\right) \\
&+ \partial_\mu \tilde t^{2(\alpha\sigma)}\partial_\beta g_{\sigma\alpha} + \partial_\alpha\Big\{ g^{\alpha\sigma}\left(2\partial_\beta \underline{t}^2{}_{(\sigma\mu)} - \partial_\sigma \underline{t}^2{}_{\mu\beta}\right) - \tilde t^{(\alpha\sigma)}\left(2\partial_\beta \underline{t}_{(\sigma\mu)} - \partial_\sigma \underline{t}_{(\mu\beta)}\right)\Big\} \\
&+ \partial_\alpha \tilde t^{2(\alpha\sigma)}\left(2\partial_\beta g_{\sigma\mu} - \partial_\sigma g_{\mu\beta}\right) + 2\partial^\alpha g_{\lambda\mu} g^{\beta\lambda}\left(2\partial^\mu \underline{t}^2{}_{(\alpha\beta)} - \partial_\alpha \underline{t}^2{}_{(\beta\rho)} g^{\mu\rho}\right) \\
&+ 2\tilde t^{2(\lambda\rho)}\partial_\lambda g_{\sigma\mu} g^{\beta\mu}\left(2\partial^\sigma g_{\beta\rho} - \partial_\rho g_{\alpha\beta}\right) - 2\tilde t^{(\lambda\rho)}\partial_\lambda g_{\sigma\mu} g^{\alpha\sigma}\left(2\partial^\mu \underline{t}_{(\rho\alpha)} - \partial_\rho \underline{t}_{(\alpha\beta)} g^{\beta\mu}\right) \\
&+ \partial^\rho \underline{t}_{(\sigma\mu)} g^{\beta\mu}\left(2\partial^\sigma \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)} g^{\alpha\sigma}\right) + \tilde t^{(\alpha\sigma)}\tilde t^{(\lambda\rho)}\Big\{\partial^\beta g_{\lambda\sigma}\partial_\beta g_{\rho\alpha} \\
&+ 2\partial_\lambda g_{\sigma\mu} g^{\mu\beta}\left(\partial_\alpha g_{\beta\rho} - \partial_\rho g_{\alpha\beta}\right)\Big\} - 2\tilde t^{(\lambda\rho)}\partial_\lambda \underline{t}_{(\sigma\mu)} g^{\beta\mu}\left(2\partial^\sigma g_{\beta\rho} - \partial_\rho g_{\alpha\beta} g^{\alpha\sigma}\right) \\
&- \partial^\rho g_{\sigma\alpha} g^{\sigma\alpha}\left(2\partial^\mu \underline{t}^2{}_{(\rho\mu)} - \partial_\rho \underline{t}^2{}_{(\mu\beta)} g^{\beta\mu}\right) - \partial^\rho \underline{t}^2{}_{(\sigma\alpha)}\left(2\partial^\mu g_{\rho\mu} g^{\sigma\alpha} - \partial^\lambda g_{\mu\beta} g^{\mu\beta}\right) \\
&- \tilde t^{2(\lambda\rho)}\partial_\lambda g_{\sigma\alpha} g^{\sigma\alpha}\left(2\partial^\mu g_{\rho\mu} - \partial_\rho g_{\mu\beta} g^{\mu\beta}\right) - \tilde t^{2(\alpha\sigma)}\partial^\rho g_{\sigma\alpha}\left(2\partial^\mu g_{\rho\mu} - \partial^\rho g_{\mu\beta} g^{\mu\beta}\right) \\
&- \tilde t^{(\lambda\rho)}\partial_\lambda g_{\sigma\alpha} g^{\alpha\sigma}\left(2\partial^\mu \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)} g^{\beta\mu}\right) - \tilde t^{(\alpha\sigma)}\partial^\rho t_{(\sigma\alpha)}\left(2\partial^\mu g_{\rho\mu} - \partial_\rho g_{\mu\beta} g^{\beta\mu}\right) \\
&+ g^{\alpha\sigma}\partial^\rho \tilde t_{(\sigma\alpha)}\left(2\partial^\mu \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)} g^{\beta\mu}\right) + \tilde t^{(\alpha\sigma)}\tilde t^{(\lambda\rho)}\partial_\lambda g_{\sigma\alpha} g^{\alpha\sigma}\left(2\partial^\mu g_{\rho\mu} - \partial_\rho g_{\mu\beta} g^{\beta\mu}\right) \\
&- \tilde t^{(\lambda\rho)}\partial_\lambda \underline{t}_{(\sigma\alpha)} g^{\alpha\sigma}\left(2\partial^\mu g_{\rho\mu} - \partial_\rho g_{\mu\beta} g^{\beta\mu}\right) - \tilde t^{(\alpha\sigma)}\partial^\rho g_{\sigma\alpha}\left(2\partial^\mu \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)} g^{\beta\mu}\right)\Big\} \\
&+ \partial^\beta\Big\{\left(\tilde t^{2(\alpha\sigma)}\partial_\beta \underline{t}_{(\sigma\alpha)} - \tilde t^{(\alpha\sigma)}\partial_\beta \underline{t}^2{}_{(\sigma\alpha)}\right)\Big\} \\
&- g^{\beta\mu}\partial_\alpha\Big\{\tilde t^{2(\alpha\sigma)}\left(2\partial_\beta \underline{t}_{(\sigma\mu)} - \partial_\sigma \underline{t}_{(\mu\beta)}\right) - \tilde t^{(\alpha\sigma)}\left(2\partial_\beta \underline{t}^2{}_{(\sigma\mu)} - \partial_\beta \underline{t}^2{}_{(\mu\beta)}\right)\Big\} \\
&+ \frac{1}{2}\Big\{ g^{\alpha\sigma}\left(\partial_\lambda g_{\sigma\mu} + \partial_\mu g_{\lambda\sigma} - \partial_\sigma g_{\mu\lambda}\right)\tilde t^{2(\lambda\rho)} g^{\beta\mu}\left(\partial_\beta \underline{t}_{(\rho\alpha)} + \partial_\alpha \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)}\right) \\
&+ g^{\alpha\sigma}\left(\partial_\lambda \underline{t}_{(\sigma\mu)} + \partial_\mu g_{\lambda\sigma} - \partial_\sigma \underline{t}_{(\mu\lambda)}\right)\tilde t^{2(\lambda\rho)}\underline{t}^{(\lambda\rho)} g^{\beta\mu}\left(\partial_\beta g_{\rho\alpha} + \partial_\alpha g_{\beta\rho} - \partial_\rho g_{\alpha\beta}\right) \\
&- g^{\alpha\sigma}\left(\partial_\lambda g_{\sigma\mu} + \partial_\mu g_{\lambda\sigma} - \partial_\sigma g_{\mu\lambda}\right)\tilde t^{(\lambda\rho)} g^{\beta\mu}\left(\partial_\beta \underline{t}^2{}_{(\rho\alpha)} + \partial_\alpha \underline{t}^2{}_{(\beta\rho)} - \partial_\rho \underline{t}^2{}_{(\alpha\beta)}\right) \\
&- g^{\alpha\sigma}\left(\partial_\lambda \underline{t}^2{}_{(\sigma\mu)} + \partial_\mu \underline{t}^2{}_{(\lambda\sigma)} - \partial_\sigma \underline{t}^2{}_{(\mu\lambda)}\right)\tilde t^{(\lambda\rho)} g^{\beta\mu}\left(\partial_\beta g_{\rho\alpha} + \partial_\alpha g_{\beta\rho} - \partial_\rho g_{\alpha\beta}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \tilde{t}^{(\alpha\sigma)} \left(\partial_\lambda g_{\sigma\mu} + \partial_\mu g_{\lambda\sigma} - \partial_\sigma g_{\mu\lambda}\right) \tilde{t}^{(\lambda\rho)} g^{\beta\mu} \left(\partial_\beta \underline{t}_{(\rho\alpha)} + \partial_\alpha \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)}\right) \\
&- g^{\alpha\sigma} \left(\partial_\lambda \underline{t}_{(\sigma\mu)} + \partial_\mu g_{\lambda\sigma} - \partial_\sigma \underline{t}_{(\mu\lambda)}\right) \tilde{t}^{(\lambda\rho)} g^{\beta\mu} \left(\partial_\beta \underline{t}_{(\rho\alpha)} + \partial_\alpha \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)}\right) \\
&- \tilde{t}^{(\alpha\sigma)} \left(\partial_\lambda \underline{t}_{(\sigma\mu)} + \partial_\mu g_{\lambda\sigma} - \partial_\sigma \underline{t}_{(\mu\lambda)}\right) g^{\lambda\rho} g^{\beta\mu} \left(\partial_\beta \underline{t}_{(\rho\alpha)} + \partial_\alpha \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)}\right) \\
&- g^{\alpha\sigma} \partial_\lambda g_{\sigma\alpha} \tilde{t}^{2\lambda\rho} g^{\beta\mu} \left(2\partial_\beta \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)}\right) + g^{\alpha\sigma} \partial_\lambda \underline{t}_{(\sigma\alpha)} \tilde{t}^{2(\lambda\rho)} \left(2\partial_\beta g_{\rho\mu} - \partial_\rho g_{\mu\beta}\right) \\
&+ g^{\alpha\sigma} \partial_\lambda g_{\sigma\alpha} \tilde{t}^{(\lambda\rho)} g^{\beta\mu} \left(2\partial_\beta \underline{t}^2{}_{(\rho\mu)} - \partial_\rho \underline{t}^2{}_{(\mu\beta)}\right) + g^{\alpha\sigma} \partial_\lambda \underline{t}^2{}_{(\sigma\mu)} \tilde{t}^{(\lambda\rho)} g^{\beta\mu} \left(2\partial_\beta g_{\rho\mu} - \partial_\rho g_{\mu\beta}\right) \\
&- \tilde{t}^{(\alpha\sigma)} \partial_\lambda g_{\sigma\alpha} \tilde{t}^{(\lambda\rho)} g^{\beta\mu} \left(2\partial_\beta \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)}\right) + g^{\alpha\sigma} \partial_\lambda \underline{t}_{(\sigma\alpha)} \tilde{t}^{(\lambda\rho)} g^{\beta\mu} \left(2\partial_\beta \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)}\right) \\
&+ \tilde{t}^{(\alpha\sigma)} \left(\partial_\lambda \underline{t}_{(\sigma\alpha)} + \partial_\alpha g_{\lambda\sigma} - \partial_\sigma \underline{t}_{(\alpha\lambda)}\right) g^{\lambda\rho} g^{\beta\mu} \left(\partial_\beta \underline{t}_{(\rho\mu)} + \partial_\mu \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\mu\beta)}\right) \Big\} \\
&- \tilde{t}^{(\beta\mu)} \Big\{ \frac{1}{2} \Big\{ \partial_\mu \tilde{t}^{(\alpha\sigma)} \partial_\beta \underline{t}^2{}_{(\sigma\alpha)} - \partial_\mu \tilde{t}^2_{(\alpha\sigma)} \partial_\beta \underline{t}_{(\sigma\alpha)} + \tilde{t}^{(\alpha\sigma)} \partial_\mu \partial_\beta \underline{t}^2{}_{(\sigma\alpha)} \\
&- \partial_\alpha \tilde{t}^{(\alpha\sigma)} \left(2\partial_\beta \underline{t}^2{}_{(\sigma\mu)} - \partial_\sigma \underline{t}^2{}_{(\mu\beta)}\right) + \partial_\alpha \tilde{t}^2_{(\alpha\sigma)} \left(2\partial_\beta \underline{t}_{(\sigma\mu)} - \partial_\sigma \underline{t}_{(\mu\beta)}\right) \\
&- \tilde{t}^{(\alpha\sigma)} \partial_\alpha \left(2\partial_\beta \underline{t}^2{}_{(\sigma\mu)} - \partial_\sigma \underline{t}^2{}_{(\mu\beta)}\right) \Big\} \\
&+ \frac{1}{4} \Big\{ g^{\alpha\sigma} \left(\partial_\lambda \underline{t}_{(\sigma\mu)} + \partial_\mu \underline{t}_{(\lambda\sigma)} - \partial_\sigma \underline{t}_{(\mu\lambda)}\right) g^{\lambda\rho} \left(\partial_\beta \underline{t}^2{}_{(\rho\alpha)} + \partial_\alpha \underline{t}^2{}_{(\beta\rho)} - \partial_\rho \underline{t}^2{}_{(\alpha\beta)}\right) \\
&- g^{\alpha\sigma} \left(\partial_\lambda \underline{t}_{(\sigma\mu)} + \partial_\mu g_{\lambda\sigma} - \partial_\sigma \underline{t}_{(\mu\lambda)}\right) \tilde{t}^{(\lambda\rho)} \left(\partial_\beta \underline{t}_{(\rho\alpha)} + \partial_\alpha \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)}\right) \\
&+ g^{\alpha\sigma} \left(\partial_\lambda \underline{t}^2{}_{(\sigma\mu)} + \partial_\mu \underline{t}^2{}_{(\lambda\sigma)} - \partial_\sigma \underline{t}^2{}_{(\mu\lambda)}\right) g^{\lambda\rho} \left(\partial_\beta \underline{t}_{(\rho\alpha)} + \partial_\alpha \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)}\right) \\
&- \tilde{t}^{(\alpha\sigma)} \left(\partial_\lambda \underline{t}_{(\sigma\mu)} + \partial_\mu \underline{t}_{(\lambda\sigma)} - \partial_\sigma \underline{t}_{(\mu\lambda)}\right) g^{\lambda\rho} \left(\partial_\beta \underline{t}_{(\rho\alpha)} + \partial_\alpha \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)}\right) \\
&- g^{\alpha\sigma} \partial_\lambda \underline{t}_{(\sigma\alpha)} g^{\lambda\rho} \left(2\partial_\beta \underline{t}^2{}_{(\rho\mu)} - \partial_\rho \underline{t}^2{}_{(\mu\beta)}\right) + g^{\alpha\sigma} \partial_\lambda \underline{t}_{(\sigma\alpha)} \tilde{t}^{(\lambda\rho)} \left(2\partial_\beta \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)}\right) \\
&- g^{\alpha\sigma} \partial_\lambda \underline{t}^2{}_{(\sigma\alpha)} g^{\lambda\rho} \left(2\partial_\beta \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)}\right) + \tilde{t}^{(\alpha\sigma)} \partial_\lambda \underline{t}_{(\sigma\alpha)} g^{\lambda\rho} \left(2\partial_\beta \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)}\right) \Big\} \Big\} \\
&+ g^{\beta\mu} \Big\{ \frac{1}{2} \Big\{ \partial_\mu \tilde{t}^{2(\alpha\sigma)} \partial_\beta \underline{t}^2{}_{(\sigma\alpha)} + \tilde{t}^{2(\alpha\sigma)} \partial_\mu \partial_\beta \underline{t}^2{}_{(\sigma\alpha)} - \partial_\alpha \tilde{t}^{2(\alpha\sigma)} \left(2\partial_\beta \underline{t}^2{}_{(\sigma\mu)} - \partial_\sigma \underline{t}^2{}_{(\mu\beta)}\right) \\
&- \tilde{t}^{2(\alpha\sigma)} \partial_\alpha \left(2\partial_\beta \underline{t}^2{}_{(\sigma\mu)} - \partial_\sigma \underline{t}^2{}_{(\mu\beta)}\right) \Big\} \\
&+ \frac{1}{4} \Big\{ g^{\alpha\sigma} \left(\partial_\lambda \underline{t}^2{}_{(\sigma\mu)} + \partial_\mu \underline{t}^2{}_{(\lambda\sigma)} - \partial_\sigma \underline{t}^2{}_{(\mu\lambda)}\right) g^{\lambda\rho} \left(\partial_\beta \underline{t}^2{}_{(\rho\alpha)} + \partial_\alpha \underline{t}^2{}_{(\beta\rho)} - \partial_\rho \underline{t}^2{}_{(\alpha\beta)}\right) \\
&- g^{\alpha\sigma} \left(\partial_\lambda \underline{t}_{(\sigma\mu)} + \partial_\mu \underline{t}_{(\lambda\sigma)} - \partial_\sigma \underline{t}_{(\mu\lambda)}\right) \tilde{t}^{(\lambda\rho)} \left(\partial_\beta \underline{t}^2{}_{(\rho\alpha)} + \partial_\alpha \underline{t}^2{}_{(\beta\rho)} - \partial_\rho \underline{t}^2{}_{(\alpha\beta)}\right) \\
&+ g^{\alpha\sigma} \left(\partial_\lambda \underline{t}_{(\sigma\mu)} + \partial_\mu \underline{t}_{(\lambda\sigma)} - \partial_\sigma \underline{t}_{(\mu\lambda)}\right) \tilde{t}^{2(\lambda\rho)} \left(\partial_\beta \underline{t}_{(\rho\alpha)} + \partial_\alpha \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)}\right) \\
&- g^{\alpha\sigma} \left(\partial_\lambda \underline{t}^2{}_{(\sigma\mu)} + \partial_\mu \underline{t}^2{}_{(\lambda\sigma)} - \partial_\sigma \underline{t}^2{}_{(\mu\lambda)}\right) \tilde{t}^{(\lambda\rho)} \left(\partial_\beta \underline{t}_{(\rho\alpha)} + \partial_\alpha \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)}\right) \\
&- \tilde{t}^{(\alpha\sigma)} \left(\partial_\lambda \underline{t}_{(\sigma\mu)} + \partial_\mu \underline{t}_{(\lambda\sigma)} - \partial_\sigma \underline{t}_{(\mu\lambda)}\right) g^{\lambda\rho} \left(\partial_\beta \underline{t}^2{}_{(\rho\alpha)} + \partial_\alpha \underline{t}^2{}_{(\beta\rho)} - \partial_\rho \underline{t}^2{}_{(\alpha\beta)}\right) \\
&+ \tilde{t}^{(\alpha\sigma)} \left(\partial_\lambda \underline{t}_{(\sigma\mu)} + \partial_\mu \underline{t}_{(\lambda\sigma)} - \partial_\sigma \underline{t}_{(\mu\lambda)}\right) \tilde{t}^{(\lambda\rho)} \left(\partial_\beta \underline{t}_{(\rho\alpha)} + \partial_\alpha \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)}\right) \\
&+ \tilde{t}^{2(\alpha\sigma)} \left(\partial_\lambda \underline{t}_{(\sigma\mu)} + \partial_\mu \underline{t}_{(\lambda\sigma)} - \partial_\sigma \underline{t}_{(\mu\lambda)}\right) g^{\lambda\rho} \left(\partial_\beta \underline{t}_{(\rho\alpha)} + \partial_\alpha \underline{t}_{(\beta\rho)} - \partial_\rho \underline{t}_{(\alpha\beta)}\right) \\
&- g^{\alpha\sigma} \partial_\lambda \underline{t}^2{}_{(\sigma\mu)} g^{\lambda\rho} \left(2\partial_\beta \underline{t}^2{}_{(\rho\mu)} - \partial_\rho \underline{t}^2{}_{(\mu\beta)}\right) + g^{\alpha\sigma} \partial_\lambda \underline{t}_{(\sigma\alpha)} \tilde{t}^{(\lambda\rho)} \left(2\partial_\beta \underline{t}^2{}_{(\rho\mu)} - \partial_\rho \underline{t}^2{}_{(\mu\beta)}\right) \\
&- g^{\alpha\sigma} \partial_\lambda \underline{t}_{(\sigma\alpha)} \tilde{t}^{2(\lambda\rho)} \left(2\partial_\beta \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)}\right) + g^{\alpha\sigma} \partial_\lambda \underline{t}^2{}_{(\sigma\mu)} \tilde{t}^{(\lambda\rho)} \left(2\partial_\beta \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)}\right) \\
&+ \tilde{t}^{(\alpha\sigma)} \partial_\lambda \underline{t}_{(\sigma\alpha)} g^{\lambda\rho} \left(2\partial_\beta \underline{t}^2{}_{(\rho\mu)} - \partial_\rho \underline{t}^2{}_{(\mu\beta)}\right) - \tilde{t}^{(\alpha\sigma)} \partial_\lambda \underline{t}_{(\sigma\alpha)} \tilde{t}^{(\lambda\rho)} \left(2\partial_\beta \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)}\right) \\
&- \tilde{t}^{2(\alpha\sigma)} \partial_\lambda \underline{t}_{(\sigma\alpha)} g^{\lambda\rho} \left(2\partial_\beta \underline{t}_{(\rho\mu)} - \partial_\rho \underline{t}_{(\mu\beta)}\right) \Big\} \Big\} \Big], \qquad (23)
\end{aligned}
$$

where $R_{\mu\nu\rho\sigma}$, $R_{\mu\nu}$ and R are the curvature tensors of Riemann space and $|w_{\text{V--A}}| = \{1 + t^a{}_a + \frac{1}{2}(t^a{}_a t^b{}_b - t^a{}_b t^b{}_a)\}$ is V–A model in two dimensional flat space.

4 Discussions

We have shown that contrary to its simple expression (1) in unified SGM space-time the complete expansion of SGM action posesses the very complicated and rich structures describing the graviton-superon interactions in Riemann space-time, even in two dimensional space-time.

The SGM in four dimensional space-time has far much more complicated structures, which may be unavoidable features for a unified theory to describe the rationale of beings of all elementary particles. Note that the total number of elementary particles in SGM is at most a few hundreds and most of them are (heavy) massive.

Here we just emphasize that SGM action in SGM space-time is a nontrivial generalization of E–H action in Riemann space-time despite the liner relation $w^a{}_\mu = e^a{}_\mu + t^a{}_\mu$. In fact, by the redefinitions(variations) $e^a{}_\mu \to e^a{}_\mu + \delta e^a{}_\mu = e^a{}_\mu - t^a{}_\mu$ and $\delta e_a{}^\mu = -e_a{}^\nu e_b{}^\mu \delta e^b{}_\nu = +t^\mu{}_a$ the inverse $w_a{}^\mu = e_a{}^\mu - t^\mu{}_a + t^\rho{}_a t^\mu{}_\rho - t^\rho{}_a t^\sigma{}_\rho t^\mu{}_\sigma + t^\rho{}_a t^\sigma{}_\rho t^\kappa{}_\sigma t^\mu{}_\kappa$ does not reduce to $e_a{}^\mu$, i.e. the nonlinear terms in $t^\mu{}_a$ in the inverse $w_a{}^\mu$ can not be eliminated. Because $t^a{}_\mu$ is not a vierbein. Such a redefinition breaks the metric properties of $w^a{}_\mu$ and $w_a{}^\mu$. Note that SGM action posesses two inequivalent flat spaces, i.e. SGM-flat $w^a{}_\mu \to \delta^a{}_\mu$ and Riemann-flat $e^a{}_\mu \to \delta^a{}_\mu$. The expansion of SGM action in terms of $e^a{}_\mu$ and $t^a{}_\mu$ is a spontaneous breakdown of space-time from SGM space-time to Riemann space-time connecting with Riemann-flat space-time.

Concerning the above-mentioned two inequivalent flat-spaces (i.e. the vacuum of the gravitational energy) of SGM action we can interprete them as follows. SGM action (1) written by the vierbein $w_a{}^\mu(x)$ and metric $s^{\mu\nu}(x)$ of SGM space-time is invariant under (besides the ordinary local $GL(4,\mathbb{R})$) the general coordinate transformation [7] with a generalized parameter $i\kappa(\bar{\zeta}\gamma^\mu\psi(x))$ (originating from the global supertranslation in SGM space-time [2]). As proved for E–H action of GR (the positive definitness of Einstein–Hilbert actionwas proved by E. Witten [10]), the energy of SGM action of E–H type is expected to be positive (for positive Λ). Regarding the scalar curvature tensor Ω for the unified metric tensor $s^{\mu\nu}(x)$ as an analogue of the Higgs potential for the Higgs scalar, we can observe that (at least the vacuum of) SGM action (i.e. SGM-flat $w^a{}_\mu(x) \to \delta^a{}_\nu$ space-time), which allows Riemann space-time and has a positive energy density with the positive cosmological constant $\frac{c^3\Lambda}{16\pi G}$ indicating the spontaneous SUSY breaking, is unstable (i.e. degenerates) against the supertransformation (3) and (4) with the global spinor parameter ζ in SGM space-time and breaks down spontaneously to Riemann space-time $w^a{}_\mu(x) = e^a{}_\mu(x) + t^a{}_\mu(x)$ with N–G fermions *superons* corresponding to $\frac{\text{super}\,GL(4,\mathbb{R})}{GL(4,\mathbb{R})}$. (Note that SGM-flat space-time allows Riemann space-time.) Remarkably the observed Riemann space-time of EGRT and matter(superons) appear simultaneously from (the vacuum of) SGM action by the spontaneous SUSY breaking.

The investigation of the structures of the vacuum of Riemann-flat space-time (described by $N = 10$ V–A action with derivative terms like (17)) plays an important role to linearize SGM and to derive SM as the low energy effective theory of SGM, which remain to be challenged. Such (higher) derivative terms can be rewritten in the tractable forms similar to (17) up to the total derivative terms.

As for the linearization, the linearization of the flat-space $N = 1$ V–A model was already carried out [9]. They proved that the linear SUSY action of a scalar supermultiplet with SUSY breaking is equivalent to V–A action under SUSY invariant constraints obtained by the systematic arguments. Recently we have shown explicitly that the action of $U(1)$ vector supermultiplet with Feyet–Iliopoulos term is equivalent to $N = 1$ V–A model [11]. It is remarkable that the renormalizable low energy effective U(1) gauge theory is derived from the highly nonlinear theory by systematic arguments. While, in the linearization of SGM (i.e. V–A model in curved space-time) it should be taken into consideration further that the algebra (gauge symmetry) would be changed from (8) to broken $SO(10)$ SP symmetry.

From the physical point of view the linearization of the flat-space $N = 2$ V–A model is very important as a toy model, for it may be equivalent to the following Higgs–Kibble–Dirac Lagrangian (composed of $N = 2$ SP off-shell multiplet)

$$L_{\mathrm{HKD}} = \frac{1}{4}{F^2}_{\mu\nu} + \bar{\psi}\gamma_\mu D^\mu\psi + \frac{1}{2}(\partial_\mu\phi_i)^2 + 2g\phi_i\bar{\psi}\psi - 2gD{\phi_i}^2 + \frac{1}{2}\left(D^2 + |F|^2\right), \tag{24}$$

where $F_{\mu\nu}$ is a gauge field, ψ is a Dirac field, ϕ_i $(i = 1, 2)$ is a real scalar field and the fields D, and F are auxiliary fields. This, speculative so far, is remarkable, for the $(U(1))$ gauge field including the gauge coupling constant is expressed in terms of the superons and the the fundamental coupling constant of V–A model including the order parameter of the symmetry breaking. A nonlinear $N = 2$ SUSY equivalent to $N = 2$ SUSY Yang–Mills theory investigated by Seiberg and Witten [12] may be a realistic case. Furthermore the baryon abundance of the universe should be explained by the spontaneous symmetry breaking of the linearized (low energy) effective theory.

Finally we just mention the hidden symmetries characteristic to SGM. It is natural to expect that SGM action may be invariant under a certain exchange between ${e^a}_\mu$ and ${t^a}_\mu$, for they contribute equally to the unified SGM vierbein ${w^a}_\mu$ as seen in (10). In fact we find, as a simple example, that SGM action is invariant under the following exchange of ${e^a}_\mu$ and ${t^a}_\mu$ [13] (in 4 dimensional space-time).

$${e^a}_\mu \longrightarrow 2{t^a}_\mu, \qquad {t^a}_\mu \longrightarrow {e^a}_\mu - {t^a}_\mu, \qquad {e_a}^\mu \longrightarrow {e_a}^\mu. \tag{25}$$

The physical meaning of such symmetries remains to be studied. Also SGM action has Z_2 symmetry $\psi^j \to -\psi^j$ but not ${e^a}_\mu \to -{e^a}_\mu$.

Beside the composite picture of SGM it is interesting to consider (elementary field) SGM with the extra dimensions and their compactifications. The compactification of ${w^A}_M = {e^A}_M + {t^A}_M$, $(A, M = 0, 1 \ldots, D - 1)$ produces rich spectrum of particles and (hidden) internal symmetries and may give a new framework for the unification of space-time and matter.

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Quantum Algebras, Particle Phenomenology, and (Quasi)Supersymmetry

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Quantum algebras $U_q(\mathrm{su}_n)$ used as the algebras of flavour symmetry (usually described by $SU(n)$) to study static properties of hadrons lead to intriguing results. In this contribution we focus on the peculiar properties manifested by different q-deformed structures (e.g., the braided line, the quantum algebras $U_q(\mathrm{su}_2)$ and $U_q(\mathrm{su}_n)$, $n \geq 3$) in the special limit of $q = -1$. Similarities (complete or partial) with supersymmetry that emerge in this special limit are discussed.

1 Introduction

Our goal is to pay special attention to the exotic situation that arises if, within the application of quantum algebras $U_q(\mathrm{su}_n)$ [1, 2] to phenomenological description (see [3, 4] and refs. therein) of basic static properties of hadrons – vector mesons as well as baryons, one restricts itself to the peculiar case $q = -1$ of the deformation parameter. In the paper, we first briefly mention the two more or less realistic appearances of supersymmetry (SUSY) algebras applied directly in the sector of hadron mass spectrum. Note that the first appearance of SUSY in the context of hadron physics goes back to Miyazawa's paper [5]. It employs a kind of superalgebra which is connected with internal symmetry and extends the usual $SU(3)$ scheme by means of baryon number changing currents. In that paper, the author has succeeded to derive, based on a superalgebra, the mass sum rules other than the celebrated Gell-Mann–Okubo (GMO) one, that is, $m_N + m_\Xi = \frac{3}{2}m_\Lambda + \frac{1}{2}m_\Sigma$. On the contrary, the spectrum generating (or dynamical) superalgebra used in [6] incorporated a superization of space-time symmetry and gave a possibility to analyse the towers of excited states, for each ground state baryon (e.g., nucleon) or vector meson (e.g., ρ-meson). We discuss these two examples in Section 2. Then, Sections 3 and 4 are devoted to the very instructive examples of q-deformed structure which, if one sends $q \to -1$, show either exact SUSY (the case of braided line whose relation to SUSY is considered in Section 3), or the features only reminiscent of supersymmetry, see Section 4. In the 5th section we deal with the peculiar case of $q = -1$ concerning the quantum algebras $U_q(\mathrm{su}_n)$ which appear in the context of their use as the algebras describing flavor symmetries of hadrons and enabling to derive new, very precise mass relations. In this scheme, the restriction to the limit $q = -1$ is physically motivated.

2 Dynamical supersymmetry and hadron mass spectrum

In [5] the two copies of superalgebra, namely,

$$\begin{aligned}
&[F_i, F_j] = \mathrm{i} f_{ijk} F_k, \qquad [F_i, G_j] = \mathrm{i} f_{ijk} G_k, \qquad \{G_i, G_j\} = d_{ijk} F_k,\\
&[\bar{F}_i, \bar{F}_j] = \mathrm{i} f_{ijk} \bar{F}_k, \qquad [\bar{F}_i, \bar{G}_j] = \mathrm{i} f_{ijk} \bar{G}_k, \qquad \{\bar{G}_i, \bar{G}_j\} = -d_{ijk} \bar{F}_k,
\end{aligned} \tag{1}$$

have been introduced. For their realization, the conventional 3×3 Hermitian matrices λ_i ($i = 0, 1, 2, 3, 8$ for the F_i, $\bar{F}_i$, and $i = 4, 5, 6, 7$ for the G_i, $\bar{G}_i$) have been utilized. By means of symmetry breaking terms (C_{3b}^{3b}, C_{a3}^{a3} and C_{33}^{33} in the notation of [5]) which provide mass splitting between quarks and diquarks (i.e., SUSY breaking), as well as splitting between isomultilplets (breaking of $SU(3)$ to $SU(2)$), instead of the standard GMO mass relation the formulas

$$m_N + m_\Xi = m_\Lambda + m_\Sigma, \qquad m_{Y_0^*} = m_\Sigma, \qquad 2m_{K^*} = m_\rho + m_\phi, \qquad m_\rho = m_\omega \tag{2}$$

for baryons and for vector mesons have been obtained.

A completely different scheme for treating hadron mass spectrum developed in [6] employs a particular *dynamical superalgebra* Osp(1|4) connected with space-time symmetries. The dynamical superalgebra with generators $S_{\mu\nu}$, Γ_μ, Q_α respects the chain

$$\mathrm{Osp}(1|4)_{S_{\mu\nu},\Gamma_\mu,Q_\alpha} \supset \mathrm{SO}(3,2)_{S_{\mu\nu},\Gamma_\mu} \supset \mathrm{SO}(3,1)_{S_{\mu\nu}},$$

where for the subalgebras $\mathrm{SO}(3,1)_{S_{\mu\nu}}$ and $\mathrm{SO}(3,2)_{S_{\mu\nu},\Gamma_\mu}$ the generators $S_{\mu\nu}$ and Γ_μ obey

$$[S_{\mu\nu}, S_{\rho\sigma}] = -i(\eta_{\mu\rho}S_{\nu\sigma} + \eta_{\nu\sigma}S_{\mu\rho} - \eta_{\nu\rho}S_{\mu\sigma} - \eta_{\mu\sigma}S_{\nu\rho}), \tag{3}$$

$$[S_{\mu\nu}, \Gamma_\rho] = -i(\eta_{\mu\rho}\Gamma_\nu - \eta_{\nu\rho}\Gamma_\mu), \qquad [\Gamma_\mu, \Gamma_\nu] = -iS_{\mu\nu}. \tag{4}$$

The relations involving anticommuting charges Q_α and $\bar{Q}_\beta = -(Q^T C)_\beta$, namely

$$[S_{\mu\nu}, Q_\alpha] = -\frac{1}{2}\left(\sigma^s_{\mu\nu}\right)_\alpha^{\ \beta} Q_\beta, \qquad [\Gamma_\mu, Q_\alpha] = -\frac{1}{2}\left(\gamma_\mu\right)_\alpha^{\ \beta} Q_\beta,$$
$$\{Q_\alpha, \bar{Q}_\beta\} = -\frac{1}{2}\left(\sigma^{\mu\nu}\right)_{\alpha\beta} S_{\mu\nu} + (\gamma^\mu)_{\alpha\beta}\Gamma_\mu,$$

along with (3), (4), complete the symmetry algebra to the superalgebra $\mathrm{Osp}(1|4)_{S_{\mu\nu},\Gamma_\mu,Q_\alpha}$.

To construct the Hamiltonian, supercharges should be incorporated (like in supersymmetric quantum mechanics), through the term $\frac{1}{2n}\sum_{\alpha=1}^{n}\{Q_\alpha, Q_\alpha^\dagger\}$. The resulting Hamiltonian

$$H = v\left(P_\mu P^\mu - \frac{1}{\alpha'}\frac{1}{4}\sum_{\beta=1}^{4}\{Q_\beta, Q_\beta^\dagger\} - \tilde{m}_0^2\right)$$

is to be completed by Casimirs of subalgebras in the chain $\mathrm{Osp}(1|4) \supset \mathrm{SO}(3,2)_{S_{\mu\nu},\Gamma_\mu} \supset \mathrm{SO}(3)_{S_{ij}} \times \mathrm{SO}(2)_{\Gamma_0}$. In its final form, the Hamiltonian reads

$$H = v\left(P_\mu P^\mu - \frac{1}{\alpha'}\hat{P}_\mu\Gamma^\mu - \lambda^2\hat{W} + \beta\hat{C}_{\mathrm{SO}(3,2)} - \tilde{m}_0^2\right) \tag{5}$$

and, correspondingly, hadron mass spectrum is described by the formula [6]

$$m^2 = -\frac{1}{\alpha'}\mu + \lambda^2 j(j+1) + \beta\left(2 - 2s^2\right) + \tilde{m}_0^2. \tag{6}$$

In this expression, $1/\alpha'$ (related to the slope of Regge trajectory), λ^2, and β are empirical system parameters; μ resp. $j(j+1)$ are eigenvalues of $\hat{P}_\mu\Gamma^\mu$ resp. $\hat{W}$; s labels $\mathrm{SO}(3,2)$ representations, and $\tilde{m}_0$ is the background mass.

Comparison of the mass formula (6) with experimental data, using the particular representation $D\left(\frac{3}{2},\frac{1}{2}\right) \oplus D(2,1)$ of the dynamical superalgebra, shows that the series (tower) of excited states over the lowest lying 1^- vector mesons ρ or ω and the $\frac{1}{2}^+$ nucleon's tower (its resonances) fit the data very well if one sets: $\frac{1}{\alpha'}(\text{meson}) \sim \frac{1}{\alpha'}(\text{nucleon})$ and $\lambda^2(\text{meson}) \sim \lambda^2(\text{nucleon})$. It is this fact that was interpreted in [6] as a kind of empirical evidence for supersymmetry in the hadron mass spectra. This observation may be considered as an extension of the well-known success of dynamical supersymmetries in nuclear physics [7] to the level of hadrons.

3 q-deformed oscillator at $q \to -1$ and supersymmetry

In [8] it was shown that the q-deformed calculus on the braided line [9] (tightly connected with q-deformed oscillator), in the nontrivial particular case of $q = -1$ exhibits supersymmetric properties. In this section we discuss some details of this correspondence, following [8].

The braided (or q-deformed) line is defined [9] in terms of a single non-commuting variable θ which obeys a Hopf algebra structure operating with coproduct,

$$\Delta\theta = \theta \otimes 1 + 1 \otimes \theta, \tag{7}$$

$$(1 \otimes \theta)(\theta \otimes 1) = q\theta \otimes \theta, \qquad (\theta \otimes 1)(1 \otimes \theta) = \theta \otimes \theta. \tag{8}$$

as well as a counit and antipode. Note that it is the first relation in (8) that determines the nontrivial (for $q \neq 1$) braiding.

With $[X, Y]_z \equiv XY - zYX$, denoting $\theta = 1 \otimes \theta$ and $\delta\theta = \epsilon = \theta \otimes 1$ as in Ref. [9], yields

$$[\epsilon, \theta]_{q^{-1}} = 0 \qquad \text{and} \qquad \Delta\theta = \epsilon + \theta.$$

Here the latter equality corresponds to (7); it encodes the action upon θ of the left translation by ϵ, $L_\epsilon\theta: \ \theta \mapsto \epsilon + \theta$. As seen, ϵ and θ anticommute when $q = -1$.

To construct a differential calculus on the braided line, one introduces a left derivation operator with respect to θ, obeying $[\epsilon\mathcal{D}_L, \theta] = \epsilon$, so that

$$[\mathcal{D}_L, \theta] = 1, \qquad \frac{d}{d\theta}\theta = 1. \tag{9}$$

Likewise, one can introduce right shifts $R_\eta\theta: \ \theta \mapsto \theta + \eta$ by odd parameter η so that $[\theta, \eta]_{q^{-1}} = [\eta, \theta]_q = 0$ (again, θ and η anticommute if $q = -1$). The right derivative operator satisfies $[\theta, \mathcal{D}_R] = 1$ and also the relation

$$\mathcal{D}_R = -q^{-(1+N)}\mathcal{D}_L \tag{10}$$

involving the number operator N defined according to

$$[N, \theta] = \theta, \qquad [N, \mathcal{D}_L] = -\mathcal{D}_L. \tag{11}$$

The differential calculus defined by (9)–(11) at generic q is called q-calculus.

With the identification $\theta = a^\dagger$, $\mathcal{D}_L = q^{N/2}a$, the q-calculus is related to the q-deformed harmonic oscillator [10]

$$aa^\dagger - q^{\mp 1/2}a^\dagger a = q^{\pm N/2}. \tag{12}$$

The entity $q^{1/2}$ and its power $\left(q^{1/2}\right)^N$ in (12) are of importance since, from (12), by exploiting Hermitian conjugacy one comes to the formulas $aa^\dagger = [N+1]_{q^{1/2}}$ and $a^\dagger a = [N]_{q^{1/2}}$ valid for the q-deformed oscillator [10] of Biedenharn and Macfarlane. Here $[A]_z \equiv \left(z^A - z^{-A}\right) / \left(z - z^{-1}\right)$.

Let $\lfloor A \rfloor_q \equiv (1 - q^A)/(1 - q)$. A function of θ given by the expansion $f(\theta) = \sum_{m=0}^{\infty} C_m\theta^m/\lfloor m \rfloor_q!$ admits the derivative $\frac{d}{d\theta}f(\theta) = \sum_{m=0}^{\infty} C_m\theta^m/\lfloor m-1 \rfloor_q!$ implying that

$$\left[\mathcal{D}_L, \frac{\theta^m}{\lfloor m \rfloor_q!}\right]_{q^m} = \frac{\theta^{m-1}}{\lfloor m-1 \rfloor_q!}.$$

The difficulties appearing in the limit $q \to -1$ already at $m = 2$ (since $\lfloor 2 \rfloor_q = 0$ in this limit) are tamed by setting $q = -1 + iy$ and letting $y \to 0$. Then, the definition

$$t := \lim_{q \to -1}\left(i\theta^2/\lfloor 2 \rfloor_q!\right) \tag{13}$$

implying that, with $\theta^2 = 0$ imposed, the limit of the ratio in (13) should be finite and nonzero, imports the additional variable t as a necessary ingredient of the braided line if $q \to -1$. As shown in [8], in this limit the terms of the form $\theta^{2r+p}/\lfloor 2r+p \rfloor_q!$ also can be handled by means of t. Due to this, any function $f(\theta)$ on the braided line (generic q), reduces in the limit $q \to -1$ to a 'superfield' given by the function $f(t, \theta)$.

It can be shown that $[\mathcal{D}_L^2, t] = i$ and, with the definition

$$\{\mathcal{D}_L, \mathcal{D}_L\} = 2i\partial_t \qquad \text{or} \qquad \partial_t = -i\mathcal{D}_L^2,$$

the relation $[\partial_t, t] = 1$ is valid. The operator $\mathcal{D}_L$ then becomes the supercharge, $\mathcal{D}_L \equiv Q$, of one-dimensional supersymmetry, and one comes to the relations:

$$Q = \partial_\theta + i\theta\partial_t, \qquad \{Q, Q\} = 2i\partial_t.$$

Likewise, the operator $D = \mathcal{D}_R = (-1)^N \mathcal{D}_L$ becomes the (super)covariant derivative so that

$$D = \partial_\theta - i\theta\partial_t, \qquad \{D, D\} = -2i\partial_t, \qquad \text{and} \qquad \{Q, D\} = 0.$$

Another interesting result derived in [8] is the coproduct for t with unusual θ-dependent term:

$$\Delta t = t \otimes 1 + 1 \otimes t + i\theta \otimes \theta.$$

Thus, proper treatment of braided line in the peculiar limit $q \to -1$ shows that, in this limit, an additional variable t related to θ^2 (see (13)), as well as to higher powers, must arise. As a result, the braided line at $q \to -1$ is made up of the two variables θ and t which span the one-dimensional superspace, SUSY being the translational invariance along this line.

4 Example of Zachos, based on the $q = -1$ limit of $U_q(\mathrm{su}_2)$

Quantum algebra $U_q(\mathrm{su}_2)$ [1, 2] is generated by the elements I_+, I_-, I_0, obeying the relations

$$\begin{aligned}
&[I_0, I_\pm] = \pm I_\pm, \qquad [I_+, I_-] = [2J_0]_q \equiv \left(q^{2J_0} - q^{-2J_0}\right) / \left(q - q^{-1}\right), \\
&\Delta(J_0) = J_0 \otimes 1 + 1 \otimes I_0, \qquad \Delta(J_\pm) = J_\pm \otimes q^{-J_0} + q^{+J_0} \otimes J_\pm
\end{aligned} \tag{14}$$

and the relations that involve antipode and counit (which will not be used here).

As shown in [11], this quantum algebra exhibits an intriguing features at the level of its representations when the deformation parameter $q = -1$. Let us consider this example.

Using coproduct, one can form composites of two spin $\frac{1}{2}$ doublets according to $\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$:

$$\begin{aligned}
&\text{singlet} \quad \longleftrightarrow \quad \alpha = |q^{1/2} \uparrow\downarrow - q^{-1/2} \downarrow\uparrow\rangle, \\
&\text{triplet} \quad \longleftrightarrow \quad \begin{cases} \beta = |\uparrow\uparrow\rangle, \\ \Delta(J_-)\beta = \frac{1}{\sqrt{2}}|q^{1/2} \uparrow\downarrow + q^{-1/2} \downarrow\uparrow\rangle, \\ (\Delta(J_-))^2\beta = |\downarrow\downarrow\rangle. \end{cases}
\end{aligned}$$

For $q = 1$, the singlet state is antisymmetric whereas each of the triplet states is symmetric. Now let $q = -1$. In this case the multiplets turn into

$$\begin{aligned}
&\alpha = |i \uparrow\downarrow - \frac{1}{i} \downarrow\uparrow\rangle && \text{(symmetric)}, \\
&\beta = |\uparrow\uparrow\rangle && \text{(symmetric)}, \\
&\Delta(J_-)\beta = \frac{1}{\sqrt{2}}|i \uparrow\downarrow + \frac{1}{i} \downarrow\uparrow\rangle && \text{(antisymmetric)}, \\
&(\Delta(J_-))^2\beta = |\downarrow\downarrow\rangle && \text{(symmetric)}.
\end{aligned} \tag{15}$$

It is seen from (15) that the coproduct operation $\Delta(J_-)$ changes the symmetry of wave function. That is, rasing and lowering operators in the coproduct act as statistics-altering operators. Although the constituents of the states haven't been converted to fermions, this alteration of the symmetry of wave function *is reminiscent of* SUSY. It is instructive to compare this structure with $N = 2$ supersymmetric quantum mechanics, stressing both similarities and peculiar features.

Consider (graded) direct product of two copies of SUSY QM algebras:

$$\begin{aligned} &SS^\dagger + S^\dagger S = 1, \qquad ss^\dagger + s^\dagger s = 1, \qquad S^\dagger S^\dagger = s^\dagger s^\dagger = SS = ss = 0,\\ &sS + Ss = 0, \qquad s^\dagger S^\dagger + S^\dagger s^\dagger = 0, \qquad sS^\dagger + S^\dagger s = 0, \qquad s^\dagger S + Ss^\dagger = 0. \end{aligned} \tag{16}$$

This graded Lie algebra can be obtained, using appropriate Wigner–Inonü contraction, from the simple Lie superalgebra $SU(2|1)$ (realizable in terms of Gell-Mann $SU(3)$ λ-matrices so that $\{\lambda_1, \lambda_2, \lambda_3, \lambda_8\}$ constitute even generators whereas $\{\lambda_4, \lambda_5, \lambda_6, \lambda_7\}$ constitute odd generators).

One can realize the algebra (16) on two boson states $|B\rangle$, $|b\rangle$, and two fermion states $|F\rangle$, $|f\rangle$, as: $S\,|B\rangle = |F\rangle$, $s|b\rangle = |f\rangle$, $S^\dagger|F\rangle = |B\rangle$, $s^\dagger|f\rangle = |b\rangle$. The (nullifying) rest of actions reads: $S\,|F\rangle = S|b\rangle = s|B\rangle = s^\dagger|F\rangle = s^\dagger|b\rangle = S^\dagger|f\rangle = S^\dagger|B\rangle = s|f\rangle = 0$. With their use,

$$s|Bb + bB\rangle = |Bf + fB\rangle, \qquad Ss|Bb + bB\rangle = |Ff - fF\rangle. \tag{17}$$

Thus, $\Delta(J_-)$ in (15) switches the symmetry of wave function like the even (bosonic) operator $Ss = -sS$, see (17), but only the latter is nilpotent due to nilpotency of S, s. The other important difference consists in the structure and dimensionality of multiplets. Namely, for $q = -1$ these remain the same as in the classical case of $su(2)$ Lie algebra. On the other hand, for graded Lie algebra the representations are of different dimensions (compare, e.g., $SU(2|1)$ and $SU(3)$). Hence, the conclusion: this $q = -1$ case implies a kind of *quasi-supersymmetry*.

5 GMO formula and $U_q(su_n)$ at $q = -1$

One can either utilize representation-theoretic aspects of the quantum algebra $U_q(su_n)$ or, alternatively, construct the mass operator using q-tensor operators. In the latter case [12], main ingredients of the Hopf algebra structure of $U_q(su_n)$ (comultiplication Δ and antipode S) play the role. The Δ and S are defined [1, 2] on the $U_q(su_n)$ generators $E_i^\pm$ and H_i as

$$\begin{aligned} &S\left(E_i^\pm\right) = -q^{H_i/2}E_i^\pm, \qquad S(H_i) = -H_i, \qquad S\left(q^{H_i/2}\right) = q^{-H_i/2}, \qquad S(1) = 1,\\ &\Delta\left(E_i^\pm\right) = E_i^\pm \otimes q^{H_i/2} + q^{-H_i/2} \otimes E_i^\pm, \qquad \Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i q^{-H_i/2}. \end{aligned} \tag{18}$$

The adjoint action of $U_q(su_n)$ defined [2] as $\mathrm{ad}_A B = \sum A_{(1)} B S(A_{(2)})$ with $A, B \in U_q(su_n)$ and $A_{(1)}$, $A_{(2)}$ determined from $\Delta(A) = \sum A_{(1)} \otimes A_{(2)}$, with the account of (18) reads:

$$\begin{aligned} &\mathrm{ad}_{H_i} B = H_i B 1 + 1 B S(H_i) = H_i B - B H_i,\\ &\mathrm{ad}_{E_i^\pm} B = E_i^\pm B q^{-H_i/2} - q^{-H_i/2} B q^{H_i/2} E_i^\pm q^{-H_i/2}. \end{aligned}$$

The q-tensor operators [13] transforming under the adjoint action of $U_q(su_3)$ as $\mathbf{3}$ and $\mathbf{3}^*$, consist of the triples (V_1, V_2, V_3) and $(V_{\bar 1}, V_{\bar 2}, V_{\bar 3})$, respectively. Let $[X, Y]_q \equiv XY - qYX$. It can be shown that the particular triple of elements from $U_q(su_4)$

$$\begin{aligned} &V_1 = [E_1^+, [E_2^+, E_3^+]_q]_q \; q^{-H_1/3 - H_2/6},\\ &V_2 = [E_2^+, E_3^+]_q \; q^{H_1/6 - H_2/6}, \qquad V_3 = E_3^+ q^{H_1/6 + H_2/3} \end{aligned} \tag{19}$$

transform as **3** under $U_q(su_3)$, V_1 corresponds to the highest weight vector, the pair (V_1, V_2) is $U_q(su_2)$ (iso)doublet and V_3 its singlet. Likewise one constructs from elements of $U_q(su_4)$ the triple $(V_{\bar{1}}, V_{\bar{2}}, V_{\bar{3}})$ that transforms as $\mathbf{3}^*$ under adjoint action of $U_q(su_3)$, where $V_{\bar{3}}$ corresponds to the highest weight vector, the pair $(V_{\bar{1}}, V_{\bar{2}})$ is isodoublet and $V_{\bar{3}}$ is $U_q(su_2)$ singlet.

The mass operator $\hat{M} = \hat{M}_0 + \hat{M}_8$ involves $\hat{M}_0$, as $U_q(su_3)$ scalar, and the term $\hat{M}_8$ transforming as the $I = 0, Y = 0$ component of tensor operator of **8**-irrep of $U_q(su_3)$. The irrep **8** occurs twice in the decomposition $\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{8}^{(1)} \oplus \mathbf{8}^{(2)} \oplus \mathbf{10}^* \oplus \mathbf{10} \oplus \mathbf{27}$. Then, usage of Wigner–Eckart theorem for $U_q(su_n)$ quantum algebras [13] applied to q-tensor operators transforming as irrep **8** of $U_q(su_3)$, turns the mass operator into $\hat{M} = \hat{M}_0 + \hat{M}_8 = M_0\mathbf{1} + \alpha V_8^{(1)} + \beta V_8^{(2)}$. Here **1** is the identity operator, $V_8^{(1)}$ and $V_8^{(2)}$ are two fixed tensor operators with non-proportional matrix elements, each transforming as the $I = 0, Y = 0$ component of irrep **8** of $U_q(su_3)$; M_0, α and β are some constants depending on details (dynamics) of the model.

If $|B_i\rangle$ is a basis vector of representation **8** space which corresponds to some $(1/2)^+$ baryon, then the mass of this baryon is calculated as

$$M_{B_i} = \langle B_i|\hat{M}|B_i\rangle = \langle B_i| \left(M_0\mathbf{1} + \alpha V_8^{(1)} + \beta V_8^{(2)} \right) |B_i\rangle. \tag{20}$$

The decompositions $\mathbf{3} \otimes \mathbf{3}^* = \mathbf{1} \oplus \mathbf{8}$, $\mathbf{3}^* \otimes \mathbf{3} = \mathbf{1} \oplus \mathbf{8}$ imply that the operators $V_3 V_{\bar{3}}$ and $V_{\bar{3}} V_3$ formed from V_3 in (19) and $V_{\bar{3}}$ are just the two isosinglets $V_8^{(1)}$, $V_8^{(2)}$ needed in (20). Hence, the mass operator in (20) can be rewritten (redefining M_0, α, β) in the equivalent form

$$\hat{M} = M_0\mathbf{1} + \alpha V_3 V_{\bar{3}} + \beta V_{\bar{3}} V_3 = \hat{M} = M_0\mathbf{1} + \alpha E_3^+ E_3^- q^Y + \beta E_3^- E_3^+ q^Y, \tag{21}$$

where the hypercharge $Y = (H_1 + 2H_2)/3$ has been introduced.

To calculate matrix elements (20) using (21) we embed the octet **8** in a particular irrep of $U_q(su_4)$; embedding it, e.g., in **15** (adjoint) irrep of $U_q(su_4)$, we get the octet baryon masses

$$M_N = M_0 + \beta q, \qquad M_\Sigma = M_0, \qquad M_\Lambda = M_0 + [2]_q[3]_q^{-1}(\alpha + \beta), \qquad M_\Xi = M_0 + \alpha q^{-1} \tag{22}$$

(obviously, the expressions for M_N, M_Ξ are not invariant under $q \to q^{-1}$). Excluding M_0, α and β from (22) results in the following q-analogue of GMO formula for octet baryons:

$$[3]_q M_\Lambda + M_\Sigma = [2]_q \left(q^{-1} M_N + q M_\Xi \right). \tag{23}$$

Using empirical masses, the deformation parameter q is fixed by fitting: for each of the $q_{1,2} = \pm 1.035$, $q_{3,4} = \pm 0.903\sqrt{-1}$, the q-deformed mass relation (23) holds within experimental uncertainty (although for q_3, q_4 the constants α and β in (22) must be pure imaginary).

The right hand side of equation (23) is invariant under $q \to q^{-1}$ only if $q = q^{-1}$, that is, if $q = \pm 1$. Behind the 'classical' GMO mass formula which obviously follows from (23) at $q = 1$ and corresponds to the nondeformed unitary symmetries $SU(4) \supset SU(3) \supset SU(2)$, there is also an unusual 'hidden symmetry' reflecting the singular $q = -1$ case of $U_q(su_4) \supset U_q(su_3) \supset U_q(su_2)$ algebras, undefined in this case. The relevant objects, however, exist as operator algebras [12]. Let us describe them in the part corresponding to $n = 2$ and $n = 3$.

At generic q, $q \neq -1$, the algebra $U_q(su_2)$ is generated by the elements E^+, E^- and H, which satisfy the relations

$$[H, E^\pm] = \pm 2E^\pm, \qquad [E^+, E^-] = [H]_q. \tag{24}$$

In the limit $q \to 1$ it reduces to the nondeformed su_2. We take the representation spaces of the latter in order to construct operator algebras for the case $q = -1$. To each su_2 representation space given by j (which takes integral or half-integral nonnegative values) with basis elements

$|jm\rangle$, $m = -j, -j+1, \ldots, j$, there corresponds an operator algebra generated by the operators defined according to the formulas

$$H|jm\rangle = 2m|jm\rangle, \qquad E^+|jm\rangle = \alpha_{j,m}|jm+1\rangle, \qquad E^-|jm\rangle = \alpha_{j,m-1}|jm-1\rangle, \tag{25}$$

where

$$\alpha_{j,m} = \begin{cases} \sqrt{-(j-m)(j+m+1)}, & j \quad \text{is an integer,} \\ \sqrt{(j-m)(j+m+1)}, & j \quad \text{is a half-integer.} \end{cases}$$

So defined operators E^+, E^- and H *on the basis elements* $|jm\rangle$ satisfy the relations (compare with (24)), one of which *depends on whether j is an integer or a half-integer*:

$$[H, E^{\pm}] = \pm 2E^{\pm}, \qquad [E^+, E^-] = \begin{cases} -H, & j \quad \text{is an integer;} \\ \ \ H, & j \quad \text{is a half-integer.} \end{cases} \tag{26}$$

To treat the (singular) case $q = -1$ of $U_q(su_3)$ it is more convenient to deal with $U_q(u_3)$. We take a representation space V_χ, labelled by $\{m_{13}, m_{23}, m_{33}\} \equiv \chi$, of the nondeformed u_3 and the Gel'fand–Tsetlin basis with the basis elements $|\chi; m_{12}, m_{22}; m_{11}\rangle$ in each V_χ. Define the operators E_1^+, E_1^-, H_1, E_2^+, E_2^-, H_2 that form the operator algebra of the χ-type by their action according to the formulas (let us denote $\sigma_{1,3} \equiv m_{11} + m_{13} + m_{23} + m_{33}$):

$$\begin{aligned}
&H_2|\chi; m_{12}, m_{22}; m_{11}\rangle = (2m_{12} + 2m_{22} - m_{13} - m_{23} - m_{33} - m_{11})|\chi; m_{12}, m_{22}; m_{11}\rangle, \\
&E_2^+|\chi; m_{12}, m_{22}; m_{11}\rangle = a_{\chi,m_{11}}(m_{12}, m_{22})\ |\chi; m_{12}+1, m_{22}; m_{11}\rangle \\
&\qquad\qquad + b_{\chi,m_{11}}(m_{12}, m_{22})\ |\chi; m_{12}, m_{22}+1; m_{11}\rangle, \\
&E_2^-|\chi; m_{12}, m_{22}; m_{11}\rangle = a_{\chi,m_{11}}(m_{12}-1, m_{22})\ |\chi; m_{12}-1, m_{22}; m_{11}\rangle r \\
&\qquad\qquad + b_{\chi,m_{11}}(m_{12}, m_{22}-1)\ |\chi; m_{12}, m_{22}-1; m_{11}\rangle,
\end{aligned}$$

where

$$a_{\chi,m_{11}}(m_{12}, m_{22}) = \left((-1)^{\sigma_{1,3}} \frac{(m_{13}-m_{12})(m_{23}-m_{12}-1)(m_{33}-m_{12}-2)(m_{11}-m_{12}-1)}{(m_{22}-m_{12}-1)(m_{22}-m_{12}-2)}\right)^{1/2},$$

$$b_{\chi,m_{11}}(m_{12}, m_{22}) = \left((-1)^{\sigma_{1,3}} \frac{(m_{13}-m_{22}+1)(m_{23}-m_{22})(m_{33}-m_{22}-1)(m_{11}-m_{22})}{(m_{12}-m_{22}+1)(m_{12}-m_{22})}\right)^{1/2}.$$

Action formulas for the operators $E_1^{\pm}$ and H_1 are completely analogous to formulas (25) above (with account of $m_{11} - m_{22} = 2j$, $2m_{11} - m_{12} - m_{22} = 2m$).

The presented action formulas for the operators that form the operator algebra of the χ-type show that their matrix elements are, to some extent, similar to the 'classical' matrix elements (i.e. to the matrix elements of the irrep χ operators for $su(n)$). However, there is an essential distinction: now we observe the important phase factors (namely, $(-1)^{m_{11}+m_{13}+m_{23}+m_{33}}$ under the square root in $a_{\chi,m_{11}}$ and $b_{\chi,m_{11}}$) which *depend on χ and a specified basis element.* No such basis-element dependent factors exist in the $su(n)$ case.

Let us illustrate such treatment with the particular example of operator algebra appearing in the singular $q = -1$ case of $U_q(su_3)$ and corresponding to the octet representation of su_3. We give here explicitly only those action formulas for $E_1^{\pm}$ and $E_2^{\pm}$ in which matrix elements differ from their corresponding 'classical' counterparts:

$$\begin{aligned}
&E_1^-|\Sigma^+\rangle = \sqrt{-2}|\Sigma^0\rangle, \qquad E_1^-|\Sigma^0\rangle = \sqrt{-2}|\Sigma^-\rangle, \qquad E_1^+|\Sigma^-\rangle = \sqrt{-2}|\Sigma^0\rangle, \\
&E_1^+|\Sigma^0\rangle = \sqrt{-2}|\Sigma^+\rangle, \qquad E_2^-|n\rangle = \frac{1}{\sqrt{-2}}|\Sigma^0\rangle + \sqrt{-3/2}|\Lambda\rangle, \qquad E_2^-|\Lambda\rangle = \sqrt{-3/2}|\Xi^0\rangle,
\end{aligned}$$

$$E_2^-|\Sigma^0\rangle = \frac{1}{\sqrt{-2}}|\Xi^0\rangle, \qquad E_2^+|\Xi^0\rangle = \frac{1}{\sqrt{-2}}|\Sigma^0\rangle + \sqrt{-3/2}|\Lambda\rangle, \qquad E_2^+|\Lambda\rangle = \sqrt{-3/2}|n\rangle,$$
$$E_2^+|\Sigma^0\rangle = \frac{1}{\sqrt{-2}}|n\rangle.$$

To complete this operator algebra, we must add the rest of action formulas for $E_1^\pm$ and $E_2^\pm$ (i.e., action on those basis elements) which coincide with the 'classical' ones, as well as the action formulas for H_1, H_2 (these latter also coincide with 'classical' formulas).

Likewise, for $U_q(su_3)$ at $q = -1$ an operator algebra corresponding to any other irrep of su_3 can be given. The treatment is obviously extendible to $U_{q=-1}(su_n)$, $n > 3$.

Let us also remark that SUSY-based mass relation $m_\rho = m_\omega$, see (2), is obtainable from a q-deformed structure. Indeed, it follows from the q-analog of vector meson mass relation,

$$m_{\omega_8} + (2[2]_q/[3]_q - 1)m_\rho = (2[2]_q/[3]_q)m_{K^*}$$

(which was derived [14] using $U_q(su_n)$ quantum algebras), if one fixes q as 4th root of unity: $q = \sqrt{-1}$ (then, $[2]_q = 0$). The intriguing interplay between SUSY and the special cases $q = -1$ and $q = \sqrt{-1}$ of the q-algebras $U_q(su_n)$ deserves further detailed study.

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Fractional Supersymmetry and $F-$fold Lie Superalgebras

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We give infinite dimensional and finite dimensional examples of $F-$fold Lie superalgebras. The finite dimensional examples are obtained by an inductive procedure from Lie algebras and Lie superalgebras.

1 Introduction

It is generally held that supersymmetry is the only non-trivial extension of the Poincaré algebra. This point of view is based on the two theorems [1, 2]. However, as usual, if some of the assumptions of these two *no-go* theorems are relaxed symmetry beyond supersymmetry can be constructed [3–23] In all these possible extensions of the Poincaré symmetry, new generators are introduced. The basic structure underlying these extensions is related to algebraic structures which are neither Lie algebras, nor Lie superalgebras. In this contribution we would like to give some results concerning fractional supersymmetry (FSUSY) [6–23], one of the possible extensions of supersymmetry, and the associated algebraic structure, the so-called $F-$Lie algebra [21]. Such a structure admits a $\mathbb{Z}_F$ grading, the zero-graded part defining a Lie algebra, and an $F-$fold symmetric product (playing the role of the anticommutator in the case $F = 2$) allows one to express the zero graded part in terms of generators of the non zero graded part. In Section 2 the basic definition of $F-$Lie algebras will be given. In Section 3, some examples of infinite dimensional $F-$Lie algebras will be explicitly constructed. In Section 4, some examples of finite dimensional $F-$Lie algebras will be explicitly constructed with the classification of the usual Lie (super)algebras as a guideline.

2 $F-$Lie algebras

A natural mathematical structure, generalizing the concept of Lie superalgebras and relevant for the algebraic description of fractional supersymmetry was introduced in [21] and called an $F-$Lie algebra. We do not want to go into the detailed definition of this structure here and will only recall the basic points, useful for our purpose. More details can be found in [21].

Let F be a positive integer and $q = e^{2i\frac{\pi}{F}}$. We consider now a complex vector space S which has an automorphism ε satisfying $\varepsilon^F = 1$. We set $A_k = S_{q^k}$, $1 \le k \le F-1$ and $B = S_1$ (S_{q^k} is the eigenspace corresponding to the eigenvalue q^k of ε). Hence,

$$S = B \oplus A_1 \oplus \cdots \oplus A_{F-1}.$$

We say that S is an $F-$Lie algebra if:

1. B, the zero graded part of S, is a Lie algebra.
2. A_i $(i = 1, \dots, F-1)$, the i graded part of S, is a representation of B.
3. There are symmetric multilinear $B-$equivariant maps

$$\{\ ,\dots,\ \}:\ \mathcal{S}^F\left(A_k\right) \to B.$$

   In other words, we assume that some of the elements of the Lie algebra B can be expressed as $F-$th order symmetric products of "more fundamental generators". It is easy to see that

$$\{\varepsilon(a_1), \dots, \varepsilon(a_F)\} = \varepsilon\left(\{a_1, \dots, a_F\}\right), \quad \forall\, a_1, \dots, a_F \in A_k.$$

The generators of S are assumed to satisfy Jacobi identities ($b_i \in B$, $a_i \in A_k$, $1 \le k \le F-1$):

$$\begin{aligned}
&\left[\left[b_1, b_2\right], b_3\right] + \left[\left[b_2, b_3\right], b_1\right] + \left[\left[b_3, b_1\right], b_2\right] = 0,\\
&\left[\left[b_1, b_2\right], a_3\right] + \left[\left[b_2, a_3\right], b_1\right] + \left[\left[a_3, b_1\right], b_2\right] = 0,\\
&\left[b, \{a_1, \dots, a_F\}\right] = \{\left[b, a_1\right], \dots, a_F\} + \cdots + \{a_1, \dots, \left[b, a_F\right]\},\\
&\sum_{i=1}^{F+1} \left[a_i, \{a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_{F+1}\}\right] = 0.
\end{aligned} \tag{1}$$

The first three identities are consequences of the previously defined properties but the fourth is an extra constraint.

More details (unitarity, representations, *etc.*) can be found in [21]. Let us first note that no relation between different graded sectors is postulated. Secondly, the sub-space $B \oplus A_k \subset S$ $(k = 1, \dots, F-1)$ is itself an $F-$Lie algebra. From now on, $F-$Lie algebras of the types $B \oplus A_k$ will be considered.

3 Examples of infinite dimensional $F-$Lie algebras

It is possible to construct an $F-$Lie algebra starting from a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}-$module $\mathcal{D}$. The basic idea is the following. We consider $\mathfrak{g}$ a semi-simple Lie algebra of rank r. Let $\mathfrak{h}$ be a Cartan sub-algebra of $\mathfrak{g}$, let $\Phi \subset \mathfrak{h}^\star$ be the corresponding set of roots and let $\mathfrak{f}_\alpha$ be the one-dimensional root space associated to $\alpha \in \Phi$. We choose a basis $\{H_i, i = 1, \dots, r\}$ of $\mathfrak{h}$ and elements $E^\alpha \in \mathfrak{f}_\alpha$ such that the commutation relations become

$$\begin{aligned}
&\left[H_i, H_j\right] = 0, \qquad \left[H_i, E^\alpha\right] = \alpha^i E^\alpha,\\
&\left[E^\alpha, E^\beta\right] = \begin{cases} \epsilon\{\alpha, \beta\} E^{\alpha+\beta} & \text{if } \ \alpha + \beta \in \Phi,\\ \dfrac{2\alpha . H}{\alpha . \alpha} & \text{if } \ \alpha + \beta = 0,\\ 0 & \text{otherwise.} \end{cases}
\end{aligned} \tag{2}$$

We now introduce $\{\alpha_{(1)}, \dots, \alpha_{(r)}\}$, a basis of simple roots. The weight lattice $\Lambda_W(\mathfrak{g}) \subset \mathfrak{h}^\star$ is the set of vectors μ such that $\frac{2\alpha.\mu}{\alpha.\alpha} \in \mathbb{Z}$ and, as is well known, there is a basis of the weight lattice consisting of the fundamental weights $\{\mu_{(1)}, \dots, \mu_{(r)}\}$ defined by $\frac{2\mu_{(i)}.\alpha_{(j)}}{\alpha_{(j)}.\alpha_{(j)}} = \delta_{ij}$. A weight $\mu = \sum\limits_{i=1}^{r} n_i \mu_{(i)}$ is called dominant if all the $n_i \ge 0$ and it is well known that the set of dominant weights is in one to one correspondence with the set of (equivalence classes of) irreducible finite dimensional representations of $\mathfrak{g}$.

Recall briefly how one can associate a representation of $\mathfrak{g}$ to $\mu \in \mathfrak{h}^\star$, $\mu = \sum\limits_{i=1}^{r} n_i \mu_i$, $n_i \in \mathbb{C}$. We start with a vacuum $|\mu\rangle$ such that

$$\begin{aligned} &E^\alpha |\mu\rangle = 0, \qquad \alpha > 0, \\ &2\frac{\alpha_{(i)}.H}{\alpha^2}|\mu\rangle = n_i|\mu\rangle, \qquad i = 1, \ldots, r. \end{aligned} \tag{3}$$

The space obtained from $|\mu\rangle$ by the action of elements of $\mathfrak{g}$:

$$\mathcal{V}_\mu = \left\{E^{-\alpha_{(i_1)}} \cdots E^{-\alpha_{(i_k)}}|\mu\rangle, \alpha_{(i_1)}, \ldots, \alpha_{(i_k)} > 0\right\},$$

clearly defines a representation of $\mathfrak{g}$. Taking the quotient of $\mathcal{V}_\mu$ by its maximal $\mathfrak{g}-$stable subspace, the representation $\mathcal{D}_\mu$ of highest weight μ is obtained. If the n_i are positive integers, this is the irreducible finite dimensional representation of $\mathfrak{g}$ corresponding to the dominant weight μ.

To come back to our original problem, consider a finite dimensional irreducible representation $\mathcal{D}_\mu$, with highest weight $\mu = \sum\limits_{i=1}^{r} n_i\mu_i, n_i \in \mathbb{N}$. The basic idea is to try to define a structure of an $F-$Lie algebra on $S = B \oplus A_1 = (\mathfrak{g} \oplus \mathcal{D}_\mu) \oplus \mathcal{D}_{\frac{\mu}{F}}$ since, roughly speaking, the representations $\mathcal{D}_\mu$ and $\mathcal{D}_{\frac{\mu}{F}}$ can be related:

$$\mathcal{S}^F\left(\mathcal{D}_{\mu/F}\right) \sim \mathcal{D}_\mu. \tag{4}$$

Indeed $|\frac{\mu}{F}\rangle^{\otimes F} \in \mathcal{S}^F\left(\mathcal{D}_{\frac{\mu}{F}}\right)$ and $|\mu\rangle \in \mathcal{D}_\mu$ both satisfy equation (3). However, the representation $\mathcal{D}_\mu$ is finite dimensional but the sub-representation of $\mathcal{S}^F\left(\mathcal{D}_{\frac{\mu}{F}}\right)$ generated by $|\frac{\mu}{F}\rangle^{\otimes F}$ is infinite dimensional [21, 22]. Thus, the main difficulty in such a construction is to do with the requirement of relating an infinite dimensional representation $\mathcal{D}_{\mu/F}$ to a finite dimensional representation $\mathcal{D}_\mu$ in an equivariant way, *i.e.* respecting the action of $\mathfrak{g}$. One possible way of overcoming this difficulty is to embed $\mathcal{D}_\mu$ into an infinite dimensional (reducible but indecomposable) representation [21, 22, 23]. Another possibility is to embed $\mathfrak{g}$ into an infinite dimensional algebra (dubbed $V(\mathfrak{g})$) [21, 23, 24]) and extend the representations $\mathcal{D}_\mu$ and $\mathcal{D}_{\frac{\mu}{F}}$ to representations of $V(\mathfrak{g})$.

There is another difficulty related to such a construction. If one starts with $\mathfrak{D}_{\mu_1}$, the vector representation of $\mathfrak{so}(1, d-1)$, the representation $\mathfrak{D}_{\frac{\mu_1}{F}}$ cannot be exponentiated (see *e.g.* [25]) and does not define a representation of the Lie group $\overline{SO(1, d-1)}$ (the universal covering group of $SO(1, d-1)$) except when $d = 3$, where such representations describe relativistic anyons [18, 26].

4 Example of finite dimensional $F-$Lie algebras

In the previous section, we indicated how one can construct infinite dimensional examples of $F-$Lie algebras. In this section, with the classification of Lie (super)algebras as a guideline, we will give an inductive construction of finite dimensional $F-$Lie algebras.

In what follows S is a $1-$Lie algebra means:

1. $S = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, with $\mathfrak{g}_0$ a Lie algebra and $\mathfrak{g}_1$ is a representation of $\mathfrak{g}_0$ isomorphic to the adjoint representation;
2. there is a $\mathfrak{g}_0-$ equivariant map $\mu : \mathfrak{g}_1 \to \mathfrak{g}_0$ such that $[f_1, \mu(f_2)] + [f_2, \mu(f_1)] = 0, f_1, f_2 \in \mathfrak{g}_1$.

The basic result is the following theorem:

Theorem 1. *Let $\mathfrak{g}_0$ be a (complex) Lie algebra and $\mathfrak{g}_1$ a representation of $\mathfrak{g}_0$. Suppose given*
(i) the structure of an F_1-Lie algebra on $S_1 = \mathfrak{g}_0 \oplus \mathfrak{g}_1$;
(ii) the structure of an F_2-Lie algebra on $S_2 = \mathbb{C} \oplus \mathfrak{g}_1$.
Then $S = \left(\mathfrak{g}_0 \otimes \mathbb{C}\right) \oplus \mathfrak{g}_1$ can be given the structure of an $(F_1 + F_2)-$Lie algebra.

Proof. There exists (i) a $\mathfrak{g}_o-$equivariant map $\mu_1 : \mathcal{S}^{F_1}(\mathfrak{g}_1) \longrightarrow \mathfrak{g}_0$ and (ii) a $\mathfrak{g}_o-$equivariant map $\mu_2 : \mathcal{S}^{F_2}(\mathfrak{g}_1) \longrightarrow \mathbb{C}$, the second map is just a symmetric $F_2^{\text{th}}-$order invariant form on $\mathfrak{g}_1$. Now, consider $\mu : \mathcal{S}^{F_1+F_2}(\mathfrak{g}_1) \longrightarrow \mathfrak{g}_0 \otimes \mathbb{C} \cong \mathfrak{g}_0$ defined by $\forall\ f_1, \ldots, f_{F_1+F_2} \in \mathfrak{g}_1$:

$$\begin{aligned}&\mu(f_1, \ldots, f_{F_1+F_2})\\ &\qquad = \frac{1}{F_1!}\frac{1}{F_2!}\sum_{\sigma\in S_{F_1+F_2}} \mu_1(f_{\sigma(1)}, \ldots, f_{\sigma(f_{F_1})}) \otimes \mu_2(f_{\sigma(f_{F_1+1})}, \ldots, f_{\sigma(f_{F_1+F_2})}).\end{aligned} \tag{5}$$

with $S_{F_1+F_2}$ the group of permutations of $F_1 + F_2$ elements. By construction, this map is a $\mathfrak{g}_o-$equivariant map from $\mathcal{S}^{F_1+F_2}(\mathfrak{g}_1) \longrightarrow \mathfrak{g}_o$, thus the first three Jacobi identities (1) are clearly satisfied. The last Jacobi identity is more difficult to check and is directly related to the last Jacobi identity for the F_1-Lie algebra S_1 by a factorisation property. Indeed (with $F = F_1 + F_2$) if one calculates:

$$\sum_{i=0}^{F} \left[f_i, \mu\left(f_0, \ldots, f_{i-1}, f_{i+1}, \ldots, f_F\right)\right],$$

and selects terms of the form (with $\sigma \in S_{F_1+F_2+1)}$)

$$\mu_1\left(f_{\sigma(1)}, \ldots, f_{\sigma(f_{F_1})}\right) \otimes \mu_2\left(f_{\sigma(f_{F_1+1})}, \ldots, f_{\sigma(f_{F_1+F_2})}\right),$$

using $\mu_2\left(f_{\sigma(f_{F_1+1})}, \cdots, f_{\sigma(f_{F_1+F_2})}\right) \in \mathbb{C}$ the identity reduces to

$$\sum_{i=0}^{F_1}\left[f_{\sigma(i)}, \mu_1\left(f_{\sigma(1)}, \ldots, f_{\sigma(i-1)}, f_{\sigma(i+1)}, \ldots, f_{\sigma(f_{F_1})}\right)\right] \otimes \mu_2\left(f_{\sigma(f_{F_1+1})}, \ldots, f_{\sigma(f_{F_1+F_2})}\right) = 0.$$

This follows from the corresponding Jacobi identity for the F_1-Lie algebra S_1. Now proceeding along the same lines for the other terms, a similar factorisation works. Thus the fourth Jacobi identity is satisfied and S is an $(F_1 + F_2)-$Lie algebra. ■

Here there are some families of examples:

1. $S_1 = \mathfrak{g} \oplus \mathrm{Ad}(\mathfrak{g})$(a 1$-$Lie algebra); $S_2 = \mathbb{C} \oplus \mathrm{Ad}(\mathfrak{g})$ (a Lie superalgebra if $\mathfrak{g}$ admits an equivariant quadratic form).

2. If $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra (basic of type I or II or $Q(n)$ [27]) we associate to $\mathfrak{g}$ an "augmented" Lie superalgebra as follows:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \longrightarrow \left\{\begin{array}{lll} S = \mathcal{B} \oplus \mathcal{F} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 & & \text{if } \mathfrak{g} \text{ is of type I},\\ S = \mathcal{B} \oplus \mathcal{F} = \mathfrak{g}_0 \oplus (\mathfrak{g}_1 \oplus \mathfrak{g}_1) & & \text{if } \mathfrak{g} \text{ is of type II},\\ S = \mathcal{B} \oplus \mathcal{F} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 & & \text{if } \mathfrak{g} = Q(n).\end{array}\right. \tag{6}$$

The non-zero graded part of these "augmented" Lie superalgebras always admits a $\mathfrak{g}_0$ invariant quadratic form and hence $S_2 = \mathbb{C} \oplus \mathcal{F}$ is a Lie superalgebra:

For the type I superalgebras we have $\mathfrak{g}_1 = \mathfrak{D} \oplus \mathfrak{D}^*$ (see [27]), and so one has a natural map: $\mathcal{S}^2\left(\mathfrak{D} \oplus \mathfrak{D}^*\right) \longrightarrow \mathbb{C}$.

For the type II superalgebras, we recall that $\mathfrak{g}_1$ admits an invariant antisymmetric bilinear form and hence $\mathfrak{g}_1 = \mathcal{D}$ is self-dual [27]. Therefore, there is an invariant quadratic form on $\mathcal{F} = \mathfrak{g}_1 \oplus \mathfrak{g}_1$.

For the strange superalgebra $Q(n)$, $\mathfrak{g}_0 = \mathfrak{sl}(n + 1)$, the representation $\mathfrak{g}_1$ is the adjoint representation of $\mathfrak{g}_0$ (see [27]) and hence admits an invariant quadratic form (the Killing form).

The existence of an invariant bilinear form on $\mathfrak{g}_1$ (*i.e.* before the "augmentation" (6)) means that there is a $\mathfrak{g}_0-$equivariant mapping ${\mathcal{S}_\pm}^2(\mathfrak{g}_1) \longrightarrow \mathbb{C}$ (where ${\mathcal{S}_+}^2(\mathfrak{g}_1)$ (resp. ${\mathcal{S}_-}^2(\mathfrak{g}_1)$) represent the two-fold symmetric (antisymmetric) tensor product of $\mathfrak{g}_1$). We denote generically this tensor by $\delta_{\alpha\beta}$ when it is symmetric and $\Omega_{\alpha\beta}$ when it is antisymmetric. This can equivalently be rewritten in a basis of $\mathfrak{g}_1, F_\alpha \in \mathfrak{g}_1$

$$\begin{aligned} \{F_\alpha, F_\beta\} &= \delta_{\alpha\beta}, && \text{for the type I superalgebras and for } Q(n), \\ [F_\alpha, F_\beta] &= \Omega_{\alpha\beta}, && \text{for the type II superalgebras.} \end{aligned} \tag{7}$$

with $\{\ ,\ \}$ (resp. $[\ ,\ ]$) the symmetric (resp. antisymmetric) bilinear forms.

However, after the "augmentation "(6) in the case of Lie superalgebra of type II, the mapping $\mathcal{S}^2(\mathcal{F}) \longrightarrow \mathbb{C}$ (*i.e.* the quadratic form on $\mathcal{F}$) reads

$$\{F_{i\alpha}, F_{j\beta}\} = \varepsilon_{ij}\Omega_{\alpha\beta}, \tag{8}$$

with $F_{i\alpha} \in \mathfrak{g}_1 \oplus \mathfrak{g}_1$. The index α represents the $\mathfrak{g}_1$ degrees of freedom, the index i ($i = -1, 1$) the two copies of $\mathfrak{g}_1$ and ε_{ij} the two dimensional antisymmetric tensor.

To conclude, we will give an example of a 3−Lie (resp. 4−Lie) algebra, associated to a 1−Lie algebra (resp. superalgebra).

Example 1. Let $\mathfrak{g}_0$ be a Lie algebra and $\mathfrak{g}_1$ the adjoint representation of $\mathfrak{g}_0$ and $S_3 = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. We introduce J_a, A_a, $a = 1, \ldots, \dim(\mathfrak{g}_0)$ a basis of S_3. We denote $\mathrm{tr}(A_a A_b) = g_{ab}$ the Killing form. The trilinear bracket of the 3−Lie algebra S_3, associated to the Lie algebra $\mathfrak{g}$, is:

$$\{A_a, A_b, A_c\} = g_{ab}J_c + g_{ac}J_b + g_{bc}J_a. \tag{9}$$

If $\mathfrak{g} = \mathfrak{sl}(2)$, this is the 3−Lie algebra constructed in [28].

Example 2. As a second example we give the formulae for the quadrilinear bracket of the 4−Lie algebra constructed from the orthosymplectic superalgebra. Starting from $\mathfrak{osp}(m|2n) = (\mathfrak{so}(m) \oplus \mathfrak{sp}(2n)) \oplus (\mathbf{m}, \mathbf{2n})$, we define $\mathfrak{osp}(m|2n; 4) = (\mathfrak{so}(m) \oplus \mathfrak{sp}(2n) \oplus \mathfrak{u}(1)) \oplus ((\mathbf{m}, \mathbf{2n})^+ \oplus (\mathbf{m}, \mathbf{2n})^-)$.

Let $F_{qi\alpha}$ ($q = -1, +1$, $1 \le i \le m$, $1 \le \alpha \le 2n$) denote the odd part, J_{ij} the $\mathfrak{so}(m)$ generators, $S_{\alpha\beta}$ the $\mathfrak{sp}(2n)$ generators and h the $\mathfrak{u}(1)$ generator (J_{ij} are antisymmetric and $S_{\alpha\beta}$ are symmetric). The invariant tensor on $\mathfrak{so}(m)$ is given by the symmetric tensor δ_{ij} and on $\mathfrak{sp}(2n)$ by the antisymmetric tensor $\Omega_{\alpha\beta}$, hence the invariant tensor for $\mathfrak{osp}(m|2n)$ is given by $\delta_{ij}\Omega_{\alpha\beta}$. Thus, the quadrilinear bracket of the 4−algebra takes the form

$$\begin{aligned} &\{F_{q_1i_1\alpha_1}, F_{q_2i_2\alpha_2}, F_{q_3i_3\alpha_3}, F_{q_4i_4\alpha_4}\} \\ &\quad = \varepsilon_{q_1q_2}\delta_{i_1i_2}\Omega_{\alpha_1\alpha_2}\left(\delta_{q_3+q_4}\delta_{i_3+i_4}S_{\alpha_3\alpha_4} + \delta_{q_3+q_4}\Omega_{\alpha_3\alpha_4}J_{i_3i_4} + a\delta_{q_3+q_4}\varepsilon_{i_3i_4}\Omega_{\alpha_3\alpha_4}h\right) \\ &\quad + \varepsilon_{q_1q_3}\delta_{i_1i_3}\Omega_{\alpha_1\alpha_3}\left(\delta_{q_2+q_4}\delta_{i_2+i_4}S_{\alpha_2\alpha_4} + \delta_{q_2+q_4}\Omega_{\alpha_2\alpha_4}J_{i_2i_4} + a\delta_{q_2+q_4}\varepsilon_{i_2i_4}\Omega_{\alpha_2\alpha_4}h\right) \\ &\quad + \varepsilon_{q_1q_4}\delta_{i_1i_4}\Omega_{\alpha_1\alpha_4}\left(\delta_{q_2+q_3}\delta_{i_2+i_3}S_{\alpha_2\alpha_3} + \delta_{q_2+q_3}\Omega_{\alpha_2\alpha_3}J_{i_2i_3} + a\delta_{q_2+q_3}\varepsilon_{i_2i_3}\Omega_{\alpha_2\alpha_3}h\right) \\ &\quad + \varepsilon_{q_2q_3}\delta_{i_2i_3}\Omega_{\alpha_2\alpha_3}\left(\delta_{q_1+q_4}\delta_{i_1+i_4}S_{\alpha_1\alpha_4} + \delta_{q_1+q_4}\Omega_{\alpha_1\alpha_4}J_{i_1i_4} + a\delta_{q_1+q_4}\varepsilon_{i_1i_4}\Omega_{\alpha_1\alpha_4}h\right) \\ &\quad + \varepsilon_{q_2q_4}\delta_{i_2i_4}\Omega_{\alpha_2\alpha_4}\left(\delta_{q_1+q_3}\delta_{i_1+i_3}S_{\alpha_1\alpha_3} + \delta_{q_1+q_3}\Omega_{\alpha_1\alpha_3}J_{i_1i_3} + a\delta_{q_1+q_3}\varepsilon_{i_1i_3}\Omega_{\alpha_1\alpha_3}h\right) \\ &\quad + \varepsilon_{q_3q_4}\delta_{i_3i_4}\Omega_{\alpha_3\alpha_4}\left(\delta_{q_1+q_2}\delta_{i_1+i_2}S_{\alpha_1\alpha_2} + \delta_{q_1+q_2}\Omega_{\alpha_1\alpha_2}J_{i_1i_2} + a\delta_{q_1+q_2}\varepsilon_{i_1i_2}\Omega_{\alpha_1\alpha_2}h\right), \end{aligned} \tag{10}$$

with $a \in \mathbb{C}$.

Remark 1. It should be noticed that F−Lie algebras associated to Lie algebras (resp. to Lie superalgebras) are of odd (resp. even) order.

5 Conclusion

In this paper a sketch of the construction of F−Lie algebras associated to Lie (super)algebras was given. More complete results, such as a criteria for simplicity, representation theory, matrix realisations *etc.*, will be given elswhere.

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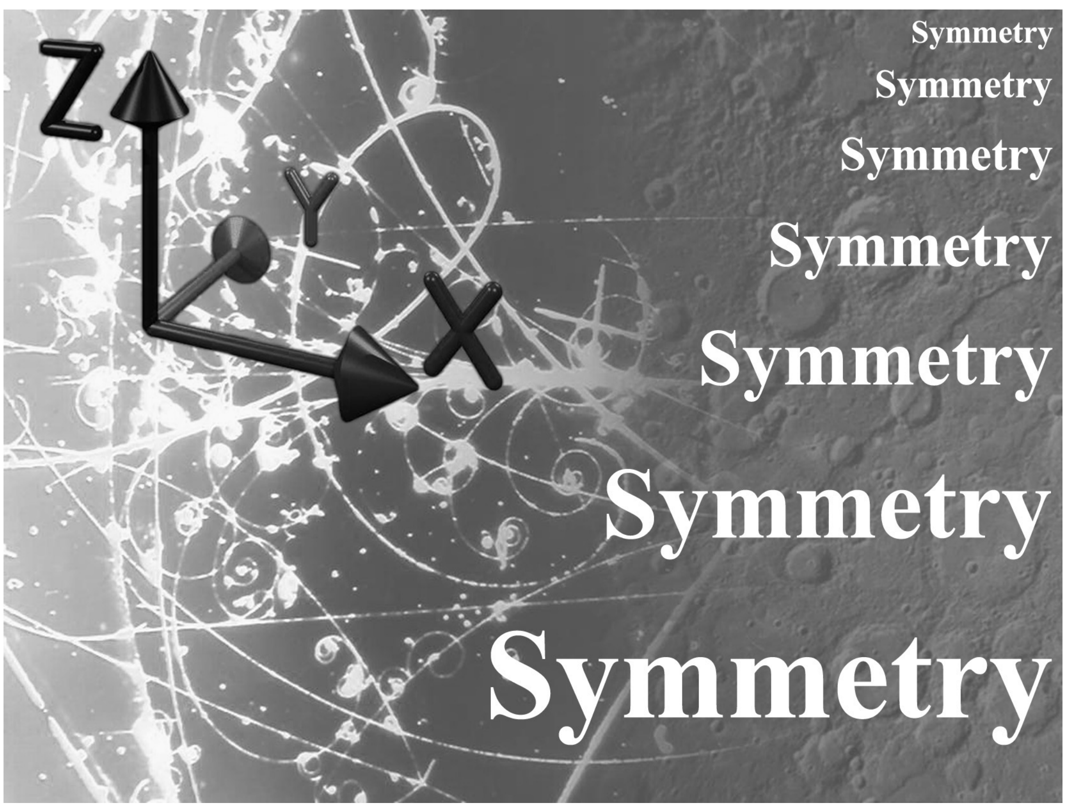
Symmetry in Physics



Symmetry in Nonlinear Mathematical Physics

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General Relativity as a Symmetry of a Unified Space–Time–Action Geometrical Space

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We derive the basic principles of Electromagnetism and general relativity from a common (geometrical) starting formulation we call START, from its geometrical structure as a Space–Time–Action Relativity Theory. Gravitation results from the epistemological approach of defining a test particle which explores the physical world in such a form that its trajectory indicates the influence of the rest of the system. Electromagnetism defines a collection of test particles, we call carriers, in interaction among themselves and with the rest of the system. General Relativity is then derived from a symmetry transformation of the quadratic space geometry corresponding to space–time and action and the philosophical principles of Einstein's general relativity theory.

1 Introduction

We present a deductive approach to General Relativity (GR) Theory, deriving it from the quadratic space geometry corresponding to the, in our approach fundamental concepts, of space, of time, and of action and from the philosophical principles of Einstein's general relativity theory. Our basic and more fundamental idea is that the physical world can be described as a distribution of action density in space–time. The properties of matter fields and their interaction are represented by the mathematical properties of this distribution. Action is considered as a fundamental variable, not as a quantity resulting from some calculation.

In [6, 7, 12] we analyzed a classical theory of fields in (complex) space–time geometry and arrived to the conclusion that this geometry corresponds to a unified space–time–action geometry. We started from three basic assumptions: a) The use of space–time as a basic frame of reference; b) The introduction of physical phenomena as an action density function over space–time; and c) The geometrical (vectorial) union of space, time, and action in a quadratic space where a relativistic condition $(dS)^2 = 0$ defines both kinematics and dynamics. The basic principles of this Space–Time–Action Relativity Theory (START) are used to derive General Relativity.

In the construction of General Relativity the procedure is to perform a symmetry transformation which modifies the space–time components, and then the metric, by the allocation of the appropriate amount of action corresponding to the additional action attributed to a test particle as a result of the interaction with the rest of the world. This transformation is made at the level of the quadratic form. This kind of transformations which transfer action into equivalent space–time to modify the metric tensor will be interpreted by the observer as those changes in the metric corresponding to its clocks running slower and the space intervals becoming larger, as is customary in the analysis of general relativity.

1.1 Carriers, Action, Space and Time

Action, as a fundamental variable, is distributed among a set of **carrier (of action) fields**. An action density $a(\boldsymbol{x}, t)$ is the fundamental concept defining all three space (parametrized by $\boldsymbol{x}$), time (parametrized by t) and action density (parametrized by an scalar analytical function $a(\boldsymbol{x}, t)$) as **primitive** concepts from which all other physical quantities will be derived or at least related directly or indirectly. The density of a carrier field can be defined through a set of scalar constants such that the integral of the product of these constants and the density gives the experimentally attributed value of a property for that carrier. A carrier field identified will have a density $\rho(\boldsymbol{x}, t)$ and if the property is Q we will obtain the definition $Q = \int q(\boldsymbol{x}, t) d\boldsymbol{x} = \int Q\rho(\boldsymbol{x}, t) d\boldsymbol{x}$ for all t, which defines that Q is a constant property (in space and time) for that field. The set of properties $\{Q\}$ characterizes a carrier field and in turn establishes the conditions for a density field to correspond to an acceptable carrier.

We already stated that in our theory space and time are fundamental, primordial, concepts. The geometrical unification of these concepts into a space–time coordinate $\boldsymbol{x} = (\boldsymbol{x}, ct)$ and an interval ds^2 requires the introduction of a universal constant: the speed of light c. As we will also use action as a fundamental concept we need another universal constant $\kappa = d_0/h$ which we will construct from a fundamental distance d_0 and a fundamental unit of action we will choose to be Planck's action constant h. In this form we will have a five dimensional, START, manifold $3 + 1 + 1$ with all dimensions in units of length. To agree with standard formulations energy $E = \partial a/\partial t$ and momentum $p_i = \partial a/\partial x^i$ are the fundamental rates of change of the primitive concept of action. It is also appropriate to say that the concept of **matter**, hitherto formally undefined, will acquire proper formal definition in the context of the different structural theories. Then the START theory presented here corresponds to a geometrization beyond Minkowski's fundamental paper. In fact that author, introducing the space-time interval squared ds^2, adds: "the axiom signifies that at any world-point the expression $c^2dt^2 - dx^2 - dy^2 - dz^2$ always has a positive value, or, what comes to the same thing, that any velocity v always proves less than c". In our full geometrization scheme action change $dK = P \cdot dX$ is introduced through a series $|dK|^2$ of quadratic terms

$$dS^2 - ds^2 = -|dK|^2 = -\left\{\left(E^2/c^2\right) c^2dt^2 - p_x^2dt^2 - p_y^2dt^2 - p_z^2dt^2\right\}, \tag{1}$$

joined in one unified geometrical quadratic form dS^2. The dK vector, the directional in space-time change of action, is a new theoretical quantity formally defined by (1). It acquires additional relevance because action density will be described as a sum of contributions over carriers, $a = \sum\limits_c a_c$ .We are constructing a systematic deductive approach to Physics and it is essential that we derive many of the basic useful structures which have been used.

For a given observer the carrier field c is defined to have an energy density $\frac{1}{N_c}\mathcal{E}_c\rho_c^{(x,t)}$ with $\mathcal{E}_c$ a constant in space and N_c the integer number of carrier units of type c. The density $\rho_c(x, t)$ is required to obey $\int_V \rho_c(x, t)dx = N_c$ in the system volume V.

Action is in our approach a coordinate (expressed in units of distance) and one of the properties of a distribution describing, in relation to an observer, the contents of the physical world in space–time. The concept of Physical Phenomena refers to the existence and change of this distribution. Physics corresponds to the description of the action distribution and its changes in relation to a given observer.

The action density function in space–time $a(X)$, or energy density in space for a given observer, can be considered as a density of action trajectories in space–time. For elementary carriers the trajectories would be elementary trajectories. Both the density function $a(X)$ and the splitting among carrier fields will be considered analytically well behaved functions.

The energy is $\mathcal{E} = \sum\limits_c \mathcal{E}_c$ a sum of constants for a given observer, assumed to be distributed among the different carriers $\{c\}$ and can furthermore be described as a sum of contributions per carrier. The simplest, almost universal, type of distribution per carrier type is into the constitution energy $\mathcal{E}_0$, the position dependent kinetic energy $\mathcal{E}_k(X)$, and the position dependent sum of potential energies $\mathcal{E}_v(X)$

$$\mathcal{E}_c = \mathcal{E}_0^c + \mathcal{E}_k^c(X) + \mathcal{E}_v^c(X) + \mathcal{E}_\Delta^c(X). \tag{2}$$

It is precisely this distribution (2) which defines the carrier for a given observer. $\mathcal{E}_0^c$ defines the basic carrier, $\mathcal{E}_k^c(X)$ the state of motion relative to the observer, and $\mathcal{E}_v^c(X)$ the relation between that carrier and the rest of the system as defined by the observer. The $\mathcal{E}_\Delta^c(X)$ term is required to make $\mathcal{E}_c$ a position independent constant, this is needed to have a meaningful definition of the carriers of type c. The action is considered to be distributed among interacting carriers, and the concept of charges of the carriers has been introduced. Consider the simplest possibility that this action does not depend on the direction, and that at a given distance from the source, in concentric spheres, the total force field per charge should be independent of the distance from the charge then

$$F_Q = F(r)4\pi r^2 = \frac{Q}{\epsilon_0 4\pi r^2}4\pi r^2 = Q/\epsilon_0 \tag{3}$$

which expresses that a definite capability is attributed to a charge Q. The factor $1/\epsilon_0$ represents any additional condition that the observer has to include to match its definitions.

We use in the analysis a tetrad of, observer dependent, basis vectors $\{e_0, e_1, e_2, e_3\}$, with $e_0^2 = -e_1^2 = -e_{2_3}^2 = -e_3^2 = 1$ and the definition property $e_\mu e_\nu = -e_\nu e_\mu$. We also use the notation $e_{0j} = e_0 e_j = \boldsymbol{e}_j$ $(i,j,k = 1,2,3)$ and $e_5 = e_0e_1e_2e_3 = e_{0123}$. For a given observer a space–time d'Alembertian operator $\Box$ has the property $e_0\Box = \frac{1}{c}\partial_t + \nabla = \frac{1}{c}\partial_t + \boldsymbol{e}_i\partial_i$, with ∇ the (ordinary space) Laplacian operator for that observer.

a) In space–time–action the action density $a(x,t)$ is inhomogeneously distributed, corresponding to the different material objects to which this action corresponds, in a possible relative motion in the spatial directions with speeds $0 \le v \le c$. Distributions which move with relative velocities $0 \le v < c$ with respect to a given observer are called matter-like.

b) The matter-like energy distributions are to be considered as sources of (infinite extension, in principle) decaying deformations of action distribution of several types. This second property is not given *a priori* but it is a consequence of the description of the objects, as developed in the previous section, shown below.

c) We introduce now a third fundamental concept: energy–momentum carriers, the definition of identical carriers as a density in a space volume V_s such that at time $t = t'$

$$\int_{V_s} \rho_b dx = n_b, \qquad h\int_{V_s} \partial_t a_b dx = \int_{V_s} \rho_b \varepsilon_b dx = n_b\varepsilon_b = E_b, \tag{4}$$

and $E_{t'} = \left[\sum_b E_b\right]_{t'}$ for a collection $\{b\}$ of (by construction) independent types of carriers. In agreement with our freedom of description we could also allow the n_b not to be integers, provided $E_{t'}$ is not changed. Here

$$\boldsymbol{\varepsilon} = \boldsymbol{m}_c c^2 + \mathbf{kin}\left[\rho_0^{(N)}(\boldsymbol{x})\right] + \boldsymbol{V}(\boldsymbol{x}) + \boldsymbol{W}_{\text{int},xc}\left[\rho_0^{(N)}(\boldsymbol{x})\right] + \boldsymbol{\varepsilon}_0\left[\rho_0^{(N)}(\boldsymbol{x})\right], \tag{5}$$

the constitutional energy of the carriers $\boldsymbol{m}_c c^2$, actual local kinetic energy per carrier $\mathbf{kin}\left[\rho_0^{(N)}(\boldsymbol{x})\right]$, external potential energy per carrier (in its simplest form) $\boldsymbol{V}(\boldsymbol{x})$. Second $\boldsymbol{W}_{\text{int},xc}\left[\rho_0^{(N)}(\boldsymbol{x})\right]$ to define independent carriers and finally a local energy term $\boldsymbol{\varepsilon}_0\left[\rho_0^{(N)}(\boldsymbol{x})\right]$ to compensate for any difference in the sum of the previous ones with respect to $\boldsymbol{\varepsilon}$. The last two terms define in practice an actual carrier in the system (a pseudo-carrier in condensed matter physics language) as different from an isolated carrier.

2 Space–Time–Action Relativity Theory

Our basic and more fundamental idea [6, 7, 8, 10, 11, 12] is that the physical world can be described as a distribution of action density in space–time. The properties of matter fields and their interaction are represented by the mathematical properties of this distribution. Action is considered as a fundamental variable, not as a quantity resulting from some calculation.

We use the traditional indices 0, 1, 2, 3 for the joint time and space coordinates x_μ, also, the vectors e_μ in the geometry of space–time. They generate $G_{\rm ST}$ a 16 dimensional space–time geometry of multivectors. A special property of the pseudo-scalar (and also volume or inverse volume) in space–time $e_5 = e_0e_1e_2e_3$, and of the linearly independent combination $e_4 \Rightarrow \boldsymbol{ij}e_5$, is that $e_5e_\mu = -e_\mu e_5$ (from $e_\mu e_\nu = -e_\nu e_\mu$, $\mu \neq \nu$). Its use allows the construction of a geometrical framework for the description of physical processes: a unified space–time–action geometry $G_{\rm STA} = G_{\rm ST} \otimes C$, mathematically a vector space with a quadratic form. The auxiliary element $\boldsymbol{j}$ commutes with all e_μ : $e_\mu \boldsymbol{j} = \boldsymbol{j} e_\mu$. In the $G_{\rm STA}$ geometry the coordinates are real numbers.

2.0.1 Formal presentation

The ideas developed in START (Space–Time–Action Relativity Theory) are derived from the systematic use of the following principles and postulates [10, 11, 12].

First Principle: *Principle of Relativity.* Constancy of the **value of the observed velocity of light** in vacuum. Independently of the state of movement of the source or of the observer (Poincaré–Einstein Relativity {Poincaré 1904, Einstein 1905 [1]}). The *Principle of Relativity* in full also requires that the laws of Physics should have the same form for all observers.

First Postulate. There is a geometry, corresponding to space–time, where the First Principle is satisfied (Minkowski space–time with local pseudo-Euclidean structure). Here it is clear that the velocity of light is a fundamental geometrical parameter and the First Principle could be rephrased to assign a unit value to it.

Second Principle: *Principle of Existence.* Constancy of the **action** corresponding to a physical system and in particular to an elementary physical phenomenon. Independently of the state of movement of the phenomenon or of the observer. Each observer considers the physical entity as an amount of action A contained in a given space-time volume $V_{\rm ST}$, A is a relativistic invariant in the sense of Minkowski's discussion.

Second Postulate. There is a geometry corresponding to the union of space–time and action where the First and Second Principles are satisfied (pseudo-Euclidean structure).

Main Theorem KT: *Complex Structure Theorem.* The geometry where the Second Postulate is satisfied is a five-dimensional basis geometry, mathematically corresponding to a particular complexification of space–time. The relation between a 5-dimensional geometry and the complexification of the basis set has been briefly presented in the introduction and will be discussed below.

Third Principle: *Principle of Quantization.* The exchanges in action always occur as integer multiples of h. (This has to be a constitutive part of the units and practical use of **KT** theorem). This makes Planck's principle a universal principle which requires the definition of the entities we have called **action carriers**, because if there are not differentiated action carriers there is not a proper definition of the exchanges of action. This is also a guide and a limitation in the definition of the action carriers and of the equivalent length associated to the time interval in which the system with total action A is defined.

Fourth Principle: *Principle of Choice.* The distribution of action in space–time corresponding to a physical system is unique and any description of this distribution should be equally acceptable. The acceptability of a description, in relation to the Third Principle, is interpreted here as implying an optimization of the action distribution among the available number of START cells of action and a mechanism to allow the system of carriers to arrive to this optimized state.

Third Postulate. The equivalent acceptable descriptions, for a physical system of carriers, are related by isometries and gauge transformations in the space–time–action geometric space corresponding to the Second Postulate.

Proof of KT. We have the kinematical concept of trajectory ($\mu, \nu = 0, 1, 2, 3$) with a quadratic form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{6}$$

generated by the dx^μ and we want to include as a fifth coordinate the dynamical concept of action and its distribution at each space–time point $X = x^\mu e_\mu$, use the real scalar quantity

$$dA(X) = p_\mu(X) dx^\mu \qquad \text{which defines} \qquad p_\mu(X) = \partial A(X)/\partial x^\mu, \tag{7}$$

here $p_\mu(X)$ is a distribution itself, write $p_\mu(X) = \tan \boldsymbol{\Theta}(X, \mu)$ and join formally, defining the (hyper)complex numbers $\boldsymbol{j}^2 = -\mathbf{1}$ and $i^2 = -1$, into

$$dS^\mu = dx^\mu (\mathbf{1} + \boldsymbol{j} \kappa_0 i \tan \boldsymbol{\Theta}(X, \mu)), \tag{8}$$

to obtain from the **real** quadratic form (in units of distance square)

$$dS^2 = g_{\mu\nu} dx^\mu dx^\nu \left(\mathbf{1} - \kappa_0^2 \tan \boldsymbol{\Theta}(X, \mu) \tan \boldsymbol{\Theta}(X, \nu)\right), \tag{9}$$

or, in five-dimensional-like formulation

$$dS^2 = g_{uv} dx^u dx^v = ds^2 - \kappa_0^2 \left|dK(X)\right|^2, \qquad u, v = 0, 1, 2, 3, 4, \tag{10}$$

where $\kappa_0^2 \left|dK(X)\right|^2$ corresponds to the sum of the squares of action contributions. Both quantities i and $\boldsymbol{j}$ are necessary to explicitly show the complex structure and give simultaneously the desired metric. This has introduced a new $4 - D$ vector function (remark: $\boldsymbol{e}_\mu$ and $i\boldsymbol{e}_\mu$ are linearly independent vectors)

$$dK(X) = dK(X)^\mu i\boldsymbol{e}_\mu = \sum_\mu \tan \boldsymbol{\Theta}(X, \mu) dx^\mu i\boldsymbol{e}_\mu,$$

$$dK^\mu = \left(\frac{\partial A}{\partial x^\mu}\right) dx^\mu = \tan \boldsymbol{\Theta}(X, \mu) dx^\mu.$$

It is important to notice that it is not the actual value of the action density which is dynamically important but its space-time variations. Even more important for dynamics is that, when the action density is considered a sum $a = \sum_c a_c$ over carriers, the contributions to $dK = \sum_c (dK)_c$ per carrier could be non-zero even is the sum could itself be null. That is the dynamics could be purely **relative dynamics**. The basic dynamical equation is proposed to be

$$\delta \int dS^2 = 0, \tag{11}$$

in a joint minimization of trajectory and action. The universal constant $\boldsymbol{\kappa}_0$, the ratio of a fundamental distance to the fundamental unit of action, expresses the action as an equivalent distance in such a form that $(dx^4)^2 = \left|(\boldsymbol{\kappa}_0 dK)\right|^2$, with $g_{mn} = \operatorname{diag}(1, -1, -1, -1, -1)$ defines the metric of the equivalent five dimensional geometry basis vectors. ■

For the units to be used in the unified geometry consider the definition (m_0 electron rest mass, c speed of light, $h = 2\pi\hbar$ Planck's constant, e electron charge, r_0 classical electron radius and $\alpha = e^2/\hbar c$)

$$r_{\text{Compton}} = \frac{\hbar}{2m_0 c} = \frac{r_0}{2\alpha}, \qquad \boldsymbol{\kappa}_0 = \frac{d_0}{h} = 4\pi r_{\text{Compton}}/h = \frac{1}{m_0 c}. \tag{12}$$

The classical limit of the unification of action to space–time is obtained when $\kappa_0 \to \infty$ in a form similar to the classical limit of the unification of space and time being obtained when $c \to \infty$. Note $\kappa_0 \gg c$.

From our definitions we are considering two quantities: energy $\int dV\, \partial_t\, a(\boldsymbol{X})$ and the corresponding momenta $\int dV\, \partial_{x_i}\, a(\boldsymbol{X})$. One of the basic relations in relativistic dynamics is the transformation of the above quantities with respect to observers in relative motion with a relative velocity v_{12}.

For observer 1 $\mathcal{E} = mc^2 = \int dV\, \partial_t\, a(\boldsymbol{X})$, if by definition for this observer that object is at rest and then the energy corresponds to a mass m and no momenta are involved.

For observer 2 the same relations hold. The energy for this observer will be $\mathcal{E}'{=}m'c^2 = \int dV'\partial_{t'}\, a'(\boldsymbol{X}')$, larger than $\mathcal{E}$

$$\mathcal{E}' = \frac{\partial}{\partial t'} A = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \to \quad \text{with } \lim(v \ll c) \quad \to mc^2 + \frac{1}{2} mv^2 + \cdots\,, \tag{13}$$

and he can call the energy $\mathcal{E}'$ the sum of the rest (mass) energy $\mathcal{E}$ and the kinetic energy $\mathcal{E}_k$.

2.1 Dynamical principles as symmetries

In space–time–action geometry the main dynamical principle should be that all elementary trajectories be minimal. That is, from our definition of carriers above where $dA = \{\epsilon_c \int \rho_c(X) dx\} dt$, a minimization of a total action A (in the case when we admit that the $\kappa_0 \gg 1$ predominates) or a minimization of a START trajectory. Defining the (square of the) differential $(dS)^2 = (ds)^2 - (da)^2$, where $(ds)^2 = g_{\mu\nu} dx^\mu dx^\nu$ is the space–time differential and $(da)^2$ the action differential, the minimal principle

$$\delta (dS)^2 = 0, \tag{14}$$

could be separated into a kinematical principle similar to the one of (general) relativity and an additional principle of minimum action

$$\delta (ds')^2 = 0, \qquad \delta (da')^2 = 0, \tag{15}$$

$$(ds')^2 = (ds)^2 - \left[(da)^2 - (da')^2\right], \tag{16}$$

as a modified space–time interval square which, in fact, corresponds to considering a curved effective space–time as will be shown below. The action interval square $(da')^2$ corresponds to some 'inactive' part of the action in relation to a given geometrical description.

3 Maxwell and Newton equations from START

Let us formally show that the Maxwell equations in their standard textbook form are analytical properties of the third derivatives of the action density attributed to a test carrier (with 'electric' charge).

In the reference frame of a given observer the induced action density, denoted by $a_e(X)$, per unit charge ($\Rightarrow$ puch) of a test carrier at space–time point $X = x^\mu e_\mu$ (here the greek indices $\mu = 0, 1, 2, 3$ and $x^0 = ct$ whereas the space vectors $\boldsymbol{q} = q^i e_i = q_i e^i$, $i = 1, 2, 3$ are written in bold face letters), the related energy density $\mathfrak{E}_e(X)$ and the total (external plus induced) momentum density $\boldsymbol{p}_e$, **per unit charge of the test carrier**, would be

$$\mathfrak{E}_e(X) = \frac{\partial a_e(X)}{\partial t}, \qquad \boldsymbol{p}_e = p_{e,i} e^i = \left(\frac{\partial a_e(X)}{\partial x^i} + \Delta_R p_{e,i} \right) e^i, \tag{17}$$

and the, by definition, electric field intensity $\boldsymbol{E}$ is the force (puch) corresponding to this terms

$$\boldsymbol{E} = \left(\frac{\partial \mathfrak{E}_e(X)}{\partial x^i} + \frac{\partial p_{e,i}}{\partial t} \right) e^i = \nabla \mathfrak{E}_e(X) + \frac{\partial \boldsymbol{p}_e}{\partial t},$$

with time dependence

$$\frac{\partial \boldsymbol{E}}{\partial t} = \left(\frac{\partial^2 \mathfrak{E}_e(X)}{\partial t \partial x^i} + \frac{\partial^2 p_{e,i}}{\partial t \partial t} \right) e^i = 2\frac{\partial^3 a_e(X)}{\partial t \partial x^i \partial t} e^i + \frac{\partial^2 \left(\Delta_R p_{e,i}\right)}{(\partial t)^2} e^i.$$

By definition of interacting carriers, we have added in (17) the term $\Delta_R p_{e,i} e^i$ as the effect of the conservation of **interaction transverse momenta** between the field representing the rest of the carriers with that sort of charges. This is by definition the origin, in START, of a magnetic field intensity $\boldsymbol{B} = B_k e^k$ that will appear as the curl of the momentum (puch) of an interaction field acting on a carrier of type b. The axial vector

$$\boldsymbol{B} = \left(\frac{\partial p_{e,i}(X)}{\partial x^j} \right) e^j \times e^i = \nabla \times \boldsymbol{p}_e, \qquad \frac{\partial \boldsymbol{B}}{\partial t} = \frac{\partial^2 p_{e,i}(X)}{\partial t \partial x^j} e^j \times e^i.$$

Otherwise the space variation of $\boldsymbol{E}$, including the **interaction transverse momenta**,

$$\nabla \boldsymbol{E} = \nabla \cdot \boldsymbol{E} + \nabla \times \boldsymbol{E}, \tag{18}$$

will then also include a transversal (rotational) term

$$\nabla \times \boldsymbol{E} = \frac{\partial^2 p_{e,j}(X)}{\partial x^i \partial t} e^i \times e^j = -\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \text{(2nd Maxwell Equation)}$$

relation which is the direct derivation in START of this well known Maxwell equation. The scalar term $\nabla \cdot \boldsymbol{E}$ being a divergency of a vector field should be defined to be proportional to a source density

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0}\rho = \sum_i \left(\frac{\partial^3 a_e(X)}{\partial x^i \partial x^i \partial t} \right) = \frac{\partial}{\partial t} \nabla^2 a_e(X), \qquad \text{(1st Maxwell Equation)}$$

and will be given full physical meaning below. For the space variation of $\boldsymbol{B}$ we have

$$\nabla \boldsymbol{B} = \nabla \cdot \boldsymbol{B} + \nabla \times \boldsymbol{B}.$$

The first term vanishes identically in our theory because it corresponds to the divergence of the curl of a vector field

$$\nabla \cdot \boldsymbol{B} = 0, \qquad \text{(3rd Maxwell Equation)}$$

while the last term, using $U \times V \times W = V(U \cdot W) - (U \cdot V)W$

$$\nabla \times \boldsymbol{B} = \nabla \left(\nabla^2 a_e(X) \right) - \nabla^2 \boldsymbol{p}_e = \mu_0 \left(\boldsymbol{J} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right), \qquad \text{(4th Maxwell Equation)}$$

where the additional dimensional constant μ_0 is needed to transform from time units (used in the conceptual definition of a current $\boldsymbol{J} = \nabla \left(\nabla^2 a_e(X) \right) / \mu_0$) into distance units and in fact $\epsilon_0 \mu_0 = c^{-2}$.

The derived Maxwell equations are formally equivalent to the original Maxwell equations.

Both the 4th Maxwell Equation, defining $\boldsymbol{J}$, related to a Lorentz transformation of the 1st Maxwell Equation, defining ρ, can immediately be integrated using geometric analysis techniques, the standard approach being of fundamental conceptual consequences in START. The space divergence of a non-solenoidal vector field like $\boldsymbol{E}$ is immediately interpreted as its 'source' given that $\Delta\boldsymbol{E} = (\nabla\cdot\boldsymbol{E})\,S\Delta\boldsymbol{x}$, and this equation is integrated using the standard geometric theorem that the volume integral of a divergency $\nabla\cdot\boldsymbol{E}$ equals the surface integral of the normal (to the surface) component of the corresponding vector field $\boldsymbol{n}\cdot\boldsymbol{E}$. Where $\boldsymbol{n}$ is a unit vector perpendicular to the surface S (in the text-book formula below $S = 4\pi r^2$ corresponding to an integration sphere of radius r containing a spherically symmetric source density $\rho(r)$ generating a force field per unit charge $\boldsymbol{E} = E(r)\frac{\boldsymbol{r}}{r}$) of the integration volume $V = 4\pi r^3/3$:

$$\int_V (\nabla\cdot\boldsymbol{E})\,dV = \int_V \frac{4\pi}{\epsilon_0}\rho(r')r'^2dr' = \frac{1}{\epsilon_0}Q \quad\Longrightarrow\quad \boldsymbol{E} = E(r)\frac{\boldsymbol{r}}{r} = \frac{Q}{4\pi\epsilon_0 r^2}\frac{\boldsymbol{r}}{r}.$$

In the case of the, generated by a current, magnetic force field intensity $\boldsymbol{B}$, being a space bivector, it is also a direct geometrical consequence that its source can (must) be a current vector density $\boldsymbol{J}$. For a small ($l \ll r$) current source at the origin of coordinates: (in the sphere $\boldsymbol{r}^t(\theta,\phi)\bullet\boldsymbol{r}^{ct} = 0$, $\left(\boldsymbol{r}^t\right)^2 = \left(\boldsymbol{r}^{ct}\right)^2 = 1$)

$$\int_V 4\pi\mu_0\boldsymbol{J}\delta\left(r'\right)r'^2dr' = \mu_0 M\boldsymbol{r}^{ct} = 4\pi r^2 fB(r)\boldsymbol{r}^{ct} \quad\Longrightarrow\quad \boldsymbol{B} = B(r)\boldsymbol{r}^t = \frac{\mu_0 M}{4\pi r^2 f}\boldsymbol{r}^t.$$

3.1 Newtonian gravity

The analysis above depends only on the assumption of the decomposition of the action and of the energy momentum into contributions per carrier. The solution of the first Maxwell equation, when applied to gravitation considering the mass $M = \mathcal{E}/c^2$, implies (as shown above) the Newtonian gravitational potential equation per unit test mass m:

$$\mathfrak{E} = V = -G\frac{M}{r}, \qquad \text{that is} \qquad \boldsymbol{E} = -G\frac{M}{r^2}$$

the usual relations in the textbook formulation of Newtonian gravity. The constant $G = 1/4\pi\epsilon_0^{(g)}$. If we define $c^2\mu_0^{(g)}\epsilon_0^{(g)} = 1$ then $\mu_0^{(g)} = 4\pi G/c^2$.

4 General relativity and the test particle

The Schwarzschild solution. In our present theory there are two fundamental carrier structures: the massless fields and the massive electron field with basic relation

$$\mathcal{E}^2 = \left(\mathcal{E}_0 + \Delta\mathcal{E}\right)^2, \qquad \mathcal{E}^2 - \mathcal{E}_0^2 = (pc)^2, \tag{19}$$

where $\Delta\mathcal{E}$ is any gauge-free energy contribution and $\mathcal{E}_0 = m_0c^2$ the energy, at rest relative to some observer, considered to be the mass of the carrier.

The concept of test particle in general relativity in the Schwarzschild solution is compatible with the Newtonian limit for the interaction gravitational energy

$$\Delta\mathcal{E}\left(r\right) = -m_0\frac{GM}{r}, \tag{20}$$

where M is the total mass of 'the system' of radius r_s we are exploring with the test particle and, conceptually, with the START use of the action square difference, writing $\mathcal{E} = \mathcal{E}_0 + \Delta\mathcal{E}$ for large (classical limit) values of $r > r_s$

$$\begin{aligned}\mathcal{E}^2 - \mathcal{E}_0^2 &= \mathcal{E}_0^2 + 2\mathcal{E}_0\Delta\mathcal{E} + \left(\Delta\mathcal{E}\right)^2 - \mathcal{E}_0^2 = (pc)^2 \\ &= 2\mathcal{E}_0\Delta\mathcal{E} + \left(\Delta\mathcal{E}\right)^2 \to -2m_0c^2m_0\frac{GM}{r} + \left(m_0\frac{GM}{r}\right)^2,\end{aligned} \tag{21}$$

this corresponds to the energy and radial momentum terms in $(da)^2-(da')^2$ if $(da')^2=(m_0c^2dt)^2$, and substituting in (16) using $\kappa_0=1/m_0c$ and space spherically symmetric coordinates t, r, θ, ϕ we obtain

$$(dS)^2=\left(1-2\frac{GM}{c^2r}+\left(\frac{GM}{c^2r}\right)^2\right)c^2\,(dt)^2 \\ -\left(1+\frac{2GM}{c^2r}-\left(\frac{GM}{c^2r}\right)^2\right)(dr)^2-r^2\left[(d\theta)^2+\sin^2\theta\,(d\phi)^2\right], \tag{22}$$

which is the Schwarzschild metric in the limit of $r\gg GM/c^2$.

It is customary to write [15] the interval square using in our case $f\,(r)=1+b^2\,(r)$ and $h\,(r)=1-b^2(r)$

$$f\,(r)=\left(1-2\frac{GM}{c^2r}+\left(\frac{GM}{c^2r}\right)^2\right)\qquad\text{and}\qquad h\,(r)=\left(1+\frac{2GM}{c^2r}-\left(\frac{GM}{c^2r}\right)^2\right), \tag{23}$$

for $c^2r\gg GM$ we obtain the Schwarzschild relation $f\cong h^{-1}$.

The result (22) shows that our approach provides a conceptual understanding of the role of sources carriers and test particles in general relativity. It also shows the possibility of extending the analysis to circumstances more difficult to consider within the traditional approaches.

Once we have obtained the Schwarzschild metric we can now find the **curved hypersurface in START corresponding to the curved space–time** where the test particles are assumed to move. Formally we need to define a set of vectors e_μ, $\mu=0,1,2,3$, $g_{\mu\nu}=\mathrm{diag}\,(1,-1,-1,-1)$, and their reciprocal, in terms of a vierbein using the Minkowski space reference vectors. From (22) it is clear (see [15]) how to construct an orthonormal system of vectors.

One of the possible symmetries in START is the transformation of position vectors $\boldsymbol{y}$ in START to a new set $\{\boldsymbol{y}=x^u e_u;\ u=0,1,2,3,4\}$

$$\boldsymbol{y}'=\boldsymbol{f}(\boldsymbol{y})=x'^u e'_u. \tag{24}$$

which describes the curvature of the space–time part necessary for representing physical interactions, at the expense of defining ‘test’ carriers.

4.0.1 General relativity in START

From our previous analysis, the structure equivalent to Einstein’s general relativity is the following:

- In the flat space–time–action geometry a distribution of action is given and analyzed as corresponding to the total matter and interaction fields (radiation) content.
- Basically one obtains the structure corresponding to general relativity by the process of transforming this $1+3+1$ geometrical description into an equivalent $1+3$ description given by a curved space–time.
- Even if the projection of the surface in five dimensions as a four-dimensional space corresponds to the curved space–time of general relativity, the physical meaning of this curved space–time is given by defining the trajectories of ‘test’ particles as the geodesics in this 4-D space.

The analysis we have presented here corresponds to changing the status of general relativity from a physical model to a part of a deductive theory.

4.0.2 A charged carrier as a test particle in general relativity

A charged particle at rest which is acted on by gravitational and electromagnetic interactions will have for the (attributed) total energy (at distances large enough such that the collection of masses with which the test carrier interacts are collectively represented by the volume integral of a mass density $\mathcal{M}(\boldsymbol{r})$) in the presence of the mass $M = \int_D \mathcal{M}(\boldsymbol{r})dv$, the following description:

$$\varepsilon = m_0 c^2 - m_0 \frac{GM}{r} + e\frac{Q}{r}.$$

Substituting this in (19)–(23) will change the functions $f(r)$ and $h(r)$ into

$$f(r) = 1 - 2\frac{GM}{c^2 r} + \left(\frac{GM}{c^2 r}\right)^2 - \frac{e}{m_0} Q \frac{GM}{c^4 r^2} + 2\frac{eQ}{m_0 c^2 r} + \left(\frac{eQ}{m_0 c^2 r}\right)^2,$$
$$h(r) = 1 + 2\frac{GM}{c^2 r} - \left(\frac{GM}{c^2 r}\right)^2 + \frac{e}{m_0} Q \frac{GM}{c^4 r^2} - 2\frac{eQ}{m_0 c^2 r} - \left(\frac{eQ}{m_0 c^2 r}\right)^2.$$

The analysis of these functions would lead to the following conclusions:

1. Besides the attractive gravitational term there is a (quadratic) repulsive term which will dominate at intermediate distances. Time coordinates do not become imaginary or discontinuous.
2. The electric part of the interaction depends explicitly in the e/m_0 ratio of the test particle, and it can then not be a universal behavior of a test particle.

Otherwise, when the relations corresponding to general relativity are derived from START, those entering into the experimental proofs of the validity of general relativity (considered this far) are not changed and retain their validation status.

4.1 The mathematical structure of general relativity from START

Once we have seen that an electron used as a test particle in the START geometry allows us to obtain the Schwarzschild metric we can now proceed to a systematic derivation of the structure of general relativity from START.

The main considerations are the following.

a) General relativity is a geometric theory describing the trajectories of test particles as the natural trajectories, geodesics, in curved space–time geometry.

b) The curved space–time is obtained by incorporating, within STA, equivalent distances from the action part into the ST part. That is, general relativity is a theory where the geometry describes everything that is to be described, through the curved space–time, and the test particle is only an auxiliary in this description.

c) The quadratic form obtained was afterwards analyzed using intrinsic geometrical techniques to have a purely geometrical theory. The basic equations, everywhere in space, are the transfer of the intervals corresponding to the relevant action (squared) to the flat quadratic form of space–time.

d) We can directly consider that the quadratic form defines the metric tensor of the new geometry, and then use the definition of the curvature from the metric tensor in the generated curved space–time, to obtain a relation between the curvature and the energy–momentum–stress tensor.

The metric in GR. Once we have created the equivalent curved space–time the metric in GR is given through the use of the line element (here $g_{\mu\nu} = g_{\mu\nu}^{\mathrm{GR}}$ from the choice of action allocation to geometry and $g_{\mu\nu}^{0}$ corresponds to flat space–time)

$$dS^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = g_{\mu\nu}^{0} \left(1 + \Delta g_{\mu\nu}\right) dx^{\mu} dx^{\nu}, \tag{25}$$

which in turn defines local vector frames (up to a gauge transformation)

$$e_{\mu}^{GR} = \boldsymbol{h}\left(e_{\mu}\right), \qquad \text{such that} \qquad g_{\mu\nu} = e_{\mu}^{\mathrm{GR}} \cdot e_{\nu}^{\mathrm{GR}},$$

with $\boldsymbol{h}(x)$ a vector-valued function of vectors usually represented through a vierbein h_{μ}^{ν}.

In practice the metric appears as an independent field in START which is defined according to the Principle of Choice of Acceptable Descriptions, then once it is chosen the condition of flat STA is that the total curvature vanishes. Otherwise (from the integral of the selected contributions to action)

$$\kappa_0 \frac{\delta A}{\delta g^{\mu\nu}(x)} \equiv \frac{\kappa_0}{2} \mathcal{T}_{\mu\nu}(x), \tag{26}$$

(the factor $\frac{1}{2}$ is needed for convention reasons); also, from the Ricci scalar curvature $\mathcal{R}$ which results from the chosen line elements

$$\frac{\delta \mathcal{R}}{\delta g^{\mu\nu}(x)} = \mathcal{R}_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} \mathcal{R}, \qquad \text{with} \qquad \mathcal{R}_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} \mathcal{R} + \frac{\kappa_0}{2} \mathcal{T}_{\mu\nu} = 0 \tag{27}$$

to obtain the equivalent to the GR basic equation.

4.2 Rumer (Kaluza–Klein) theory deduced from START

The Rumer form of the Kaluza–Klein–Einstein–Bergmann theory is deduced from START when besides deriving the metric tensor from the square of the line element dS, as the symmetric part of dS^2, the antisymmetric, then imaginary, elements are kept and considered in turn as as real elements of an extended metric tensor in a 5-D geometry. That is consider again the complex line element and compute again the complex square, keeping now the scalar and the bivector parts

$$(dS)^2 = dS^2 + e_{\mu\nu} dx^{\mu} dx^{\nu} \boldsymbol{ij} \kappa_0 (p(X,\mu) - p(X,\nu))$$

where the antisymmetric product of two vectors, the bivectors $e_{\mu\nu}$ are also the generators of spin angular momentum.

From the principles of General Relativity of considering the changes in energy-momentum for the test particle, consider that in the case of an electromagnetic interaction the test particle of charge e receives and additional energy momentum $p(X,\mu) = eA_{\mu}(X)$

$$\boldsymbol{ij} \kappa_0 p(X,\mu) = \boldsymbol{ij} \frac{e}{m_0 c} A_{\mu}(X)$$

using the action equivalent distance $\kappa_0 = 1/m_0 c$.

Besides the many papers which have been written about the Kaluza–Klein proposition and their extension to the idea of hyper-space with one additional dimension (at least) for each additional interaction included, the direct inclusion of action as a fifth dimension was proposed as early as the 1949–1956 by the Russian physicist Y.B. Rumer [13, 14] under the name of "Action as a spatial coordinate. I–X". In the work of Rumer the main foreseen application is to the case of optics in what he called 5-optics. We should remember that in this case the action $dA = 0$ and then the fifth coordinate turns out to be identically null.

5 Hypothesis and principles in START

The set of hypothesis and principles which are explicitly included are:

Physics is the science which describes the basic phenomena of Nature within the procedures of the Scientific Method.We consider that the mathematization of the anthropocentric primary concepts of space, time and the existence of the physical objects (action carriers), is a suitable point of departure for creating intellectual structures which describe Nature.

We introduce a set of principles: Relativity, Existence, Quantization and Choice as the operational procedure, and a set of 3 mathematical postulates to give this principles a formal, useful, structure.

In START, because of its equivalent complex structure and its quadratic forms we have, besides the geometrical space–time Poincaré group $\mathcal{P}$ of transformations leaving the finite difference $\left(dx^0\right)^2 - (d\boldsymbol{x})^2$ invariant, an additional set of transformations related to the complex structure. The additional operations are: a translation in the e^4 direction, three rotations in the e^ie^4, $i = 1, 2, 3$ planes and one 'boost' in the e^0e^4 plane.

It is clear that most of the here presented relations are known relations as far as we are **deriving** the structures and theories from START.

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Point Form Relativistic Quantum Mechanics and an Algebraic Formulation of Electron Scattering

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Electron scattering off hadronic systems is used to motivate an algebraic approach to hadronic physics. Point form relativistic quantum mechanics, in which all interactions are in the four-momentum operator, along with current operators, is shown to form an infinite dimensional algebra, the representations of which would then generate the observables in electron scattering, namely form factors and structure functions. Several examples of such algebras are given.

1 Electron scattering and point form relativistic quantum mechanics

Electron scattering provides an important tool for investigating the structure of hadrons, both at the nuclear and quark levels. The cross sections measured by experimentalists, for example in inclusive scattering, are exhibited in such a way that they indicate the degree to which an object being probed is not a point particle. The measure of an object not being pointlike is given by sums of squares of form factors. It is well known that the number of independent form factors in elastic scattering is equal to $2s + 1$, where s is the spin of the object. Associated with each of the $2s + 1$ form factors is a static property of the object, such as its charge or magnetic moment, obtained by evaluating the appropriate form factor at zero momentum transfer, $Q^2 = 0$. If every form factor for a spin s object were a constant as a function of Q^2, the object would be a genuine point particle. If form factors are not constant as functions of Q^2, the object has an internal structure, which is revealed in electron scattering experiments as deviations from pointlike cross sections.

Form factors are matrix elements of electromagnetic current operators and provide the link to electron scattering experiments. In order to compute such form factors it is necessary to know both how the objects are described in terms of their constituents, and the nature of the currents of the constituents. Two examples of form factor calculations will be given in the following two sections, one the elastic deuteron form factor in terms of proton and neutron constituents, the other nucleon form factors with three quark constituents.

The goal of this paper is to present an algebraic formulation of electron scattering, algebraic in the sense that the operators describing hadronic dynamics and currents close under commutation. The context for such a formulation is point form relativistic quantum mechanics [1], in which all of the hadronic dynamics is put into the four-momentum operator and the Lorentz generators are all kinematic.

It is then convenient to write the Poincaré commutation relations, necessary for the theory to be properly relativistic, not in terms of the ten generators, but rather in terms of the four-momentum operators that contain the interactions, and global Lorentz transformations:

$$[P_\mu,\ P_\nu] = 0, \tag{1}$$

$$U_\Lambda P_\mu U_\Lambda^{-1} = \left(\Lambda^{-1}\right)_\mu^{\ \nu} P_\nu, \tag{2}$$

where U_Λ is a unitary operator representing the Lorentz transformation Λ. These rewritten Poincaré relations will be called the point form equations, and are the fundamental equations that have to be satisfied for the system of interest. The mass operator is given by $M = \sqrt{P \cdot P}$ and must have a spectrum that is bounded from below.

The simplest example of the point form equations is given by the irreducible representations of the Poincaré group for a single particle of mass m and spin j. If $|p, \sigma\rangle$ is an eigenstate of four-momentum p (with $p \cdot p = m^2$) and spin projection σ, then

$$P_\mu |p, \sigma\rangle = p_\mu |p, \sigma\rangle, \tag{3}$$
$$U_\Lambda |p, \sigma\rangle = \sum |\Lambda p, \sigma'\rangle D^j_{\sigma',\sigma}(R_W), \tag{4}$$

with R_W a Wigner rotation defined by $R_W = B^{-1}(\Lambda v)\Lambda B(v)$, and $B(v)$ a canonical spin (rotationless) boost (see reference [2]) with argument $v = p/m$. Many-particle operators with the same transformation properties as the single particle ones are conveniently obtained by introducing creation and annihilation operators. Let $a^\dagger(p, \sigma)$ be the operator that creates the state $|p, \sigma\rangle$ from the vacuum. If $a(p, \sigma)$ is its adjoint, these operators must satisfy the following relations:

$$[a(p, \sigma), a^\dagger(p', \sigma')]_\pm = E\delta^3(p - p')\delta_{\sigma,\sigma'}, \tag{5}$$
$$U_a a^\dagger(p, \sigma) U_a^{-1} = e^{ip\cdot a} a^\dagger(p, \sigma), \tag{6}$$
$$P_\mu(\mathrm{fr}) = \sum \int \frac{d^3p}{E} p_\mu a^\dagger(p, \sigma) a(p, \sigma), \tag{7}$$
$$U_\Lambda a^\dagger(p, \sigma) U_\Lambda^{-1} = \sum a^\dagger(\Lambda p, \sigma') D^j_{\sigma',\sigma}(R_W). \tag{8}$$

Here $P_\mu(\mathrm{fr})$ is the free four-momentum operator and plays a role analogous to the free Hamiltonian in nonrelativistic quantum mechanics. Again it is straightforward to show that P_μ satisfies the point form equations. U_a in equation (6) is the unitary operator representing the four-translation a.

To prepare for the construction of interacting four-momentum operators, out of which the interacting mass operators will be built, it is convenient to introduce velocity states, states with simple Lorentz transformation properties. If a Lorentz transformation is applied to a many-particle state, $|p_1, \sigma_1, \ldots, p_n, \sigma_n\rangle = a^\dagger(p_1, \sigma_1) \cdots a^\dagger(p_n, \sigma_n)|0\rangle$, then it is not possible to couple all the momenta and spins together to form spin or orbital angular momentum states, because the Wigner rotations associated with each momentum are different. However, velocity states, defined as n-particle states in their overall rest frame boosted to a four-velocity v will have the desired Lorentz transformation properties:

$$|v, \vec{k}_i, \mu_i\rangle := U_{B(v)}|k_1, \mu_1, \ldots, k_n, \mu_n\rangle \tag{9}$$
$$= \sum |p_1, \sigma_1, \ldots, p_n, \sigma_n\rangle \prod D^{j_i}_{\sigma_i,\mu_i}(R_{W_i}), \tag{10}$$
$$U_\Lambda |v, \vec{k}_i, \mu_i\rangle = U_\Lambda U_{B(v)}|k_1, \mu_1, \ldots, k_n, \mu_n\rangle = U_{B(\Lambda v)} U_{R_W}|k_1, \mu_1, \ldots, k_n, \mu_n\rangle$$
$$= \sum |\Lambda v, R_W \vec{k}_i, \mu_i'\rangle \prod D^{j_i}_{\mu_i',\mu_i}(R_W). \tag{11}$$

Unlike the Lorentz transformation of an n-particle state, where all the Wigner rotations of the D functions are different, in equation (11) it is seen that the Wigner rotations in the D functions are all the same and given by equation (4). Moreover the same Wigner rotation also multiplies the internal momentum vectors, which means that for velocity states, spin and orbital angular momentum can be coupled together exactly as is done nonrelativistically (for more details see reference [2]). The relationship between single particle and internal momenta is given by $p_i = B(v)k_i$, $\sum \vec{k}_i = 0$ and R_{W_i} in equation (10) by replacing p by k_i and Λ by $B(v)$

in equation (4). From the definition of velocity states it then follows that

$$V_\mu|v,\vec{k}_i,\mu_i\rangle = v_\mu|v,\vec{k}_i,\mu_i\rangle, \tag{12}$$

$$M_{fr}|v,\vec{k}_i,\mu_i\rangle = m_n|v,\vec{k}_i,\mu_i\rangle, \tag{13}$$

$$P_\mu(fr)|v,\vec{k}_i,\mu_i\rangle = m_n v^\mu|v,\vec{k}_i,\mu_i\rangle, \tag{14}$$

with $m_n = \sum\sqrt{m_i^2+\vec{k}_i^{\,2}}$ the mass of the n-particle velocity state and $P_\mu(\mathrm{fr}) = M_{\mathrm{fr}}V_\mu$. On velocity states the free four-momentum operator has been written as the product of the four-velocity operator times the free mass operator, which is the so-called Bakamjian–Thomas construction [3] in the point form.

To introduce interactions, write $P_\mu = MV_\mu$, $M = M_{\mathrm{free}} + M_I$. Such a four-momentum operator will satisfy the point form equations if the velocity state kernel, $\langle v',\vec{k}_i^{\,\prime},\mu_i'|M_I|v,\vec{k}_i,\mu_i\rangle$ is independent of v and rotationally invariant (which is the same as the nonrelativistic condition on potentials). With such a four-momentum operator, the point form equations become a mass eigenvalue equation:

$$M\Psi = m\Psi, \tag{15}$$

which gives the bound and scattering wavefunctions.

Besides the mass operator, the other quantity needed to compute form factors is a current operator. Current operators must satisfy general properties such as Poincaré covariance and current conservation. In the point form the current operator at the space-time point 0 plays a special role in that it determines the Poincaré covariance and conservation properties at an arbitrary space-time point x. In fact it is easy to see that if $J_\mu(0)$ satisfies

$$U_\Lambda J_\mu(0)U_\Lambda^{-1} = (\Lambda^\nu_\mu)^{-1}J_\nu(0),$$
$$[P^\mu, J_\mu(0)] = 0,$$

then $J_\mu(x) := e^{iP\cdot x}J_\mu(0)e^{-iP\cdot x}$ is Poincaré covariant and is conserved.

Form factors are current operator matrix elements. If the states are chosen to be eigenstates of the four-momentum operator, then the covariance properties of the states and current operators make it possible to greatly simplify the structure of the form factors. As shown in reference [4] current operators are irreducible tensor operators of the Poincaré group, so that a generalized Wigner–Eckart theorem can be used to decompose current matrix elements into Clebsch–Gordan coefficients times reduced matrix elements, which are the invariant form factors. There is a natural frame in which the Clebsch–Gordan coefficients are one, namely the Breit frame, indicated by $p(\mathrm{st})$ (st=standard=Breit) below:

$$\langle p'j'\sigma'I'|J^\mu(0)|pj\sigma I\rangle = \sum\Lambda^\mu_\nu(p',p)D^{j'}_{\sigma'r'}(R_W')F^\nu_{r'r}\left(Q^2\right)D^j_{r\sigma}(R_W^{-1}), \tag{16}$$

$$\langle p'(\mathrm{st})j'r'I'|J^\mu(0)|p(\mathrm{st})jrI\rangle = F^\mu_{r'r}(Q^2), \tag{17}$$

$$p'(\mathrm{st}) = m'(\cosh\Delta,0,0,\sinh\Delta), \qquad p(\mathrm{st}) = m(\cosh\Delta,0,0,-\sinh\Delta),$$
$$Q^2 = (p'(\mathrm{st})-p(\mathrm{st}))^2 = (m'-m)^2 - 4m'm\sinh\Delta^2,$$
$$p' = \Lambda(p',p)p'(\mathrm{st}), \qquad p = \Lambda(p',p)p(\mathrm{st}).$$

$\Lambda(p',p)$ is a Lorentz transformation that carries the two standard four-momenta to arbitrary four-momenta, while the Wigner rotations in equation (16) are formed from these four-momenta with $\Lambda(p',p)$.

It can then be shown that the invariant form factors in equation (17), indexed by the spin projection labels r' and r, always give the correct number of independent form factors [4]. In fact

$F^{\mu=0}_{r'r}(Q^2)$ is a diagonal matrix giving the electric form factors, $F^{\mu=1,2}_{r'r}\left(Q^2\right)$ is an off-diagonal matrix giving the magnetic form factors, and $F^{\mu=3}_{r'r}\left(Q^2\right)=0$ is an expression of current conservation in the Breit frame. To actually compute an invariant form factor using equation (17) a choice for the current operator must be made; usually one begins with a one-body current operator, resulting in what is called the point form spectator approximation (PFSA) [5]. This means that the four-momenta of the unstruck constituents do not change, which has the consequence that the momentum transfer to the struck constituent is greater than the momentum transfer to the object as a whole. As will be seen in the next sections, this has important consequences for the behavior of the form factors as a function of the momentum transfer Q^2.

With the assumption of a one-body current operator, equation (17) can be written more explicitly as

$$\begin{aligned} F^{\mu}_{r'r}\left(Q^2\right) = \sum \int & \mathcal{J} d^3\vec{k}_i \mathcal{J}' d^3\vec{k}'_i \Psi^{*}_{m'j'r'}(\vec{k}'_i,\mu'_i) u(p'_1\sigma'_1)\gamma^{\mu} u(p_1\sigma_1) F\left((p'_1-p_1)^2\right) \\ & \times E_{i\neq 1}\delta^3(p'_i-p_i)\delta_{\sigma'\sigma}\Psi_{mjr}(\vec{k}_i\mu_i), \end{aligned} \tag{18}$$

where Ψ is an eigenfunction of the mass operator, $\mathcal{J}$ and $\mathcal{J}'$ are Jacobian factors, and the delta functions express the fact that the momenta of the unstruck constituents do not change. The one-body current matrix element in equation (18) has been chosen for a spin 1/2 particle with form factor F.

2 Elastic deuteron form factors

To compute elastic deuteron form factors using the point form it is necessary to have a mass operator that will generate the deuteron wave functions. To make use of the many nonrelativistic potentials that are able to give good deuteron wave functions, the mass operator, a sum of relativistic kinetic energy and interaction, is squared and then rewritten in the form of a nonrelativistic Schrödinger equation [6]:

$$M = 2\sqrt{m_N^2+\vec{k}^2} + M_{\text{int}}, \qquad M^2 = 4\left(m_N^2+\vec{k}^2\right) + 4m_N V_{N-N}, \tag{19}$$

$$\begin{aligned} & M^2\Psi = \left(4m_N^2 + 4\vec{k}^2 + 4m_N V_{N-N}\right)\Psi = m_D^2\Psi, \\ & \left(\frac{\vec{k}^2}{m_N} + V_{N-N}\right)\Psi = \left(\frac{m_D^2}{4m_n} - m_N\right)\Psi; \end{aligned} \tag{20}$$

in this work the Argonne v_{18} and Reid'93 potentials were used to obtain the deuteron wave functions.

Since the nucleons that make up the deuteron themselves have internal structure, it is necessary to choose form factors for them. In this calculation the one-body current operators were determined by form factors given by Gari, Krümpelmann [7] and Mergell, Meissner, and Drechsel [8].

The results of these calculations have been published in reference [5]. Collaborators are T. Allen and W. Polyzou, with much help from F. Coester and G. Payne. A comparison of our results with those of other calculations is given by F. Gross [9], where it is seen that the structure function falls off too fast in comparison with experimental data, while the results for the tensor polarization agree reasonably well with data. These results show the need for including two-body currents in the form factor calculations, a subject which is discussed elsewhere [10].

3 Nucleon form factors

To compute nucleon form factors the mass operator is obtained in a rather different way as compared with the deuteron mass operator. In this case the three quark mass operator comes from a "semi-relativistic" Hamiltonian, the sum of relativistic kinetic energy, linear confinement potential and hyperfine interaction (Goldstone Boson Exchange model [11]):

$$H \longrightarrow M = \sum \sqrt{m^2 + \vec{k}_i^2} + \sum V(\mathrm{conf}) + \sum V(HF).$$

That is, the "semi-relativistic" Hamiltonian can be reinterpreted as a point form mass operator and the eigenfunctions previously calculated can be used to compute form factors. Thus, the bound state problem for three quarks, $M\Psi = m\Psi$, gives the wave functions and a good spectroscopic fit (Glozman, et al [12]). Finally the current operator is a point-like Dirac current with no anomalous magnetic moment.

When these eigenfunctions and current operators are put into equation (18), excellent agreement with data is obtained. The form factor graphs and static properties can be found in reference [13]. Collaborators in this project include S. Boffi, L. Glozman, W. Plessas, M. Radici, and R. Wagenbrunn. It should be noted that form factors for the weak interactions have also been calculated and give excellent agreement with experiment [14].

4 Algebraic formulation of electron scattering

As shown in previous sections the two quantities needed to calculate electron scattering observables in the point form are the hadronic four-momentum operator P_μ, satisfying $P_\mu^\dagger = P_\mu$ and the electromagnetic current operator $J_\mu(0)$. To rewrite these quantities in an algebraic form it is more convenient to work with the Fourier transform of the current operator

$$J_\mu(Q) = \int d^4x e^{iQ\cdot x} J_\mu(x), \tag{21}$$

with $J_\mu^\dagger(Q) = J_\mu(-Q)$, for then two of the fundamental commutation relations are

$$[P_\mu, P_\nu] = 0, \tag{22}$$
$$[P_\mu, J_\nu(Q)] = Q_\mu J_\nu(Q). \tag{23}$$

These operators must also satisfy Lorentz transformation properties, with the four-momentum operator transforming as a four-vector (equation (2)) and the current operator as a four-vector density. To get an algebraic structure, $[J_\mu(Q), J_\nu(Q')]$ should close. Since equation (22) is a point form equation required by Poincaré covariance, and equation (23) is a consequence of the translational covariance of current operators it is clear that the crucial commutator relation is the one involving the two currents. While the commutator of two currents closing is reminiscent of the current algebra of the 1960's (see for example [15]), the crucial difference is that the currents are not regarded as the fundamental degrees of freedom, to be used in a Hamiltonian; rather in combination with equations (22), (23), the four-momentum operator and the current operator form a closed algebraic system, the representations of which should give the observables of electron scattering.

These observables include the structure tensor for inclusive scattering,

$$W_{\mu\nu}(p, Q) = \sum \int d^4Q' \langle pj\sigma|[J_\mu(Q), J_\nu(Q')]|pj\sigma\rangle, \tag{24}$$

and form factors for exclusive scattering,

$$F_\mu(Q^2) = \langle p'j'\sigma'|J_\mu(Q)|pj\sigma\rangle. \tag{25}$$

Further, it follows from equation (23) that $J_\mu(Q)$ acts as a raising (lowering) operator on eigenstates of P_μ, and for certain values of the momentum transfer Q, acts as an annihilation operator on the ground state:

$$J_\mu(Q)|p_{\text{gnd}}j\sigma\rangle = 0 \tag{26}$$

for $(p_{\text{gnd}} + Q)^2 < m^2_{\text{gnd}}$.

Such an algebraic structure of $\{P_\mu, J_\mu(Q)\}$ is reminiscent of a Cartan algebra, with the diagonal operators and the raising and lowering operators (see for example [16, 17]). Such algebras also have automorphism groups; in the case of the present algebra, the Lorentz transformations are a subgroup of the automorphism group. Finally the annihilation property, equation (26) is analogous to positive energy, or discrete series representations of finite dimensional Lie algebras.

A well known example from two dimensional field theory is the Virasoro algebra [17]:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m\left(m^2-1\right)\delta_{m+n,0}, \tag{27}$$

where $L_0 \approx P_\mu$ is interpreted as a mass or energy and $L_{m\neq 0} \approx J_\mu(Q)$ the space component of a current in discrete variables. There is no analogue of equation (22) unless the product of two Virasoro algebras is used, in which case the interpretation of L_0 becomes the light front operators $P_\pm = P_0 \pm P_1$. But equation (23) is already contained in equation (27) when the index m is set equal to zero. When both m and n are nonzero in equation (27), the commutator gives the closure of two current operators.

As a second example consider a "$U(N)$" model for spinless particles of mass m, with creation and annihilation operators satisfying $[a(p), a^\dagger(p')] = E\delta^3(p-p')$, and out of which the following operators can be built:

$$P_\mu = \int \frac{d^3p}{E} p_\mu a^\dagger(p)a(p), \tag{28}$$

$$J_\mu(x) = \int \frac{d^3p_1}{E_1}\frac{d^3p_2}{E_2} F\left((p_1-p_2)^2\right)(p_1+p_2)_\mu e^{i(p_1-p_2)\cdot x}a^\dagger(p_1)a(p_2), \tag{29}$$

$$J_\mu(Q) = F\left(Q^2\right)\int \frac{d^3p_1}{E_1}\frac{d^3p_2}{E_2}\delta^4(p_1-p_2-Q)(p_1+p_2)_\mu a^\dagger(p_1)a(p_2); \tag{30}$$

both the free four-momentum operator (28) and the Fourier transform of the current operator (30) are formed from operators of the form $a_i^\dagger a_j$, which forms the Lie algebra of the unitary group, hence the name "$U(N)$" model. From the definition given of these operators, it can now be shown by direct calculation that equations (22), (23) hold, for an arbitrary form factor $F\left(Q^2\right)$.

The key equation is the commutator of the two currents. Using equation (30) suggests the following possibility:

$$[J^\dagger_\mu(Q), J_\nu(Q)] = 4F\left(Q^2\right)^2(P_\mu Q_\nu + P_\nu Q_\mu), \tag{31}$$

$$[J_\mu(Q), J_\nu(Q')] = 2\frac{F\left(Q^2\right)F\left(Q'^2\right)}{F\left((Q+Q')^2\right)}\left(Q'_\mu J_\nu\left(Q+Q'\right) - Q_\nu J_\mu\left(Q+Q'\right)\right), \tag{32}$$

for $Q + Q' \neq 0$. Equation (31) is the analogue of the Cartan algebra commutator, where the commutator of a raising operator with its adjoint gives a diagonal operator, while equation (32) is similar to the Virasoro algebra (27), when $m+n \neq 0$. Note also that equation (31) has no central extension, as is the case with the Virasoro algebra.

5 Conclusion

Motivated by the analysis of electron scattering experiments, an algebraic formulation of hadronic systems has been given in the context of point form relativistic quantum mechanics. The point form is one of the forms of relativistic quantum mechanics proposed by Dirac, in which all of the interactions are in the four-momentum operator, and the Lorentz generators are all kinematic (free of interactions). As shown in the introduction the other operator besides the four-momentum operator needed to compute form factors and structure functions that provide the link to experimental data is the electromagnetic current operator. While it suffices to know the current operator at the space-time point zero for computing form factors, to uncover an algebraic structure, it is more useful to consider the Fourier transform of the current operator $J_\mu(Q)$, where the independent variable Q is the four-momentum transfer. From the definition of $J_\mu(Q)$ it follows that it acts as a raising or lowering operator on eigenstates of the four-momentum operator.

The key commutator is between two current operators, and here there is no direct guide from hadronic physics. Two examples of algebraic structures were given in the previous section, but there is much work to be done to find physically interesting examples. One possibility is to work with free hadronic systems and then deform the current commutators to produce interactions.

Both P_μ and $J_\mu(Q)$ have definite transformation properties under Lorentz transformations, which suggests that the Lorentz transformations belong to an automorphism group, just as the symmetric group is the automorphism group for the $U(N)$ algebras.

A final important issue concerns the representations of such algebraic structures, for it is the representations that provide the actual form factors and structure functions. Since for certain values of Q, $J_\mu(Q)$ acts as an annihilation operator on the ground state (see equation (26)), the representations of interest should be "discrete series" types of representations (see for example [16, 17]) and it should be possible to generalize the techniques for generating such representations to those needed here.

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Relativity without the First Postulate

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We are changing Einstein's axiom system for special relativity and propose a new fundamental theory in relativistic physics. We do not assume that all inertial coordinate systems are equivalent (Einstein's first postulate), but we keep the second axiom, that the speed of light c is the same in all inertial frames. Some key results are [7]:
- The limiting energy and momentum of any particle as its speed approaches the speed of light, are *finite* and proportional to its rest mass.
- These upper bounds give the minimum lengths and time intervals of a particle in the sense of uncertainty and the ultraviolet cutoffs in the renormalization in quantum field theories.
- Photons, and all other particles moving at the speed of light have *nonzero* rest mass. They, however, obey the corresponding (modified) equations with *vanishing mass terms.*
We prove results of the new theory which can give answers and solutions to the following problems and difficulties in modern physics: Divergence difficulties in quantum field theories, "zero over zero" operations in momenta-energy calculations, failure in finding Higgs particles in gauge theories, singularities in general relativity.

1 Introduction

In modern physics, there exist some problems and difficulties:

(1) Singularities in general relativity. "*... it is my opinion that singularities must be excluded*" (Einstein [4, p. 164]). And because of this, he underlines that "*One may not therefore assume the validity of the equations for very high density of field and of matter ...*" (Einstein [4, p. 129]).

(2) Divergence difficulties in quantum field theories, which, "*... are symptomatic of a chronic disorder in the small-distance behavior of the theory*" (Bjorken and Drell [2, p. 4]), and "*In any case the existence of divergent quantities leads one to suspect trouble in the theory at large momenta or, equivalently, small distances*" (Bjorken and Drell [1, p. 154]). Because of the irrational calculation, $\infty - \infty$, in renormalizations, Dirac, a founder of renormalizations, repeatedly asserted [3] that fundamental physics, relativity and quantum theory, must be reformed.

(3) The finite momenta-energies of particles moving at speed of light are given by the operation "zero over zero". For example, the finite energies of photons are given by

$$E = h\nu = \frac{0}{\sqrt{1 - c^2/c^2}}\, c^2 = \frac{0}{0}\, c^2$$

which can be of any value for ν can be of any value.

(4) The inconsistency of gauge invariances for short range interactions with nonzero rest masses of the corresponding gauge particles. The Higgs mechanism seems to be helpful in trying to reach consistency, but there is no experimental evidence which indicates the existence of Higgs particles. Moreover, too many parameters caused by the mechanism make the theory look like a phenomenological rather than a basic theory, as T.D. Lee [5] pointed out. S. Weinberg [8] as the first person who used the Higgs mechanism to establish a unified gauge theory for electromagnetic and weak interactions, proposed a model without the Higgs mechanism in 1981, several years after he won the Nobel prize for that work. Some physicists believe that they will be able to

find Higgs particles in the superconducting super collider. But Weinberg [9], speaking of the proposed SSC accelerator, says "*I refuse to believe that fundamental physics will stop at that point ... We do not know these underlying laws ... We may never know the ultimate laws of nature*".

In this paper, we will change Einstein's axiom system and propose a new fundamental physics based on a new axiom structure. Einstein's theory of relativity is based in its entirety on two postulates [4]:

P1: The laws of physics take the same form in all inertial frames.

P2: The speed of light c is the same in all inertial frames.

From these two postulates Einstein derived that the laws of motion are invariant under Lorentz transformations, in particular

$$dS'^2 = dS^2 \quad \text{for any inertial frames } S,\ S', \qquad \text{and} \qquad dx'^\mu = \alpha^\mu_\nu dx^\nu,$$

where α^μ_ν are the matrix elements of the Lorentz transformation from S to S'. Einstein pointed out [4, p. 35] that assuming only P2 one can allow more general transformations than Lorentz transformations of the form $dS'^2 = \lambda(v)dS^2$, where $\lambda(v)$ is a function of the relative velocity v of the inertial frames. If in addition one assumes P1 he showed that $\lambda(v) = \text{const} = 1$. In other words the Lorentz transformations are a necessary result of Einstein's axioms P1 and P2.

In this paper we study the consequences if we abandon the first postulate P1 but keep the second postulate P2. We do not assume that two inertial coordinate systems are still "equivalent" when their relative velocity is high enough, and we allow the limiting deviations (as $V \to c$) of the new theory from the current one to be large enough. This will lead us to more general linear transformations which leave the speed of light invariant and the new equations of laws of physics will be invariant under these transformations, called *c-invariant transformations*. We will derive the corresponding c-invariant equations of particle mechanics, the c-invariant Klein–Gordon, Proca and Maxwell equations and their interactions. No matter whether the deviations from the classical Einstein theory can be verified experimentally or not, we prove theoretical results of the new theory which can give us answers and solutions to the problems and difficulties mentioned above. For details we refer to [7].

2 *c*-invariant groups

We introduce a new type of general linear transformations leaving the speed of light invariant. Their algebraic structure is *not* the one of a group but of a groupoid (see Section 3); we call it a *c-invariant group* . In Section 3 we will discuss this algebraic structure. These transformations will generalize the Lorenz transformations from the classical theory.

Let Σ be the set of all inertial coordinate system and set for $S \in \Sigma$

$$dS^2 = dx^\mu dx^\mu = \left(dx^1\right)^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2 - c^2dt^2. \tag{1}$$

We are not considering dS as a distance element as in general geometrical models of flat space-time but rather as a formal definition because the generalized transformations we will consider will not be a symmetry of this dS^2 but for a different quantity which will define our geometry.

Consider the following coordinate transformations connecting two inertial coordinate systems $S, S' \in \Sigma$:

$$dx'^\mu = f_S^{1/2}(\vec{V}_{S'S})\alpha_{(\mu\nu)}(\vec{V}_{S'S})dx^\nu \equiv T^\mu_\nu(S'S)dx^\nu, \tag{2}$$

where $\vec{V}_{S'S}$ is the velocity of S' relative to S (measured in S) and $f_S(\vec{V}_{S'S})$ is a positive function, called the *transformation factor* from S to S', and $\alpha_{(\mu\nu)}(\vec{V}_{S'S})$ are the matrix elements of the

Lorentz transformation from S to S' (we use the index notation $\alpha_{(\mu\nu)}$ to indicate that $\alpha_{(\mu\nu)}$ is not covariant under the new transformations) and $T^{\mu}_{\nu}(S'S) \equiv f_S(\vec{V}_{S'S})\alpha_{(\mu\nu)}(\vec{V}_{S'S})$ are the matrix elements of the corresponding more general linear transformation. In general the map f_S might depend on S, especially when the relative velocity of the coordinate systems is high enough. Thus we have the transformation rule

$$dS'^2 = f_S(\vec{V}_{S'S})dS^2, \qquad \text{for all} \quad S, S' \in \Sigma. \tag{3}$$

These transformations leave the speed of light invariant; indeed let $u^i = dx^i/dt$, $i = 1, 2, 3$ be the coordinate velocity in S then for $u = c$ in S and $u' = c'$ in S' we have $dS'^2 = dx'^{\mu}dx'^{\mu} = (c'_1 dt')^2 + (c'_2 dt')^2 + (c'_3 dt')^2 - c^2(dt')^2 = \left(c'^2 - c^2\right)(dt')^2 = f_S(\vec{V}_{S'S})dS^2 = f_S(\vec{V}_{S'S})dx^{\mu}dx^{\mu} = 0$, hence $c' = c$.

Remark 1. These transformations are *not* conformal transformations, because dS^2 and dS'^2 are not two metrics; see Definition 1.

We now study the important properties of these new transformations $T^{\mu}_{\nu}(S'S)$. For any $S, S', S'' \in \Sigma$ we have $dS''^2 = f_{S'}(\vec{V}_{S''S'})dS'^2 = f_{S'}(\vec{V}_{S''S'})f_S(\vec{V}_{S'S})dS^2$ and $dS''^2 = f_S(\vec{V}_{S''S})dS^2$, hence

$$f_{S'}(\vec{V}_{S''S'})f_S(\vec{V}_{S'S}) = f_S(\vec{V}_{S''S}). \tag{4}$$

In particular, we get $f_S(\vec{V}_{S'S})f_{S'}(\vec{V}_{SS'}) = 1$, $(dS'^2 = f_S(\vec{V}_{S'S})dS^2 = f_S(\vec{V}_{S'S})f_{S'}(\vec{V}_{SS'})dS'^2)$ which implies $f_S^{-1}(V_{S'S}) = f_{S'}(V_{SS'})$. For the matrix representation we find that they satisfy the *consistency condition*

$$T^{\mu}_{\sigma}(S''S')T^{\sigma}_{\nu}(S'S) = T^{\mu}_{\nu}(S''S), \qquad \text{for all} \quad S, S', S'' \in \Sigma, \tag{5}$$

that means $f_{S'}^{1/2}(\vec{V}_{S''S'})\alpha_{(\mu\sigma)}(\vec{V}_{S''S'})f_S^{1/2}(\vec{V}_{S'S})\alpha_{(\sigma\nu)}(\vec{V}_{S'S}) = f_S^{1/2}(\vec{V}_{S''S})\alpha_{(\mu\nu)}(\vec{V}_{S''S})$ for all S, S', $S'' \in \Sigma$.

More abstractly we write the consistency condition (5) as

$$T(S''S')T(S'S) = T(S''S), \qquad \text{for all} \quad S, S', S'' \in \Sigma, \tag{6}$$

whose algebraic meaning we will explain in Section 3.

Let $S_0 \in \Sigma$ be a fixed but arbitrary inertial frame, then for any $S \in \Sigma$,

$$dS_0^2 = f_S(\vec{V}_{S_0S})\delta_{(\mu\nu)}dx^{\mu}dx^{\nu}$$

(the Kronecker symbol $\delta_{(\mu\nu)}$ is not covariant under the general transformations). Define

$$\sigma_{\mu\nu} \equiv f_S(\vec{V}_{S_0S})\delta_{(\mu\nu)} \tag{7}$$

then from straightforward calculations we have

Proposition 1. *$\sigma_{\mu\nu}dx^{\mu}dx^{\nu}$ is invariant under the transformations $T(SS_0)$ for all $S \in \Sigma$, i.e.*

$$\sigma_{\mu\nu}dx^{\mu}dx^{\nu} = dS_0^2, \qquad \textit{for all} \quad S \in \Sigma. \tag{8}$$

Definition 1. Let S_o be a fixed absolutely isotropic inertial frame, i.e. there exists a function g such that $f_{S_o}(\vec{X}) \equiv g(|\vec{X}|)$. We define the *distance element* by

$$dS_o^2 = f_S(\vec{V}_{S_oS})dS^2 = f_S(\vec{V}_{S_oS})\delta_{(\mu\nu)}dx^{\mu}dx^{v} \equiv \sigma_{\mu\nu}dx^{\mu}dx^{\nu}, \tag{9}$$

where $\sigma_{\mu\nu} \equiv f_S(\vec{V}_{S_oS})\delta_{(\mu v)}$ is the metric tensor for general flat space-time. Note that Minkowski's space time is a special case with $f_S(\vec{V}_{S_oS}) \equiv 1$ for all $S \in \Sigma$.

The Lorentz transformations (both homogeneous and inhomogeneous) are special cases of our more general transformations, namely those with transformation factors equal to 1.

The transformation factor in (3) depends on the velocity $\vec{V}_{S'S}$ between S and S', so we regard f_S as a function on $\mathbb{R}^3$ for any given inertial frame $S \in \Sigma$. We call f_S the *factor function* of S. More precisely, for $S \in \Sigma$ let $f_S : \mathbb{R}^3 \to \mathbb{R}_+$, $f_S(\vec{V}) = f_S(v^1, v^2, v^3)$, $\vec{V} = v^i \vec{e}_i$, be a function on $\mathbb{R}^3$, where $\{\vec{e}_i, i = 1, 2, 3\}$ is the orthonormal basis of S. If for any $S' \in \Sigma$, $f_S(\vec{V}_{S'S}) = f_S(\vec{V})_{|\vec{V}=\vec{V}_{S'S}}$, then f_S is called the factor function of S, which gives the transformation factors from S to all other inertial coordinate systems $S' \in \Sigma$.

Theorem 1. *If the factor function of one $S \in \Sigma$ is given then the factor functions of all other inertial coordinate systems $S' \in \Sigma$ are determined.*

Proof. Let $\oplus$ denote the addition of velocity vectors. For any $S, S', S'' \in \Sigma$ we have $\vec{V}_{S''S} = \vec{V}_{S'S} \oplus \vec{V}_{S''S'}$ and the consistency condition becomes $f_{S'}(\vec{V}_{S''S'}) = \frac{f_S(\vec{V}_{S''S'} \oplus \vec{V}_{S'S})}{f_S(\vec{V}_{S'S})}$. With $\vec{V}' \equiv \vec{V}_{S''S'}$, we find $f_{S'}(\vec{V}') = f_S(\vec{V}' \oplus \vec{V}_{S'S})/f_S(\vec{V}_{S'S})$, for all $\vec{V}'$, $0 \le V' < c$, and all $S, S' \in \Sigma$. This expresses the factor function $f_{S'}$ of any $S' \in \Sigma$ in terms of the factor function f_S of S. ■

Let $\mathrm{Vec} = \{\vec{V} \in \mathbb{R}^3 | 0 \le V < c\}$ and $S \in \Sigma$. We denote the set of all factor functions generated by f_S by

$$\mathbf{F}_S = \left\{ f_{S'} \in C(\mathrm{Vec}, \mathbb{R}_+) | S' \in \Sigma, f_{S'}(\vec{V}') = \frac{f_S(\vec{V}' \oplus \vec{V}_{S'S})}{f_S(\vec{V}_{S'S})}, \quad \text{for all } \vec{V}' \in \mathrm{Vec} \right\}.$$

Proposition 2. *For any $S, S' \in \Sigma$ with $\vec{V}_{S'S} \neq 0$ we have $f_S = f_{S'}$ if and only if $f_S(\vec{V}) \equiv 1$.*

3 Algebraic structure of the transformations $T(S'S)$

Let $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times (ic\mathbb{R})$ and let $\varepsilon = \{\varepsilon_\alpha | \alpha \in J\}$ be the collection of all events, where J is some index set. The event ε_α has coordinates X_α in the S frame: $X_\alpha \in \mathbb{R}^4$, $X_\alpha = (x_\alpha^1, x_\alpha^2, x_\alpha^3, x_\alpha^4)$, $x_\alpha^4 \equiv ict_\alpha$. Denote $\mathbf{X} = \{X_\alpha \in \mathbb{R}^4 | \alpha \in J\}$ and let $T(S'S) : \mathbf{X} \to \mathbf{X}' = \{X_\alpha' \in \mathbb{R}^4 | \alpha \in J\}$ be a mapping such that $X_\alpha' = T(S'S)(X_\alpha)$ for all $\alpha \in J$. Then the consistency condition for the set $\{T(S'S) | S, S' \in \mathbf{\Sigma}\}$ is $T(S''S')T(S'S) = T(S''S)$, for all $S, S', S'' \in \Sigma$. With this (2) becomes $dX' = T(S'S)dX = f_S^{1/2}(\vec{V}_{S'S})\alpha(\vec{V}_{S'S})dX$ where $\alpha(\vec{V}_{S'S})$ is the matrix for the Lorentz transformation from S to S'. We call the matrix $[T_\nu^\mu(S'S)] = \left[f_S^{1/2}(\vec{V}_{S'S})\alpha_{(\mu\nu)}(r\vec{V}_{S'S})\right] = \left[f_S^{1/2}(\vec{V}_{S'S})\alpha(\vec{V}_{S'S})\right]$ the *matrix representation* for the mapping $T(S'S)$.

More abstractly we have the following algebraic situation: Let $\mathbf{A}$ be a collection of sets and for any $A_\alpha, A_\beta \in \mathbf{A}$ let $T(A_\alpha A_\beta)$ be a transformation from A_β to A_α. Denote by $\mathbf{T_A}$ the set of all such transformations defined in $\mathbf{A}$. The product of two such transformations $T(A_\alpha A_\beta)T(A_\sigma A_\gamma)$ is only defined if $A_\beta = A_\sigma$, in which case $T(A_\alpha A_\beta)T(A_\beta A_\gamma)$ is called the *physical product* where the two transformations are successive from A_γ to A_β, then from A_β to A_α.

Definition 2. A set of transformations $\mathbf{T_A}$ is called a *physical group* if it is closed under the physical product; in other words if the transformations satisfy the *consistency condition*

$$T(A_\alpha A_\beta)T(A_\beta A_\gamma) = T(A_\alpha A_\gamma), \qquad \text{for all} \quad A_\alpha, A_\beta, A_\gamma \in \mathbf{A}. \tag{10}$$

Proposition 3. *A) For every α the set $\mathbf{T}_\alpha \equiv \{T(A_\alpha A_\beta) | A_\beta \in \mathbf{A}\} \subset \mathbf{T}_A$, has a left unit element and for every β the set $\mathbf{T}_\beta \equiv \{T(A_\alpha A_\beta) | A_\alpha \in \mathbf{A}\}$ has a right unit element.*

B) For every element of a physical group there exist a right and a left inverse, which are identical to each other.

C) The physical product is associative.

Theorem 2. *Let $S \in \Sigma$ and f_S be given. The set* $\mathbf{T}_S = \{T(S'S'') = f_{S''}^{1/2}(\vec{V}_{S'S''})\alpha(\vec{V}_{S'S''}) \mid S', S'' \in \Sigma, f_{S''} \in \mathbf{F}_S\}$ *(where $\alpha(\vec{V}_{S'S''})$ is the matrix of the Lorentz transformation from S'' to S') is a physical group, called a c-invariant group.*

4 Model case of factor functions

We give an example of factor functions which shows how these ideas can be realised and which can serve as a model. Let $S_0 \in \Sigma$ be a fixed inertial frame and fix a parameter $N > 0$. Define $f_{S_0}(\vec{V}) = f_{S_0}(V) \equiv \left(1 - (V/c)^N\right)^{-1}$, $V = |\vec{V}| < c$. This factor function of S_0 generates a set $\mathbf{F}_{S_0} = \left\{ f_S(\vec{V}) = \frac{f_{S_0}(\vec{V} \oplus \vec{V}_{SS_0})}{f_{S_0}(\vec{V}_{SS_0})} \;\Big|\; S \in \Sigma \right\}$, where $f_{S_0}\left(\vec{V} \oplus \vec{V}_{SS_0}\right) = \left(1 - B^N\right)^{-1}$ with $B = \left(1 + \vec{V} \cdot \vec{V}_{SS_0}/c^2\right)^{-1} \left[\left(1 + \vec{V} \cdot \vec{V}_{SS_0}/c^2\right)^2 - \left(1 - V_{SS_0}^2/c^2\right)\left(1 - V^2/c^2\right)\right]^{1/2}$.

The Lorentz model is nothing but the limiting case of this model as $N \to \infty$.

5 Dynamics

Now let us derive the equations of fundamental laws of nongravitational physics which are invariant under the c-invariant groups. We call these equations *c-invariant*. We will see that the transformation factors will appear in these equations. When we let all the transformation factors be 1, then all the equations will go back to their counter-parts in the Lorentz invariant theory. When we take some c-invariant groups with transformation factors having the same limiting behavior, some important theoretical results will be obtained.

5.1 *c*-invariant classical mechanics

Let S^* be the instantaneous rest frame of a particle and let $\vec{u} = \vec{V}_{S^*S}$ be the instantaneous velocity of the particle measured in the S-frame. The interval of proper time is

$$d\tau = \sqrt{-dS^{*2}/c^2} = \left[-f_S(\vec{V}_{S\Sigma})dS^2/c^2\right]^{1/2} = f_S^{1/2}(\vec{u})\gamma^{-1}dt, \qquad \text{where} \quad \gamma \equiv \frac{1}{\sqrt{1 - u^2/c^2}}$$

is called the *Lorentz factor*. Define the 4-velocity $\mathbf{U}^\mu \equiv dx^\mu\ /d\tau = f^{-1/2}(\vec{u})\gamma dx^\mu\ /dt$ and the 4-momentum $P^\mu \equiv m_o \mathbf{U}^\mu$.

The particle mechanics invariant under the c-invariant groups is

$$\mathbf{F}^\mu = m_o \frac{d\mathbf{U}^\mu}{d\tau} = \frac{dP^\mu}{d\tau}, \tag{11}$$

where m_o is the rest mass of the particle and $\mathbf{F}^\mu$ is the 4-force determined by electromagnetical and gravitational fields through the corresponding formula. Denote $\vec{P} = (P^1,\ P^2, P^3)$ and $P^4 = iE/c$, then $\vec{P} = m_o f_S^{-1/2}(\vec{u})\gamma\vec{u}$ and with $\lambda \equiv f_S^{-1/2}(\vec{u})$

$$E = m_o f_S^{-1/2}(\vec{u})\gamma c^2 = m_o \lambda \gamma c^2. \tag{12}$$

Thus, $P^\mu P^\mu = P^2 - E^2/c^2 = -m_o^2 c^2 f_S^{-1}(\vec{u})$, and hence $E^2 = P^2 c^2 + m_o^2 c^4 f_S^{-1}(\vec{u})$.

5.2 *c*-invariant quantum mechanics

The de Broglie wave of a free particle is

$$\psi = A \exp(i\hbar^{-1}\Lambda^{-1}P^\mu x^\mu) \tag{13}$$

where the transformation property of A is determined by the spin of the particle and Λ is defined as $\Lambda \equiv f_{S_0}(\vec{V}_{SS_0})$ with S_0 being a fixed absolutely isotropic inertial frame, (i.e. there exists a function g such that $f_{S_o}(\vec{X}) = g(|X|)$). The phase is invariant under c-invariant groups; indeed $\Lambda^{-1}P^\mu x^\mu = f_S(\vec{V}_{S_0S})P^\mu x^\mu = f_S(\vec{V}_{S_0S})\delta_{(\mu\nu)}P^\mu x^\nu = \sigma_{\mu\nu}P^\mu x^\nu$, where $\Lambda^{-1} \equiv f_{S_0}^{-1}(\vec{V}_{SS_0}) = f_S(\vec{V}_{S_0S})$.

Generally, we have

Theorem 3. *Let A^μ and B_μ be a contravariant and a covariant 4-vector under c-invariant groups respectively. Then $\Lambda^{-1}A^\mu$ and ΛB_μ are covariant and contravariant 4-vectors under c-invariant groups respectively.*

Proposition 4. *The c-invariant Klein–Gordon equation for free spin zero particles in the observer-frame S is*

$$\left(\Box - c^2\hbar^{-2}\underline{m}^2\right)\psi = 0, \qquad \textit{where} \quad \Box = \partial^\mu\partial_\mu, \quad \textit{with} \quad \partial^\mu \equiv \Lambda\partial_\mu \tag{14}$$

and

$$\underline{m} \equiv m_o\lambda\Lambda^{-1/2}, \qquad \textit{with} \quad \lambda \equiv f_S^{-1/2}(\vec{u}), \quad \textit{and} \quad \Lambda \equiv f_{S_0}(\vec{V}_{S_0S}). \tag{15}$$

We call $\underline{m}$ the *apparent mass* of the particle in S.

Proof. For any free particle with spin zero, we have $\psi = A\exp\left(i\hbar^{-1}\Lambda^{-1}P^\mu x^\mu\right)$, where A is a scalar. It is clear that ψ obeys (14) for one can easily check

$$\Box\psi = \Lambda\left(i\hbar^{-1}\Lambda^{-1}\right)^2 P^\mu P^\mu\psi = \hbar^{-2}m_0^2c^2\lambda^2\Lambda^{-1}\psi = \underline{m}^2c^2\hbar^{-2}\psi \tag{16}$$

which holds for any $S \in \Sigma$ since $\Lambda^{-1}P^\mu x^\mu$ is an invariant and A is a scalar. ■

Moreover, we have

Theorem 4. *The apparent mass $\underline{m}$ is an invariant under c-invariant groups, i.e. $\underline{m}' = \underline{m}$.*

Proof. This is true simply because

$$\underline{m} = m_o\lambda\Lambda^{-1/2}m_o f_S^{-1/2}(\vec{u})f_{S_0}^{-1/2}(\vec{V}_{SS_0}) = m_o f_{S_0}^{-1/2}(\vec{V}_{S^*S_o}),$$

which is independent of the choices of the observer-frame S. ■

For the Lorentz group we have $\lambda = 1$ and $\Lambda = 1$ for all $S \in \Sigma$, hence $\underline{m} = m_o$ and the c-invariant Klein–Gordon equation goes back to the Lorentz invariant Klein–Gordon equation.

Proposition 5. *The c-invariant Proca equation for free particles with spin 1 is*

$$\left(\Box - c^2\hbar^{-2}\underline{m}^2\right)\psi_\mu = 0, \qquad \textit{where} \quad \psi_\mu = A_\mu\exp\left(i\hbar^{-1}\Lambda^{-1}P^\mu x^\mu\right) \tag{17}$$

with A_μ being a 4-vector.

We see that $m_o\lambda$ instead of m_o appears in c-invariant equations of law of motion, Klein–Gordon and Proca equations (later we will see also in the c-invariant Dirac equation), where $\lambda = f_S^{-1/2}(\vec{u}) = f_S^{-1/2}(\vec{V}_{S^*S})$ and S^* is the instantaneous rest frame of the particle.

5.3 Limit $u \to c$

Now we consider the limit as $u \to c$, i.e. write $\vec{c} = c\vec{n}$ ($|\vec{n}| = 1$) and let

$$N_S(\vec{n}) \equiv \lim_{\vec{u}\to c\vec{n}} \lambda\gamma, \qquad \lambda \equiv f_S^{-1/2}(\vec{u}). \tag{18}$$

Our fundamental assumption is the following: There exists an inertial frame $S \in \Sigma$, such that

$$N_S(\vec{n}) \equiv \lim_{\vec{u}\to c\vec{n}} \lambda\gamma \equiv \lim_{\vec{u}\to c\vec{n}} f_S^{-1/2}(\vec{u})\gamma < \infty. \tag{19}$$

Under this assumption we have the Theorems 5, 6, 7 and the Results 1, 2, 3.

Theorem 5. *If $N_S(\vec{n}) < \infty$ for all $\vec{n}$ with $|\vec{n}| = 1$ for some $S \in \Sigma$, then $N_{S'}(\vec{n}') = \lim_{\vec{u}'\to c\vec{n}'} \lambda'\gamma' < \infty$ for all $\vec{n}'$ with $|\vec{n}'| = 1$, for all $S' \in \Sigma$, where $\lambda' = f_{S'}^{-1/2}(\vec{u}')$ and $\gamma' = \left(1 - u'^2/c^2\right)^{-1/2}$.*

Theorem 6. *Let $S \in \Sigma$ and denote $\Sigma_S = \{S' \in \Sigma | V_{S'S} = 0\}$. Then whenever there exists an $S_o \in \Sigma$ such that f_{S_o} is isotropic, then $f_{S'_o}$ is isotropic for each $S'_o \in \Sigma_{S_o}$ and f_S is not isotropic for each $S \in \Sigma \setminus \Sigma_{S_o}$.*

In case $N_S(\vec{n}) < \infty$ for all $\vec{n}$ with $|\vec{n}| = 1$ and all $S \in \Sigma$, we get the following results:

Result 1. The contravariant ultraviolet cut-offs are

$$\Lambda^\mu(\vec{n}) \equiv \lim_{\vec{u}\to c\vec{n}} P^\mu = m_o c N_S(\vec{n})(\vec{n}, i),$$

where we use the notion $A^\mu = (\vec{a}, b)$ to indicate that $A^i = a^i$, $i = 1, 2, 3$ and $b = A^4$.

We assume that there exists an $S_o \in \Sigma$ such that f_{S_o} is isotropic. Then in case $\lambda\gamma$ is bounded the upper bound of the 4-momentum for a particle observed in any $S \in \Sigma$ exists, and at least in case $\lambda\gamma$ is nondecreasing, it is given by

$$\Lambda^\mu \equiv \max_{\substack{\vec{n} \in \mathbb{R}^3 \\ |\vec{n}| = 1}} \Lambda^\mu(\vec{n}) = f_{S_o}^{1/2}(\vec{V}_{SS_o}) \frac{\sqrt{1 - V_{SS_o}^2/c^2}}{1 - V_{SS_o}/c} N_0$$

which gives the minimum nonzero lengths and time-intervals for particles in the sense of uncertainty, indicating a true meaning of "discrete" or "quantized" space-time and of any model for non-pointlike elementary particles, e.g. strings. Furthermore, photons and all the particles moving at speed c must have nonzero rest masses which are given by a l'Hospital type limit. For example, consider a photon with energy E in $S \in \Sigma$ which moves along the direction $\vec{n}$. Then its nonzero rest mass is

$$m_o = \frac{Ec^2}{\lim\limits_{\vec{u}\to c\vec{n}} \lambda\gamma} = \frac{Ec^2}{N_S(\vec{n})} \neq 0.$$

In Einstein's relativity, $\lambda \equiv 1$, $N_S(\vec{n}) = \infty$ and $E = h\nu$, hence $m_o = 0$ while $E = h\nu = 0c^2/\sqrt{1 - c^2/c^2} = 0c^2/0$ can be of any value for ν can be of any value. There is no limit process as there is in our theory.

In case $N_S(\vec{n}) < \infty$ for all $\vec{n}$ and all $S \in \Sigma$, for photons, $E = \Lambda h\nu$ (we will show this later), which can be given by (12) through a l'Hospital-type limit process:

$$E = \Lambda h\nu = m_o c^2 \lim_{\vec{u}\to c\vec{n}} \lambda\gamma = m_o c^2 N_S(\vec{n}),$$

where $m_o = c^{-2}(N_S(\vec{n}))^{-1}\Lambda h\nu$ is the nonzero rest mass of the photons with frequency ν and moving along the unit direction $\vec{n}$. It is impossible to make photons and any particles moving with speed c be at rest. Hence, the so-called "rest mass" of a particle moving with speed c is just a coefficient of proportionality between $N_S(\vec{n})$ and the energy of the particle measured in the S frame, and is independent of S for it is a scalar under c-invariant groups but dependent on both its energy and direction of motion.

Result 2. Since $\gamma \to \infty$ as $u \to c$, our assumption $N_S(\vec{n}) < \infty$ leads to

$$\lim_{u\to c} \lambda \equiv \lim_{u\to c} f_S^{-1/2}(\vec{u}) = 0 \qquad \text{for all} \quad S \in \Sigma. \tag{20}$$

Thus, every scalar or vector particle moving at speed c has *zero apparent mass*, i.e., $\underline{m} \equiv m_o\lambda\Lambda^{-1/2} = 0$. Then they obey the corresponding equations with vanishing mass terms by which the gauge invariances are characterized.

5.4 c-invariant classical electrodynamics

We can now study the equations of classical electrodynamics. The c-invariant classical electrodynamics is given by

$$\partial_\mu F^{\mu\nu}_{em} = -4\pi c^{-1} J^\nu \qquad \text{and} \qquad f^\mu_{em} = c^{-1} F^{\mu\nu}_{em} J_\nu, \tag{21}$$

where

$$F^{\mu\nu}_{em} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu = \Lambda^2(\partial_\mu A_\nu - \partial_\nu A_\mu) \equiv \Lambda^2 F^{em}_{\mu\nu}$$

and $J_\mu = \Lambda^{-1}J^\mu$, $J^\mu \equiv \rho^* U^\mu$ with $\rho^* \equiv dq^*/dV^*$, where dq^*, dV^* and ρ^* are charge element, volume element and charge density measured in $S^* \in \Sigma$ (the instantaneous rest frame of the charged particle). We call ρ^* the *proper charge density* which is frame-invariant as the proper time interval. We keep the assumption that charges are frame-invariant. Then $dq^* = dq$ and $\rho^* = dq/dV^*$. The kinematic effects of moving rods under c-invariant groups give $dV^* = (f_S^{1/2}(\vec{V}_{S^*S}))^3\gamma dV = \lambda^{-3}\gamma dV$. Hence $\rho^* = \lambda^3\gamma^{-1}\rho$. Now denote $A^\mu \equiv (\vec{A}, i\phi)$ which means $\vec{A} \equiv (A^1, A^2, A^3)$, $i\phi \equiv A^4$ and $\vec{E} \equiv -\nabla\phi - c^{-1}\partial\vec{A}/\partial t$, $\vec{B} \equiv \nabla \times \vec{A}$.

Then it is easy to check that (21) leads to

Proposition 6. *The c-invariant Maxwell equations are:*

$$\begin{aligned}
&\nabla \cdot \vec{B} = 0,\\
&\nabla \times \vec{E} = -c^{-1}\partial\vec{B}/\partial t,\\
&\nabla \cdot \vec{E} = 4\pi\underline{\rho},\\
&\nabla \times \vec{B} = c^{-1}\partial\vec{E}/\partial t + 4\pi c^{-1}\underline{\vec{J}},
\end{aligned} \tag{22}$$

where $\underline{\rho} = \Lambda^{-1}\lambda\gamma\rho^* = \Lambda^{-1}\lambda^4\rho$, *and* $\underline{\vec{J}} \equiv \underline{\rho}\vec{u}$.

Also (21) leads to a Lorentz-type force

$$F^{em}_{(\mu)} = q(\vec{E} + \vec{\beta} \times \vec{B}, i\vec{E}\cdot\vec{\beta}), \qquad \vec{\beta} \equiv \vec{u}/c, \tag{23}$$

where

$$F^{em}_{(\mu)} \equiv \frac{d\tau}{dt}\mathbf{F}^\mu_{em} \equiv \lambda^{-1}\gamma^{-1}\int f^\mu_{em} dV^*.$$

Without difficulty, we obtain $F^{\text{space}}_{(\mu)} = m_o c(1 + \vec{A}_S\cdot\vec{\beta})^{-1}\frac{d\lambda}{dt}\gamma^{-1}(-\vec{A}_S, i)$, $\vec{A}_S \equiv \vec{V}_{SS_o}/c$, with $F^{\text{space}}_{(\mu)} \sim N_o^{-2}$. When N_o is large enough, $F^{\text{space}}_{(\mu)}$ does not cause any practically measurable effect.

5.5 c-invariant quantum electrodynamics

We can now combine the previous results.

Proposition 7. *The c-invariant Dirac equation for a free particle with spin $\frac{1}{2}$ is*

$$\left(\gamma_\mu \partial_\mu + c\hbar^{-1}\Lambda^{-1/2}\underline{m}\right)\psi = 0. \tag{24}$$

where γ_μ are the Dirac matrices.

Result 3. Equation (24) tells us that neutrinos moving at speed c must have vanishing apparent mass and nonzero rest mass in every c-invariant theory with finite $N_S(\vec{n})$ and hence satisfy the c-invariant Dirac equation with vanishing mass term and a two-component theory.

We now study the electromagnetic coupling. In the presence of electromagnetic fields we obtain

$$\left[\gamma_\mu\left(\partial_\mu - iqc^{-1}\hbar^{-1}A_\mu\right) + c\hbar^{-1}\Lambda^{-1/2}\underline{\hat{m}}\right]\psi = 0, \tag{25}$$

where $\underline{\hat{m}} \equiv m_o\Lambda^{-1/2}\hat{\lambda} \equiv m_o\Lambda^{-1/2}f_S^{-1/2}(\hat{\vec{u}})$.

The replacement $P^\mu \mapsto -i\hbar\Lambda\partial_\mu - qc^{-1}\Lambda A_\mu = -i\hbar\partial^\mu - qc^{-1}A^\mu$ and the identity $\vec{u} = c^2\vec{P}E^{-1}$ give $\hat{\vec{u}} = -c^2(i\hbar\nabla + qc^{-1}\Lambda^{-1}\vec{A})/(i\hbar\partial_t - \Lambda^{-1}q\phi)$ $=c^2\vec{P}_m E_m^{-1}$, where the two operators $\vec{P}_m$ and E_m^{-1} must be regarded as *commutative*. For an eigenfunction of the energy operator E_m^{-1} with eigenvalue E we have $E_m^{-1}\psi = (E - q\phi)^{-1}\psi$ and

$$\vec{P}_m E_m^{-1}\psi = (E-q\phi)^{-1}\vec{P}_m\psi, \qquad \left(\vec{P}_m E_m^{-1}\right)^2\psi = (E-q\phi)^{-2}\vec{P}_m^2\psi, \qquad \text{ect.}$$

For the electron in a hydrogen atom being at rest in any $S \in \Sigma$, we have $\hat{\vec{u}} = cR\nabla$ where $R \equiv -i\hbar\Lambda c/(E + \Lambda^{-1}e^2/r)$. For example, taking our model case

$$\begin{aligned}\lambda &\equiv f_S^{-1/2}(\vec{u}) = f_S^{-1/2}(\vec{V}_{S^*S}) = f_{S_o}^{-1/2}(\vec{V}_{S^*S_o})/f_{S_o}^{-1/2}(\vec{V}_{SS_o})\\ &= \Lambda^{1/2}\sqrt{1-\beta_o^N} = \Lambda^{1/2}\left\{1 - \left[1 - \Gamma^{-2}(1+\vec{A}_S\cdot\vec{u}/c)^{-2}\left(1-u^2/c^2\right)\right]^{N/2}\right\}^{1/2}\end{aligned}$$

one finds

$$\hat{\lambda} \equiv f_S^{-1/2}(\hat{\vec{u}}) = \Lambda^{1/2}\left\{1 - \left[1 - \Gamma^{-2}(1+R\vec{A}_S\cdot\nabla)^{-2}\left(1-R^2\nabla^2\right)\right]^{N/2}\right\}^{1/2},$$

where $\Gamma \equiv (1-A_S^2)^{-1/2}$. The first approximation is $\hat{\lambda} = 1-\frac{N}{2}\Lambda\Gamma^{-2}A_s^{N-2}R\vec{A}_S\cdot\nabla = 1-\vec{D}_S\cdot(R\nabla)$, where $\vec{D}_S \equiv \frac{1}{2}N\Lambda A_S^{N-2}\left(1-A_S^2\right)\vec{A}_S$. Since $f_S(\vec{V})$ is almost 1 within the velocity range of the particles in the accelerators and $u \ll c$ for electrons bound in atoms, we have a very good approximation:

$$\hat{\lambda} \approx 1, \qquad \underline{\hat{m}} \approx m_o\Lambda^{-1/2}.$$

So we obtain an approximate equation for electrons bound in an atom which is at rest in an inertial frame S:

$$\left(-i\hbar c\vec{\alpha}\cdot\nabla + \gamma_4 m_o c\Lambda^{-1}\right)\psi(\vec{r}) = \left(\Lambda^{-1}E + \Lambda^{-2}Ze^2/r\right)\psi(\vec{r}),$$

whose $N\cdot R$ approximation is $E = m_oc^2\left(1 - \frac{1}{2n^2}\Lambda^{-4}Z^2\alpha^2\right)$, $n = 1, 2, 3, \ldots$ and

$$\nu(i\to f) = m_oc^2h^{-1}\Lambda^{-5}\frac{1}{2}Z^2\alpha^2\left(n_f^{-2} - n_i^{-2}\right), \qquad (\alpha \equiv e^2\hbar^{-1}c^{-1}) \tag{26}$$

which gives the frequency spectrum of the photons emitted from the atom and observed in S; (the notion $(i \to f)$ meaning from initial to final state).

Now let us consider frequency shifts. Label

$$K_\mu \equiv (\vec{K}, i\omega/c) \equiv \hbar^{-1}\Lambda^{-1}P^\mu \equiv \hbar^{-1}\Lambda^{-1}(\vec{P}, iE/c), \qquad \omega = 2\pi\nu,$$

which is the covariant wave vector of a particle. By use of the transformation rule for a covariant 4-vector, we obtain

$$\nu/\nu' = f_S^{1/2}(\vec{V}_{S'S})\left(1 - \vec{n}\cdot\vec{V}_{S'S}/c\right)^{-1}\left(1 - V_{S'S}^2/c^2\right)^{1/2}, \qquad (\vec{n} = \vec{P}/|\vec{P}|). \tag{27}$$

If the particle is a photon, (27) gives the formula of frequency shifts. It is interesting that taking the point of view of an emission theory can also give (27). Let $\vec{u}$ and $\vec{u}'$ be the velocities of the same photon (as a "bullet") observed in the S and S' respectively. From the velocity addition law, we know that $u = u' = c$. However, the same velocity addition law gives

$$\lim_{\vec{u}\to c\vec{n}}\left(1 - u^2/c^2\right)\left(1 - u'^2/c^2\right)^{-1} = \left(1 - \vec{n}\cdot\vec{V}_{S'S}/c\right)^2\left(1 - V_{S'S}^2/c^2\right)^{-1}. \tag{28}$$

Thus, (12), (27), and (28) give

$$\nu/\nu' = f_S^{1/2}(\vec{V}_{S'S})(1 - \vec{n}\cdot\vec{V}_{S'S}/c)^{-1}\sqrt{1 - V_{S'S}^2/c^2},$$

which is identical with (27). Of course, light sources are not Galileo–Newton's "guns".

Let S_1 be the instantaneous rest frame of a moving atom and $\nu_{1(S_1)}(i \to f)$ be the spectrum of photons emitted from the atom and observed in the S_1-system. Using (26) one can write

$$\nu_{1(S_1)}(i \to f) = m_o c^2 h^{-1}\Lambda_1^{-5}\frac{1}{2}Z^2\alpha^2\left(n_f^{-2} - n_i^{-1}\right), \qquad \Lambda_1 \equiv f_{S_o}(\vec{V}_{S_1S_o}).$$

Let $\nu_{(S_1)}(i \to f)$ be the spectrum of the same photons observed in S. Using (27) one can obtain the frequency shifts of the spectrum:

$$\nu_{(S_1)}(i \to f) = f_S(\vec{V}_{S_1S})\left(1 - \vec{n}\cdot\vec{V}_{S_1S}/c\right)^{-1}\left(1 - V_{S_1S}^2/c^2\right)^{1/2}\nu_{1(S_1)}(i \to f).$$

The new formula for *Doppler shifts* is given by

$$\frac{\nu_{(S_1)}(i \to f)}{\nu(i \to f)} = f_S^{-9/2}(\vec{V}_{S_1S})\left(1 - \vec{n}\cdot\vec{V}_{S_1S}/c\right)^{-1}\left(1 - V_{S_1S}^2/c^2\right)^{1/2}. \tag{29}$$

When the source-speed V_{S_1S} is high enough, (29) is most sensitive by comparison to the possible deviation of the values of transformation factors from 1, because the transformation factor $f_S(\vec{V}_{S_1S})$ appears in the formula with the exponent $-9/2$. To test (29), we suggest accelerating lithium ions to sufficiently high speed and then observing their light spectrum. This will be a crucial test if $N_S(\vec{n})$ is not too large. In principles, such experiment will find the function form of the factor function of a laboratory-system S. Then according to Theorem 1 the factor function of S will determine the function forms of the factor functions of all inertial coordinate systems. In particular, letting $S' = S_o$ and $\vec{V}_{S''S_o} = \vec{V}_o$, we find the consistency condition that

$$f_{S_o}(\vec{V}_o) = f_S(\vec{V}_o \oplus \vec{V}_{S_oS})/f_S(\vec{V}_{S_oS}), \tag{30}$$

which must be exactly independent of the direction of $\vec{V}_o$ since f_{S_o} is isotropic. The unique solution $\vec{V}_{S_oS}$ which makes the right side of (30) independent of the direction of $\vec{V}_o$ is the velocity of the special and exactly isotropic inertial frame S_o relative to S and $-\vec{V}_{S_oS}$ is just the special velocity of the laboratory-system S relative to the special and exactly isotropic inertial frame S_o. Furthermore, the following theorem is trivially true.

Theorem 7. *If* $N_S(\vec{n}) \equiv \lim_{\vec{u}\to c\vec{n}} \Lambda\gamma \equiv \lim_{\vec{u}\to c\vec{n}} f_S^{-1/2}(\vec{u})\gamma < \infty$*, then there exists a* $u_b < c$ *such that* $f_S(\vec{V})$ *deviates markedly from* 1 *when* $V > u_b$*, i.e. the deviation of* $f_S(\vec{V})$ *from* 1 *is large enough in the case* $V > u_b$ *and will easily be tested by experiment if only the corresponding energy* E_b *is within the power of accelerators man can or will be able to build.*

Our unique results are mainly described in Result 1, 2 and 3, which are based on the assumption

$$\lim_{\vec{u}\to c\vec{n}} f_S^{-1/2}(\vec{u})\gamma < \infty. \tag{31}$$

We emphasize the following

Theorem 8. *If* u_b *is high enough and* E_b *is large enough, then there will never be direct crucial experimental evidence except for indirect evidence, which would be able to tell us Einstein relativity principle, Einstein symmetry and the relevant results, and our assumption with Result 1, 2 and 3 are true or not.*

6 Conclusion

The equations of laws of physics invariant under c-invariant groups are the analogue of the classical Lorentz invariant equations but with the transformation factors appearing in the equations; especially those from the instantaneous rest frames of particles and the special inertial coordinate system to an arbitrarily given observer – inertial-system. All the equations will go back to their counterparts in the Lorentz invariant theory, if one takes the Lorentz group. The Lorentz invariant theory is that with all the transformation factors equal to 1.

In comparison with Newton's principle, Einstein's theory of relativity is a refinement of the classical Newton theory. It is necessary to know what are the phenomena which are most sensitive to the change of an axiom and those which are not affected at all, in order to avoid doing useless experiments and center attention on those phenomena which are proved to be most sensitive in comparison to the change. Evidently, one wants to verify ones faith in the Einstein symmetry and his first postulate, which claims that when

$$\frac{m_o c^2}{\sqrt{1 - V_{S'S}^2/c^2}} = \tilde{E} = \tilde{M} c^2$$

the S and S'-system are still equivalent, one needs a new theory based on the changed axiom-structure. Only such a new theory can provide the information about sensitivity of various phenomena to the change of the axiom system and a possibility to examine different faiths carefully by indicating the most sensitive phenomena by comparison phenomena and relevant crucial tests. Even if in Theorem 8 u_b is too close to c such that E_b is too large to give any practically measurable deviations of the new theory from the current one within the energy region the objects of experiment will be able to reach, the following theoretical results are still valued if only

$$\lim_{\vec{u}\to c\vec{n}} f_S^{-1/2}(\vec{u})\gamma \neq \infty.$$

(1) The upper limit of the momentum-energy of a particle in any observer-inertial-system is finite. The upper bounds give natural and real ultraviolet cut-offs manifestly contravariant, and the minimum lengths and time intervals of particles in the sense of uncertainty indicating a true meaning of "discrete" or "quantized" space-time and of any model for non-pointlike

elementary particles. The unreasonable operations, "infinities minus infinities", will become "finite quantities minus finite ones" in the renormalizations due to the cut-offs.

(2) All particles with nonzero energies must have nonzero rest masses. The finite 4-momenta of the particles moving with speed c are given by l'Hospital-type limits rather than by the irrational calculation, "zero over zero", without acceptable limit process in Einstein's theory. All the particles moving with speed c obey the corresponding equations with vanishing mass terms. The nonzero rest masses of neutrinos consist with a two-component theory and the nonzero rest masses of photons and other free gauge particles consist with the gauge invariances characterized by vanishing mass terms if these particles move with speed c.

Generally, in any theory invariant under a c-invariant group there are gauge invariances for the gauge particles moving with speed c, which are characterized by the vanishing mass terms, no matter whether "the first postulate" is absolutely valid and the free gauge particles possess zero rest masses. The gauge invariances root in the frame-invariance of the finite transmission rate of interactions. The gauge particles obeying the corresponding gauge invariances possess their nonzero rest masses only in the theories invariant under those c-invariant groups which give the ultraviolet cut-offs.

A new gravitational theory whose zero-field limitation will give the non-Minkowski metric will be established in the future. The difference between it and the general relativity will not be big, but hopeful of success in removing the singularities (which, according to Einstein, must be removed), due to the upper bounds of the densities of particle groups.

Einstein underlines that one should not extend his general relativity to where the gravitational field is very strong and the density of matter is very large [4, p. 129]. In the absence of gravitation, is the Lorentz invariance an absolute truth? We certainly do not assert so. W. Rindler [6] expounds profoundly the nature of physical laws: "*...even the best of physical laws do not assert an absolute truth, but rather an approximation to the truth ... no amount of experimental agreement can ever "prove" a theory, partly because no experiment can ever be infinitely accurate, and partly because we can evidently not test all relevant instances ... Although special relativity is today one of the most firmly established theories in physics ... it is well to keep an open mind even here. ...some law of special relativity may one day be found to fail ... every theory is only a model ... theories should not stagnate in complacency.*" Also, T.D. Lee [5] said that it seems more than likely that our present understanding is transitory and our basic concepts and theories will further undergo major changes. Because of the irrational operations, infinities minus infinities in the renormalizations of the Lorentz invariant quantum field theories, Dirac [3] asserted that the foundations of the current theory must be reformed. In this paper, we actually reform the corner-stone of the current fundamental theory.

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New Relationships and Measurements for Gravity Physics

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The suggested formulation of the laws governing the physics of gravitation provides new phenomenological considerations for a *mathematical method of elucidating and measuring phenomena*. A systematic treatment with broader conceptual framework, than the conventional formalism is presented advancing new physical relationships and fundamental constants that are based on known fundamental constants, physical relationships and high precision measurements.

1 Introduction

We discuss foundational principles of characterizing and formulating the laws governing the physics of gravity. The suggested formulation of the laws governing the physics of gravitation provides new phenomenological considerations/correlations for a *mathematical method of elucidating and measuring phenomena*. A systematic treatment with broader conceptual framework, than the conventional formalism is presented advancing new physical relationships and fundamental constants that are based on known fundamental constants, physical relationships and high precision measurements. Theory (symmetries, scale-invariance, singularities, the Principle of the One-and-the Many) [1], measurements (fundamental constants in quantum electrodynamics [2], and nucleon-meson dynamics [3]) are united. Simple algebra of logical/measurable evidence to predict: 1) physical relationships/quantities, 2) fundamental physical constants, and 3) the basic units of quantities, for the laws governing the physics of gravity are utilized. The suggested representation permits mathematical characterization/testing of new phenomena based on measurements [2], enabling one to calculate/determine new physical relationships [1, 3] and the nature of unmeasured reality. *New quantities* (**) and *relationships* (*) for the: (gravitational flux density, penetrability, potential density, field quantum, resonance condition, gravitance)** and (gravitational field strength, Newton's gravitational constant, mass [flux], gravitational potential, gravitational force)* are suggested. For comparative purposes, the Earth's dimensions and the values of the electron, proton, and neutron constants, as they relate to the fundamental equations of gravitation are given.

Symmetry is a very useful tool in the group theoretical physics [4]. It has been suggested by some authors (Lie [5], Lorenz [6], Einstein [7], Poincaré [8], Heaviside [9], Bateman [10], Cunningham [11], Rainich [12]), that symmetries of Maxwell and Dirac equations, as well as, supersymmetry (a symmetry that connects elementary particles of integer/half-integer spin in common symmetry multiplets, Weinberg [13]), and other differential equations of quantum mechanics [14], produce immensely valuable fundamental results. We suggest: in addition to [4–14] approaches, symmetries (in particular group symmetries) be integrated with fundamental constants and the laws of physics in scale-invariant relationships [1] (see Tables 2, 3, 4), resulting in a very effective phenomenological means to: 1) discover new phenomena; 2) formulate, verify, and elucidate the foundation of physics and astrophysics in general [15], and in particular the broad range nature of gravitation [1, 2, 3]. The suggested formulation places a restriction on the possible solutions of the laws governing the physics of 'gravity', permitting general relationships than those allowable by usual interpretation. Facilitating system of equations within the formalism of wave mechanics [16, 17], an "observer", for continuous characterization and measurement of phenomena.

The validity of any fundamental equation rests in its agreement with experiment. The severe constraint of invariance, normalization and scale changes [3, 15], and the symmetry principle, enables one to advance toward the *invariable foundation of physics* (the Highest Common Factor: nature of invariable/unmeasured reality, where, as we have seen [1, 2, 3], the standard formalism is incomplete) and compute equations from a wide range of probabilities. There is only one system of Poincaré-invariant partial differential equations of first order, for two real vectors $\boldsymbol{E}$ and $\boldsymbol{H}$. This is the system, which translates to Maxwell's equations [4]. It is feasible to "derive" the Dirac, Schrödinger, electromagnetic field [14], and other equations [2, 16, 17] in a comparable manner. It is this rigorous constraint that causes energy quantization. Correspondingly, the equations of Newton, Maxwell, Poincaré, Laplace, d'Alembert, Euler–Lagrange, Lame, and Hamilton–Jacobi have a very high symmetry [4]. It is this high symmetry which is the property distinguishing these equations from other ones considered by physicists and mathematicians.

2 General principles

Central to our methodology is the *singularity* '**1**' (the Principle of the Initial Conditions of measurement: the *dimensionless point*, discussed in [1, 2, 3, 18, 19]. Just as each number, on the mathematical scale, has a unifying principle (zero) as its' *starting frame of reference*, so each physical quantity (and the laws of physics), on the *physics scale of quantities* ([1] equation (2), or [18] equation (4)), has a unifying principle as its' starting frame of reference: the initial conditions of measurement, as in $\mathbf{1} = E/mc^2$. The '**1**' serves also as the experimental underpinning, normalization condition, and the scale-invariant equilibrium frame of reference for E and mc^2 [1]. In this same vein, E serves as surrogate (proxy) equilibrium, and scale-invariant frame of reference for m and c^2 [18]. The *surrogate equilibrium* frame of reference (singularity) defines the *Principle of the Final Condition* of measurement, with an equality ('='), as in $E = mc^2$.

The '**1**' and the equality '=' are *dimensionless points* (law of physics singularities $\gamma = 0$) with vast power to describe the nature of invariable/unmeasured reality. The '**1**' and the '=' represent the *a priori principle of physics* (invariance), and a natural location for the "collapse of the wave function", the *points of inversion* and measurement, also called the "quantum jump" or the point of amplification, which manifests a sharp increase in output signal when (via variation of the magnetic field) the Zeeman splitting frequency is varied through the cavity resonant frequency. It is the '**1**' and the '=' that place a restriction on the possible solutions of the Schrödinger equation: a restriction [3] that leads to energy quantization. In the logarithmic, or the natural log scale, the equilibrium frame of reference (invariance) '**1**' $\to 0$, i.e. $\gamma = 0$ ($\mathbf{1} = 10^0 = e^0 = x^0$) [1].

The concept of the *zero* ('0'), goes back around 300 BC to Babylonians, who used two slanted wedges to represent an *empty space*. In mathematics the concept of the zero has been developed. However, in the formalism of physics, the notion of the singularity (*a priori*) has not been adequately defined/developed with mathematical/experimental formalism. Namely, what its positional notation (the Principle of Position [18]) or fundamental nature is, how it behaves with physical quantities/fundamental constants, or how it may be generalized. The quantum interpretation is not characterizing the *nature of singularity*, but the relationship between reality and its representation, the proxy wave ψ.

Mathematics is our universal language. When *validated by experiment*, mathematics becomes our generalized language of the laws of physics. We know how zero interrelates with numbers, and those numbers with one another. These descriptions take form of the laws governing their interactions. The effect of such laws brings zero and numbers closer together. It changes our understanding of numbers themselves. If you look at a *singularity* $\gamma = 0$ you see a single *dimensionless point*; but glimpse through the singularity and you will see the universe [1, equations (2)–(12)]. At $\gamma = 0$ the concept of space-time loses its meaning. Einstein's equations are violated (i.e., collapse of the wave function: the essence of Gödel's Incompleteness Theorem of 1930), with reality becoming *indeterminable* to the observer, and that human beings will ever expose all ultimate secrets of the universe. For zero (singularity) to be the possibility of universal significance with what it gives power to, we must understand how to add/subtract with it, for a start, replacing it with variety of words for the same thing with concise rules for zero/numbers.

In the Copenhagen interpretation, all the unexplained transitions among the classical/quantum physics occur at the *boundary* connecting measuring/quantum system. We suggest that physical quantities, atoms, and galaxies are the 'quantum entities' and 'observers' (John Wheeler's participators): Georg Cantor's sets, with their own structure and physical laws, that have order (the Principle of Order [18]), endless hierarchy of infinities and *sequence* (the Principle of Position [18]). As a final point, the whole universe may be drawn in as observers: participators: sets: physical quantities, while the boundary between measuring and quantum system is the '**1**' and the '=' points in the unmeasured reality.

Einstein (1924), Dirac (1937), Teller (1948), Landau (1955), Brans and Dicke (1961), DeWitt (1964), Isham, Salam and Stratdhdee (1971), Salam and Wigner (1972), and others have suggested a variety of approaches leading to a relation between gravitation, electromagnetism, and cosmology. To formulate the nature of 'gravitation' (so that from any given physical conditions equations relating the physical quantities may be deduced or vice versa), we systematize the laws of physics and the fundamental physical constants (of quantum electrodynamics and nucleon-meson dynamics [2, 3, 15]) through the singularity '**1**' in the *Principle of the One-and-the Many* and the Logarithmic Slide-Rule for Physical Relationships (LSPR) [1, 18, 19, 20].

3 The Principle of the One-and-the-Many

The Principle of the One-and-the-Many rests on the Principle of the Initial Conditions and the Principle of the Final Conditions of measurement wherein conceivable property of the *one* (individual quantity q_k) is also a property of the many (a number of q_k's: group quantity Q_k). If we regard a number of identical balls as many (Georg Cantor's *sets*), having a unity between them, then it is feasible to roll up the balls (or the *null, empty sets*) and mathematically unite them together, thereby moving from the many into the one. Indeed, *unity and multiplicity are two inverse views* of the same phenomena (Table 1). It is instructive to consider that in any *equilibrium*, it is impossible to have a group of balls without having individual balls and vice versa ([1, equation (2)] or [18, equation (4)]).

Note 1. The nature of space has dominated our thinking. Customarily, a discrete bundle of energy is called a quantum. Our work [1, 18] indicates that physical quantities are comprised of discrete bundles of the '**1**' (singularities): a number invariant q_k points of *empty space*: Final Equilibrium (Invariance) state. In suggested formalism, empty space (the Highest Common Factor) is an *invariant physical structure* with properties of its own. In addition, each singularity has two inverse points of view. That is, as in (1), the q_k of '**1**': the *point*: singularity, and the Q_k of '**1**': the *empty space*: Cantor's/Gödel's/Cohen's *Continuum* (where the universe and the laws of physics materialize: Plato's "receptacle, and in a manner the nurse, of all generation": Einstein's (1924) "Continuum which is equipped with physical properties; for the general theory of relativity") are *inverted viewpoints* of the same reality. Namely, the inverse of *one* zero dimension point (where the log of 1 is zero) is the *many* zero dimension points (where the inverse of log 1 is the *Absolute Infinity* [1, 18]. Between these two inverse landscapes encoded potential possibilities exist. That is, in the Principle of the One/Many, the individual phenomenon q_k is an inverted group phenomena Q_k, where

$$\mathbf{1} = Q_k q_k, \tag{1}$$

and q_k is either equal to, or less than ($\leq$) $\mathbf{1}$, or $\mathbf{1}$ is equal to, or less than ($\leq$) Q_k, where

$$q_k \leq \mathbf{1} \leq Q_k. \tag{2}$$

The q_k and Q_k values are determined by physical constants. Consider the following illustrative examples, in Table 1, of individual and collective phenomena in the One-and-the-Many Principle (proposed new gravitation quantities are highlighted with bold letters):

Table 1. The "**1**" and the One-and-the-Many principle through the laws of physics.

q_k: Individual Quantity	Q_k: Group Quantity
Period of harmonic motion T	Frequency f $(= \mathbf{1}/T)$, where $Tf = \mathbf{1}$
Conductance G	Resistance R $(= \mathbf{1}/G)$, where $GR = \mathbf{1}$
Inductance L	Reluctance r $(= \mathbf{1}/L)$, where $Lr = \mathbf{1}$
Resistivity ρ	Conductivity σ $(= \mathbf{1}/\rho)$, where $\rho\sigma = \mathbf{1}$
Compton wavelength λ_c	Number of waves n $(= \mathbf{1}/\lambda_c)$, where $\lambda_c n = \mathbf{1}$
Magnetic flux quantum $\Phi_0 = h/2e$	Josephson constant $2e/h$, where $(\Phi_0)(2e/h) = \mathbf{1}$
Quantized Hall conductance e^2/h	von Klitzing constant $R_K = h/e^2$, where $(e^2/h)(R_K) = \mathbf{1}$
Gravitational penetrability z_0	Gravitational Constant G $(= \mathbf{1}/z_0)$, where $(z_0)(G) = \mathbf{1}$
Gravitational field quantum $\acute{\Gamma}$	Gravitational field strength g $(= \mathbf{1}/\acute{\Gamma})$, where $(\acute{\Gamma})(g) = \mathbf{1}$
Gravitational resonance cond. LC	**Gravitational potential density** ϑ $(= \mathbf{1}/LC)$, where $(LC)(\vartheta) = \mathbf{1}$

The singularity/Continuum are of *absolute uncertainty* (point of inversion: a natural location for the collapse of the wave function, while the mathematical formalism of the Heisenberg uncertainty relationships (expressed in terms of the building blocks of nature, i.e., energy/time) is of *relative uncertainty.* At a singularity the laws of science and our ability to predict, break down [1, 18]. The Heisenberg uncertainty relationships demonstrate the workings of a singularity, as expressed in equations (1) and (2). Similarly, because of the '**1**'/'=' in (1), a particle cannot be expressed by a wave packet, in which both the momentum and the position have arbitrary ranges. They must be scale-invariant [2, equations (36)–(40)]. As we make the range of one of them larger the range of the other becomes smaller, according to equation (1). Here the quantum uncertainty is not tied to one particular quantity but slides from quantum entity to quantum entity (the Principle of Position [18]: a rainbow appears at a different time in a different place with different intensity for *each* observer [19].

Physical quantities (or meaning), on the other hand, are created by *limits.* Namely, *meaning illustrates the unit interval between* ('**1**'/'=') *points* (singularities) [18]. To represent a physical quantity: natural unit-of-measurement (the Principle of Natural-Unit-of-Measurement [18]: Georg Cantor's non-empty set) in the Continuum *two* scale-invariant points in equilibrium ('**1**'/'=') are required. The unit interval (i.e., of open and closed lines, or Edward Witten's strings) between points (nodes) in the Continuum can be established via the fundamental physical constants, or with two quantities in terms of which a third quantity is described. For example, velocity is characterized in terms of m/sec. By analogy to mathematical zero's role, as a *shifter in value* (i.e., 83 to 80003), the '**1**'/'=' states in the Continuum are said to be quantized (natural grouping: the Principle of Quantization [18]). This means, ultimately the '**1**' (the Continuum) and the equality '=' states are scale-invariant, dimensionless, and quantized. It also suggests: Coleman et al. [21], *the invariance of the Continuum is the invariance of the Universe*, which is discussed elsewhere [18].

The concept of *dimension* is fundamental to all of mathematics and physics. With a series leaps of insight, the work of Euclid (defining a *point*); Eudoxus (introducing the concept of a *potential infinity*: enabling Newton, Leibniz, Gauss, Euler, and others approach zero/infinity, thus facilitating development of a *limit*); Bolzano–Weierstrass (showing that infinite sequences in a bounded space contain *limit points*); Galileo (leaping from potential infinity to *actual infinity*: an infinite set can be equal in number of elements to the smaller subset of itself); Cantor (arriving at *actual infinity*, and learning important truths about it, starting with sets); Peano (characterized as the empty set); Hahn–Banach (giving conditions where a linear functional can be extended to the full space that shares boundedness conditions with the functional); Zermelo (helping to design axioms of set theory), Gödel (proposing the incompleteness theorem); Cohen (concluding that the Continuum is beyond the lower infinities), and others, unlocked a door to the Absolute Infinity.

Cantor gave his sequence $\varkappa_0, \varkappa_1, \varkappa_2, \varkappa_3, \ldots$ of alephs (infinities) the name taf, π, to mean finality: every infinite cardinal had to be an aleph — belonging to the system π that includes all alephs. From the Principle of the One/Many, we see through the laws of physics, equation (1), and Table 1, a point and the Absolute Infinity are the inverse of each other. Using Cantor's sets, in the One/Many group, we have: $\varkappa_0 = 1/\pi$, where the point is '**1**' : $\varkappa_0$, and the Continuum is '**1**' : π. Therefore, as with zero/numbers, and numbers with one another, the suggested interpretation brings singularity, physical quantities and the fundamental physical constants closer together: changing our understanding of the quantities themselves. The observer's experience is expressed in the classical language of actualities (physical quantities), while the invariable/unmeasured quantum realm is not represented as a wavewise superposition of possibilities, or taf, π, but with the '**1**'. This offers a more general framework, than that provided by the standard interpretation for 'gravity' physics. In particular, when we consider the gravitational/mass relationships in nuclear/earth's dimensions, gravity no longer is gravity, because of scale invariance, becomes strong or weak force [1, equation (30)].

The LSPR operates on the same mathematical principle as a logarithmic slide-rule for numbers [1, 19, 20]. A logarithmic slide-rule has zero/numbers. A LSPR has the '**1**' and physical quantities. The LSPR generates experimentally verified equations of the laws of physics and fundamental physical constants. Furthermore, a LSPR will, in simple equations, give form to (i.e., predict) new (unknown: 'hidden' variables, Bohm [22]) physical relationships/constants. A more comprehensive discussion on the Principle of the Initial Conditions ('**1**'), the Principle of the One-and-the-Many, and the LSPR, can be found in [1, 18]. Conventional symbols and the SI units (in which they are usually quoted) are deployed.

Measuring a physical quantity signifies comparing the quantity with a standard quantity (unit of measurement) of the same scale and nature. To make the relationships more accessible to physicists who work with gravitational phenomena, model building, and to facilitate new/more encompassing gravitational experiments and precision measurements (of the basic physical relations/constants), we restate the meaning of these equations by providing several relationships for the same phenomena. These equations advance the development through direct observation of new experimental investigations in gravity, enabling one to formulate additional systems of units and their conversion factors of physics.

4 The gravitational field strength $\boldsymbol{g}$

The gravitational field strength $\boldsymbol{g}$ $(= \boldsymbol{F}/m)$, in $\mathrm{m \cdot s^{-2}}$, at a point of the gravitational *force* $\boldsymbol{F}$, in N, per unit mass m, in kg, can be written as the *angular gravitational potential* [1, equation (37)]

$$\boldsymbol{g} = V_g/S, \tag{3}$$

where V_g is the gravitational potential, in $\mathrm{J \cdot kg^{-1}}$, and S is the *length* unit, in $\mathrm{m \cdot rad^{-1}}$.

Note 2. The unit radian (rad), for plane angle, has historically been designated as a supplementary unit [1]. In 1980, the International Committee for Weights and Measures determined that the unit radian and steradian are equivalent to the number one 1 and may be omitted in the expression for derived units. For completeness of presentation, due to the angle of rotation, expressed in radiant per cycle, is a physical quantity, which like other quantities enters into physical relationships, it is included here. Furthermore, as stated earlier, to represent a quantity, two dimensionless points are necessary.

Notice, deploying constants from [2] in equation (2), the $\boldsymbol{g}$ and V_g quantities, in equation (3), for the electron, are larger than '**1**', therefore, they are Q_{kq} quantities. Additionally, the S quantity, for the electron in [2], is smaller than '**1**', therefore, it is a q_k quantity. To make equation (3) for students more logical, we could describe (3) in Q_{kq} terms (i.e., $\boldsymbol{g} = V_g$ times the number of separation points $1/S$), or as a quantum q_k expression.

Therefore, following this approach, we can write equation (3) as a relationship of the *gravitational flux density* $\boldsymbol{M}$, where $\boldsymbol{M} = m/S^2$, $\mathrm{kg} \cdot \mathrm{m}^{-2} \cdot \mathrm{rad}^{-1}$, and the *Newtonian constant of gravitation* G, in $\mathrm{m}^3 \cdot \mathrm{kg}^{-1} \cdot \mathrm{s}^{-2}$, whereas

$$\boldsymbol{g} = \boldsymbol{M}G. \tag{4}$$

Moreover, entering equation (4) as a relationship of *pressure* P, in Pa, and the gravitational flux density, gives us

$$\boldsymbol{g} = P/\boldsymbol{M}. \tag{5}$$

Combining equations (4) and (5), provides

$$\boldsymbol{g} = (GP)^{1/2}. \tag{6}$$

5 The gravitational constant G

The Gravitational Constant G, in $\mathrm{m}^3 \cdot \mathrm{kg}^{-1} \cdot \mathrm{s}^{-2}$, is routinely stated in terms of the gravitational force $\boldsymbol{F}$, that two particles of masses m_1 and m_2 separated by a distance S exert on each other, where $\boldsymbol{F} = Gm_1m_2/S^2$. We characterize the universal constant G by way of the gravitational field strength $\boldsymbol{g}$ and the gravitational flux density $\boldsymbol{M}$, as shown in equation (4), $G = \boldsymbol{g}/\boldsymbol{M}$. The Newtonian constant of gravitation G can also be given in terms of the gravitational potential V_g and the *linear mass density* μ $(= m/S)$, where the mass per-unit-length of μ is in $\mathrm{kg} \cdot \mathrm{m}^{-1}$. Thus,

$$G = V_g/\mu. \tag{7}$$

This then leads us to a third method representing equation (7) using the length S and the *gravitance* $\Omega = m/V_g$, in $\mathrm{kg}^2 \cdot \mathrm{J}^{-1}$, equation (21). Accordingly,

$$G = S/\Omega. \tag{8}$$

Combining equations (4) with (5) gives

$$G = P/\boldsymbol{M}^2 = 1/\chi\boldsymbol{M}^2, \tag{9}$$

whereby χ is compressibility in $\mathrm{m}^2 \cdot \mathrm{N}^{-1}$.

Note 3. The Newtonian constant of universal gravitation is the constant of proportionality within an equation relating to the attraction force between any two bodies (particles) separated by distance S. In a scale-invariant setting of nuclear dimensions, *transformation* of physical quantities and scale changes (renormalizability) take place [1, 3, 15, 16]. The dimensional values of quantities (i.e., G, S, m, $\boldsymbol{F}$) are no longer gravitational values of the Earth, but nuclear values. This can be seen, by defining the gravitational constant G from the law of periods.

Note 4. As suggested earlier, the Continuum is invariant. Separation between points, in the Continuum, determines the meaning of the natural unit-of-measurement (i.e., physical quantity) [18], making the laws of physics invariant. Considering Note 3 and the lack of scale-invariance in general relativity [18], reduce the general relativity to a limited construct, that is, says Hoyle, Burbidge and Narlikar [23]: "the equations of general relativity are not scale-invariant. They are the special form to which the scale-invariant equations reduce with respect to a particular scale, namely that in which particle masses are everywhere the same".

6 The gravitational potential V_g

The Gravitational Potential V_g, at a point, is the potential energy per unit test mass, in $\mathrm{J \cdot kg^{-1}}$. The gravitational potential is usually determined using $V_g = -Gm/S$. The V_g can also be depicted as the linear stopping power, where $V_g = \boldsymbol{g}S$. Furthermore, the gravitational potential can be expressed as an *area* $\boldsymbol{A}$, in m^2, and the *angular speed* (rotation rate) ω, in $\mathrm{rad \cdot s^{-1}}$, where

$$V_g = \boldsymbol{A}\omega^2. \tag{10}$$

Equation (10) is, in addition related to purely electric and magnetic quantities by

$$V_g = 1/\varepsilon_0\mu_0. \tag{11}$$

For ε_0 and μ_0 terms see the gravitational penetrability, in Section 7. Equation (10) can also be characterized by the gravitational field strength and the *gravitational potential density* ϑ, in $\mathrm{rad^2 \cdot s^{-2}}$ (Section 10), where

$$V_g = \boldsymbol{g}^2/\vartheta. \tag{12}$$

Also,

$$V_g = \boldsymbol{g}^2/\omega^2. \tag{13}$$

7 The gravitational penetrability

Analogous to the (electric) permittivity of vacuum ε_0 $(= C/S)$, in $\mathrm{F \cdot m^{-1}}$, where C is the capacitance in farad (F), and the (magnetic) permeability of vacuum μ_0 $(= L/S)$, in $\mathrm{H \cdot m^{-1}}$, where L is the inductance in henry (H), we propose a new physical quantity, the (gravitational) *penetrability of free space* (vacuum) z_0 $(= \Omega/S)$, equation (22), expressed in $\mathrm{kg \cdot s^2 \cdot m^{-3}}$. The gravitational penetrability is the inverse of the gravitational constant G

$$z_0 = 1/G. \tag{14}$$

Combining the angular rotation rate and the density d $(= m/V_0)$, in $\mathrm{kg \cdot m^{-3}}$, where V_0 is volume, in $\mathrm{m^3}$. We can write equation (14) as

$$z_0 = d/\omega^2. \tag{15}$$

Equation (14) can also be stated in terms of the mass m, gravitational potential V_g, and the separation S, where

$$z_0 = m/V_gS. \tag{16}$$

Or, in terms of work W, in J, volume and the gravitational field strength,

$$z_0 = W/V_0g^2. \tag{17}$$

8 The gravitational flux density

Comparable to the electric flux density $\boldsymbol{D}$ $(= \varepsilon_0 \boldsymbol{E})$, in $\mathrm{C} \cdot \mathrm{m}^{-2}$, and the magnetic flux density $\boldsymbol{B}$ $(= \mu_0 \boldsymbol{H})$, in T ($\mathrm{Wb} \cdot \mathrm{m}^{-2}$), we derived the *gravitational flux density* $\boldsymbol{M}$, in $\mathrm{kg} \cdot \mathrm{m}^{-2} \cdot \mathrm{rad}^{-1}$, where

$$\boldsymbol{M} = z_0 \boldsymbol{g}. \tag{18}$$

Equation (18) can be represented as a relationship of the pressure and the gravitational field strength, $\boldsymbol{M} = P/\boldsymbol{g}$, or the gravitational field strength and the gravitational constant G, where $\boldsymbol{M} = \boldsymbol{g}/G$. In addition, the gravitational flux density can be given by the mass m and a unit of length S, where,

$$\boldsymbol{M} = m/S^2. \tag{19}$$

Equation (19) can also be described in terms of the Hooke's law proportionality *spring constant* k, in force per unit length, and the gravitational potential, where

$$\boldsymbol{M} = k/V_g. \tag{20}$$

9 The gravitance

Comparable to the capacitance $(C = e/V)$ in the electric domain and the inductance $(L = \phi/i)$ in the magnetic domain we suggest the *gravitance* Ω, stated in $\mathrm{kg}^2 \cdot \mathrm{J}^{-1}$, in the gravitational domain to be:

$$\Omega = m/V_g, \tag{21}$$

where e is the electric flux (elementary charge), in C, ϕ is the magnetic flux, in Wb, V is the electric potential, in V, and i is the magnetic potential, in A. The gravitance can also be found from (8), where $\Omega = S/G$. Combining (8) with (14) we obtain,

$$\Omega = S z_0. \tag{22}$$

Also, the gravitance can be written by uniting equations (11) and (21), where

$$\Omega = \varepsilon_0 \mu_0 m. \tag{23}$$

10 The gravitational potential density

Comparable to the magnetic potential (current) density $\boldsymbol{j}$ $(= i/\boldsymbol{A})$, in $\mathrm{A} \cdot \mathrm{m}^{-2}$, and the electric potential (voltage) density $(= V/\boldsymbol{A})$, in $\mathrm{V} \cdot \mathrm{m}^{-2}$, we suggest the *gravitational potential density* ϑ, expressed in $\mathrm{rad}^2 \cdot \mathrm{s}^{-2}$, where

$$\vartheta = V_g/\boldsymbol{A}. \tag{24}$$

Equation (24) can, in addition, be characterized via the gravitational field strength and S, where

$$\vartheta = \boldsymbol{g}/S. \tag{25}$$

Furthermore, equation (25) can be written as the time t, in $\mathrm{s} \cdot \mathrm{rad}^{-1}$, where

$$\vartheta = 1/t^2. \tag{26}$$

We can also find the gravitational potential density through the gravitational flux density and gravitance

$$\vartheta = \boldsymbol{M}/\Omega. \tag{27}$$

11 The gravitational force

Analogous to the electromagnetic field which exerts sideways electric $\boldsymbol{F}_y = VS \times \boldsymbol{D}$ and magnetic $\boldsymbol{F}_z = iS \times \boldsymbol{B}$ forces, we suggest that a sideways *gravitational force* $\boldsymbol{F}_x$, in N, is exerted in the gravitational field subjected to the gravitational flux density, whereby,

$$\boldsymbol{F}_x = V_g S \times \boldsymbol{M}, \tag{28}$$

where, $VS = e/\varepsilon_0$; $iS = \phi/\mu_0$; and $V_g S = m/zo = mG$. For a second approach, consider that $\boldsymbol{F}_y = e\boldsymbol{H}$, $\boldsymbol{F}_z = \phi\boldsymbol{H}$, then

$$\boldsymbol{F}_x = m\boldsymbol{g}, \tag{29}$$

where the electric field strength $\boldsymbol{E}$ is in $\mathrm{V \cdot m^{-1}}$, and the magnetic field strength $\boldsymbol{H}$ is in $\mathrm{A \cdot m^{-1}}$. Notice that the electric, magnetic and gravitational forces are in equilibrium at singularity ('='): $\boldsymbol{F}_y = \boldsymbol{F}_z = \boldsymbol{F}_x$. Further, in electromagnetic traveling waves when the lines of $\boldsymbol{E}$ are parallel to the y-axis, and the lines of $\boldsymbol{B}$ are parallel to the z-axis, we suggest that, the lines of $\boldsymbol{M}$ are parallel to the x-axis. Our observation indicates that $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{M}$ are perpendicular to one another. In addition, consistent with equation (11) $V_g = 1/\varepsilon_0\mu_0$, and therefore is on the velocity axis. Furthermore, $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{M}$ are in phase (they achieve their maxima at the identical time, and they are zero at the same time).

12 The gravitational resonance condition

In our discussion of resonances, a damped LC circuit oscillating at natural frequency $\omega = (LC)^{-1/2}$, is described in terms of gravitational quantities as the *gravitational resonance condition*. We derived the LC condition by way of the gravitance and the gravitational flux density, where

$$LC = \Omega/\boldsymbol{M}. \tag{30}$$

Also, equation (30) can be written as a relationship of the area and the gravitational potential [1, equation (23)],

$$LC = A/V_g. \tag{31}$$

In addition, equation (30) can be expressed by the gravitational potential and the gravitational field strength, where

$$LC = V_g/\boldsymbol{g}^2. \tag{32}$$

Combining equations (31) with (32) gives

$$LC = S/\boldsymbol{g}. \tag{33}$$

Furthermore,

$$LC = 1/\vartheta. \tag{34}$$

13 The gravitational field quantum

The One-and-the-Many Principle, can be utilized with the gravitational field strength $\boldsymbol{g}$ to find the *gravitational field quantum* $\acute{\Gamma}$, in $\mathrm{s}^2 \cdot \mathrm{m}^{-1}$, where

$$\acute{\Gamma} = 1/\boldsymbol{g}. \tag{35}$$

Earlier we expressed the gravitational field strength in Q_k language. Equation (36) is a *quantum* q_k expression, where the gravitational field quantum equals the *gravitational potential quantum* $(1/V_g)$ times the separation quantum S,

$$\acute{\Gamma} = (1/V_g)S. \tag{36}$$

Additionally, the gravitational field quantum $\acute{\Gamma}$ and the velocity v can be used to describe the time quantum t, we find

$$t = \acute{\Gamma}v. \tag{37}$$

For the electron $t = 1.1812 \times 10^{-22}$ seconds^2. Derivation/higher accuracy constants can be found in [2, equation (10)]. In Tables 2, 3 and 4 we list gravitational (x), electric (y) and magnetic (z) correlation between the quantities and their relationships.

Table 2. Symmetries of electric, magnetic and gravitational quantities.

1	$\boldsymbol{F}/e =$ Electric field strength	$\boldsymbol{E}$
	$\boldsymbol{F}/\phi =$ Magnetic field strength	$\boldsymbol{H}$
	$\boldsymbol{F}/m =$ Gravitational field strength	$\boldsymbol{g} = S/LC$
2	$W/e =$ Electric potential	$V = \boldsymbol{E}S$
	$W/\phi =$ Magnetic potential	$i = \boldsymbol{H}S$
	$W/m =$ Gravitational potential	$V_g = \boldsymbol{g}S = A/LC$
3	$e/A =$ Electric flux density	$\boldsymbol{D} = \boldsymbol{Y}C$
	$\phi/A =$ Magnetic flux density	$\boldsymbol{B} = \boldsymbol{J}L$
	$m/A =$ Gravitational flux density	$\boldsymbol{M} = \vartheta\Omega = \Omega/LC$
4	$V/A =$ Electric potential (voltage) density	$\boldsymbol{Y} = \boldsymbol{E}/S$
	$i/A =$ Magnetic potential (current) density	$\boldsymbol{J} = \boldsymbol{H}/S$
	$V_g/A =$ Gravitational potential (m) density	$\vartheta = \boldsymbol{g}/S = 1/LC$
5	$e/V =$ Capacitance	$C = e^2/W = \boldsymbol{D}/\boldsymbol{Y}$
	$\phi/i =$ Self inductance	$L = \phi^2/W = \boldsymbol{B}/\boldsymbol{J}$
	$m/V_g =$ Gravitance	$\Omega = m^2/W = \boldsymbol{M}LC = \boldsymbol{M}/\vartheta$
6	$\boldsymbol{F}/\boldsymbol{E} =$ Electric flux	$e = W/V$
	$\boldsymbol{F}/\boldsymbol{H} =$ Magnetic flux	$\phi = W/i$
	$\boldsymbol{F}/\boldsymbol{g} =$ Gravitational flux (mass)	$m = W/V_g$

Table 3. Symmetries of gravitational ($\boldsymbol{x}$), electric ($\boldsymbol{y}$) and magnetic ($\boldsymbol{z}$) quantities.

	x	y	z
1	$\Omega = m/V_g$	$C = e/V$	$L = \phi/i$
2	$zo = \Omega/S$	$\varepsilon_0 = C/S$	$\mu_0 = L/S$
3	$\boldsymbol{M} = z_0\boldsymbol{g}$	$\boldsymbol{D} = \varepsilon_0\boldsymbol{E}$	$\boldsymbol{B} = \mu_0\boldsymbol{H}$
4	$G = \boldsymbol{g}/\boldsymbol{M}$	$1/\varepsilon_0 = \boldsymbol{E}/\boldsymbol{D}$	$1/\mu_0 = \boldsymbol{H}/\boldsymbol{B}$
5	$\boldsymbol{F}_x = m\boldsymbol{g}$	$\boldsymbol{F}_y = e\boldsymbol{E}$	$\boldsymbol{F}_z = \phi\boldsymbol{H}$

14 Results and discussion

One of the most significant advances in the field of physics was the scientific method: the procedure physicist use to gain knowledge. To quantify the experiment's results, measurement of phenomena has been essential to the scientific method. We suggest comparatively simple systematic

Table 4. Comparison of the electron, proton, neutron, and earth calculations and constants.

Quantity	Symbol	Earth	Electron	Proton	Neutron	Units
Mass	m	5.972×10^{24}	9.109×10^{-31}	1.673×10^{-27}	1.675×10^{-27}	kg
Gravitational constant	G	6.674×10^{-11}	3.494×10^{33}	7.922×10^{29}	7.909×10^{29}	$\mathrm{m}^3 \cdot \mathrm{kg}^{-1} \cdot \mathrm{s}^{-2}$
Gravitational field strength	g	9.807	2.538×10^{30}	1.262×10^{27}	1.260×10^{27}	$\mathrm{m} \cdot \mathrm{s}^{-2}$
Gravitational potential	V_g	6.252×10^{7}	8.988×10^{16}	1.293×10^{15}	1.292×10^{15}	$\mathrm{J} \cdot \mathrm{kg}^{-1}$
Gravitational penetrability	z_0	1.498×10^{10}	2.862×10^{-34}	1.262×10^{-30}	1.264×10^{-30}	$\mathrm{kg} \cdot \mathrm{s}^2 \cdot \mathrm{m}^{-3}$
Gravitational flux density	M	1.469×10^{11}	7.265×10^{-4}	1.593×10^{-3}	1.593×10^{-3}	$\mathrm{kg} \cdot \mathrm{rad}^2 \cdot \mathrm{m}^{-2}$
Gravitance	Ω	9.553×10^{16}	1.014×10^{-47}	1.294×10^{-42}	1.296×10^{-42}	$\mathrm{kg}^2 \cdot \mathrm{J}^{-1}$
Gravitational potential density	ϑ	1.538×10^{-6}	7.1673×10^{43}	1.232×10^{39}	1.229×10^{39}	$\mathrm{rad}^2 \cdot \mathrm{s}^{-2}$
Gravitational force	F_x	5.857×10^{25}	2.312×10^{0}	2.111×10^{0}	2.110×10^{0}	N
Gyroradius	S	6.378×10^{6}	3.541×10^{-14}	1.025×10^{-12}	1.025×10^{-12}	$\mathrm{m} \cdot \mathrm{rad}^{-1}$
Gravitational field quantum	$\acute{\Gamma}$	1.019×10^{-1}	3.940×10^{-31}	7.924×10^{-28}	7.937×10^{-28}	$\mathrm{s}^2 \cdot \mathrm{m}$

treatment/methodology of how, by unifying *theory* (algebra of logical and measurable evidence) with *measurement* (known fundamental constants/laws of physics), and an indirect procedure of physical constant and fundamental quantity formation (symmetry, scale-invariance, et al.), improved understanding in the observation/measurement, the formulation of physical laws and the development of a theory that is used to predict new phenomena can yield otherwise unobtainable results. Fortunately, the *Continuum is invariant* '**1**'. Known laws of physics and fundamental physical constants (obtained through high precision measurements of the Continuum, i.e., separation between '**1**' and '=') are reducible to mathematical relations/operations, which are *constant*, and which can be used to penetrate, define, calculate, and predict more accurate measurements [2] of fundamental quantities (Table 2), values of the physical constants, '*a priori*' numerical computations, discovery of new phenomenon, and looking or thinking about a problem in a totally different way.

Based on these elementary considerations and systematic procedures, one of the important objects of this note consists in suggesting very simple formulas, physical relationships, fundamental constants, and experimental tests for gravity physics. As far as we know, there are less than 5000 physical relationships that have been verified, in the last 400 years by science. Using five constants [2], computers, and the suggested methodology, we obtained over 100,000 physical relationships, and more than 20 new physical quantities, some of which are presented here as gravity physics. Whether the formulas, elucidation of their properties, and correlation depicted in the present note is consistent with experimental facts is an open question. However, the approach is based on experimental data (known constants and laws of physics) and the agreement of the values in Appendix A and B is very strong evidence in support of this methodology of *measurement-based, mathematical procedure* for obtaining these results. Also, it might be pointed out, the remarkable agreement between other analogous formulas generated in similar manner [2, 4], and the experiments, can leave but little doubt that the suggested phenomenological relationships constitute gravity phenomena.

Further, the experimental support of the approach indicates very convincingly that the integration of a scattered and immense body of fundamental physical phenomena into a more systematic order is possible [1, 19, 20]. It should be noted, at the present stage of physics we are not able to predict accurately new gravitational quantities or fundamental constants of nucleon-meson dynamics [3]. Should experiments corroborate the suggested relationships/constants, physicist would gain a phenomenological leap in our understanding of the Continuum '**1**', through a simple mathematical method for the measurements of phenomena, formulation of physical laws from the generalization of the phenomena and the development of theories that is used to predict new phenomena. Furthermore, because the Continuum is invariant and there are infinite potentialities within it, measurements between '**1**' and '=' seems likely to continue (characterizing fundamental quantities/constants, revealing hitherto ignored physical effects, inducing *inverted* populations to radiate in concert (through the Principle of One/Many), i.e., generate coherent states of the gravitational field, laser technology, et al.), I see no let up in the bread-and-butter business of measuring and theoretical prediction of phenomena. Conversely, I expect accelerated advancement and greater opportunities in all branches of science.

Appendix A. Sample derivations and calculations

In [2] we have provided the derivation of the fundamental constants in quantum electrodynamics, and in [3] the fundamental physical constants of nucleon-meson dynamics. Now, we will discuss the derivation of Earth's dimensions.

To obtain the standard acceleration of gravity $\boldsymbol{g}$ we utilized the 1998 CODATA (the Committee on Data for Science and Technology, the international arbiter of metrology) set of recommended values of the basic constants and conversion factors of physics [24]. The preliminary value of G comes from higher precision measurement of the gravitational constant by Jens Gundlach and Stephen Merkowitz at the University of Washington, (Seattle), reported at APS April 2000 meeting in Long Beach (Ca) [25]. The gravitational flux density $\boldsymbol{M}$ is a derivative of $\boldsymbol{M} = \boldsymbol{g}/G$, while the gravitational penetrability, of equation (14), is $z_0 = 1/G$.

The Earth has an equatorial radius of 6.378×10^6 m, a polar radius of 6.357×10^6 m, and a mean radius of 6.371×10^6 m. The Earth's gravitational field (polar surface gravity) varies from place to place on it's surface, with the main variation occurring with latitude, averaging approximately $9.8322\ \mathrm{m \cdot s^{-2}}$at the poles, and at the Equator (equatorial surface gravity) $9.7303\ \mathrm{m \cdot s^{-2}}$ (includes rotation). We deployed the standard acceleration of gravity value $9.80665\ \mathrm{m \cdot s^{-2}}$, the Earth mass m of $5.97223(\pm 0.00008) \times 10^{24}$ kg, and the Newtonian constant of gravitation values of G $6.674215 \pm 0.000092 \times 10^{-11}\ \mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}}$, to find the Earth's value of the length of the semi-major axis S, via $S^2 = m/\boldsymbol{M}$ and $\boldsymbol{M} = \boldsymbol{g}/G$. This method yields the length of the semi-major axis of S 6.37541×10^6 m. The length of the semi-major axis can customarily be determined by the arc method. The determinations of dimensions of the Earth ellipsoid from arc measurements yield 6.37816×10^6 m for the semi-major axis. Similarly, utilizing the G, $\boldsymbol{g}$, and the Earth ellipsoid from arc measurements, of 6.37816×10^6 m value in equations (19) and (4), where $m = S^2\boldsymbol{g}/G$, we attain 5.97734×10^{24} kg for the Earth's mass. Because we were studying the same problem from two different points, for the two approaches to be compatible, the present measurements (of the length of the semi-major axis or the Earth's mass) could be refined, i.e., $\boldsymbol{g}$, G, and S measurements would have to use identical initial conditions ('**1**').

Appendix B. Comparison of the electron, proton, neutron, and earth calculations and constants

For comparative purposes we have computed the electron [2], proton and neutron [3] length units. Notice the proton and the neutron length of the semi-major axis S is approximately 30 times larger than that of the electron.

The gravitance value is obtained via equation (22) ($\Omega = Sz_0$), the gravitational potential via ($V_g = S\boldsymbol{g}$), and the gravitational potential density by equation (25) ($\vartheta = \boldsymbol{g}/S$). In Table 4 we list values of the electron, proton, neutron, and Earth calculations and constants, as they relate to 'gravitational' relationships.

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D-branes, B Fields and Deformation Quantization

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The worldvolume geometry of flat and curved Dp-branes embedded in flat and curved background spaces in the zero slope limit of Seiberg and Witten is studied.

1 Introduction

Dp branes in the Type II superstring theory with nonzero NS-NS B fields have the interesting features. When B field is switched off the system of the D0-D6 BPS branes is nonsupersymmetric. But it becomes supersymmetric when a suitable constant B field is turned on.

In the absence of the B field the system of D0-D4 branes is supersymmetric. But the presence of the B field changes the properties of supersymmetry. The D0-D4 system of branes remaines supersymmetric only if the B field is anti-self-dual [1].

An identification of the Dp-brane charges with K-theory classes holds in the case of vanishing B field. In the presence of a B field the arguments of [2] have to be modified. The point is that a gauge field in the presence of a B field is rather a connection over a noncommutative algebra than over a vector bundle. Therefore it is natural to suspect that Dp-brane charges must be identified with K-theory classes of some noncommutative algebra. It is the principal property of Dp-branes with switched B fields is following: theirs worldvolume geometry is noncommutative.

The noncommutative geometry studies geometric spaces (and their generalizations) using noncommutative algebras of functions on them. The noncommutative torus is one of the most important examples of the manifolds in noncommutative geometry. The noncommutative geometry of the worldsheet plays an important role in the study of the string theory. These problems have attached much attention [3, 4, 5, 6].

But most of them were dealing with the case of a constant B field in the flat background. Connes, Douglas, Schwarz [3] have shown that the matrix theory of M theory compactified on a T^2 with a background three form potential, C_{-12} is related to gauge theory on a noncommutative torus. Douglas and Hull [4] have studied Dp-branes on T^2 with the constant NS-NS two form field, B, and have shown that the effective worldvolume theory will be noncommutative gauge theory on the noncommutative torus.

When a Dp-brane is placed into a background which carries a non-vanishimg constant B field the algebra of functions on its classical worldvolume is deformed. The involving of this constant B field background can be described by replacing the ordinary product of functions on the worldvolume of the Dp-branes by the Moyal product, which is associative and noncommutative. This case corresponds to the embedding of a flat Dp-brane into a flat background.

In the zero slope limit $\alpha' \to 0$ of Seiberg and Witten this case is extended to the one in which $\omega = B + F$ is such that $d\omega = 0$. Here F is the strength of some $U(1)$ gauge field on a Dp-brane. In this case the ordinary product of functions on the worldwolume of a Dp-brane is replaced by the Kontsevich star product which is also associative and noncommutative. This case corresponds to the embedding of a curved Dp-brane into a flat background.

There are several attempts to extend this consideration to open strings in a general background. In the terminology of Dp-branes it corresponds to the embedding of the curved Dp-branes in the curved backgrounds. The last corresponds to the case $d\omega = dB = H \neq 0$. The ordinary product of the algebra of functions on a Dp-brane is replaced by the Kontsevich star product. But in this case it is noncommutative and nonassociative. The algebra of functions on it defines "a noncommutative and nonassociative manifold".

In this article we shall study the noncommutative and homotopy associative algebras of functions on Dp-branes defining on them noncommutative and homotopy associative structures of the manifold. We shall clarify the role of their K-theory classes in the labeling of unequivalent unstable Dp-brane configurations.

2 Open string description of Dp-brane in parallelizable backgrounds

The bosonic part of the action for a fundamental open string ending on a Dp-brane in the background of a NS-NS B field is

$$\begin{aligned} S = &\frac{1}{4\pi\alpha'}\int_\Sigma g_{ab}(X)dX^a\wedge\star dX^b \\ &+\frac{i}{4\pi\alpha'}\int_\Sigma B_{ab}(X)dX^a\wedge dX^b+\frac{i}{2\pi\alpha'}\int_{\partial\Sigma}ds(\partial_s X^a A_a(X)), \end{aligned} \tag{1}$$

or

$$S=\frac{1}{4\pi\alpha'}\int_\Sigma g_{ab}(X)dX^a\wedge\star dX^b+\frac{i}{4\pi\alpha'}\int_\Sigma(B_{ab}(X)+F_{ab}(X))dX^a\wedge dX^b, \tag{2}$$

where $F(X)=dA(X)$. The action (2) of the open string is invariant under both gauge transformations for the one-form gauge field $A\to A+d\Lambda$, $B\to B$ and for the two-form gauge field $B\to B+d\Lambda$, $A\to A-\Lambda$.

From now on we will consider the Dp-brane in the weakly curved backgrounds [7]. We shall restrict ourselves to the case of maximal branes and assume that Σ has the topology of the disk.

In order to use (2) for calculation of the correlation functions it is useful to introduce Riemann normal coordinates at the origin in which we have [7]

$$g_{ab}(x)=g_{ab}-\frac{1}{3}R_{abcd}x^cx^d+\cdots, \tag{3}$$

$$B_{ab}(x)=B_{ab}+\frac{1}{3}H_{abc}x^c+\frac{1}{4}\nabla_d H_{abc}x^cx^d+\cdots. \tag{4}$$

With the help of (3) and (4) the action (2) can be represented in approximation in small curved devivation from the flat closed string background

$$S_B=S_0+S_1, \tag{5}$$

$$S_0=\frac{1}{2}g_{ab}\int_\Sigma dX^a\wedge\star dX^b+i\int_\Sigma(B_{ab}+F_{ab}(X))dX^a\wedge dX^b, \tag{6}$$

$$S_1=\frac{i}{6}H_{abc}\int_\Sigma X^a dX^b\wedge X^c. \tag{7}$$

Let us denote $\omega(x)=B_{ab}+F_{ab}(x)$ and consider

$$S_0+S_1=\frac{1}{2}g_{ab}\int_\Sigma dX^a\wedge\star dX^b+i\int_\Sigma\omega_{ab}dX^a\wedge dX^b+\frac{i}{6}H_{abc}\int_\Sigma X^a dX^b\wedge X^c. \tag{8}$$

The simplest way to prove the noncommutativity of a Dp-brane is to quantize the open string ending on it. In the zero slope limit $\alpha'\to 0$ [1] the closed string metric g scales to zero and since $d\omega = 0$ we can use the Cattaneo and Felder path integral representation of the Kontsevich deformation quantization product [8] to calculate of the correlation functions corresponding to S_0. If we choose n functions $f_1, \dots, f_n$ positioned at ordered points $\tau_1, \dots, \tau_n$ on the boundary $\partial\Sigma$ of the string worldsheet, then path integral

$$\int [dX] e^{-S_0(X)} f_1(X(\tau_1)) \cdots f_n(X(\tau_n)) \tag{9}$$

defines the n-point correlation functions corresponding S_0 [6, 9]

$$\langle f_1 \cdots f_n \rangle = \int V(B) dx (f_1 * \cdots * f_n), \tag{10}$$

$V(B) = \sqrt{\det B}$ and $*$ is the Moyal star product (we put $F(x) = 0$)

$$f \star g = fg + \frac{i}{2}\alpha^{ab}\partial_a f \partial_b g - \frac{i}{8}\alpha^{ac}\alpha^{bd}\partial_a\partial_b f \partial_c\partial_d g + \mathcal{O}\left(\alpha^3\right), \qquad \alpha = B^{-1}.$$

Analogously, the path integral

$$\int [dX] e^{-S_0(X) - S_1(X)} f_1(X(\tau_1)) \cdots f_n(X(\tau_n)) \simeq \int [dX] e^{-S_0(X)}[1 + \nu + \mu], \tag{11}$$

$$\nu = -\frac{i}{6} H_{abc} x^c \int_\Sigma d\zeta^a \wedge d\zeta^b, \tag{12}$$

$$\mu = -\frac{i}{6} H_{abc} \int_\Sigma \zeta^a d\zeta^b \wedge d\zeta^c, \tag{13}$$

defines n-point correlation functions corresponding to $S_0 + S_1$ [7]. The integral (11) as a result of the path integration is decomposed into three parts. The first part gives the nonperturbed correlation function

$$\int f_1 * \cdots * f_n, \tag{14}$$

the second ones coming from the two-vertex ν gives

$$V + \sum_{i<j} V_{ij}, \tag{15}$$

$$V = \frac{1}{3} H_{abc}\theta_{bc} \int x^a * (f_1 * \cdots * f_n), \tag{16}$$

$$V_{ij} = \frac{i}{6} H_{abc}\theta^{a\tilde{a}}\theta^{a\tilde{b}} \int x^c * (f_1 * \cdots * \partial_{\tilde{a}} f_i * \cdots * \partial_{\tilde{b}} f_j * \cdots * f_n). \tag{17}$$

The third part coming from the three-vertex μ is given by the expression

$$\sum_{i<j<k} S\left(\frac{\tau_{ji}}{\tau_{ik}}\right) W_{i<j<k}, \tag{18}$$

$$W_{ijk} = -\frac{1}{12} H_{abc}\theta^{a\tilde{a}}\theta^{b\tilde{b}}\theta^{c\tilde{c}} \int f_1 * \cdots * \partial_{\tilde{a}} f_i * \cdots * \partial_{\tilde{b}} f_j * \cdots * \partial_{\tilde{c}} f_k * \cdots * f_n, \tag{19}$$

$S(x) = 1 - 2L(x)$, and $L(x)$ is the Rogers dilogarithm [10].

It is useful to change the notation and represent functions as operators, the $*$ product as the operator multiplication and integral $\int$ as Tr:

$$x^a \to X^a, \quad f_i \to F_i, \quad \int V(\omega) \to \mathrm{Tr}, \quad \theta^{a\tilde{a}}\partial_{\tilde{a}} f \to -i[X^a, F], \quad \theta^{ab} \to -i[X^a, X^b].$$

In these notations the formulas (16), (17), (19) are represented as

$$V = -\frac{2i}{3} H_{abc} \operatorname{Tr} \left(X^a X^b X^c F_1 \cdots F_n \right), \tag{20}$$

$$V_{ij} = -\frac{i}{6} H_{abc} \operatorname{Tr} \left(X^c F_1 \cdots [X^a, F_i] \cdots [X^b, F_j] \cdots F_n \right), \tag{21}$$

$$W_{ijk} = -\frac{i}{12} H_{abc} \operatorname{Tr} \left(F_1 \cdots [X^a, F_i] \cdots [X^b, F_j] \cdots [X^c, F_k] \cdots F_n \right). \tag{22}$$

The foregoing construction can be generalized to the case where $\omega_{ab} = B_{ab} + F_{ab}(x)$. As has shown in [7] to this end it is necessary to replace

$$W_{ijk} \to \mathbf{W}_{ijk} = W_{ijk} - \frac{1}{n}(W_{ij} - W_{ik} + W_{jk}), \tag{23}$$

$$\begin{aligned} W_{ij} = &\frac{i}{24} H_{abc} \operatorname{Tr} \left(F_1 \cdots [[X^a, X^b], F_i] \cdots [X^c, F_j] \cdots F_n \right) \\ &- \frac{i}{24} H_{abc} \operatorname{Tr} \left(F_1 \cdots [X^a, F_i] \cdots [[X^b, X^c], F_j] \cdots F_n \right). \end{aligned} \tag{24}$$

The correct generalization $\mathbf{V}$ of $V(F_1, \ldots, F_n)$ [7] together with (23) gives the final result for the n-point correlation function in this case

$$\mathbf{V} = \sum_{i,j,k} S_{ijk} \mathbf{W}_{ijk}. \tag{25}$$

3 Nonassociative algebra of functions on worldvolume of Dp-brane

According to [8] the generalization of the symplectic form $\omega_{ab} = B_{ab}$ to the one $\tilde{\omega}_{ab}(x) = \omega_{ab} + F_{ab}(x)$ gives in the zero slope limit [1] the correlation functions

$$\langle f_1(X(\tau_1)) \cdots f_n(X(\tau_n)) \rangle = \int V(\omega) d^{n+1} x f_1 * \cdots * f_n, \tag{26}$$

where now $*$ is the Kontsevich star product (because $d\omega(x) = 0$) [11]

$$(f * g)(x) = \exp \left[\frac{i}{2} \alpha^{ab} \frac{\partial}{\partial x^a} \frac{\partial}{\partial y^b} \right] f(x) g(y)|_{x=y} \tag{27}$$

or

$$\begin{aligned} f * g = &fg + \frac{i}{2} \alpha^{ab} \partial_a f \partial_b g - \frac{1}{8} \alpha^{ac} \alpha^{bd} \partial_a \partial_b f \partial_c \partial_d g \\ &- \frac{1}{12} \alpha^{ad} \partial^d \alpha^{bc} (\partial_a \partial_b f \partial_c g - \partial_b f \partial_a \partial_c g) + \mathcal{O}\left(\alpha^3\right). \end{aligned} \tag{28}$$

Hence,

$$(f * g) * h - f * (g * h) = \frac{1}{6} \left(\alpha^{il} \partial_l \alpha^{jk} + \alpha^{jl} \partial_l \alpha^{ki} + \alpha^{kl} \partial_l \alpha^{ij} \right) \partial_i f \partial_j g \partial_k h + \mathcal{O}\left(\alpha^3\right). \tag{29}$$

If α is invertible and $d\left(\alpha^{-1}\right) = d\omega = 0$, from (29) it follows the associativity of the $*$ product.

As it was shown in the previous section the $*$ product (28) with $\tilde{\omega}_{ab}(x) = B_{ab} + \frac{1}{3} H_{abc} x^c$ gives the correlation functions defined by (26). But in this case the product becomes nonassociative (because $d\tilde{\alpha} = H \neq 0$). It is denoted by $\bullet$. The associator of the product $\bullet$ of functions is defined by the equation

$$(f \bullet g) \bullet h - f \bullet (g \bullet h) = \frac{1}{6}\alpha^{ia}\alpha^{jb}\alpha^{kc}H_{abc}\partial_i f \partial_j g \partial_k h + \cdots . \tag{30}$$

Because $H_{abc} \neq 0$ the product $\bullet$ is not associative. The two products $*$ and $\bullet$ given by the Kontsevich expansion (28) in terms of ω_{ab} and $\tilde{\omega}_{ab}(x) = \omega_{ab}(x) + \frac{1}{3}H_{abc}x^c$, respectively, are connected between themselves by the relation

$$f \bullet g = f * g + \frac{i}{12}H_{abc}\left\{x^c, [x^a, f]_* * [x^b, g]_*\right\}_*, \tag{31}$$

or in the operator form

$$F \bullet G = FG - \frac{i}{12}H_{abc}\left\{X^c[X^a, F][X^b, G]\right\}. \tag{32}$$

One can find more general relation

$$f_1 \bullet (f_2 \bullet \cdots \bullet (f_{n-1} \bullet f_n) \cdots) = f_1 * f_2 * \cdots * f_n + \sum_{i<j} V_{ij}. \tag{33}$$

4 Homotopy associative structure of algebra of functions on Dp-brane

With the help of the methods of the preceding section we can obtain the exact expressions of the first correlation functions of the model. The two-point correlation function is given by

$$P_2(f_1, f_2) = \int f_1 f_2 \left(1 + \frac{1}{3}H_{abc}x^a\theta^{bc}\right). \tag{34}$$

The three-point correlation function is written

$$\begin{aligned} P_3(f_1, f_2, f_3) = &\int f_1 * f_2 * f_3 + \frac{1}{3}B_{bc}K^{abc}\int f_1 * y_a * f_2 * f_3 \\ &- \frac{i}{6}K^{abc}\int \left(-\partial_a f_1 * y_c * \partial_b f_2 * f_3 - \partial_a f_1 * y_c * f_2 * \partial_b f_3 + f_1 * y_c * \partial_a * f_2 \partial_b * f_3\right), \end{aligned} \tag{35}$$

where $K^{abc} = \theta^{a\tilde{a}}\theta^{b\tilde{b}}\theta^{c\tilde{c}}$ and $y_a = B_{ab}x^b$. Every n-point correlation function can be represented in the operator form $P_n[F_1, \ldots, F_n]$. It depends on the $n-3$ conformal moduli of the insertion points $\tau_1, \ldots, \tau_n$ of the functions $F_1, \ldots, F_n$. The one-point correlation function

$$P_1[F] = \mathrm{Tr}\,(F) + \frac{2i}{3}H_{abc}\,\mathrm{Tr}\left(X^a X^b X^c F\right) \tag{36}$$

defines operator $P[F] := P_1[F]$. With the help of the operator P one can represent the correlation functions

$$\begin{aligned} &P_1[F_1] = P[O_1(F_1)(\tau_1)], \qquad P_2[F_1, F_2] = P[O_2(F_1, F_2)(\tau_1, \tau_2)], \\ &P_3[F_1, F_2, F_3] = P[O_3(F_1, F_2, F_3)(\tau_1\tau_2\tau_3)], \\ &P_4[F_1F_2F_3F_4] = P[O_4(F_1, F_2, F_3, F_4)(\tau_1, \tau_2, \tau_3, \tau_4)], \end{aligned}$$

where

$$O_1[F_1] = F_1, \tag{37}$$

$$O_2[F_1, F_2](\tau_1, \tau_2) = F_1 \bullet F_2, \tag{38}$$

$$O_3[F_1,F_2,F_3](\tau_i)=L(1-x)(F_1\bullet F_2)\bullet F_3+L(x)F_1\bullet(F_2\bullet F_3),\qquad x=\frac{\tau_{21}}{\tau_{31}}, \tag{39}$$

$$\begin{aligned}O_4[F_1,\dots,F_4](\tau_i)&=L\left[\left(1-\frac{x}{y}\right)\left(1-\frac{1-y}{1-x}\right)\right](F_1\bullet F_2)\bullet(F_3\bullet F_4)\\&+L\left[\left(1-\frac{x}{y}\right)\left(\frac{1-x}{1-y}\right)\right]((F_1\bullet F_2)\bullet F_3)\bullet F_4+L\left[\frac{x}{y}\left(\frac{1-y}{1-x}\right)\right]F_1\bullet(F_2\bullet(F_3\bullet F_4))\\&+L\left[\frac{x}{y}(1-y)\right](F_1(\bullet F_2\bullet F_3))\bullet F_4+L\left[x\left(\frac{1-y}{1-x}\right)\right]F_1((\bullet F_2\bullet F_3)\bullet F_4),\\x&=\frac{\tau_{21}}{\tau_{41}},\qquad y=\frac{\tau_{31}}{\tau_{41}}.\end{aligned} \tag{40}$$

The function $Q_5[F_1,F_2,F_3,F_4,F_5](x,y,z)$ is written by means of the sum of product of functions F_1, F_2, F_3, F_4, F_5 of 14 terms corresponding the different ways to insert parenthesis:

$$\begin{aligned}&\{(((F_1\bullet F_2)\bullet F_3)\bullet F_4)\bullet F_5,\ ((F_1\bullet F_2)\bullet F_3)(\bullet F_4\bullet F_5),\ (F_1\bullet F_2)((\bullet F_3\bullet F_4)\bullet F_5),\\&((F_1\bullet F_2)\bullet(F_3\bullet F_4))\bullet F_5,\ (F_1\bullet(F_2\bullet F_3))\bullet F_4)\bullet F_5,\ (F_1\bullet(F_2\bullet F_3))\bullet(F_4\bullet F_5),\\&(F_1\bullet((F_2\bullet F_3)\bullet F_4))\bullet F_5,\ F_1\bullet(((F_2\bullet F_3)\bullet F_4)\bullet F_5),\ (F_1\bullet(F_2\bullet(F_3\bullet F_4))\bullet F_5,\\&F_1\bullet((F_2\bullet(F_3\bullet F_4))\bullet F_5),\ F_1\bullet(F_2((\bullet F_3\bullet F_4)\bullet F_5)),\ (F_1\bullet F_2)\bullet(F_3\bullet(F_4\bullet F_5)),\\&F_1\bullet(F_2\bullet(F_3\bullet(F_4\bullet F_5))),\ F_1\bullet((F_2\bullet F_3)\bullet(F_4\bullet F_5))\}.\end{aligned} \tag{41}$$

For correlation functions of the higher order this procedure can be continued. There exists conjecture that every correlation function $P_n[F_1,\dots,F_n]$ can be represented in the form

$$P_n[F_1,\dots,F_n](\tau_i)=P[O_n(F_1,\dots,F_n)(\tau_i)]. \tag{42}$$

The functions $O_n(F_1,\dots,F_n)(\tau_i)$ define mappings of the algebra of functions on the Dp-brane onto itself. For example, the role of the mappings O_n we can see, for example, in the case O_3. The homotopy properties of the mapping

$$O_3[F_1,F_2,F_3](x):[0,1]\times C^\infty(M)^{\times 3}\to C^\infty M \tag{43}$$

is ensured by the equation (39) and by the properties of the Rogers dilogarithm $L(x)$:

$$L(x)+L(1-x)=0,\qquad L(0)=0,\qquad L(1)=1. \tag{44}$$

The mapping (43) connecting of the products $(F_1\bullet F_2)\bullet F_3$ and $F\bullet(F_3\bullet F_3)$, corresponding by two different ways to stay the parenthesis in the product $F_1\bullet F_2\bullet F_3$ is the homotopy equivalence. The mapping $O_3(m)$ defines the A_3 homotopy associative structure on the nonassociative algebra $C^\infty(M)$. It is obvious that homotopies $O_n[F_1,\dots,F_n]$ play the same role for higher product as $O_3[F_1,F_2,F_3]$ for $F_1\bullet F_2\bullet F_3$.

The concepts of the homotopy spaces and the strong homotopy algebras are due to Stasheff [12], where it is shown that a topological space has homotopy type of a loop space if and only if it is strong homotopy associative one. The strong homotopy algebras has been found at number of the unexpected places: in the topological conformal field theory, in Morse theory. The "nonassociative manifold" is defined by means of a strong homotopy algebra of functions on them.

5 Charges of Dp-branes

Soon after Polshinski's identification of Dp-branes as nonperturbative objects in the perturbative string theory that carry R-R charge, Witten [2] suggest that the D-branes charges should take the values in a K-theory of the spacetime. The groups $K^0(M)$, $K^1(M)$ are associated with Dp-branes in IIA, IIB string theory, respectively. The presence of the B field introduces the corrections in evaluation of the charges. The charges of the Dp-branes in the topological case are dependent on the cohomolgy class $[H] \in H^3(M, Z)$ of the strength $H = dB$.

Let $K^j(M, \mathcal{E}_H) = K_j(C_0(M, \mathcal{E}_H))$, $j = 0, 1$, denotes $K_\bullet$-groups of C^*-algebra $C_0(M, \mathcal{E}_H)$ generated by the continuous sections vanishing on the infinity of the unique local trivial gauge bundle $\mathcal{E}_H$ whose structure depends on Dixmier–Douady invariant $[H]$.

It is destinguished three cases:

1. $[H] = 0$, $(B = 0, H = 0)$ [2].

   $\mathcal{E}_H$ is the gauge bundle with fibre C^n and gauge group $\operatorname{Aut} C^n = U(n)$.

2. $[H] = 0$, $(B \neq 0,\ H = 0)$ [15].

   $\mathcal{E}_H$ is the gauge bundle with fibre $M_n(C)$, the matrix algebra of $n \times n$ dimension and the gauge group $\operatorname{Aut} M_n(C) = SU(n)/Z_n$. In this case $C_0(M, \mathcal{E}_H)$ is called by the Azumaya algebra.

3. $[H] \neq 0$, $(B \neq 0,\ H \neq 0)$ [15].

   $\mathcal{E}_H$ is the gauge bundle with fibre $\mathcal{K}$, algebra of compact operators in a Hilbert space and the gauge group $\operatorname{Aut} \mathcal{K} = \lim\limits_{n\to\infty} SU(n)/Z_m$. In this case $C_0(\mathcal{K}, \mathcal{E}_H)$ is called by the Rosenberg algebra.

In the bosonic string theory the physical interpretation of K-theory classes is less clear then in type II superstring theory, since the branes carry no conserved charges and, likely, are unstable. Conjecture is that these K-theory classes of the algebra of functions on the Dp-branes label unequivalent unstable Dp-brane configurations.

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The Euclidean Propagator in Quantum Models with Non-Equivalent Instantons

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We consider in detail the Euclidean propagator in quantum-mechanical models which include the existence of non-equivalent instantons. For such a purpose we resort to the semiclassical approximation in order to take into account the fluctuations over the instantons themselves. The physical effects of the multi-instanton configurations appear in terms of the alternate dilute-gas approximation.

1 Introduction

The tunnelling phenomenon represents one of the most outstanding effects in quantum theory. Starting from the pioneering work of Polyakov on the subject [1], the semiclassical treatment of the tunnelling is presented in terms of the Euclidean version of the path-integral formalism. The basis of this approach relies on the so-called instanton calculus. As usual the instantons themselves correspond to localised finite-action solutions of the Euclidean equation of motion where the time variable is essentially imaginary. In short, one finds the appropiate classical configuration to evaluate the term associated with the quadratic fluctuations. On the other hand the functional integration is solved by means of the gaussian scheme except for the zero-modes which appear in connection with the translational invariances of the system. As expected one introduces collective coordinates so that ultimately the gaussian integration is performed along the directions orthogonal to the zero-modes. A functional determinant includes an infinite product of eigenvalues so that a highly divergent result appears in this context. However one can regularize the fluctuation factors by means of the conventional ratio of determinants.

Next let us describe in brief the instanton calculus for the one-dimensional particle as can be found in [2]. Our particle moves under the action of a confining potential $V(x)$ which yields a pure discrete spectrum of energy eigenvalues. If the particle is located at the initial time $t_i = -T/2$ at the point x_i while one finds it when $t_f = T/2$ at the point x_f, the well-known functional version of the non-relativistic quantum mechanics allows us to write the transition amplitude in terms of a sum over all paths joining the world points with coordinates $(-T/2, x_i)$ and $(T/2, x_f)$. If we incorporate the change $t \to -i\tau$, known in the literature as the Wick rotation, the Euclidean formulation of the path-integral reads

$$\langle x_f|\exp(-HT)|x_i\rangle = N(T)\int [dx]\exp\{-S_e[x(\tau)]\},$$

where H represents as usual the Hamiltonian, the factor $N(T)$ serves to normalize the amplitude while $[dx]$ indicates the integration over all functions which fulfil the corresponding boundary conditions. In addition the Euclidean action S_e corresponds to

$$S_e = \int_{-T/2}^{T/2}\left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)\right]d\tau \tag{1}$$

whenever the mass of the particle is set equal to unity. Now we take care of the octic potential $V(x)$ given by

$$V(x) = \frac{\omega^2}{2}\left(x^2 - 1\right)^2\left(x^2 - 4\right)^2 .$$

When considering that $\omega^2 \gg 1$ the energy barriers are high enough to split the physical system into a sum of harmonic oscillators. The particle executes small oscillations around each minima of the potential located at $x = \pm 1$ and $\tilde{x} = \pm 2$. The second derivative of the potential at these points, i.e. $V''(x = \pm 1) = 36\omega^2$ and $V''(\tilde{x} = \pm 2) = 144\omega^2$, gives the frequencies of the harmonic oscillators at issue.

As regards the discrete symmetry $x \to -x$ which the potential $V(x)$ enjoys, we observe how the four minima are non-equivalent since no connection is possible between the two sets represented by $x = \pm 1$ and $\tilde{x} = \pm 2$. We would like to make the description of the tunnelling phenomenon to describe how the symmetry cannot appear spontaneously broken at quantum level. The expectation value of the coordinate x evaluated for the ground-state is zero as corresponds to the even character of the potential $V(x)$.

2 The one-instanton amplitude

In this section we would like to discuss the transition amplitude between the points $x_i = 1$ and $x_f = 2$. For such a purpose we need the explicit form of the topological configuration with $x_i = 1$ at $t_i = -T/2$ while $x_f = 2$ when $t_f = T/2$. To get the instanton $x_{c1}(\tau)$ which connects the points $x_i = 1$ and $x_f = 2$ with infinite euclidean time, we can resort to the well-grounded Bogomol'nyi bound [3]. The situation is solved by integration of a first-order differential equation which derives from the zero-energy condition for the motion of a particle under the action of $-V(x)$. In short

$$x_{c1}(\tau) = 2\cos\left[\frac{\pi}{3} - \frac{1}{3}\arccos\left(\frac{e^{-12\omega(\tau-\tau_c)} - 1}{e^{-12\omega(\tau-\tau_c)} + 1}\right)\right], \tag{2}$$

where τ_c indicates the point at which the instanton makes the jump. Equivalent solutions are obtained by means of the transformations $\tau \to -\tau$ and $x_{c1}(\tau) \to -x_{c1}(\tau)$. The instanton calculus allows the connection between adjoint minima of the potential. We notice therefore the existence of a second instanton interpolating between $x_i = -1$ and $x_f = 1$. The classical Euclidean action S_1 associated with the topological configuration at issue is computed according to (1) so that $S_1 = 22\omega/15$. Next the standard description of the one-instanton amplitude between $x_i = 1$ and $x_f = 2$ takes over

$$\begin{aligned}&\langle x_f = 2|\exp(-HT)|x_i = 1\rangle\\ &\quad= N(T)\left\{\mathrm{Det}\left[-\frac{d^2}{d\tau^2} + \nu^2\right]\right\}^{-1/2}\left\{\frac{\mathrm{Det}\left[-\left(d^2/d\tau^2\right) + V''[x_{c1}(\tau)]\right]}{\mathrm{Det}\left[-\left(d^2/d\tau^2\right) + \nu^2\right]}\right\}^{-1/2}\exp(-S_1),\end{aligned}$$

where as usual we have multiplied and divided by the determinant of a generic harmonic oscillator of frequency ν. The so-called regularization term is interpreted as a new amplitude given by

$$\langle x_f = 0|\exp(-H_{\mathrm{ho}}T)|x_i = 0\rangle = N(T)\left\{\mathrm{Det}\left[-\frac{d^2}{d\tau^2} + \nu^2\right]\right\}^{-1/2}. \tag{3}$$

Now the explicit evaluation of (3) is made according to the method explained in [4]. To sum up

$$\langle x_f = 0|\exp(-H_{\mathrm{ho}}T)|x_i = 0\rangle = \left(\frac{\nu}{\pi}\right)^{1/2}(2\sinh\nu T)^{-1/2}.$$

The existence of a zero-mode $x_0(\tau)$ in the spectrum of the stability equation requires the introduction of a collective coordinate. The zero eigenvalue reflects the translational invariance of the system so that there is one direction in the functional space of the second variations which results incapable of changing the action. The explicit form of the zero-mode $x_0(\tau)$ corresponds to the derivative of the topological configuration itself, i.e.

$$x_0(\tau) = \frac{1}{\sqrt{S_1}} \frac{dx_{c1}}{d\tau}.$$

The integral over the zero-mode becomes equivalent to the integration over the center of the instanton τ_c. If the change of variables is incorporated our ratio of determinants corresponds to [2]

$$\left\{ \frac{\mathrm{Det}\left[-\left(d^2/d\tau^2\right) + V''[x_{c1}(\tau)]\right]}{\mathrm{Det}\left[-\left(d^2/d\tau^2\right) + \nu^2\right]} \right\}^{-1/2} = \left\{ \frac{\mathrm{Det}'\left[-\left(d^2/d\tau^2\right) + V''[x_{c1}(\tau)]\right]}{\mathrm{Det}\left[-\left(d^2/d\tau^2\right) + \nu^2\right]} \right\}^{-1/2} \sqrt{\frac{S_1}{2\pi}} d\tau_c,$$

where as usual Det' stands for the so-called reduced determinant once the zero-mode has been removed. Next we take advantage of the Gelfand–Yaglom method of computing ratios of determinants where only the knowledge of the large-τ behaviour of the classical solution $x_{c1}(\tau)$ is necessary [2]. If $\hat{O}$ and $\hat{P}$ represent a couple of second order differential operators whose eigenfunctions vanish at the boundary, the quotient of determinants is given in terms of the zero-energy solutions $f_0(\tau)$ and $g_0(\tau)$ so that

$$\frac{\mathrm{Det}\,\hat{O}}{\mathrm{Det}\,\hat{P}} = \frac{f_0(T/2)}{g_0(T/2)}$$

whenever the eigenfunctions fulfil the initial conditions

$$f_0(-T/2) = g_0(-T/2) = 0, \qquad \frac{df_0}{d\tau}(-T/2) = \frac{dg_0}{d\tau}(-T/2) = 1.$$

As the zero-mode $g_0(\tau)$ of the harmonic oscillator of frequency ν is given by

$$g_0(\tau) = \frac{1}{\nu}\ \sinh[\nu(\tau + T/2)]$$

we need the form of the solution $f_0(\tau)$ associated with the topological configuration written in (2). Starting from $x_0(\tau)$ we can write a second solution $y_0(\tau)$ according to

$$y_0(\tau) = x_0(\tau) \int_0^\tau \frac{ds}{x_0^2(s)}.$$

As regards the asymptotic behaviour of $x_0(\tau)$ and $y_0(\tau)$ we have that

$$x_0(\tau) \sim C\exp(-12\omega\tau) \quad \text{if} \quad \tau \to \infty,$$
$$x_0(\tau) \sim D\exp(6\omega\tau) \quad \text{if} \quad \tau \to -\infty$$

together with

$$y_0(\tau) \sim \exp(12\omega\tau)/24\omega C \quad \text{if} \quad \tau \to \infty,$$
$$y_0(\tau) \sim -\exp(-6\omega\tau)/12\omega D \quad \text{if} \quad \tau \to -\infty,$$

where the constants C and D can be obtained from the derivative of (2). Taking the linear combination of $x_0(\tau)$ and $y_0(\tau)$ given by

$$f_0(\tau) = Ax_0(\tau) + By_0(\tau)$$

the incorporation of the initial conditions allows us to write that

$$f_0(\tau) = x_0(-T/2)y_0(\tau) - y_0(-T/2)x_0(\tau).$$

Now we can extract the asymptotic behaviour of $f_0(\tau)$, i.e.

$$f_0(T/2) \sim \frac{D}{24\omega C} \exp(3\omega T) \quad \text{if} \quad T \to \infty.$$

Next we need to consider the lowest eigenvalue of the stability equation. From a physical point of view we can explain the situation as follows: the derivative of the topological solution does not quite satisfy the boundary conditions for the interval $(-T/2, T/2)$. When enforcing such a behaviour, the eigenstate is compressed and the energy shifted slightly upwards. In doing so the zero-mode $x_0(\tau)$ is substituted for the $f_\lambda(\tau)$, i.e.

$$-\frac{d^2 f_\lambda(\tau)}{d\tau^2} + V''[x_{c1}(\tau)]f_\lambda(\tau) = \lambda f_\lambda(\tau)$$

whenever

$$f_\lambda(-T/2) = f_\lambda(T/2) = 0.$$

Going to the lowest order in perturbation theory we find

$$f_\lambda(\tau) \sim f_0(\tau) + \lambda \left.\frac{df_\lambda}{d\lambda}\right|_{\lambda=0}$$

so that

$$f_\lambda(\tau) = f_0(\tau) + \lambda \int_{-T/2}^{\tau} [x_0(\tau)y_0(s) - y_0(\tau)x_0(s)]f_0(s)ds.$$

The asymptotic behaviour of the zero-modes together with the condition $f_\lambda(T/2) = 0$ allow us to write that

$$\lambda = 12\omega D^2 \exp(-6\omega T).$$

The evaluation of this quotient of determinants requires a choice for the parameter ν so that the frequency of the harmonic oscillator of reference is the average of the frequencies over the non-equivalent minima located at $x = 1$ and $\tilde{x} = 2$. In other words $\nu = 9\omega$. When considering the well-grounded double-well model the two minima of the potential are equivalent so that the aforementioned average is not necessary. However in this case the Gelfand–Yaglom method fixes the frequency ν in order the ratio of determinants to be finite. In addition we have that (see (2))

$$C = \frac{4\sqrt{3}\omega}{\sqrt{S_1}}, \qquad D = \frac{16\omega}{3\sqrt{S_1}}.$$

Now we can write the one-instanton amplitude between the points $x_i = 1$ and $x_f = 2$, i.e.

$$\langle x_f = 2|\exp(-HT)|x_i = 1\rangle = \left(\frac{9\omega}{\pi}\right)^{1/2} (2\sinh 9\omega T)^{-1/2} \sqrt{S_1} K_1 \exp(-S_1)\omega d\tau_c,$$

where K_1 represents a numerical factor given by

$$K_1 = 16\sqrt{\frac{15\sqrt{3}}{11\pi}}.$$

In doing so we get a transition amplitude just depending on the point τ_c at which the instanton makes the jump. This regime seems plausible whenever

$$\sqrt{S_1} K_1 \exp(-S_1)\omega T \ll 1$$

a nonsense condition when T is large enough. However in this situation we can accommodate configurations constructed of instantons and anti-instantons which mimic the behaviour of a trajectory just derived from the euclidean equation of motion.

To finish this section we take care of the second instanton of the octic model. The one-instanton amplitude between $x_i = -1$ and $x_f = 1$ is based on the topological configuration $x_{c2}(\tau)$

$$x_{c2}(\tau) = 2\cos\left[\frac{\pi}{3} + \frac{1}{3}\arccos\left(\frac{e^{12\omega\tau} - 1}{e^{12\omega\tau} + 1}\right)\right]$$

whose classical euclidean action corresponds to $S_2 = 76\omega/5$. This second instanton reminds the case of the double-well potential since connects equivalent minima of the potential. The form of the ratio of determinants at issue should be

$$\left\{\frac{\mathrm{Det}'\left[-\left(d^2/d\tau^2\right) + V''[x_{c2}(\tau)]\right]}{\mathrm{Det}\left[-\left(d^2/d\tau^2\right) + 36\omega^2\right]}\right\}^{-1/2} = \sqrt{S_2} K_2 \omega d\tau_c,$$

where K_2 corresponds to

$$K_2 = 12\sqrt{\frac{15}{38\pi}}.$$

3 The multi-instanton amplitude

In this section we discuss the complete amplitude which incorporates the physical effect of a string of instantons and anti-instantons along the τ axis. The octic potential represents a more complicated case since we need to include the whole scheme of non-equivalent instantons. We wish to evaluate the functional integral by summing over all such configurations with n instantons and anti-instantons centered at points $\tau_1, \ldots, \tau_n$ whenever

$$-\frac{T}{2} < \tau_1 < \cdots < \tau_n < \frac{T}{2}.$$

We can carry things further and assume as usual that the action of the string of instantons and anti-instantons is given by the sum of the n individual actions. This method is well-known in the literature where it appears with the name of dilute-gas approximation [5]. The translational degrees of freedom yield an integral like

$$\int_{-T/2}^{T/2} \omega d\tau_n \int_{-T/2}^{\tau_n} \omega d\tau_{n-1} \cdots \int_{-T/2}^{\tau_2} \omega d\tau_1 = \frac{(\omega T)^n}{n!}.$$

When considering the transition amplitude between $x_i = 1$ and $x_f = 2$ the total number n of topological configurations must be odd. We can split n (odd) into the sum of two contributions n_1 (odd) and n_2 (even) which represent the different possibilities associated with the existence of non-equivalent instantons. Then we have n_1 topological configurations just interpolating between $x = 1$ and $\tilde{x} = 2$ or $x = -1$ and $\tilde{x} = -2$. Identical situation appears in connection with n_2 where now the initial and final points of the trip are $x = \pm 1$. Now we need to include a combinatorial factor F to count the different possibilities that we have of distributing the n instantons. Except for the last step which corresponds to the instanton analyzed in the previous section, we deal with a closed path of topological configurations starting and coming back to the point $x = 1$. As regards the instantons (anti-instantons) belonging to the first type we observe the formation of pairs due to the location of the four minima of the potential along the real axis. Therefore we have $(n_1 - 1)/2 + n_2$ holes to fill bearing in mind that once the $(n_1 - 1)/2$ pairs of instantons and anti-instantons are distributed no freedom at all remains to locate the topological configurations associated with n_2. In short

$$F = \begin{pmatrix} (n_1 - 1)/2 + n_2 \\ (n_1 - 1)/2 \end{pmatrix}.$$

At this point we can discuss the complete transition amplitude we are looking for in terms of the so-called instanton density, i.e.

$$d_i = \sqrt{S_i} K_i \exp(-S_i), \qquad i = 1, 2.$$

To be precise

$$\langle x_f = 2|\exp(-HT)|x_i = 1\rangle \\ = \left(\frac{9\omega}{\pi}\right)^{1/2} (2\sinh 9\omega T)^{-1/2} \sum_{n_1,n_2} [d_1\omega T]^{n_1} [d_2\omega T]^{n_2} \frac{F}{n!}. \tag{4}$$

The best way of dealing with the double sum of (4) should be the following

$$S = \sum_{r=0}^{\infty} \frac{[d_1\omega T]^{2r+1}}{(2r+1)!} \sum_{q=0}^{r} \begin{pmatrix} r+q \\ r-q \end{pmatrix} (d_2/d_1)^{2q},$$

where we can handle the sum $\tilde{S}$ concerning the variable q taking advantage of [6]

$$\sum_{q=0}^{r} (-1)^q \begin{pmatrix} r+q \\ 2q \end{pmatrix} = \sec[\arcsin(x/2)] \cos[(2r+1)\arcsin(x/2)]$$

including the transformation $x \to ix$ to obtain that

$$\tilde{S} = \frac{\cosh[(2r+1)\arg\sinh(s/2)]}{\cosh[\arg\sinh(s/2)]},$$

where s stands for the relative instanton density given by $s = d_2/d_1$. In terms of a new variable z defined as

$$z = \arg\sinh(s/2)$$

it is the case that a typical value of r provides us with the final expression for $\tilde{S}$, i.e.

$$\tilde{S} = \frac{\exp[(2r+1)z]}{\sqrt{4+s^2}}.$$

In other words

$$S = \frac{\sinh[d_1\omega T\exp(z)]}{\sqrt{4+s^2}}.$$

In doing so the complete amplitude between the points $x_i = 1$ and $x_f = 2$ reads

$$\langle x_f = 2 | \exp(-HT) | x_i = 1 \rangle = \left(\frac{9\omega}{\pi} \right)^{1/2} (2 \sinh 9\omega T)^{-1/2} \frac{\sinh[d_1 \omega T \exp(z)]}{\sqrt{4 + s^2}}.$$

To sum up, we have explained the method of dealing with quantum-mechanical models which exhibit a more complicated structure of non-equivalent classical vacua in comparison with the well-grounded cases of the double-well or periodic sine-Gordon potentials where the equivalence of all the minima of $V(x)$ is taken for granted [5]. As regards the octic potential the topological solutions of the system inherit the property of non-equivalence. The global effect of the multi-instanton configurations is discussed in terms of the alternate dilute-gas approximation.

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Equation for Particles of Spin $\frac{3}{2}$ with Anomalous Interaction

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We consider tensor-bispinor equation, which describes doublets of particles with spin $\frac{3}{2}$, nonzero masses and anomalous interaction with electromagnetic field. Using this equation we find the energy levels of the particle with spin $\frac{3}{2}$ in constant magnetic field, crossed electric and magnetic fields. We show that it is possible to introduce anomalous interaction in such way that energy levels are real for all values of magnetic field and arbitrary gyromagnetic ratio g.

1 Introduction

After the discovery of the acausality of the Rarita–Schwinger equation [1] various equations have been adopted for the description of spin $\frac{3}{2}$ particles. But these equations also have different defects. Some of these defects are acausal propagation of solutions, complex energies, incorrect value of the gyromagnetic ratio etc [2, 3, 4].

In the present paper we propose the equation for doublets of massive particles with spin $\frac{3}{2}$ interacting with external electromagnetic field, which do not have many of defects enumerated above. In our case the wave function of particle with spin $\frac{3}{2}$ is described by irreducible anti-symmetric tensor-bispinor of rank 2. We generalize results [5, 6, 7] for the case of linear and quadratic anomalous interactions with electromagnetic field. Finally we consider the motion of particle with spin $\frac{3}{2}$ in crossed magnetic and electric fields.

2 Tensor-bispinor equation with minimal interaction in electromagnetic field

Here we describe particles with spin $\frac{3}{2}$ in terms of irreducible antisymmetric tensor-bispinor $\Psi^{\mu\nu}$ ($\Psi^{\mu\nu} = -\Psi^{\nu\mu}$). In the case of free particle, $\Psi^{\mu\nu}$ is a 24-component tensor-bispinor, $\Psi^{\mu\nu}_{\alpha} = -\Psi^{\mu\nu}_{\alpha}$, which has two tensorial indexes $\mu, \nu = 0, 1, 2, 3$ and a spinorial index $\alpha = 0, 1, 2, 3$. We suppose $\Psi^{\mu\nu}_{\alpha}$ satisfies the following covariant condition

$$\gamma_\mu \gamma_\nu \Psi^{\mu\nu} = 0, \tag{1}$$

where γ^μ are Dirac matrices acting on spinor index α which is omitted. $\Psi^{\mu\nu}$ must also satisfy the Dirac equation

$$(\gamma_\lambda p^\lambda - m)\Psi^{\mu\nu} = 0, \tag{2}$$

where $p_\mu = i\frac{\partial}{\partial x^\mu}$. Commuting $\gamma_\mu\gamma_\nu$ and $(\gamma_\lambda p^\lambda - m)$ we come to the secondary constraint for $\Psi^{\mu\nu}$

$$p_\mu \gamma_\nu \Psi^{\mu\nu} = 0. \tag{3}$$

It is possible to show that conditions (1), (3) reduce the number of independent components of $\Psi^{\mu\nu}$ to 16. Equations (1)–(3) describe a doublet of particles with spin $\frac{3}{2}$ and mass m [8, 9].

In order to introduce interaction with electromagnetic field, we rewrite system (1)–(3) as a single equation

$$\begin{aligned}&(\gamma_\lambda p^\lambda - m)\Psi^{\mu\nu} + \frac{1}{12}(p^\mu\gamma^\nu - p^\nu\gamma^\mu)[\gamma_\rho,\gamma_\sigma]\Psi^{\rho\sigma}\\&\quad - \frac{1}{12}[\gamma^\mu,\gamma^\nu](p_\rho\gamma_\sigma - p_\sigma\gamma_\rho)\Psi^{\rho\sigma} + \frac{1}{24}[\gamma^\mu,\gamma^\nu]\gamma_\lambda p^\lambda[\gamma_\rho,\gamma_\sigma]\Psi^{\rho\sigma} = 0.\end{aligned} \tag{4}$$

The minimal interaction can be introduced in equation (4) in standard way

$$p_\mu \;\longrightarrow\; \pi_\mu = p_\mu - eA_\mu, \tag{5}$$

where A_μ is the vector-potential of electromagnetic field. As a result we obtain

$$\begin{aligned}&(\gamma_\lambda \pi^\lambda - m)\Psi^{\mu\nu} + \frac{1}{12}(\pi^\mu\gamma^\nu - \pi^\nu\gamma^\mu)[\gamma_\rho,\gamma_\sigma]\Psi^{\sigma\rho}\\&\quad - \frac{1}{12}[\gamma^\mu,\gamma^\nu](\pi_\rho\gamma_\sigma - \pi_\sigma\gamma_\rho)\Psi^{\rho\sigma} + \frac{1}{24}[\gamma^\mu,\gamma^\nu]\gamma_\lambda \pi^\lambda[\gamma_\rho,\gamma_\sigma]\Psi^{\rho\sigma} = 0.\end{aligned} \tag{6}$$

Contracting (6) with $\gamma_\mu\gamma_\nu$ and with $\pi_\mu\gamma_\nu - \pi_\nu\gamma_\mu$ we come to the constraints

$$\gamma_\mu\gamma_\nu\Psi^{\mu\nu} = 0, \qquad \pi_\mu\gamma_\nu\Psi^{\mu\nu} = \frac{ie}{m}(F_{\mu\nu} - \gamma^\lambda\gamma_\nu F_{\mu\lambda})\Psi^{\mu\nu}, \tag{7}$$

where $F_{\mu\nu}$ is the tensor of electromagnetic field $F_{\mu\nu} = i(p_\mu A_\nu - p_\nu A_\mu)$. Substituting (7) in (6) we obtain an equation for $\Psi^{\mu\nu}$

$$(\gamma_\lambda\pi^\lambda - m)\Psi^{\mu\nu} - \frac{ie}{6m}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(F_{\rho\lambda} - \gamma^\sigma\gamma_\lambda F_{\rho\sigma})\Psi^{\rho\lambda} = 0. \tag{8}$$

Equation (8) is equivalent to introduced in [7]. It can be shown by using the substitution

$$\Psi^{ab} = \frac{1}{2}\epsilon_{abc}\left(\Phi_c^{(1)} + \Phi_c^{(2)}\right), \qquad \Psi^{0c} = \frac{i}{2}\left(\Phi_c^{(2)} - \Phi_c^{(1)}\right), \tag{9}$$

$(a, b, c = 1, 2, 3)$, $\Phi_c^{(1)}$ and $\Phi_c^{(2)}$ are bispinors.

3 Anomalous interaction

In this section we generalize equation (6) by adding the terms $T^{\mu\nu}_{\rho\sigma}\Psi^{\rho\sigma}$ and $T^{\mu\nu}_{\rho\sigma}T^{\rho\sigma}_{\delta\varepsilon}\Psi^{\delta\varepsilon}$ [8, 9], which are linear and quadratic in $F^{\mu\nu}$ correspondingly, i.e. consider both minimal and anomalous interactions [2]

$$\begin{aligned}&(\gamma_\lambda \pi^\lambda - m)\Psi^{\mu\nu} + \frac{1}{12}(\pi^\mu\gamma^\nu - \pi^\nu\gamma^\mu)[\gamma_\rho,\gamma_\sigma]\Psi^{\rho\sigma} - \frac{1}{12}[\gamma_\mu,\gamma_\nu](\pi_\rho\gamma_\sigma - \pi_\sigma\gamma_\rho)\Psi^{\rho\sigma}\\&\quad + \frac{1}{24}[\gamma_\mu,\gamma_\nu]\gamma_\lambda\pi^\lambda[\gamma_\rho,\gamma_\sigma]\Psi^{\rho\sigma} + T^{\mu\nu}_{\rho\sigma}\Psi^{\rho\sigma} + \tilde{T}^{\mu\nu}_{\rho\sigma}\tilde{T}^{\rho\sigma}_{\delta\epsilon}\Psi^{\delta\epsilon} = 0.\end{aligned} \tag{10}$$

We suppose the following relations are satisfied

$$\gamma_\mu T^{\mu\nu}_{\rho\sigma} = 0, \qquad \gamma_\mu \tilde{T}^{\mu\nu}_{\rho\sigma} = 0. \tag{11}$$

It is possible to show [8, 9] that (11) are the necessary and sufficient conditions to obtain consistent equation (3) whose solutions propagate with the velocity less then velocity of light. Using $F^{\mu\nu}$, $\epsilon^{\mu\nu\rho\sigma}$, $g_{\mu\nu}$ and γ_μ one can construct the basis antisymmetric tensor-bispinors linear in $F^{\mu\nu}$:

$$T^{\mu\nu}_{1\,\rho\sigma} = F^\nu_\rho\gamma^\mu\gamma_\sigma - F^\mu_\rho\gamma^\nu\gamma_\sigma - F^\nu_\sigma\gamma^\mu\gamma_\rho + F^\mu_\sigma\gamma^\nu\gamma_\rho,$$

$$
\begin{aligned}
T^{\mu\nu}_{2\ \rho\sigma} &= F^{\mu}_{\rho}\delta^{\nu}_{\sigma} - F^{\nu}_{\rho}\delta^{\mu}_{\sigma} - F^{\mu}_{\sigma}\delta^{\nu}_{\rho} + F^{\nu}_{\sigma}\delta^{\mu}_{\rho},\\
T^{\mu\nu}_{3\ \rho\sigma} &= \gamma^{\nu}\gamma^{\lambda}F_{\rho\lambda}\delta^{\mu}_{\sigma} - \gamma^{\mu}\gamma^{\lambda}F_{\rho\lambda}\delta^{\nu}_{\sigma} - \gamma^{\nu}\gamma^{\lambda}F_{\sigma\lambda}\delta^{\mu}_{\rho} + \gamma^{\mu}\gamma^{\lambda}F_{\sigma\lambda}\delta^{\nu}_{\rho}\\
&\quad + \gamma_{\rho}\gamma^{\lambda}F^{\nu}_{\lambda}\delta^{\mu}_{\sigma} - \gamma_{\rho}\gamma^{\lambda}F^{\mu}_{\lambda}\delta^{\nu}_{\sigma} + \gamma_{\sigma}\gamma^{\lambda}F^{\mu}_{\lambda}\delta^{\nu}_{\rho} - \gamma_{\sigma}\gamma^{\lambda}F^{\nu}_{\lambda}\delta^{\mu}_{\rho},\\
T^{\mu\nu}_{4\ \rho\sigma} &= (\delta^{\mu}_{\sigma}\delta^{\nu}_{\rho} - \delta^{\nu}_{\sigma}\delta^{\mu}_{\rho})\gamma_{\alpha}\gamma_{\beta}F^{\alpha\beta},\\
T^{\mu\nu}_{5\ \rho\sigma} &= \gamma_4(\tilde{F}^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \tilde{F}^{\nu}_{\rho}\delta^{\mu}_{\sigma} - \tilde{F}^{\mu}_{\sigma}\delta^{\nu}_{\rho} + \tilde{F}^{\nu}_{\sigma}\delta^{\mu}_{\rho}),\\
T^{\mu\nu}_{6\ \rho\sigma} &= \gamma_4(F^{\mu}_{\alpha}\epsilon^{\alpha\nu}_{\rho\sigma} - F^{\nu}_{\alpha}\epsilon^{\alpha\mu}_{\rho\sigma} + F_{\alpha\rho}\epsilon^{\alpha\mu\nu}{}_{\sigma} - F_{\alpha\sigma}\epsilon^{\alpha\mu\nu}{}_{\rho}),\\
T^{\mu\nu}_{7\ \rho\sigma} &= (\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})F_{\rho\sigma} + F^{\mu\nu}(\gamma_{\rho}\gamma_{\sigma} - \gamma_{\sigma}\gamma_{\rho}),\\
T^{\mu\nu}_{8\ \rho\sigma} &= \gamma^{\nu}\gamma^{\lambda}F_{\rho\lambda}\delta^{\mu}_{\sigma} - \gamma^{\mu}\gamma^{\lambda}F_{\rho\lambda}\delta^{\nu}_{\sigma} - \gamma^{\nu}\gamma^{\lambda}F_{\sigma\lambda}\delta^{\mu}_{\rho} + \gamma^{\mu}\gamma^{\lambda}F_{\sigma\lambda}\delta^{\nu}_{\rho}\\
&\quad - \gamma_{\rho}\gamma^{\lambda}F^{\nu}_{\lambda}\delta^{\mu}_{\sigma} + \gamma_{\rho}\gamma^{\lambda}F^{\mu}_{\lambda}\delta^{\nu}_{\sigma} - \gamma_{\sigma}\gamma^{\lambda}F^{\mu}_{\lambda}\delta^{\nu}_{\rho} + \gamma_{\sigma}\gamma^{\lambda}F^{\nu}_{\lambda}\delta^{\mu}_{\rho},\\
T^{\mu\nu}_{9\ \rho\sigma} &= \gamma_4(F^{\mu}_{\alpha}\epsilon^{\alpha\nu}_{\rho\sigma} - F^{\nu}_{\alpha}\epsilon^{\alpha\mu}_{\rho\sigma} - F_{\alpha\rho}\epsilon^{\alpha\mu\nu}{}_{\sigma} + F_{\alpha\sigma}\epsilon^{\alpha\mu\nu}{}_{\rho}),\\
T^{\mu\nu}_{10\ \rho\sigma} &= (\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu})F_{\rho\sigma} - F^{\mu\nu}(\gamma_{\rho}\gamma_{\sigma} - \gamma_{\sigma}\gamma_{\rho}),
\end{aligned}
$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu}_{\rho\sigma}F^{\rho\sigma}$.

Then the general form of $T^{\mu\nu}_{\rho\sigma}$ and $\tilde{T}^{\mu\nu}_{\rho\sigma}$ is the following

$$
T^{\mu\nu}_{\rho\sigma} = \sum_{i=1}^{10}\alpha_i T^{\mu\nu}_{i\ \rho\sigma}, \qquad \tilde{T}^{\mu\nu}_{\rho\sigma} = \sum_{i=1}^{10}\tilde{\alpha}_i \tilde{T}^{\mu\nu}_{i\ \rho\sigma},
$$

where α_i, $\tilde{\alpha}_i$ are arbitrary constants.

Using (11) and asking for existence of real Lagrangian correspondingly to (3)

$$
\begin{aligned}
\Im &= \bar{\Psi}_{\mu\nu}(\gamma_{\lambda}\pi^{\lambda} - m)\Psi^{\mu\nu} + \frac{1}{12}\bar{\Psi}_{\mu\nu}(\pi^{\mu}\gamma^{\nu} - \pi^{\nu}\gamma^{\mu})[\gamma_{\rho},\gamma_{\sigma}]\Psi^{\rho\sigma} - \frac{1}{12}\bar{\Psi}_{\mu\nu}[\gamma_{\mu},\gamma_{\nu}]\\
&\quad \times(\pi_{\rho}\gamma_{\sigma} - \pi_{\sigma}\gamma_{\rho})\Psi^{\rho\sigma} + \frac{1}{24}\bar{\Psi}_{\mu\nu}[\gamma_{\mu},\gamma_{\nu}]\gamma_{\lambda}\pi^{\lambda}[\gamma_{\rho},\gamma_{\sigma}]\Psi^{\rho\sigma} + \bar{\Psi}_{\mu\nu}T^{\mu\nu}_{\rho\sigma}\Psi^{\rho\sigma} + \bar{\Psi}_{\mu\nu}\tilde{T}^{\mu\nu}_{\rho\sigma}\tilde{T}^{\rho\sigma}_{\delta\epsilon}\Psi^{\delta\epsilon}
\end{aligned}
$$

we come to the conditions

$$
\alpha_1 = \alpha_6 = -\alpha_9 = \frac{\lambda}{2}, \qquad \alpha_2 = 2\lambda, \qquad \alpha_3 = \alpha_8 = \frac{\lambda}{4}, \qquad \alpha_4 = \alpha_5 = \alpha_7 = \alpha_{10} = 0,
$$

where λ is an arbitrary constant. The analogous relations are valid for $\tilde{\alpha}_i$.

Substituting (9) into (3) we can express (3) in the Dirac-like form [7]

$$
\left(\Gamma_{\mu}\pi^{\mu} - m + \frac{e}{4m}(1 - i\Gamma_4)\left(\left(\frac{i}{4}(g-2)[\Gamma_{\mu},\Gamma_{\nu}] + g\tau_{\mu\nu}\right)F^{\mu\nu} + g_1(S_{\mu\nu}F^{\mu\nu})^2\right)\right)\Psi^{(1)} = 0, \quad (12)
$$

$$
\left(\Gamma_{\mu}\pi^{\mu} - m + \frac{e}{4m}(1 + i\Gamma_4)\left(\left(\frac{i}{4}(g-2)[\Gamma_{\mu},\Gamma_{\nu}] + g\tau_{\mu\nu}\right)F^{\mu\nu} + g_1(S_{\mu\nu}F^{\mu\nu})^2\right)\right)\Psi^{(2)} = 0, \quad (13)
$$

where $g = \frac{2}{3}\left(1 - \frac{\lambda}{2}\right)$, $g_1 = \frac{\tilde{\lambda}^2}{9}$, $\Psi^{(1)} = \left(\Phi^{(1)}_1, \Phi^{(1)}_2, \Phi^{(1)}_3\right)^T$, $\Psi^{(2)} = \left(\Phi^{(2)}_1, \Phi^{(2)}_2, \Phi^{(2)}_3\right)^T$, $S_{\mu\nu} = \frac{i}{4}[\Gamma_{\mu},\Gamma_{\nu}] + \tau_{\mu\nu}$ and $\tau_{\mu\nu}$ satisfy the relations $\tau_{ab} = \epsilon_{abc}\tau_c$; $\tau_{0a} = i\tau_a$, $\tau_a\tau_a = \tau(\tau+1)$; $[\tau_a,\tau_b] = i\epsilon_{abc}\tau_c$, $a,b,c = 1,2,3$.

Matrices Γ_{μ} and τ_a can be represented in the following forms; $\Gamma_{\mu} = \gamma_{\mu}\otimes I_3$, $\tau_a = I_4\otimes\hat{\tau}_a$, symbol $\otimes$ denotes the direct product of matrices, $\hat{\tau}_a$ are 3×3 matrices, realizing the representation $D(3)$ of the algebra $AO(3)$, I_3 and I_4 are the unit 3×3 and 4×4 matrices correspondingly.

In the representation (9) constraints (7) are reduced to the forms

$$
[(\Gamma_{\mu}\pi^{\mu} + m)(1 + i\Gamma_4)(S_{\mu\nu}S^{\mu\nu} - 3)]\Psi^{(1)} = 24m\Psi^{(1)}, \quad (14)
$$

$$
[(\Gamma_{\mu}\pi^{\mu} + m)(1 - i\Gamma_4)(S_{\mu\nu}S^{\mu\nu} - 3)]\Psi^{(2)} = 24m\Psi^{(2)}. \quad (15)
$$

We see that the value $g = 2$ corresponds to the most simple form of equations (12)–(13). Moreover, using Foldy–Wouthysen transformation [7] it can be shown that the Hamiltonian of the equation (12) or (13) in quasiclassical approximation is Hermitian, when $g = 2$.

4 Particle with spin $\frac{3}{2}$ in homogeneous magnetic field

Now let us use proposed equations to solve the problem of motion of charged particle with spin $\frac{3}{2}$ in constant magnetic field.

We start with equation (12) which can be written in the following equivalent form

$$\left(\pi_\mu\pi^\mu - m^2 + \frac{eg}{2}S_{\mu\nu}F^{\mu\nu} + \frac{eg_1}{2}(S_{\mu\nu}F^{\mu\nu})^2\right)\Psi_+^{(1)} = 0, \tag{16}$$

$$(S_{\mu\nu}S^{\mu\nu} - 15)\Psi_+^{(1)} = 0, \tag{17}$$

$$\Psi_-^{(1)} = \frac{1}{m}\Gamma_\mu\pi^\mu\Psi_+^{(1)}, \qquad \Psi^{(1)} = \Psi_+^{(1)} + \Psi_-^{(1)} \tag{18}$$

(the similar equation can be obtained for (13)).

The tensor $F^{\mu\nu}$ corresponding to the constant and homogeneous magnetic field can be chosen in the form

$$F_{0a} = F_{23} = -F_{32} = F_{31} = -F_{13} = 0, \qquad a = 1,2,3, \qquad F_{12} = -F_{21} = H_3 = H, \qquad H \geq 0,$$

where H is the strength of the magnetic field.

The solution of equation (17), (18) is of the form

$$\Psi_+^{(1)} = \begin{pmatrix} \Phi_{\frac{3}{2}}^{(1)} \\ \hat{0} \\ \frac{1}{m}\left(\varepsilon + \frac{2}{3}S_a\pi_a\right)\Phi_{\frac{3}{2}}^{(1)} \\ -\frac{2}{3m}K_a^{\frac{3}{2}}\pi_a\Phi_{\frac{3}{2}}^{(1)} \end{pmatrix}, \tag{19}$$

where

$$\left(K_3^{\frac{3}{2}}\right)_{mm'} = \delta_{mm'}\sqrt{\frac{9}{4} - m^2}; \quad \left(K_1^{\frac{3}{2}}\right)_{mm'} \pm i\left(K_2^{\frac{3}{2}}\right)_{mm'} = \pm\delta_{m\pm m'}\sqrt{\frac{3}{2} \mp m(m\mp 1) \pm 3m},$$

$m, m' = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$; $\hat{0} = (0,0)^T$, $\Phi_{\frac{3}{2}}^{(1)}$ is a 4 component spinor which satisfies the equation

$$\left[p^2 + e^2H^2x_2^2 - eH\left(gS_3 + 2g_1S_3^2H + 2x_2p_1\right)\right]\Phi_{\frac{3}{2}}^{(1)} = \left(\varepsilon^2 - m^2\right)\Phi_{\frac{3}{2}}^{(1)}. \tag{20}$$

So the problem of describing the motion of particle with spin $\frac{3}{2}$ reduces to the solution of equation (20).

Using the eigenvectors $\Omega_\nu^{\frac{3}{2}}$ of matrix S_3 ($\nu = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ are eigenvalues of S_3) we can represent $\Phi_{\frac{3}{2}}$ in the form

$$\Phi_{\frac{3}{2}}^{(1)} = \exp(ip_1x_1 + ip_3x_3)\sum_{\nu=-\frac{3}{2}}^{\frac{3}{2}} f_\nu^{\frac{3}{2}}(x_2)\Omega_\nu^{\frac{3}{2}}, \tag{21}$$

here $f_\nu^{\frac{3}{2}}(x_2)$ are unknown functions, $\Omega_\nu^{\frac{3}{2}}$ are 4 components spinor eigenvectors of S_3 and $\Omega_\nu^{\frac{3}{2}}$ are 4 components spinors

$$\Omega^{\frac{3}{2}}_{-\frac{3}{2}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \Omega^{\frac{3}{2}}_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \Omega^{\frac{3}{2}}_{\frac{1}{2}} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \Omega^{\frac{3}{2}}_{\frac{3}{2}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Substituting (21) into (20) we come to the equation for $f_\nu^{\frac{3}{2}}$

$$\left(-\frac{d^2}{dy^2} + y^2\right) f_\nu^{\frac{3}{2}}(y) = \eta f_\nu^{\frac{3}{2}}(y),$$

where $y = \frac{1}{\sqrt{eH}}(eHx_2 - p_1)$ and $\eta = \frac{1}{eH}(\varepsilon^2 - m^2 - p_3^2 + eH(\nu g + 2g_1\nu H))$. It is the equation for the harmonics oscillator. So, requiring that function $f_\nu^{\frac{3}{2}} \to 0$ when $x_2 \to \pm\infty$, we obtain the condition for η

$$\eta = 2n + 1, \qquad n = 0, 1, 2, 3, \ldots,$$

then the energy levels for the particle with spin $\frac{3}{2}$ in constant magnetic field can be written in the following form

$$\varepsilon^2 = m^2 + p_3^2 + eH(2n + 1 - \nu(g + 2g_1\nu H)), \qquad n = 0, 1, 2, 3, \ldots. \tag{22}$$

Function $f_\nu^{\frac{3}{2}}$ has the form

$$f_\nu^{\frac{3}{2}}(x_2) = \exp\left(-\frac{eHx_2 - p_1)}{2eH}\right) h_n\left(\frac{eHx_2 - p_1}{\sqrt{eH}}\right),$$

where $h_n(y) = \frac{H_n(y)}{||H_n(y)||}$, $H_n(y)$ are Hermitian polynomials.

We note that for the case of anomalous interaction linear in $F^{\mu\nu}$ (i.e. for $g_1 = 0$) ε^2 can be negative, provided $n = p_3 = 0$, $(\nu g - 1)eH < m^2$. Thus the difficulty with complex energies indicated earlier for spin-1 equation [7] appears also for spin $\frac{3}{2}$ equation [9]. However, this difficulty is overcome introducing anomalous interaction quadratic in $F^{\mu\nu}$ (i.e. choosing $g_1 \neq 0$ in (16)), namely

$$g_1 \leq -\frac{(3g - 2)^2 e}{72m^2}. \tag{23}$$

5 The charged particle with spin $\frac{3}{2}$ in electric and magnetic fields

In this case, as it was shown in [10, 11], we can confine our attention to the parallel and orthogonal configurations of electric and magnetic fields (all others configurations can be obtained ones mentioned in the above using Lorentz transformation)

a) $\vec{E}||\vec{H}$. For constant, uniform $\vec{E}$ and $\vec{H}$ directed along z axis we may choose $\vec{E} = (0, 0, E)$, $\vec{H} = (0, 0, H)$, $A_\mu = (x_3E, x_2H, 0, 0)$ and $E \neq H$. After substituting (19) into (16), equation (16) takes the form

$$\begin{aligned}&\left[(p_0 - ex_3E)^2 - (p_1 - ex_2H)^2 - p_2^2 - p_3^2 - m^2\right.\\ &\qquad \left.+egS_3(H - iE) + 2eg_1S_3^2(H - iE)^2\right] \Phi^{(1)}_{\frac{3}{2}} = 0.\end{aligned} \tag{24}$$

Choosing $\Phi^{(1)}_{\frac{3}{2}}$ in the form [10, 11]

$$\Phi^{(1)}_{\frac{3}{2}} = \exp(ip_1x_1 - \varepsilon x_0) f(x_2) \sum_{\nu=-\frac{3}{2}}^{\frac{3}{2}} g_\nu(x_3)\Omega^{\frac{3}{2}}_\nu, \tag{25}$$

where $f(x_2)$ and $g_\nu(x_3)$ are unknown functions, we can decompose equation (24) into two separate equations for $f(x_2)$ and $g_\nu(x_3)$. After solving these equations, $\Phi^{(1)}_{\frac{3}{2}}$ takes the form [10, 11]

$$\Phi^{(1)}_{\frac{3}{2}} = \exp(ip_1x_2 - \varepsilon x_0)\exp\left(-\frac{(p_1 + ex_2H)^2}{2eH}\right) h_n(x_2)$$
$$\times \exp\left(\frac{iz^2}{2}\right) \sum_{\nu=-\frac{3}{2}}^{\frac{3}{2}} G^j_\nu(-i\delta_\nu,\ -iz^2)\Omega^{\frac{3}{2}}_\nu, \qquad j = 1, 2, \tag{26}$$

where $p_1, \varepsilon = \text{const}$, $h_n(x_2) = \frac{H_n(x_2)}{||H_n(x_2)||}$, $H_n(x_2)$ are Hermitian polynomials, $z = \frac{1}{\sqrt{|eH|}}(\varepsilon - ex_2E)$, $\delta_\nu = \frac{m^2 - e\nu g(H-iE) - 2e\nu^2 g_1(H-iE)^2}{|eH|} - (2n+1)$, $n = 0, 1, 2, 3, \ldots$, $G^1_\nu\left(-i\delta_\nu,\ -iz^2\right) = F\left(\frac{1}{4}(1 - i\delta_\nu),\ \frac{1}{2}, -iz^2\right)$ and $G^2_\nu\left(-i\delta_\nu, -iz^2\right) = F\left(\frac{1}{4}(1 - i\delta_\nu) + \frac{1}{2},\ \frac{3}{2},\ iz^2\right)\sqrt{iz^2}$. F is the confluent hypergeometric function. So, in this case, the energy levels are not quantized.

b) $\vec{E} \perp \vec{H}$. Setting $\vec{E} = (0, E, 0)$, $\vec{H} = (0, 0, H)$, $A_\mu = (x_2E, x_2H, 0, 0)$ we obtain following equation for $\Phi^{(1)}_{\frac{3}{2}}$ instead of (24)

$$\big[(p_0 - ex_2E)^2 - (p_1 - ex_2H)^2 - p_2^2 - p_3^2 - m^2$$
$$+ eg(S_3H - iS_2E) + 2eg_1(S_3H - iS_2E)^2\big]\,\Phi^{(1)}_{\frac{3}{2}} = 0. \tag{27}$$

Representing $\Phi^{(1)}_{\frac{3}{3}}$ in the form

$$\Phi^{(1)}_{\frac{3}{2}} = \exp(ip_1x_1 + ip_3x_3 - i\varepsilon x_0) \sum_{\nu=-\frac{3}{2}}^{\frac{3}{2}} P_\nu(x_2)\Omega^{\frac{3}{2}}_\nu, \tag{28}$$

substituting (28) into (5) and using transformation $\hat{P}_\nu = U_{\nu\nu'}P_{\nu'}(x_2)$ we come to the equation

$$\left[(\varepsilon - ex_2E)^2 - (p_1 - ex_2H)^2 + \frac{d^2}{dx_2^2} - p_3^2 - m^2\right]\hat{P}_\nu(x_2) = eU_{\nu\nu'}\Lambda_{\nu'\nu''}U^{-1}_{\nu''\nu'''}\hat{P}_{\nu'''}(x_2). \tag{29}$$

where $\Lambda_{\nu\nu'} = eg(S_3H - iS_2E)_{\nu\nu'} + 2eg_1(S_3H - iS_2E)^2_{\nu\nu'}$ and $U_{\nu\nu'}\Lambda_{\nu'\nu''}U^{-1}_{\nu''\nu'''} = \lambda_\nu\delta_{\nu\nu'}$.

The solution of equation (29) has the following form

$$E = H : \hat{P}_\nu(x_2) = \Phi(\alpha - 2eH\gamma(p_1 - \varepsilon)x_2), \tag{30}$$

where Φ is Airy function, $\alpha = \left(p_3^2 + p_1^2 + m^2 - \varepsilon^2\right)\gamma$ and $\gamma = \left(4e^2h^2(p_1 - \varepsilon)^2\right)^{-\frac{1}{3}}$. So, the energy levels are not quantized

$$E \neq H : \hat{P}_\nu(x_2) = \exp\left(\frac{iz^2}{2}\right) G^j_\nu(-ia_\nu,\ -iz^2), \qquad j = 1, 2. \tag{31}$$

Here

$$a_\nu = -\left(\frac{p_3^2 + p_1^2 + m^2 - \varepsilon^2 + e\lambda_\nu}{e\eta} + \frac{(p_1H - \varepsilon E)^2}{e\eta^3}\right), \qquad \eta = \sqrt{E^2 - H^2},$$

$$\lambda_\nu = -ig\nu\eta - 2g_1\nu^2\eta^2, \qquad z = \sqrt{e\eta}\left(x_2 + \frac{p_1 H - \varepsilon E}{e\eta^2}\right).$$

When $E > H$, iz^2 of (31) becomes purely imaginary. So the energy levels are not quantized. In the case $E < H$, iz^2 becomes purely real and energy levels are quantized

$$\left(\varepsilon - \frac{p_1 E}{H}\right)^2 = \left(\frac{\eta'}{H}\right)^2 \left((2n+1)\eta' + e\lambda_\nu + p_3^2 + m^2\right), \tag{32}$$

where $\eta' = -i\eta$, $n = 0, 1, 2, 3, \ldots$. If $E \to 0$ we come to formula (22).

The exact form of matrix $U_{\nu\nu'}$ which must diagonalize matrix $\Lambda_{\nu\nu'}$ ($U_{\nu\nu'}\Lambda_{\nu'\nu''}U^{-1}_{\nu''\nu'''} = \lambda_\nu\delta_{\nu\nu'''}$, $\lambda_\nu = -ig\nu\eta - 2g_1\nu^2\eta^2$) can be obtained from the equation

$$\sqrt{\left(\frac{3}{2} - \nu\right)\left(\frac{5}{2} + \nu\right)}U_{\nu\nu'+1} + 2(\nu - \lambda_\nu)U_{\nu\nu'} + \sqrt{\left(\frac{3}{2} + \nu\right)\left(\frac{5}{2} - \nu\right)}U_{\nu\nu'-1} = 0,$$
$$\nu = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \qquad U_{\nu,\frac{5}{2}} = U_{\nu,-\frac{5}{2}} = 0.$$

6 Discussion

Thus we present the equation for particles with spin $\frac{3}{2}$ interacting with electromagnetic field, which is casual (it can be proved using approach proposed in [7]), i.e. their solutions are propagated with the velocity smaller than the light velocity. We also find the solutions of this equation in constant magnetic field and in static electric field inclined at an arbitrary angle to a static magnetic field. As it was shown above the corresponding constant g_1 can be chosen in such form in which the energies levels of charged particle with spin $\frac{3}{2}$ in constant magnetic field are not complex for arbitrary values H and g.

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Mirror Symmetry: Algebraic Geometric and Lagrangian Fibrations Aspects

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We survey some algebraic geometric aspects of mirror symmetry and duality in string theory.

1 Introduction

Symmetry principles always played an important role in mathematics and physics. Development of these sciences in direction of string theory enlarged the context of symmetry considerations and included in it the notion of duality. String theory has following ingredients: (i) base space (open or closed string) Σ; (ii) target space M; (iii) fields: $X \to \Sigma \to M$; (iv) action $S = \int \mathcal{L}(X, \varphi)$. where $\mathcal{L}$ is a Lagrangian [1]. Let G be a group such that $G \supset SU(3) \times SU(2) \times U(1)$. Recall that if $\mathcal{L}(G\Phi) = \mathcal{L}(\Phi)$ then $\mathcal{L}$ is G-invariant, or G-symmetry. In string theory [1] one of the beautiful symmetries is the radius symmetry $R \to 1/R$ of circle, known as T-duality [2, 3] and [4] and references there in. Authors of papers [5, 6] conjectured that a similar duality might exist in the context of string propagation on Calabi–Yau (CY) manifolds, where the role of the complex deformation on one manifold gets exchanged with the Kähler deformation on the dual manifold. A pair of manifolds satisfying this symmetry is called *mirror pair*, and this duality is called *mirror symmetry*.

From the point of view of physicists which did the remarkable discovery, mirror symmetry is a type of duality that means that we may take two types of string theory and compactify them in two different ways and achieve "isomorphic" physics [7]. Or in the case of a pair of Calabi–Yau threefolds (X, Y) P. Aspinwall are said [8] that X and Y to be a *mirror pair* if and only if the type IIA string compactified on X is "isomorphic" to the $E_8 \times E_8$ heterotic string compactified on Y. In the case that X is Calabi–Yau threefold Y will be the product of a $K3$ surface and elliptic curve. C. Vafa defines the notion of mirror of a Calabi–Yau manifold with a stable bundle. Lagrangian and special Lagrangian submanifolds appear in this situation. Mathematicians also work hard upon the problems of mirror symmetry, although it is difficult in some cases to attribute to a researcher the identifier "mathematician" or "physicist". V. Batyrev gives construction of mirror pairs using Gorenstein toric Fano varieties and Calabi–Yau hypersurfaces in these varieties [9]. M. Kontsevich in his talk at the ICM'94 gave a conjecturel interpratation of mirror symmetry as a "shadow" of an equivalence between two triangulated categories associated with A_∞-categories [10]. His conjecture was proved in the case of elliptic curves by A. Polishchuk and E. Zaslow [11]. The aim of the paper is to provide a short and gentle survey of some algebraic aspects of mirror symmetry, duality and special lagrangian fibrations with examples – without proofs, but with (a very restricted) guides to the literature.

2 Preliminaries

We shall use in contrast to [1] some another definition of Calabi–Yau (CY) manifold. The definition based on the theorem of Yau who proved Calabi's conjecture that a complex Kähler manifold of vanishing first Chern class admits a Ricci-flat metric.

Definition 1. A complex Kähler manifold is called Calabi–Yau (CY) manifold if it has vanishing first Chern class.

Examples of the CY-manifolds include, in particular, elliptic curves E, $K3$-surfaces and their products $E \times K3$.

2.1 Vector bundles

Local chart or a *system of coordinates* on a topological space M is a pair (U, φ) where U is an open set in M and $\varphi : U \to \mathbb{R}^m$ is a homeomorphism from U to an open set $\varphi(U)$ in $\mathbb{R}^m$. An *atlas* Φ of dimension m is a collection of local charts whose domains cover M and such that if $(U, \varphi), (U_1, \varphi_1) \in \Phi$ and $U \cap U_1 \neq 0$ then the map

$$\varphi_1 \circ \varphi^{-1} : \varphi(U \cap U_1) \to \varphi_1(U \cap U_1)$$

is a C^r-diffeomorphism between open sets in $\mathbb{R}^m$.

Fibre space is the object (E, p, B), where p is the continuous surjective (= on) mapping of a topological space E onto a space B and $p^{-1}(b)$ is called the *fibre* above $b \in B$. Both the notation $p : E \to B$ and (E, p, B) are used to denote a *fibration*, a *fibre space*, a *fibre bundle* or a *bundle*.

Vector bundle is fibre space each fibre $p^{-1}(b)$ of which is endowed with the structure of a (finite dimension) vector space V over skew-field K such that the following local triviality condition is satisfied: each point $b \in B$ has an open neighborhood U and a V -isomorphism of fibre bundles $\phi : p^{-1}(U) \to U \times V$ such that $\phi\mid_{p^{-1}(b)}: p^{-1}(b) \to b \times V$ is an isomorphism of vector spaces for each $b \in B$. $\dim V$ is said to be the dimension of the vector bundle.

An *Hermitian bundle* over algebraic variety X consists of a vector bundle over X and a choice of C^∞ Hermitian metric on the vector bundle over complex manifold $X(\mathbb{C})$, which is invariant under antiholomorphic involution of $X(\mathbb{C})$.

The *tangent space* to a differentiable manifold M at point $a \in M$ can be defined as the set of tangency classes of smooth paths in M based at a. It will be denoted by T_aM. Elements of T_aM are called tangent vectors to M at a.

The *tangent bundle* of M, denoted by TM, is the union of the tangent spaces at all the points of M. By well known way TM can be made into a smooth manifold. Recall well known facts about TM :

(i) if M is C^r then TM is C^{r-1};

(ii) if M is C^∞ or C^ω then the same holds for TM;

(iii) if M has dimension n then TM has dimension $2n$;

(iv) there is a natural map $p : TM \to M$ called the *projection* map, taking T_aM to a for each a in M, i.e. p takes all tangent vectors at a to the point a itself. Thus $p^{-1}(a) = T_aM$ (fibre of the bundle over a). The projection p is a smooth map C^{r-1} if M is C^r.

A *vector field* on a smooth manifold M is a map $F : M \to TM$ which satisfies $p \circ F = id_M$, where p is the natural projection $TM \to M$. By its definition a vector field is a *section* of the bundle TM.

2.2 Blow-ups

Blowing up is a well known method of constructing complex manifolds M. There are points on the manifolds that are not divisors on M. Blow up is the construction that transforms points of complex manifolds to divisors. For instance in the case of two dimensional complex manifolds (complex surface) N it consists of replacing a point $p \in N$ by a projective line $\mathbb{CP}(1)$ considered as the set of limit directions at p.

Example 1. Let $\pi : M_2 \to \mathbb{C}^2$ be the blow-up of $\mathbb{C}^2$ at the point $0 \in \mathbb{C}^2$. Then M_2 is a two dimensional complex manifold that defined by two local charts. In coordinates $\mathbb{C}^2 = (z_1, z_2), \mathbb{CP}(1) = [l_0, l_1)$ manifold M_2 is defined in $\mathbb{CP}(1) \times \mathbb{C}^2$ by quadratic equations $z_i l_j = z_j l_i$. Thus M_2 is a line bundle over Riemann sphere $\mathbb{CP}(1)$. $\pi^{-1}(0) = \mathbb{CP}(1)$ is called *the divisor of the blow up (the exceptional divisor).*

Recently a large class of CY orbifolds in weighted projective spaces was suggested. C. Vafa have predicted and S. Roan [14] have computed the Euler number of all the resolved CY hypersurfaces in a weighted projective space $\mathbb{WCP}(4)$.

2.3 Vector bundles over projective algebraic curves

Let X be a projective algebraic curve over algebraically closed field k and g the genus of X. Let $\mathcal{VB}(X)$ be the category of vector bundles over X. Grothendieck showed that for a rational curve every vector bundle is a direct sum of line bundles. Atiyah classified vector bundles over elliptic curves. The main result is

Theorem 1. *Let X be an elliptic curve, A a fixed base point on X. We may regard X as an abelian variety with A as the zero element. Let $\mathcal{E}(r,d)$ denote the the set of equivalence classes of indecomposable vector bundles over X of dimension r and degree d. Then each $\mathcal{E}(r,d)$ may be identified with X in such a way that* $\det : \mathcal{E}(r,d) \to \mathcal{E}(1,d)$ *corresponds to $H : X \to X$, where $H(x) = hx = x + x + \cdots + x$ (h times), and $h = (r,d)$ is the highest common factor of r and d.*

Curve X is called a *configuration* if its normalization is a union of projective lines and all singular points of X are simple nodes [16]. For each configuration X can assign a non-oriented graph $\Delta(X)$, whose vertices are irreducible components of X, edges are its singular and an edge is incident to a vertex if the corresponding component contains the singular point. Drozd and Greuel have proved:

Theorem 2. *1. $\mathcal{VB}(X)$ contains finitely many indecomposable objects up to shift and isomorphism if and only if X is a configuration and the graph $\Delta(X)$ is a simple chain (possibly one point if $X = \mathbb{P}^1$).*

2. $\mathcal{VB}(X)$ is tame, i.e. there exist at most one-parameter families of indecomposable vector bundles over X, if and only if either X is a smooth elliptic curve or it is a configuration and the graph $\Delta(X)$ is a simple cycle (possibly, one loop if X is a rational curve with only one simple node).

3. Otherwise $\mathcal{VB}(X)$ is wild, i.e. for each finitely generated k-algebra Λ there exists a full embedding of the category of finite dimensional Λ-modules into $\mathcal{VB}(X)$.

Let X be an algebraic curve. How to normalize it? There are several methods, algorithms and implementations for this purpose. A new algorithm and implementation is presented in [17].

2.4 Connection

Consider the connection in the context of algebraic geometry. Let S/k be the smooth scheme over field k, U an element of open covering of S, $\mathcal{O}_S$ the structure sheaf on S, $\Gamma(U, \mathcal{O}_S)$ the sections of $\mathcal{O}_S$ on U. Let $\Omega^1_{S/k}$ be the sheaf of germs of 1-dimension differentials, $\mathcal{F}$ be a coherent sheaf. The *connection* on the sheaf $\mathcal{F}$ is the sheaf homomorphism

$$\nabla : \mathcal{F} \to \Omega^1_{S/k} \otimes \mathcal{F},$$

such that, if $f \in \Gamma(U, \mathcal{O}_S)$, $g \in \Gamma(U, \mathcal{F})$ then

$$\nabla(fg) = f\nabla(g) + df \otimes g.$$

There is the dual definition. Let $\mathcal{F}$ be the locally free sheaf, $\Theta^1_{S/k}$ the dual to sheaf $\Omega^1_{S/k}$, $\partial \in \Gamma\left(U, \Theta^1_{S/k}\right)$. The *connection* is the homomorphism

$$\rho : \Theta^1_{S/k} \to \mathrm{End}_{\mathcal{O}_S}(\mathcal{F}, \mathcal{F}), \qquad \rho(\partial)(fg) = \partial(f)g + f\rho(\partial).$$

3 Moduli spaces in string theory

Mirror symmetry connects with geometrical deformations of complex and Kähler structures on CY-manifolds. So we have to know moduli spaces of complex and Kähler structures on CY-manifolds.

3.1 Moduli spaces

The theory of moduli spaces [12, 13] has, in recent years, become the meeting ground of several different branches of mathematics and physics-algebraic geometry, instantons, differential geometry, string theory and arithmetics. Here we recall some underlieing algebraic structures of the relation. In previous section we have reminded the situation with vector bundles on projective algebraic curves X. On X any first Chern class $c_1 \in H^2(X, \mathbb{Z})$ can be realized as c_1 of vector bundle of prescribed rank (dimension) r. How to classify vector bundles over algebraic varieties of dimension more than 1? This is one of important problems of algebraic geometry and the problem has closed connections with gauge theory in physics and differential geometry. Mamford [12] and others have formulated the problem about the determination of which cohomology classes on a projective variety can be realized as Chern classes of vector bundles? Moduli spaces are appeared in the problem. What is moduli? Classically Riemann claimed that $3g-3$ (complex) parameters could be for Riemann surface of genus g which would determine its conformal structure (for elliptic curves, when $g = 1$, it is needs one parameter). From algebraic point of view we have the following problem: given some kind of variety, classify the set of all varieties having something in common with the given one (same numerical invariants of some kind, belonging to a common algebraic family). For instance, for an elliptic curve the invariant is the modular invariant of the elliptic curve.

Let $\mathbb{B}$ be a class of objects. Let S be a scheme. A family of objects parametrized by the S is the set of objects

$$X_s : \quad s \in S, \quad X_s \in \mathbb{B}$$

equipped with an additional structure compatible with the structure of the base S. Parameter varieties is a class of moduli spaces. These varieties is a very convenient tool for computer algebra investigation of objects that parametrized by the parameter varieties. We have used the approach for investigation of rational points of hyperelliptic curves over prime finite fields [21].

Example 2. Let ω_1, $\omega_2 \in \mathbb{C}$, $\mathrm{Im}\,(\omega_1/\omega_2) > 0$, $\Lambda = n\omega_1 + m\omega_2$, $n, m \in \mathbb{Z}$ be a lattice. Let H be the upper half plane. Then $H/\Lambda = E$ be the elliptic curve. Let

$$y^2 = x^3 + ax + b = (x-e_1)(x-e_2)(x-e_3), \qquad 4a^3 + 27b^2 \neq 0,$$

be the equation of E. Then the differential of first kind on E is defined by formula

$$\omega = dx/y = dx/\left(x^3 + ax + b\right)^{1/2}.$$

Periods of E:

$$\pi_1 = 2\int_{e_1}^{e_2} \omega, \qquad \pi_2 = 2\int_{e_2}^{e_3} \omega.$$

The space of moduli of elliptic curves over $\mathbb{C}$ is $\mathbb{A}^1(\mathbb{C})$. Its completion is $\mathbb{CP}(1)$.

For $K3$-surfaces the situation is more complicated but in some case is analogous [18].

Theorem 3. *The moduli space of complex structure on market $K3$-surface (including orbifold points) is given by the space of possible periods.*

Some computational aspects of periods and moduli spaces are considered in author's notes [22, 23].

4 Some categorical constructions

Let (X, ω, Ω) be a complex manifold (real dimension $=2n$) with

$$\omega^n/n! = (-1)^{n(n-1)/2}(i/2)^n \cdot \Omega \wedge \overline{\Omega}.$$

It is said that a n-dimensional submanifold $L \subset X$ is *special Lagrangian (s-lag)* $\Leftrightarrow$

$$\mathrm{Re}\,(\Omega|_L) = \mathrm{Vol}|_L \;\Leftrightarrow\; \omega|_L = 0, \qquad \mathrm{Im}\,(\Omega|_L) = 0.$$

Example 3. Let X be an elliptic curve E. Then $\omega = c(i/2)dz \wedge d\overline{z}$, $\Omega = cdz$. S-lag $L \subset E$ are straight lines with slope determined by $\arg c$.

Every compact symplectic manifold Y, ω with vanishing first Chern class, one can associate a A_∞-category whose objects are essentially the Lagrangian submanifolds of Y, and whose morphisms are determined by the intersections of pairs of submanifolds. This category is called Fukaya's category and is denoted by $\mathcal{F}(Y)$ [10]. Let (X, Y) be a mirror pair. Let M be any element of the mirror pair. The bounded derived category $D^b(M)$ of coherent sheaves on M is obtained from the category of bounded complexes of coherent sheaves on M [19]. In the case of elliptic curves A. Poleshchuk and E. Zaslov have proved [11]:

Theorem 4. *The categories $D^b(E_q)$ and $\mathcal{F}^0(\overline{E}^q)$ are equivalent.*

Recently A. Kapustin and D. Orlov have suggested that Kontsevich's conjecture must be modified: coherent sheaves must be replaced with modules over Azumaya algebras, and the Fukaya category must be "twisted" by closed 2-form [20].

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Symmetries and Dynamical Symmetry Breaking of General n-Dimensional Self-Consistently Renormalized Spinor Diangles

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Using the self-consistent renormalization (SCR), a careful study of complicated tangle of problems associated on the one hand with renormalizations and on the other with symmetries conservation, their breaking, the Ward identities (WIs) behavior, the Schwinger terms contributions (STCs), and quantum anomalies is performed for some set of UV-divergent Feynman amplitudes (FAs) connected with general mass-anysotropic spinor diangles in any space-time dimension $n = 2r + \delta_n$, $\delta_n = 0, 1$. It is shown that the WIs involving SCR FAs do retain (or imitate) the canonical WIs (CWIs). In this context quantum anomalies reveal themselves either as an oversubtraction effect for a non-chiral case and for chiral limits (in these cases the STCs are zero) or as nonzero STCs for the chiral case. Effective formulae for general quantum corrections (QCs) to the CWIs and primitive "daughter reduction identities" (DRIs) are derived for any dimension n. For an anysotropic case ($m_1 \neq m_2$, $m_l \neq 0$), the QCs are the zero degree homogeneous functions of masses and are expressed in terms of hypergeometric functions ${}_2F_1$. For the degenerate nonchiral case ($m_1 = m_2 = m \neq 0$), these QCs either are equal to zero for vector WIs or reduce to mass-independent expressions for axial-vector WIs. The Schwinger-Johnson anomaly for $n = 2$ is a particular case of general formulas obtained. Conditions under which the nonzero STCs exist are obtained and the role of the STCs in the QCs are revealed. The behavior of FAs and QCs in the chiral case ($m = 0$) and in the symmetric chiral limit ($m \to 0$) is different. In the chiral case only the "left-handed vector" current may be conserved and hence it may be more fundamental than vector or axial-vector currents.

1 Introduction

In the perturbation theory, quantum anomalies manifest themselves as breakdown of the canonical WIs (CWIs) at a level of regular (finite) values of FAs involved in them. Therefore, modes and interpretations of these CWIs violations are extremely important as for the quantum field theory itself and for physical applications [1–8]. Despite the large number of papers which have been written on quantum anomalies, surprisingly many facets of this problem have not been adequately described, if at all. We hope to clarify some obscure points in these violations by employing the SCR [9–14] to general spinor diangle FAs, being very important objects for physical applications [15, 16], and to illustrate possibilities of the SCR. Subjects which will be raised here are: i) mass dependence of quantum anomalies; ii) distinction between chiral and chiral limit anomalies; iii) relation between the Schwinger terms contributions and quantum anomalies. Previously, we have carried out similar investigation for the spinor triangle FAs [17–22] in which new features of quantum anomalies have been exhibited. Recall that the SCR is an effective realization of the Bogoliubov–Parasiuk R-operation [23–26] which is complemented with recurrence, compatibility, and differential relations fixing a renormalization arbitrariness of the R-operation in some universal way based on mathematical properties of FAs only.

2 General spinor diangle amplitudes and their identities

2.1. The main Feynman amplitude corresponding to the spinor diangle graph of the most general kind (different masses, arbitrary Clifford structure of vertices, n-dimensional world with (q,p)-signature of a nondegenerate metric g, where q and p are respectively the number of negative and positive squares in g, i.e. $q+p=n=2r+\delta_n$, $\delta_n=0,1$) looks as follows:

$$\begin{aligned}
&I^{\gamma_1\gamma_2}(m,k) := \int_{-\infty}^{\infty}(d^np)\,\delta(p,k)\frac{\mathrm{tr}[\gamma_1(m_1+\hat{p}_1)\gamma_2(m_2+\hat{p}_2)]}{(\mu_1-p_1^2)(\mu_2-p_2^2)},\\
&(d^np) := d^np_1d^np_2, \quad \hat{p}_l := \gamma^\sigma p_{l\sigma}, \quad m := (m_1,m_2), \quad k := (k_1,k_2),\\
&\delta(p,k) := \delta_1(-k_1+p_2-p_1)\,\delta_2(-k_2+p_1-p_2), \quad \mu_l := m_l^2 - i\epsilon_l. \qquad (1)
\end{aligned}$$

The matrices γ_i, γ_σ, I_g, act in the N_g-dimensional space of the faithful representation $\pi(g)$ of the lowest dimension for the Clifford algebra $Cl(g)_{\mathbb{K}}$, $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, with $\gamma^\sigma \in \Lambda^1(g)$, $\sigma = 1,\dots,n$, being the generating elements of the $Cl(g)_{\mathbb{K}}$-algebra in its matrix representation $\pi(g)$, i.e. $\gamma^\sigma\gamma^\tau+\gamma^\tau\gamma^\sigma = 2g^{\sigma\tau}I_g$; γ_i, $i=1,2$, are, as a rule, some k-degree ($k=0,1,\dots,n$) homogeneous elements of the $Cl(g)_{\mathbb{K}}$-algebra in the $\pi(g)$-representation or some linear combination of such elements; I_g is the N_g-dimensional unit matrix. The n-degree element $\gamma^* \in \Lambda^n(g)$, i.e. the dual conjugation matrix, with the obvious but important properties:

$$\begin{aligned}
&\gamma^* := \gamma^1\gamma^2\cdots\gamma^n, \quad (\gamma^*)^2 = \varepsilon(g)I_g, \quad \gamma^\sigma\gamma^* = (-1)^{n+1}\gamma^*\gamma^\sigma, \quad \sigma=1,\dots,n,\\
&\varepsilon(g) := (-1)^q(-1)^{n(n-1)/2} = (-1)^{\varkappa(\varkappa+1)/2}, \quad \varkappa := (q-p)\ (\mathrm{mod}\,8), \qquad (2)
\end{aligned}$$

is the natural analog of the Dirac γ^5-matrix. For more details on properties of the γ^*-matrix and on the self-consistent version of the dimensional regularization with the γ^*-matrix see [27, 28].

2.2. The UV-divergent FAs (1) satisfy formally the *canonical Ward identities* (CWIs):

$$\begin{aligned}
k_{1\mu}I^{(\gamma^\mu\gamma)\gamma_2}(m,k) &= D_1^{\dot{\gamma}\gamma_2}(m,k)\\
&= (-1)^{\pi_1}P_1^{\gamma\gamma_2}(m,k) - P_2^{\gamma\gamma_2}(m,k) + (m_2-(-1)^{\pi_1}m_1)\,I^{\gamma\gamma_2}(m,k),\\
k_{2\alpha}I^{\gamma_1(\gamma^\alpha\gamma)}(m,k) &= D_2^{\gamma_1\dot{\gamma}}(m,k)\\
&= (-1)^{\pi_2}P_2^{\gamma_1\gamma}(m,k) - P_1^{\gamma_1\gamma}(m,k) + (m_1-(-1)^{\pi_2}m_2)\,I^{\gamma_1\gamma}(m,k). \qquad (3)
\end{aligned}$$

Here the quantities $D_1^{\dot{\gamma}\gamma_2}(m,k)$, $D_2^{\gamma_1\dot{\gamma}}(m,k)$, $P_l^{\gamma_1'\gamma_2'}(m,k)$, $l=1,2$, $\gamma_i'=\gamma_i$ or γ, $i=1,2$, are similar to the main amplitude $I^{\gamma_1\gamma_2}(m,k)$ and differ from it only in polynomials of the integrand:

$$\begin{aligned}
&I^{\gamma_1\gamma_2}(m,k) \longleftrightarrow \mathrm{tr}\,[\mathcal{I}^{\gamma_1\gamma_2}(m,p)] := \mathrm{tr}\,[\gamma_1\,(m_1+\hat{p}_1)\,\gamma_2\,(m_2+\hat{p}_2)]\,; \qquad (4)\\
&D_1^{\dot{\gamma}\gamma_2}(m,k) \longleftrightarrow \mathrm{tr}\left[\mathcal{D}_1^{\dot{\gamma}\gamma_2}(m,p)\right] := (p_2-p_1)_\mu\mathrm{tr}\left[\mathcal{I}^{(\gamma^\mu\gamma)\gamma_2}(m,p)\right],\\
&D_2^{\gamma_1\dot{\gamma}}(m,k) \longleftrightarrow \mathrm{tr}\left[\mathcal{D}_2^{\gamma_1\dot{\gamma}}(m,p)\right] := (p_1-p_2)_\alpha\mathrm{tr}\left[\mathcal{I}^{\gamma_1(\gamma^\alpha\gamma)}(m,p)\right]; \qquad (5)\\
&P_1^{\gamma_1'\gamma_2'}(m,k) \longleftrightarrow \mathrm{tr}\left[\mathcal{P}_1^{\gamma_1'\gamma_2'}(m,p)\right] := \mathrm{tr}\left[\gamma_1'\left(m_1^2-p_1^2\right)\gamma_2'\,(m_2+\hat{p}_2)\right],\\
&P_2^{\gamma_1'\gamma_2'}(m,k) \longleftrightarrow \mathrm{tr}\left[\mathcal{P}_2^{\gamma_1'\gamma_2'}(m,p)\right] := \mathrm{tr}\left[\gamma_1'\,(m_1+\hat{p}_1)\,\gamma_2'\left(m_2^2-p_2^2\right)\right]. \qquad (6)
\end{aligned}$$

In equations (3) the vector CWIs ($\gamma = I_g$) and the axial-vector CWIs ($\gamma=\gamma^*$) are represented in the uniform manner. The factors

$$(-1)^{\pi_i} = \begin{cases} 1, & \text{if } \gamma = I_g,\ \forall\, n, \text{ or } \gamma=\gamma^*,\ n=2r+1;\\ -1, & \text{if } \gamma=\gamma^*,\ n=2r, \end{cases}$$

stem from the commutation relations $\gamma^\sigma\gamma = (-1)^\pi\gamma\gamma^\sigma$, $\sigma=1,\dots,n$.

The quantities $D_1^{\dot\gamma\gamma_2}(m,k)$, $D_2^{\gamma_1\dot\gamma}(m,k)$, correspond to divergencies of current density T-products $\langle 0|\,\partial_{1\mu}T(J^{O_1}(x_1)J^{O_2}(x_2))\,|0\rangle$, $\langle 0|\,\partial_{2\alpha}T(J^{O_1}(x_1)J^{O_2}(x_2))\,|0\rangle$, where $\partial_{i\sigma}\equiv\partial/\partial x_i^\sigma$, $J^{O_i}(x_i)= :\overline{\Psi}(x_i)O_i\Psi(x_i):$, $O_i:=h_i\otimes\gamma_i$, and $h_i=\tau_i\otimes\lambda_i$ are matrices specifying flavor-color structure of current densities. The quantities $P_l^{\gamma_1'\gamma_2'}(m,k)$ are associated with current density commutators and consequently with possible contributions of the Schwinger terms in them.

2.3. Let us consider the obvious identities:

$$P_{l\epsilon}^{\gamma_1\gamma_2}(m,k)=\overline{P}_{l\epsilon}^{\gamma_1\gamma_2}(\overline{m}_{(l)},k),\quad l=1,2,\quad \overline{m}_{(1)}\equiv m_2,\quad \overline{m}_{(2)}\equiv m_1, \tag{7}$$

$$\frac{\partial}{\partial\mu_l}P_{l\epsilon}^{\gamma_1\gamma_2}(m,k)=0,\quad l=1,2, \tag{8}$$

in which the simple idea of cancelling the equal factors in factorized polynomials in numerators and the denominator of integrands is used. Therefore, these identities are named as *reduction identities* (RIs). The *nonreduced* FAs in the l.h.s. of equation (7) are defined as:

$$\begin{bmatrix}P_{1\epsilon}^{\gamma_1\gamma_2}(m,k)\\ P_{2\epsilon}^{\gamma_1\gamma_2}(m,k)\end{bmatrix}:=\int_{-\infty}^{\infty}\frac{(d^np)\,\delta(p,k)}{(\mu_1-p_1^2)(\mu_2-p_2^2)}\begin{bmatrix}\mathrm{tr}\left[\gamma_1\left(\mu_1-p_1^2\right)\gamma_2\left(m_2+\hat p_2\right)\right]\\ \mathrm{tr}\left[\gamma_1\left(m_1+\hat p_1\right)\gamma_2\left(\mu_2-p_2^2\right)\right]\end{bmatrix} \tag{9}$$

and the *reduced* FAs in the r.h.s. of equation (7) are:

$$\begin{bmatrix}\overline{P}_{1\epsilon}^{\gamma_1\gamma_2}(\overline{m}_{(1)},k)\\ \overline{P}_{2\epsilon}^{\gamma_1\gamma_2}(\overline{m}_{(2)},k)\end{bmatrix}:=\int_{-\infty}^{\infty}(d^np)\,\delta(p,k)\begin{bmatrix}\mathrm{tr}\left[\gamma_1\gamma_2\left(m_2+\hat p_2\right)\right]/\left(\mu_2-p_2^2\right)\\ \mathrm{tr}\left[\gamma_1\left(m_1+\hat p_1\right)\gamma_2\right]/\left(\mu_1-p_1^2\right)\end{bmatrix}, \tag{10}$$

which are well known as "tadpole" amplitudes.

The RIs (7) are closely related to the CWIs (3). Indeed, due to equation (6) the amplitudes $P_l^{\gamma_1'\gamma_2'}(m,k)$ involving in equations (3) are very similar to the nonreduced FAs in equation (9). The only difference between them is the $i\epsilon_l$-terms in numerator polynomials of integrands in equation (9). But exactly these terms permit to perform identical cancellations of factors and to obtain independent on $\mu_l=m_l^2-i\epsilon_l$ expressions that are reflected clearly in equation (8).

The RIs (7) induce primitive *daughter* RIs (DRIs) via decompositions involving: i) the numeric Clifford tensors $\mathrm{tr}[\gamma_1\gamma_2 m_2]$, $\mathrm{tr}[\gamma_1\gamma_2\gamma_\sigma]$, $\mathrm{tr}[\gamma_1 m_1\gamma_2]$, $\mathrm{tr}[\gamma_1\gamma_\sigma\gamma_2]$; ii) the irreducible tensor structures constructed by means of independent external momenta (e.g., k_2, or k_1). Altogether for general spinor diangles, there are 4 primitive DRIs taken two $\forall\; l=1,2$.

3 General spinor diangle amplitudes and identities in the SCR

3.1. The amplitude $I^{\gamma_1\gamma_2}(m,k)$ has the divergence index $\nu=n-2$ whereas the amplitudes $D_1^{\dot\gamma\gamma_2}(m,k)$, $D_2^{\gamma_1\dot\gamma}(m,k)$, $P_l^{\gamma_1'\gamma_2'}(m,k)$, $P_{l\epsilon}^{\gamma_1\gamma_2}(m,k)$, $\overline{P}_{l\epsilon}^{\gamma_1\gamma_2}(\overline{m}_{(l)},k)$, $l=1,2$, have the divergence index $\nu+1=n-1$. The regular values for all of them are obtained according to the SCR [9, 10, 11, 13]. Some of them have be given in [29] as the net results. Here once again, we present only some of them, but in such a form permitting to evaluate all unavailable amplitudes.

So, the regular values $(R^\nu I)^{\gamma_1\gamma_2}(m,k)$, $(R^{\nu+1}D_1)^{\dot\gamma\gamma_2}(m,k)$, and $(R^{\nu+1}P_{1\epsilon})^{\gamma_1\gamma_2}(m,k)$ of the amplitudes given by equations (1), (5), and (9) have the following α-parametric integral representation, shown here in a form best suitable for general FAs:

$$\begin{bmatrix}(R^\nu I)^{\gamma_1\gamma_2}(m,k)\\ (R^{\nu+1}D_1)^{\dot\gamma\gamma_2}(m,k)\\ (R^{\nu+1}P_{1\epsilon})^{\gamma_1\gamma_2}(m,k)\end{bmatrix}=(2\pi)^n\delta(k)b(g)\int_{\Sigma^1}\frac{d\mu(\alpha)}{\Delta^{n/2}}\sum_{s=0}^{3}\sum_{j=0}^{[s/2]}\begin{bmatrix}\mathrm{tr}[\mathcal{I}_{sj}^{\gamma_1\gamma_2}](R^\nu\mathcal{F})_{sj}\\ \mathrm{tr}[\mathcal{D}_{1;sj}^{\dot\gamma\gamma_2}](R^{\nu+1}\mathcal{F})_{sj}\\ \mathrm{tr}[\mathcal{P}_{1\epsilon;sj}^{\gamma_1\gamma_2}](R^{\nu+1}\mathcal{F})_{sj}\end{bmatrix}. \tag{11}$$

The integration measure $d\mu(\alpha)$, the integration region Σ^1, the metric dependent constant $b(g)$, and the overall δ-function $\delta(k)$ are defined as

$$d\mu(\alpha) := \delta(1-\alpha_1-\alpha_2)d\alpha_1\, d\alpha_2, \quad \Sigma^1 := \{\alpha_l |\, \alpha_l \geq 0,\ \forall\, l,\ \alpha_1+\alpha_2 = 1\},$$
$$b(g) := (\pi^{n/2} i^p)/(2\pi)^n, \quad \delta(k) := \delta(-k_1-k_2), \tag{12}$$

and p is the number of positive squares in a space-time metric g.

The explicit form of the basic functions $(R^\nu\mathcal{F})_{sj}$, $(R^{\nu+1}\mathcal{F})_{sj}$, and the determining numbers ν_{sj}, λ_{sj}, ν^1_{sj}, λ^1_{sj}, and ω appearing in them are as follows:

$$(R^\nu\mathcal{F})_{sj} := M_\epsilon^{\omega+j}\,\Gamma(\lambda_{sj})/\Gamma(2+\nu_{sj}) Z_\epsilon^{1+\nu_{sj}}\, {}_2F_1(1,\lambda_{sj};\, 2+\nu_{sj};\, Z_\epsilon),$$
$$\nu_{sj} := [(\nu-s)/2] + j, \quad \lambda_{sj} := -\omega - j + 1 + \nu_{sj}, \quad \omega := n/2-2, \tag{13}$$

$$(R^{\nu+1}\mathcal{F})_{sj} := M_\epsilon^{\omega+j}\,\Gamma(\lambda^1_{sj})/\Gamma(2+\nu^1_{sj}) Z_\epsilon^{1+\nu^1_{sj}}\, {}_2F_1(1,\lambda^1_{sj};\, 2+\nu^1_{sj};\, Z_\epsilon),$$
$$\nu^1_{sj} := [(\nu+1-s)/2] + j, \quad \lambda^1_{sj} := -\omega - j + 1 + \nu^1_{sj}, \quad \omega := n/2-2. \tag{14}$$

The $[s/2]$, $[(\nu-s)/2]$, and $[(\nu+1-s)/2]$ in equations (11), (13)–(14) are integral parts of the numbers $s/2$, $(\nu-s)/2$, and $(\nu+1-s)/2$ respectively. The subscripts (s,j) of the basic functions $(R^\nu\mathcal{F})_{sj}$ and $(R^{\nu+1}\mathcal{F})_{sj}$ just mean that these functions are attached to the homogeneous k-polynomials $\mathcal{I}^{\gamma_1\gamma_2}_{sj}$, $\mathcal{D}^{\dot\gamma\gamma_2}_{1;sj}$, and $\mathcal{P}^{\gamma_1\gamma_2}_{1\epsilon;sj}$ of the degree $s-2j$ in external momenta. The latter are α-images of the homogeneous p-polynomials $\mathcal{I}^{\gamma_1\gamma_2}_s(m,p)$, $\mathcal{D}^{\dot\gamma\gamma_2}_{1;s}(m,p)$, and $\mathcal{P}^{\gamma_1\gamma_2}_{1\epsilon;s}(m,p)$ of the degree s appearing in $\mathcal{I}^{\gamma_1\gamma_2}(m,p)$, $\mathcal{D}^{\dot\gamma\gamma_2}_1(m,p)$, and $\mathcal{P}^{\gamma_1\gamma_2}_{1\epsilon}(m,p)$ given by equations (4)–(6) and equation (9):

$$\begin{aligned}
&\mathcal{I}^{\gamma_1\gamma_2}_{00} := \gamma_1 m_1 \gamma_2 m_2, && \mathcal{P}^{\gamma_1\gamma_2}_{1\epsilon;00} := \gamma_1\mu_1\gamma_2 m_2,\\
&\mathcal{I}^{\gamma_1\gamma_2}_{10} := \gamma_1\widehat{Y}_1\gamma_2 m_2 + \gamma_1 m_1\gamma_2\widehat{Y}_2, && \mathcal{P}^{\gamma_1\gamma_2}_{1\epsilon;10} := \gamma_1\mu_1\gamma_2\widehat{Y}_2,\\
&\mathcal{I}^{\gamma_1\gamma_2}_{20} := \gamma_1\widehat{Y}_1\gamma_2\widehat{Y}_2, && \mathcal{P}^{\gamma_1\gamma_2}_{1\epsilon;20} := \gamma_1(-Y_1^2)\gamma_2 m_2,\\
&\mathcal{I}^{\gamma_1\gamma_2}_{21} := (-1/2)X_{12}\gamma_1\gamma^\sigma\gamma_2\gamma_\sigma, && \mathcal{P}^{\gamma_1\gamma_2}_{1\epsilon;30} := \gamma_1(-Y_1^2)\gamma_2\widehat{Y}_2,\\
&\mathcal{I}^{\gamma_1\gamma_2}_{30} = \mathcal{I}^{\gamma_1\gamma_2}_{31} := 0; && \mathcal{P}^{\gamma_1\gamma_2}_{1\epsilon;21} := (-1/2)(-nX_{11})\gamma_1\gamma_2 m_2,\\
&\mathcal{D}^{\dot\gamma\gamma_2}_{1;00} = \mathcal{D}^{\dot\gamma\gamma_2}_{1;21} := 0, && \mathcal{P}^{\gamma_1\gamma_2}_{1\epsilon;31} := (-1/2)[(-nX_{11})\gamma_1\gamma_2\widehat{Y}_2 + (-2X_{12})\gamma_1\gamma_2\widehat{Y}_1],\\
&\mathcal{D}^{\dot\gamma\gamma_2}_{1;sj} := (Y_2-Y_1)_\mu\mathcal{I}^{(\gamma^\mu\gamma)\gamma_2}_{s-1,j}, && \text{if } (s,j)\neq(0,0),(2,1); \quad \mu_1 := m_1^2 - i\epsilon_1.
\end{aligned} \tag{15}$$

The α-parametric functions $Z_\epsilon \equiv Z_\epsilon(\alpha,m,k)$, $M_\epsilon \equiv M_\epsilon(\alpha,m)$, $A\equiv A(\alpha,k)$, $\Delta\equiv\Delta(\alpha)$, $Y_l \equiv Y_l(\alpha,k)$ and $X_{ll'} \equiv X_{ll'}(\alpha)$ incoming in equations (11)–(15) have the form:

$$Z_\epsilon := A/M_\epsilon, \quad M_\epsilon := \alpha_1\mu_1 + \alpha_2\mu_2, \quad A := \frac{\alpha_1\alpha_2}{\Delta}\, k_2^2 = \alpha_1 Y_1^2 + \alpha_2 Y_2^2, \quad \Delta := \alpha_1+\alpha_2,$$
$$Y_1 := \beta_2\, k_2, \quad Y_2 := -\beta_1\, k_2, \quad X_{ll'} := \Delta^{-1},\ l,l'\in\{1,2\}, \quad Y_2 - Y_1 = -k_2 = k_1,$$
$$Y_l^2 = -\frac{A}{\Delta} + (1-\beta_l)k_2^2, \quad Y_1\cdot Y_2 = -\frac{A}{\Delta}, \quad \alpha_l Y_l^2 = (1-\beta_l)A, \quad \beta_l := \frac{\alpha_l}{\Delta}, \quad l = 1,2. \tag{16}$$

Similar considerations for the reduced amplitudes (10) give rise to the zero values:

$$(R^{\nu+1}\overline{P}_{l\epsilon})^{\gamma_1\gamma_2}(\overline{m}_{(l)}, k) = 0, \qquad \forall\, n\geq 1, \quad l = 1,2, \tag{17}$$

confirming once again but in another way the well known result for “tadpole” amplitudes.

3.2. In the SCR, there exist the following compatibility and recurrence relations:

$$(R^\nu\mathcal{F})_{sj} = \mathcal{F}_{sj} := M_\epsilon^{\omega+j}(1-Z_\epsilon)^{\omega+j}\,\Gamma(-\omega-j), \quad \text{if} \quad \nu_{sj}\leq -1, \tag{18}$$
$$(R^\nu\mathcal{F})_{sj} = (R^{\nu+1}\mathcal{F})_{s+1,j}, \tag{19}$$
$$M_\epsilon\,(R^\nu\mathcal{F})_{00} - A\,(R^\nu\mathcal{F})_{20} + (\omega+1)\,(R^\nu\mathcal{F})_{21} = 0,$$
$$M_\epsilon\,(R^{\nu+1}\mathcal{F})_{10} - A\,(R^{\nu+1}\mathcal{F})_{30} + (\omega+1)\,(R^{\nu+1}\mathcal{F})_{31} = 0, \tag{20}$$

between the basic functions $(R^\nu\mathcal{F})_{sj}$, $(R^{\nu+1}\mathcal{F})_{sj}$. In fact, due to compatibility relations (19) both recurrence relations (20) are different forms of the common one.

3.3. From equations (11), (15), and (16) follow more specific formulae for our quantities. So, the regular value of the main FA defined by equation (1) takes the form:

$$\begin{aligned}(R^\nu I)^{\gamma_1\gamma_2}(m,k) &= (2\pi)^n\delta(k)\,b(g)\int_{\Sigma^1}\frac{d\mu(\alpha)}{\Delta^{n/2}}\Big\{\mathrm{tr}[\gamma_1 m_1\gamma_2 m_2](R^\nu\mathcal{F})_{00}\\ &+\mathrm{tr}[\gamma_1\hat{k}_2\gamma_2 m_2\beta_2-\gamma_1 m_1\gamma_2\hat{k}_2\beta_1](R^\nu\mathcal{F})_{10}\\ &+\mathrm{tr}[\gamma_1\hat{k}_2\gamma_2\hat{k}_2](-\beta_1\beta_2)(R^\nu\mathcal{F})_{20}+\mathrm{tr}[\gamma_1\gamma^\sigma\gamma_2\gamma_\sigma](-1/2)\Delta^{-1}(R^\nu\mathcal{F})_{21}\Big\}.\end{aligned} \tag{21}$$

The regular values of convolutions and divergence contributions involved in the CWIs (3) and defined by equations (5) and (1) look as follow:

$$\begin{bmatrix}k_{1\mu}(R^\nu I)^{(\gamma^\mu\gamma)\gamma_2}(m,k)\\ k_{2\alpha}(R^\nu I)^{\gamma_1(\gamma^\alpha\gamma)}(m,k)\end{bmatrix}=\begin{bmatrix}(R^{\nu+1}D_1)^{\dot\gamma\gamma_2}(m,k)\\ (R^{\nu+1}D_2)^{\gamma_1\dot\gamma}(m,k)\end{bmatrix}=(2\pi)^n\delta(k)\,b(g)\int\limits_{\Sigma^1}\frac{d\mu(\alpha)}{\Delta^{n/2}} \tag{22}$$

$$\times\begin{bmatrix}\mathrm{tr}[\gamma\gamma_2](R^{\nu+1}\mathcal{D}_1)_{\{0\}}(m,\alpha,k)-\mathrm{tr}[\gamma\gamma_2\hat{k}_2](R^{\nu+1}\mathcal{D}_1)_{\{1\}}(m,\alpha,k)\\ \mathrm{tr}[\gamma_1\gamma](R^{\nu+1}\mathcal{D}_2)_{\{0\}}(m,\alpha,k)+\mathrm{tr}[\gamma_1\hat{k}_2\gamma](R^{\nu+1}\mathcal{D}_2)_{\{1\}}(m,\alpha,k)\end{bmatrix}, \tag{23}$$

$$(R^{\nu+1}\mathcal{D}_i)_{\{0\}}(m,\alpha,k):=k_2^2\,[\,m_i\beta_i-(-1)^{\pi_i}m_j\beta_j\,](R^{\nu+1}\mathcal{F})_{20},\quad i,j\in\{1,2\},\quad j\neq i; \tag{24}$$

$$\begin{aligned}(R^{\nu+1}\mathcal{D}_i)_{\{1\}}(m,\alpha,k)&:=m_1m_2(R^{\nu+1}\mathcal{F})_{10}-(-1)^{\pi_i}(A/\Delta)(R^{\nu+1}\mathcal{F})_{30}\\ &+(-1)^{\pi_i}(n/2-1)\Delta^{-1}(R^{\nu+1}\mathcal{F})_{31}\\ &=[\,m_1m_2-(-1)^{\pi_i}(M_\epsilon/\Delta)\,](R^{\nu+1}\mathcal{F})_{10},\quad i=1,2.\end{aligned} \tag{25}$$

The regular values of the nonreduced FAs in the RIs (7) defined by equation (9) take the form:

$$\begin{bmatrix}(R^{\nu+1}P_{1\epsilon})^{\gamma_1\gamma_2}(m,k)\\ (R^{\nu+1}P_{2\epsilon})^{\gamma_1\gamma_2}(m,k)\end{bmatrix}=\begin{bmatrix}\mathrm{tr}[\gamma_1\gamma_2]m_2(R^{\nu+1}P_{1\epsilon})_{\{0\}}-\mathrm{tr}[\gamma_1\gamma_2\hat{k}_2](R^{\nu+1}P_{1\epsilon})_{\{1\}}(m,k)\\ \mathrm{tr}[\gamma_1\gamma_2]m_1(R^{\nu+1}P_{2\epsilon})_{\{0\}}+\mathrm{tr}[\gamma_1\hat{k}_2\gamma_2](R^{\nu+1}P_{2\epsilon})_{\{1\}}(m,k)\end{bmatrix}, \tag{26}$$

$$\begin{bmatrix}(R^{\nu+1}P_{l\epsilon})_{\{0\}}(m,k)\\ (R^{\nu+1}P_{l\epsilon})_{\{1\}}(m,k)\end{bmatrix}:=(2\pi)^n\delta(k)\,b(g)\int\limits_{\Sigma^1}\frac{d\mu(\alpha)}{\Delta^{n/2}}\begin{bmatrix}(R^{\nu+1}\mathcal{P}_{l\epsilon})_{\{0\}}(m,\alpha,k)\\ (R^{\nu+1}\mathcal{P}_{l\epsilon})_{\{1\}}(m,\alpha,k)\end{bmatrix}, \tag{27}$$

$$\begin{aligned}(R^{\nu+1}\mathcal{P}_{l\epsilon})_{\{0\}}(m,\alpha,k)&:=\mu_l(R^{\nu+1}\mathcal{F})_{00}-Y_l^2(R^{\nu+1}\mathcal{F})_{20}+(n/2)\Delta^{-1}(R^{\nu+1}\mathcal{F})_{21}\\ &=(R^{\nu+1}\mathcal{P}_l)_{\{0\}}(m,\alpha,k)-i\epsilon_l(R^{\nu+1}\mathcal{F})_{00};\end{aligned} \tag{28}$$

$$\begin{aligned}(R^{\nu+1}\mathcal{P}_{l\epsilon})_{\{1\}}(m,\alpha,k)&:=\beta_l\mu_l(R^{\nu+1}\mathcal{F})_{10}-\beta_lY_l^2(R^{\nu+1}\mathcal{F})_{30}\\ &+[\,(n/2)\beta_l-(1-\beta_l)\,]\Delta^{-1}(R^{\nu+1}\mathcal{F})_{31}=(R^{\nu+1}\mathcal{P}_l)_{\{1\}}(m,\alpha,k)-i\epsilon_l\beta_l(R^{\nu+1}\mathcal{F})_{10}.\end{aligned} \tag{29}$$

The first equality in equation (22) holds due to compatibility relations (19). The second expression of equation (25) follows from the first one due to the second recurrence relation (20).

3.4. The tensor structure of regular values $(R^{\nu+1}\overline{P}_{l\epsilon})^{\gamma_1\gamma_2}(\overline{m}_{(l)},k)$, $l=1,2$, is the same as that of $(R^{\nu+1}P_{l\epsilon})^{\gamma_1\gamma_2}(m,k)$, $l=1,2$, given by the r.h.s. of equation (26), and consequently from equations (7), (17), and (26) we obtain four very practical primitive DRIs at the regular level:

$$(R^{\nu+1}P_{l\epsilon})_{\{0\}}(m,k)=0,\qquad (R^{\nu+1}P_{l\epsilon})_{\{1\}}(m,k)=0,\qquad \forall\, n\geq 1,\quad l=1,2, \tag{30}$$

where the l.h.s. are given by equations (27)–(29).

Similarly, the regular values $(R^{\nu+1}P_l)^{\gamma_1'\gamma_2'}(m,k)$, $l=1,2$, of FAs involved in CWIs (3) and defined by equations (6) and (1) are almost the same as the $(R^{\nu+1}P_{l\epsilon})^{\gamma_1\gamma_2}(m,k)$, $l=1,2$, given by equations (26)–(29). The connection between them is determined completely by the second expressions of equations (28)–(29). Taking into account equations (30) one obtains very simple representations:

$$\begin{bmatrix}(R^{\nu+1}P_l)_{\{0\}}(m,k)\\ (R^{\nu+1}P_l)_{\{1\}}(m,k)\end{bmatrix} := (2\pi)^n\delta(k)\,b(g)\int_{\Sigma^1}\frac{d\mu(\alpha)}{\Delta^{n/2}}\begin{bmatrix}i\epsilon_l(R^{\nu+1}\mathcal{F})_{00}\\ i\epsilon_l\beta_l(R^{\nu+1}\mathcal{F})_{10}\end{bmatrix},\quad l=1,2, \tag{31}$$

for quantities $(R^{\nu+1}P_l)_{\{\kappa\}}(m,k)$ in terms of which the regular values $(R^{\nu+1}P_l)^{\gamma_1'\gamma_2'}(m,k)$ are initially expressed by using of equations (26)–(29) (and replacing μ_l by m_l^2).

By virtue of properties of the hypergeometric function ${}_2F_1$ from equation (31) the important limiting values of these quantities for any space-time dimension $n = 2r+\delta_n$, $\delta_n = 0,1$, follow:

$$\lim_{\epsilon_1,\epsilon_2\to 0,\ \exists\, m_s\neq 0;\ \text{or}\ (\epsilon,m)\to 0}(R^{\nu+1}P_l)_{\{\kappa\}}(m,k) = 0, \qquad \forall\, n\geq 1,\quad l=1,2,\quad \kappa=0,1, \tag{32}$$

$$\lim_{(m,\epsilon)\to 0}\begin{cases} m_{l'}(R^{\nu+1}P_l)_{\{0\}}(m,k) = 0, \quad \forall\, n\geq 1, \quad l',l\in\{1,2\}, \quad l'\neq l,\\ (R^{\nu+1}P_l)_{\{1\}}(m,k) = (2\pi)^n\delta(k)\,b(g)(1-\delta_n)(k_2^2)^{r-1}\Gamma(r)/(2\Gamma(2r)).\end{cases} \tag{33}$$

Hereafter the (ϵ,m)-limit means first $\epsilon_l\to 0$ and then $m_l = m\to 0$, $l=1,2$, i.e. it is equivalent to the symmetric chiral limit case ($m_1 = m_2 = m\to 0$). Analogously, the (m,ϵ)-limit means first $m_l\to 0$ and then $\epsilon_l = \epsilon\to 0$, $l=1,2$, i.e. it is equivalent to the chiral case ($m_1 = m_2 = 0$).

3.5. It turns out that so calculated regular values of the FAs defined by equations (1) and (4)–(6) satisfy the identities [29]:

$$\begin{aligned} &k_{1\mu}(R^\nu I)^{(\gamma^\mu\gamma)\gamma_2}(m,k) = (R^{\nu+1}D_1)^{\dot\gamma\gamma_2}(m,k)\\ &\qquad = (-1)^{\pi_1}(R^{\nu+1}P_1)^{\gamma\gamma_2}(m,k) - (R^{\nu+1}P_3)^{\gamma\gamma_2} + (m_2-(-1)^{\pi_1}m_1)\,(R^{\nu+1}I)^{\gamma\gamma_2}(m,k),\\ &k_{2\alpha}(R^\nu I)^{\gamma_1(\gamma^\alpha\gamma)}(m,k) = (R^{\nu+1}D_2)^{\gamma_1\dot\gamma}(m,k)\\ &\qquad = (-1)^{\pi_2}(R^{\nu+1}P_2)^{\gamma_1\gamma}(m,k) - (R^{\nu+1}P_1)^{\gamma_1\gamma} + (m_1-(-1)^{\pi_2}m_2)\,(R^{\nu+1}I)^{\gamma_1\gamma}(m,k), \end{aligned} \tag{34}$$

which are referred to as *the regular analog of the CWIs* (3) (or the quantum Ward identities (QWIs)). The latter name may be more adequately depict their physical meaning. The first rows of equations (34) are due to the compatibility relations (19). It is important to note also that the last terms in the identities (34) are calculated by the renormalization index $\nu+1$, although their proper divergence index is ν. It is this peculiarity that permits to the regular analogs of the CWIs (34) both to imitate (or to retain) the CWIs (3) and to differ from them simultaneously. It is this peculiarity that permits to obtain some effective formulae for calculating of *the quantum corrections* (QCs) to the CWIs in the most general nonchiral case [29].

3.6. As a result the regular analogs of the CWIs (34) are equivalent to four scalar equations:

$$\begin{aligned} &(R^{\nu+1}D_1)_{\{\kappa\}}(m,k) = (R^{\nu+1}P_{\bar 1-2})_{\{\kappa\}}(m,k) + (R^{\nu+1}I_{2-\bar 1})_{\{\kappa\}}(m,k), \quad \kappa=0,1,\\ &(R^{\nu+1}D_2)_{\{\kappa\}}(m,k) = (R^{\nu+1}P_{\bar 2-1})_{\{\kappa\}}(m,k) + (R^{\nu+1}I_{1-\bar 2})_{\{\kappa\}}(m,k), \quad \kappa=0,1, \end{aligned} \tag{35}$$

$$\begin{bmatrix}(R^{\nu+1}D_i)_{\{\kappa\}}(m,k)\\ (R^{\nu+1}P_{\bar i-j})_{\{\kappa\}}(m,k)\\ (R^{\nu+1}I_{j-\bar i})_{\{\kappa\}}(m,k)\end{bmatrix} := (2\pi)^n\delta(k)\,b(g)\int_{\Sigma^1}\frac{d\mu(\alpha)}{\Delta^{n/2}}\begin{bmatrix}(R^{\nu+1}\mathcal{D}_i)_{\{\kappa\}}(m,\alpha,k)\\ (R^{\nu+1}\mathcal{P}_{\bar i-j})_{\{\kappa\}}(m,\alpha,k)\\ (R^{\nu+1}\mathcal{I}_{j-\bar i})_{\{\kappa\}}(m,\alpha,k)\end{bmatrix}, \tag{36}$$

$$\begin{aligned} &(R^{\nu+1}\mathcal{D}_i)_{\{0\}}(m,\alpha,k) = k_2^2\,[\,m_i\beta_i - (-1)^{\pi_i}m_j\beta_j\,](R^{\nu+1}\mathcal{F})_{20},\\ &(R^{\nu+1}\mathcal{P}_{\bar i-j})_{\{0\}}(m,\alpha,k) := (-1)^{\pi_i}m_j(R^{\nu+1}\mathcal{P}_i)_{\{0\}}(m,\alpha) - m_i(R^{\nu+1}\mathcal{P}_j)_{\{0\}}(m,\alpha)\\ &\qquad \cong [\,(-1)^{\pi_i}m_j(i\epsilon_i) - m_i(i\epsilon_j)\,](R^{\nu+1}\mathcal{F})_{00}, \end{aligned}$$

$$
\begin{aligned}
&(R^{\nu+1}\mathcal{I}_{j-\bar{i}})_{\{0\}}(m,\alpha,k) := (m_j-(-1)^{\pi_i}m_i)\big[\,m_1m_2(R^{\nu+1}\mathcal{F})_{00}-\\
&\quad-(-1)^{\pi_i}(A/\Delta)(R^{\nu+1}\mathcal{F})_{20}-(-1)^{\pi_i}(n/2)\Delta^{-1}(R^{\nu+1}\mathcal{F})_{21}\,\big]\\
&\quad= m_i(R^{\nu+1}\mathcal{P}_j)_{\{0\}}(m,\alpha)-(-1)^{\pi_i}m_j(R^{\nu+1}\mathcal{P}_i)_{\{0\}}(m,\alpha)\\
&\quad+k_2^2\,[\,m_i\beta_i-(-1)^{\pi_i}m_j\beta_j\,](R^{\nu+1}\mathcal{F})_{20}\cong[\,m_i\,(i\epsilon_j)-(-1)^{\pi_i}m_j\,(i\epsilon_i)\,](R^{\nu+1}\mathcal{F})_{00}\\
&\quad+k_2^2\,[\,m_i\beta_i-(-1)^{\pi_i}m_j\beta_j\,](R^{\nu+1}\mathcal{F})_{20},\qquad i,j\in\{1,2\},\quad j\neq i;
\end{aligned}\tag{37}
$$

$$
\begin{aligned}
&(R^{\nu+1}\mathcal{D}_i)_{\{1\}}(m,\alpha,k) := [\,m_1m_2-(-1)^{\pi_i}(M_\epsilon/\Delta)\,](R^{\nu+1}\mathcal{F})_{10},\\
&(R^{\nu+1}\mathcal{P}_{\bar{i}-j})_{\{1\}}(m,\alpha,k) := (-1)^{\pi_i}\big[\,(R^{\nu+1}\mathcal{P}_i)_{\{1\}}(m,\alpha,k)+(R^{\nu+1}\mathcal{P}_j)_{\{1\}}(m,\alpha,k)\,\big]\\
&\quad=(-1)^{\pi_i}(iE/\Delta)(R^{\nu+1}\mathcal{F})_{10},\\
&(R^{\nu+1}\mathcal{I}_{j-\bar{i}})_{\{1\}}(m,\alpha,k) := (m_j-(-1)^{\pi_i}m_i)[\,m_i\beta_i-(-1)^{\pi_i}m_j\beta_j\,](R^{\nu+1}\mathcal{F})_{10}\\
&\quad=[\,m_1m_2-(-1)^{\pi_i}(M/\Delta)\,](R^{\nu+1}\mathcal{F})_{10},\qquad i,j\in\{1,2\},\quad j\neq i.
\end{aligned}\tag{38}
$$

Equations (35) are obeyed for all values of k_2, m_l, ϵ_l, $l=1,2$, and any space-time dimension n. But limiting values of quantities in them depend strongly on the limit employed. Hereafter the quantities M and E are defined as $M:=\alpha_1m_1^2+\alpha_2m_2^2$, $E:=\alpha_1\epsilon_1+\alpha_2\epsilon_2$, and hence $M_\epsilon=M-iE$, and the congruence relation $\mathcal{A}(m,\alpha,k)\cong\mathcal{B}(m,\alpha,k)$ denotes the equality of the integrals $\int_{\Sigma^1}d\mu(\alpha)\Delta^{-n/2}\mathcal{A}(m,\alpha,k)=\int_{\Sigma^1}d\mu(\alpha)\Delta^{-n/2}\mathcal{B}(m,\alpha,k)$. See also equations (25), (28)–(29), (31), and the relations $Y_l^2=-A/\Delta+(1-\beta_l)k_2^2$, $l=1,2$, in equations (16).

4 Quantum corrections to the CWIs and STCs in the SCR

4.1. Now we investigate equations (34)–(38) more closely. Let us first consider a general mass-anysotropic nonchiral case. Then, from equations (32) follow $\lim_{\epsilon_1,\epsilon_2\to 0}(R^{\nu+1}P_l)_{\{\kappa\}}(m,k)=0$ if $m_1,\ m_2\neq 0$, $\forall\ l=1,2$, $\kappa=0,1$, and consequently we also obtain $\lim_{\epsilon_1,\epsilon_2\to 0}(R^{\nu+1}P_{\bar{i}-j})_{\{\kappa\}}(m,k)$ $=0$, $\forall\ l=1,2$, $\kappa=0,1$. The *quantum corrections* (QCs) (or anomalous contributions in usual nomenclature) to the CWIs appear now as an oversubtraction effect and take the form:

$$
\begin{aligned}
\begin{bmatrix} a_1^{\gamma\gamma_2}(m,k)\\ a_2^{\gamma_1\gamma}(m,k)\end{bmatrix} &:= \begin{bmatrix}(m_2-(-1)^{\pi_1}m_1)\,[(R^{\nu+1}I)^{\gamma\gamma_2}(m,k)-(R^{\nu}I)^{\gamma\gamma_2}(m,k)]\\ (m_1-(-1)^{\pi_2}m_2)\,[(R^{\nu+1}I)^{\gamma_1\gamma}(m,k)-(R^{\nu}I)^{\gamma_1\gamma}(m,k)]\end{bmatrix}\\
&=\begin{bmatrix}\mathrm{tr}[\gamma\gamma_2]a_{1\{0\}}(m,k)-\mathrm{tr}[\gamma\gamma_2\hat{k}_2]a_{1\{1\}}(m,k)\\ \mathrm{tr}[\gamma_1\gamma]a_{2\{0\}}(m,k)+\mathrm{tr}[\gamma_1\hat{k}_2\gamma]a_{2\{1\}}(m,k)\end{bmatrix},
\end{aligned}\tag{39}
$$

where the scalar functions $a_{i\{\kappa\}}(m,k)$ have the integral representations:

$$
a_{i\{\kappa\}}(m,k) := (2\pi)^n\delta(k)\,b(g)\int_{\Sigma^1}\frac{d\mu(\alpha)}{\Delta^{n/2}}a_{i\{\kappa\}}(m,\alpha,k),\qquad i=1,2,\quad \kappa=0,1,\tag{40}
$$

$$
\begin{aligned}
&a_{i\{0\}}(m,\alpha,k) := (m_j-(-1)^{\pi_i}m_i)\big[\,m_1m_2(\Delta\mathcal{F})_{00}-(-1)^{\pi_i}(A/\Delta)(\Delta\mathcal{F})_{20}\\
&\quad-(-1)^{\pi_i}(n/2)\Delta^{-1}(\Delta\mathcal{F})_{21}\,\big]\cong[\,m_i\,(i\epsilon_j)-(-1)^{\pi_i}m_j\,(i\epsilon_i)\,](\Delta\mathcal{F})_{00}\\
&\quad+k_2^2\,[\,m_i\beta_i-(-1)^{\pi_i}m_j\beta_j\,](\Delta\mathcal{F})_{20},\qquad i,j\in\{1,2\},\quad j\neq i;
\end{aligned}\tag{41}
$$

$$
\begin{aligned}
&a_{i\{1\}}(m,\alpha,k) := (m_j-(-1)^{\pi_i}m_i)[\,m_i\beta_i-(-1)^{\pi_i}m_j\beta_j\,](\Delta\mathcal{F})_{10}\\
&\quad=[\,m_1m_2-(-1)^{\pi_i}(M/\Delta)\,](\Delta\mathcal{F})_{10},\quad i,j\in\{1,2\},\ j\neq i.
\end{aligned}\tag{42}
$$

The quantities $(\Delta\mathcal{F})_{sj}$ appearing in equations (41)–(42) are defined as

$$
\begin{aligned}
&(\Delta\mathcal{F})_{sj} := (R^{\nu+1}\mathcal{F})_{sj}-(R^{\nu}\mathcal{F})_{sj}=(-1)\Theta_{sj}\left(\Gamma(\lambda_{sj})A^{1+\nu_{sj}}\right)/\left(\Gamma(2+\nu_{sj})M_\epsilon^{\lambda_{sj}}\right),\\
&\Theta_{sj} := H(\nu_{sj}^1)\theta_s,\qquad \theta_s := \nu_{sj}^1-\nu_{sj}=(\nu-s)\ (\mathrm{mod}\,2),
\end{aligned}\tag{43}
$$

and $H(x)$ is the Heaviside step function such that $H(x)=0$, $x<0$, $H(x)=1$, $x\geq 0$.

4.2. According to equations (13)–(14) we find that $\lambda_{00} = 2 - \delta_n/2$, $\nu_{00} = r-1$, $\Theta_{00} = \delta_n H(r-1+\delta_n)$; $\lambda_{20} = 1-\delta_n/2$, $\nu_{20} = r-2$, $\Theta_{20} = \delta_n H(r-2+\delta_n)$; $\lambda_{21} = 1-\delta_n/2$, $\nu_{21} = r-1$, $\Theta_{21} = \delta_n H(r-1+\delta_n)$; $\lambda_{10} = 1+\delta_n/2$, $\nu_{10} = r-2+\delta_n$, $\Theta_{10} = (1-\delta_n)H(r-1)$, and taking into account the representation:

$$\Phi\Big(\lambda\,\Big|\,{}^{a_1,\,a_2}_{\mu_1,\,\mu_2}\Big) := \int_{\Sigma^1} \frac{d\mu(\alpha)}{\Delta^b}\frac{\alpha_1^{a_1-1}\alpha_2^{a_2-1}}{M_\epsilon^\lambda} = \begin{cases} B(a_1,a_2)/\mu_2^\lambda\, {}_2F_1(\lambda, a_1;\, a_1+a_2;\, 1-\xi_{1/2}), \\ B(a_1,a_2)/\mu_1^\lambda\, {}_2F_1(\lambda, a_2;\, a_1+a_2;\, 1-\xi_{2/1}), \end{cases}$$
$$B(a_1,a_2) := \Gamma(a_1)\Gamma(a_2)/\Gamma(a_1+a_2), \qquad b := n/2+N,\ N\in\mathbb{Z}_+, \quad \xi_{l/s} := \mu_l/\mu_s, \tag{44}$$

the scalar functions $a_{i\{\kappa\}}(m,k)$ given by equations (40)–(43) acquire the form:

$$a_{i\{0\}}(m,k) = (2\pi)^n\delta(k)\,b(g)\,\delta_n(k_2^2)^r a_{i\{0\}}(m_1,\epsilon_1;\, m_2,\epsilon_2), \qquad i=1,2,$$
$$a_{i\{1\}}(m,k) = (2\pi)^n\delta(k)\,b(g)\,(1-\delta_n)(k_2^2)^{r-1} a_{i\{1\}}(m_1,\epsilon_1;\, m_2,\epsilon_2), \qquad i=1,2, \tag{45}$$

in which the mass dependent functions $a_{i\{\kappa\}}(m_1,\epsilon_1;\, m_2,\epsilon_2)$ with ϵ-damping look as follow:

$$a_{i\{0\}}(m_1,\epsilon_1;\, m_2,\epsilon_2) := (-1)^i\frac{\Gamma(3/2)}{\Gamma(r+1)}(m_2-m_1)\big\{m_1m_2\,\Phi\big(3/2\,\big|\,{}^{r+1,\,r+1}_{\mu_1,\ \mu_2}\big)$$
$$-\,(4r+1)\Phi\big(1/2\,\big|\,{}^{r+1,\,r+1}_{\mu_1,\ \mu_2}\big)\big\}, \qquad \mu_l := m_l^2 - i\epsilon_l, \quad i=1,2, \tag{46}$$
$$a_{i\{1\}}(m_1,\epsilon_1;\, m_2,\epsilon_2) := \frac{(-1)}{\Gamma(r)}\big\{m_1m_2\,\Phi\big(1\,\big|\,{}^{r,\ r}_{\mu_1,\,\mu_2}\big)$$
$$-\,(-1)^{\pi_i}\big[m_1^2\,\Phi\big(1\,\big|\,{}^{r+1,\ r}_{\mu_1,\ \mu_2}\big) + m_2^2\,\Phi\big(1\,\big|\,{}^{r,\ r+1}_{\mu_1,\ \mu_2}\big)\big]\big\}, \qquad i=1,2. \tag{47}$$

Since in equation (45) the $a_{i\{0\}}(m,k)\neq 0$ only for $\delta_n = 1$, i.e. for $n = 2r+1$, we put in equation (46) the factors $(-1)^{\pi_i} = 1$, $i=1,2$, both for vectors ($\gamma = I_g$) and axial-vectors ($\gamma=\gamma^*$) cases.

The functions $a_{i\{\kappa\}}(m_1,\epsilon_1;\, m_2,\epsilon_2)$ have the symmetry properties

$$a_{i\{\kappa\}}(m_2,\epsilon_2;\, m_1,\epsilon_1) = (-1)^{\kappa+1}a_{i\{\kappa\}}(m_1,\epsilon_1;\, m_2,\epsilon_2), \qquad \kappa=0,1, \tag{48}$$

and in the limit $\epsilon_l \to 0$ tend to homogeneous functions $a_{i\{\kappa\}}(x)$ of the zero degree in masses which are named as *mass functions* of the QCs to the CWIs. Using the relation

$$\mu_1\Phi\big(\lambda\,\big|\,{}^{a_1+1,\,a_2}_{\mu_1,\ \mu_2}\big) + \mu_2\Phi\big(\lambda\,\big|\,{}^{a_1,\,a_2+1}_{\mu_1,\ \mu_2}\big) = \Phi\big(\lambda-1\,\big|\,{}^{a_1,\,a_2}_{\mu_1,\,\mu_2}\big),$$

(which is a consequence of equation (44) and of the identity $M_\epsilon/M_\epsilon^\lambda = M_\epsilon^{\lambda-1}$), and other properties of the function ${}_2F_1$, from equations (46)–(47) follow the explicit form of the mass functions $a_{i\{\kappa\}}(x)$:

$$a_{i\{0\}}(x) := \lim_{\epsilon_1,\epsilon_2\to 0} a_{i\{0\}}(m_1,\epsilon_1;\, m_2,\epsilon_2) = A_{2r+1}\overline{a}_{i\{0\}}(x), \quad x := m_1/m_2,$$
$$\overline{a}_{i\{0\}}(x) := (-1)^{i-1}(1-x)/C_{2r+1}\left[\,(4r+1)\alpha\left(1/2, r+1;\, x^2\right) - x\alpha\left(3/2, r+1;\, x^2\right)\right], \tag{49}$$
$$a_{i\{1\}}(x) := \lim_{\epsilon_1,\,\epsilon_2\to 0} a_{i\{1\}}(m_1,\epsilon_1;\, m_2,\epsilon_2) = A_{2r}\overline{a}_{i\{1\}}(x), \quad x := m_1/m_2,$$
$$\overline{a}_{i\{1\}}(x) := \left[\,(-1)^{\pi_i} - a_{2r}(x)\right], \qquad a_{2r}(x) := x\alpha\left(1, r;\, x^2\right),$$
$$\alpha(\lambda, b;\, x^2) := {}_2F_1\left(\lambda, b;\, 2b;\, 1-x^2\right). \tag{50}$$

In equations (49)–(50), C_{2r+1} is the normalization constant (that gives $\overline{a}_{i\{0\}}(0) = (-1)^{i-1}$, $\forall\, r$), and A_{2r+1}, A_{2r} denote the magnitudes of the mass functions of QCs:

$$C_{2r+1} := (4r+1)\alpha(1/2, r+1;\, 0) = 2\Gamma(2r+2)\Gamma(r+1/2)/(\Gamma(2r+1/2)\Gamma(r+1)),$$
$$A_{2r+1} := \Gamma(1/2)\Gamma(r+1/2)/\Gamma(2r+1/2), \qquad A_{2r} = \Gamma(r)/\Gamma(2r). \tag{51}$$

There exists the relation $A_{2r+1} = \Gamma(3/2)A_{2r+2}\,C_{2r+1}$ between them. The magnitudes A_{2r} and A_{2r+1} are monotonically decreasing functions of the variable r, varying from $A_2 = 1$ and $A_1 = (2/3)\sqrt{\pi}$ to $\lim_{r\to\infty} A_{2r+\delta_n} = 0$. Therefore, when $r \to \infty$ the QCs go to zero very rapidly.

Similarly, the $\overline{a}_{i\{\kappa\}}(x)$ denote normalized mass functions which determine a shape of the $a_{i\{\kappa\}}(x)$. As far as $(-1)^{\pi_i} = \pm 1$, from equations (49)–(50) follow three primitive mass functions

$$a_{2r+1}(x) := \overline{a}_{1\{0\}}(x); \qquad a_{2r}^{(-)}(x) := 1 - a_{2r}(x), \qquad a_{2r}^{(+)}(x) := 1 + a_{2r}(x), \tag{52}$$

in term of which all mass functions $a_{i\{\kappa\}}(x)$ of the QCs are finally expressed. The properties of the functions $a_{2r+1}(x)$, $a_{2r}^{(\mp)}(x)$, $a_{2r}(x)$, $\forall r \geq 1$, which may be physical important, are related to the reciprocity relations, the values at $x = 0$, $x = \infty$, and at $x = 1$ (the latter corresponds to the degenerate nonchiral case ($m_1 = m_2 = m \neq 0$)), the range of values for real $0 \leq x \leq \infty$, zeros, extrema, and intervals of monotonicity, are as follows:

$$a_{2r+1}(x) = -a_{2r+1}(1/x); \qquad a_{2r}^{(\mp)}(x) = a_{2r}^{(\mp)}(1/x), \qquad a_{2r}(x) = a_{2r}(1/x); \tag{53}$$

$$a_{2r+1}(0) = -a_{2r+1}(\infty) = 1; \qquad a_{2r}^{(\mp)}(0) = a_{2r}^{(\mp)}(\infty) = 1, \qquad a_{2r}(0) = a_{2r}(\infty) = 0; \tag{54}$$

$$a_{2r+1}(1) = 0; \qquad a_{2r}^{(-)}(1) = 0, \qquad a_{2r}^{(+)}(1) = 2, \qquad a_{2r}(1) = 1, \tag{55}$$

$$1 \geq a_{2r+1}(x) \geq -1; \qquad 0 \leq a_{2r}^{(-)}(1) \leq 1, \qquad 1 \leq a_{2r}^{(+)}(x) \leq 2, \qquad 0 \leq a_{2r}(x) \leq 1.$$

The values at $x = 1$ are: the unique zero for $a_{2r+1}(x)$, the unique zero which is the unique minimum for $a_{2r}^{(-)}(x)$, the unique maxima for $a_{2r}^{(+)}(x)$ and $a_{2r}(x)$. The $a_{2r+1}(x)$ are monotonically decreasing on $0 \leq x \leq \infty$; the $a_{2r}^{(-)}(x)$ are monotonically decreasing on $0 \leq x \leq 1$ and are monotonically increasing on $1 \leq x \leq \infty$; the $a_{2r}^{(+)}(x)$ and $a_{2r}(x)$ are monotonically increasing on $0 \leq x \leq 1$ and are monotonically decreasing on $1 \leq x \leq \infty$.

Taking into consideration equations (32), (36)–(43), (45)–(52), one obtains for the regular analogs of the CWIs (35)–(38) the following expressions ($i = 1, 2$, $j \in \{1, 2\}$, $j \neq i$):

$$\begin{aligned}
&(R^{\nu+1}D_i)_{\{0\}}(m,k) = (R^{\nu+1}I_{j-\bar{i}})_{\{0\}}(m,k) = (R^{\nu}I_{j-\bar{i}})_{\{0\}}(m,k) + a_{i\{0\}}(m,k),\\
&a_{i\{0\}}(m,k) = \tilde{b}(g,k)\,\delta_n (k_2^2)^r A_{2r+1}(-1)^{i-1} a_{2r+1}(x), \quad \tilde{b}(g,k) := (2\pi)^n \delta(k)\, b(g),\\
&(R^{\nu+1}D_i)_{\{1\}}(m,k) = (R^{\nu+1}I_{j-\bar{i}})_{\{1\}}(m,k) = (R^{\nu}I_{j-\bar{i}})_{\{1\}}(m,k) + a_{i\{1\}}(m,k),\\
&a_{i\{1\}}(m,k) = \tilde{b}(g,k)\,(1-\delta_n)(k_2^2)^{r-1} A_{2r}[(-1)^{\pi_i} - a_{2r}(x)], \qquad x := m_1/m_2,
\end{aligned} \tag{56}$$

which are valid both for general and degenerate nonchiral cases.

4.3. Now we pass to the chiral behavior. Let us consider some possible ways tending to the chiral state in renormalized amplitudes at hand: i) the symmetric chiral limit ($m_1 = m_2 = m \to 0$), accomplishing as the $(\epsilon_{1,2}, m)-$limit, when first ϵ_1, $\epsilon_2 \to 0$, and then $m_l = m \to 0$, $\forall\, l$; ii) the nonsymmetric chiral limit ($m_1 \to 0$, $m_2 \to 0$), accomplishing as the $(\epsilon_{1,2}, m_1, m_2)$ – limit, when first ϵ_1, $\epsilon_2 \to 0$, then $m_1 \to 0$, and lastly $m_2 \to 0$; iii) the nonsymmetric chiral limit ($m_2 \to 0$, $m_1 \to 0$), accomplishing as the $(\epsilon_{1,2}, m_2, m_1)$ – limit, when first ϵ_1, $\epsilon_2 \to 0$, then $m_2 \to 0$, and lastly $m_1 \to 0$; iv) the chiral case ($m_1 = m_2 = 0$), accomplishing as the $(m_{1,2}, \epsilon)$ – limit, when first $m_1 = m_2 = 0$, and then $\epsilon_l = \epsilon \to 0$, $\forall\, l$.

Equations (53)–(55) imply that the symmetric chiral limit $m_1 = m_2 = m \to 0$ differs essentially from the nonsymmetric chiral limits $m_1 \to 0$, $m_2 \to 0$ or $m_2 \to 0$, $m_1 \to 0$ for primitive mass functions (52). In addition, for the $a_{2r+1}(x)$ the last two limits are also different.

From equations (36)–(39), (13) and properties of the function ${}_2F_1$ it follows that for all chiral limits, i.e. for i), ii), and iii) cases, $\lim_{m_s\to 0}(R^{\nu}I_{j-\bar{i}})_{\{\kappa\}}(m,k) = 0$, $\kappa = 0, 1$, and equations (56) take the form:

$$\lim_{m_s\to 0}(R^{\nu+1}D_i)_{\{\kappa\}}(m,k) = \tilde{b}(g,k)\begin{cases}\delta_n (k_2^2)^r A_{2r+1}(-1)^{i-1}\{\,0,\,1,\,-1\}, & \kappa = 0;\\ (1-\delta_n)(k_2^2)^{r-1}A_{2r}\{[(-1)^{\pi_i} - 1],\,(-1)^{\pi_i}\}, & \kappa = 1,\end{cases} \tag{57}$$

where the $\{0, 1, -1\}$ in the first row of equation (57) corresponds to the {i), ii), iii)}-cases, and the $\{[(-1)^{\pi_i}-1], (-1)^{\pi_i}\}$ in the second row corresponds to the {i), ii) or iii)}-cases, respectively.

In the chiral case ($m_1 = m_2 = 0$), due to equations (36)–(38), one has:

$$\begin{aligned}
&(R^{\nu+1}I_{j-\bar{i}})_{\{\kappa\}}(0,k) = 0, \qquad \kappa = 0,1, \qquad \epsilon_1,\, \epsilon_2 \neq 0,\\
&(R^{\nu+1}D_i)_{\{0\}}(0,k) = 0, \qquad (R^{\nu+1}P_{\bar{i}-j})_{\{0\}}(0,k) = 0, \qquad \epsilon_1,\, \epsilon_2 \neq 0,
\end{aligned} \tag{58}$$

and equations (35),(33) and (51) give rise to the following nontrivial identities

$$\begin{aligned}
&(R^{\nu+1}D_i)_{\{1\}}(0,k) = (R^{\nu+1}P_{\bar{i}-j})_{\{1\}}(0,k) \neq 0, \quad \epsilon_1\, \epsilon_2 \neq 0,\\
&\lim_{\epsilon_1=\epsilon_2=\epsilon\to 0}(R^{\nu+1}D_i)_{\{1\}}(0,k) = \lim_{\epsilon_1=\epsilon_2=\epsilon\to 0}(R^{\nu+1}P_{\bar{i}-j})_{\{1\}}(0,k)\\
&\qquad = \tilde{b}(g,k)\,(1-\delta_n)(k_2^2)^{r-1}A_{2r}(-1)^{\pi_i},
\end{aligned} \tag{59}$$

which are caused by the nonzero Schwinger terms contributions of current density commutators.

From the previous, it follows that for general spinor diangles the STCs may be nonzero only in the chiral case, for even space-time dimension $n = 2r$, for non light-like momenta $k_2^2 \neq 0$, and if $n = 2$ for the light-like momenta $k_2^2 = 0$ also. The dimension $n = 2$ is the unique one for which STCs are nonzero for light-like momenta $k_2^2 = 0$. Clearly, this fact is connected with the well known dynamical mass generation for the two-dimensional vector boson [30, 31].

From equations (57)–(59) also imply that the chiral case and the chiral limit cases in general do not coincide. For example, the expression in equation (59) coincides with that of corresponding to the nonsymmetric chiral limits in equation (57) for ii) and iii) cases and differs from that of corresponding to the symmetric chiral limit in equation (57) for i) case. Similar conclusion follows also from equation (58) and from the first row of equation (57).

5 Conclusions

From the above we have come to the important conclusions:

• There is the technique (SCR) in framework of which the WIs involving regular values of quantities do retain (or imitate) the CWIs. Quantum anomalies reveal themselves either as an oversubtraction effect for a non-chiral case and for the symmetric and nonsymmetric chiral limits (in these cases the STCs are zero) or as nonzero STCs for the chiral case.

• Quantum anomalies are more general phenomena than the well known mass-independent axial-vector and conformal anomalies. The related conclusion has been obtained as early as 1970 by Kummer and Schweda [6, 7, 8]. Our investigations show that canonically non-conserved vector and axial-vector currents can have mass-dependent anomalies. Furthermore, in the chiral case vector and axial-vector currents have the same anomaly (up to the factor $\varepsilon(g) = (-1)^q(-1)^{n(n-1)/2}$). It is the STCs that are responsible for these anomalies. STCs may be nonzero only in the chiral case, for even $n = 2r$, for non light-like momenta $k_2^2 \neq 0$, and if $n = 2$ for the light-like momenta $k_2^2 = 0$ also. The dimension $n = 2$ is the unique one for which STCs and quantum anomalies are nonzero for light-like momenta $k_2^2 = 0$. This fact is connected with the well known dynamical mass generation for the two-dimensional vector boson [30, 31].

• For the complex Clifford algebra $Cl(g)_{\mathbb{C}}$ the matrix dual conjugation γ^* may be always redefined as $\gamma^* := i^{(1-\varepsilon(g))/2}\gamma^1\gamma^2\cdots\gamma^n$, $(\gamma^*)^2 = I_g$. Then from equations (2) and equations (39)–(40) it follows that in the chiral case the QCs to the vector and axial-vector CWIs are the same exactly. Therefore, in this case “left-handed vector” current can be conserved and hence it can be more fundamental than vector or axial-vector currents. This may give some insight into why just the left-handed neutrino exists in Nature.

• No universal modified operator expressions for divergencies of axial-vector and vector currents exist, even in the framework of some fixed model. Modes of quantum anomalies strongly depend on the type of quantum field quantities under consideration. Moreover, the behavior of FAs and quantum anomalies in the chiral case ($m = 0$) and in the symmetric chiral limit ($m \to 0$) is different. The same is also true for the Schwinger terms of current commutators.

• A mass spectrum of fermions, appearing in the quantum anomalies, increases the predictive power of formulas widely used in the low energy phenomenological physics, e.g., for describing particle decays [1–5].

• A nontrivial mass dependence of the QCs to the CWIs prevents the standard mechanism of anomaly cancellation and requires a revision of some orthodox ideas of the counter-term renormalization.

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On a CFT Prediction in the Sine-Gordon Model

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A quantitative prediction of Conformal Field Theory (CFT), which relates the second moment of the energy-density correlator away from criticality to the value of the central charge, is verified in the sine-Gordon model. By exploiting the boson-fermion duality of two-dimensional field theories, this result also allows to show the validity of the prediction in the strong coupling regime of the Thirring model.

Some time ago Cardy [1] derived a quantitative prediction of conformal invariance [2, 3] for 2D systems in the scaling regime, away from the critical point. Starting from the so called 'c-theorem' [4], he was able to relate the value of the conformal anomaly c, which characterizes the model at the critical point, to the second moment of the energy-density correlator in the non-critical theory:

$$\int d^2x\,|x|^2\,\langle\varepsilon(x)\varepsilon(0)\rangle = \frac{c}{3\,\pi\,t^2\,(2-\Delta_\varepsilon)^2}, \tag{1}$$

where ε is the energy-density operator, Δ_ε is its scaling dimension and $t \propto (T-T_c)$ is the coupling constant of the interaction term that takes the system away from criticality. It is interesting to notice that a similar sum rule has been recently obtained by Jancovici [5] in the context of Classical Statistical Mechanics. This author considered the correlations of the number-density of particles in a 2D two-component plasma (Coulomb gas).

The validity of (1) has been explicitly verified for the Ising model [1], for 2D self-avoiding rings [6], and for the Baxter model [7]. In this last case the formula could be checked only in the weak-coupling limit, by describing the system in terms of a massive Thirring model [8] and performing a first-order perturbative computation.

The main purpose of this note is to show that (1) also holds for a bosonic QFT with highly non-trivial interactions, the well-known sine-Gordon (SG) model with Euclidean Lagrangian density given by

$$L = \frac{1}{2}(\partial_\mu\Phi)^2 - \frac{\alpha}{\lambda}\cos\left(\sqrt{\lambda}\Phi\right) + \frac{\alpha}{\lambda}, \tag{2}$$

where α and λ are real constants.

In this work we shall perform a perturbative computation up to second order in λ. We are then naturally led to consider the renormalization of this theory. Fortunately this issue has been already analyzed by many authors [9–14]. One of the main conclusions is that for $\lambda < 8\pi$ a normal order procedure that eliminates the contributions of the tadpoles is enough to have a finite theory. The only effect of this prescription is to renormalize the constant α. We shall then restrict our study to this case.

Since we want to verify equation (1) for the model given by (2), we will take $\varepsilon = (1-\cos\sqrt{\lambda}\Phi)/\lambda$ and $t=\alpha$. Taking into account that the model of free massless scalars has a conformal charge $c=1$ and that the scaling dimension of ε for this case is equal to $\lambda/4\pi$, (1) reads

$$F(\alpha,\lambda) = \int d^2x\,|x|^2\,\langle\varepsilon(x)\varepsilon(0)\rangle = \frac{1}{3\,\pi\,\alpha^2\left(2-\frac{\lambda}{4\pi}\right)^2}. \tag{3}$$

Expanding the interaction term up to order λ^2 one has $\varepsilon = \frac{1}{2}\Phi^2 - \frac{\lambda}{4!}\Phi^4 + \frac{\lambda^2}{6!}\Phi^6$. Replacing this expression in (3) we obtain

$$F(\alpha,\lambda) = A(\alpha,\lambda) + B(\alpha,\lambda) + C(\alpha,\lambda) + D(\alpha,\lambda),$$

where

$$A(\alpha,\lambda) = \frac{1}{4}\int d^2x\,|x|^2\,\langle\Phi^2(x)\Phi^2(0)\rangle_\alpha, \qquad B(\alpha,\lambda) = -\frac{\lambda}{4!}\int d^2x\,|x|^2\,\langle\Phi^2(x)\Phi^4(0)\rangle_\alpha,$$
$$C(\alpha,\lambda) = \frac{\lambda^2}{(4!)^2}\int d^2x\,|x|^2\,\langle\Phi^4(0)\Phi^4(x)\rangle_\alpha, \qquad D(\alpha,\lambda) = \frac{\lambda^2}{6!}\int d^2x\,|x|^2\,\langle\Phi^2(x)\Phi^6(0)\rangle_\alpha.$$

At this point we notice that $D(\alpha,\lambda)$, up to this order, contains only tadpoles which, as explained above, were already considered in the renormalization of α. Then we must disregard this contribution in the present context. Now, in order to illustrate the main features of the computation, we shall briefly describe the evaluation of $A(\alpha,\lambda)$, which involves both analytical and numerical procedures. In the above equations $\langle\ \rangle_\alpha$ means v.e.v. with respect to the SG Lagrangian expanded up to second order in λ. From now on we will decompose this Lagrangian into free and interaction pieces as

$$L_0 = \frac{1}{2}(\partial_\mu\Phi)^2 + \frac{\alpha}{2}\,\Phi^2, \qquad L_{\text{int}} = -\frac{\alpha\lambda}{4!}\,\Phi^4 + \frac{\alpha\lambda^2}{6!}\,\Phi^6.$$

Using Wick's theorem and the well-known expression for the free bosonic propagator $\langle\Phi(x)\Phi(0)\rangle_0 = 1/(2\pi)K_0(\sqrt{\alpha}|x|)$ (K_0 is a modified Bessel function of zeroth order), after a convenient rescaling of the form $x \to \frac{x}{\sqrt{\alpha}}$, we obtain

$$\begin{aligned}A(\alpha,\lambda) = {} & \frac{1}{2\alpha^2(2\pi)^2}\int d^2x\,|x|^2\,K_0^2(|x|) + \frac{\lambda}{4\alpha^2(2\pi)^4}\iint d^2x\,d^2x_1\,|x|^2\,K_0^2(|x_1-x|)K_0^2(|x_1|)\\ & + \frac{\lambda^2}{\alpha^2(2\pi)^6}\iiint d^2x\,d^2x_1\,d^2x_2\,|x|^2\left[\frac{1}{8}K_0^2(|x_2-x_1|)K_0^2(|x_1|)K_0^2(|x_2-x|)\right.\\ & + \frac{1}{6}K_0^3(|x_2-x_1|)K_0(|x_2|)K_0(|x_1-x|)K_0(|x|)\\ & \left. + \frac{1}{4}K_0^2(|x_2-x_1|)K_0(|x_1|)K_0(|x_2-x|)K_0(|x_2|)K_0(|x_1-x|)\right].\end{aligned}$$

The first two terms are related to tabulated integrals [15] yielding the first order analytical expression $F(\alpha,\lambda) = (1+\lambda/(4\pi))/\left(12\pi\alpha^2\right)$ which can be shown to verify (3) in a straightforward way. Concerning the remaining second order terms we have three contributions corresponding to the prefactors 1/8, 1/6 and 1/4 in the last integrand. Due to its symmetry, the first one can be analytically computed by repeatedly using

$$\int d^2x\,|x|^2\,K_0^2(|x-y|) = \frac{2\pi}{3} + \pi|y|^2.$$

The result is $2\pi^3$. The computation of the other two contributions is more involved. In fact we could not find analytical results in these cases. However we were able to considerably simplify these multiple integrals in order to facilitate their numerical evaluation. Indeed, using the Fourier transform of K_0 and the integral representation of the first kind Bessel function J_0:

$$J_0(|x|) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\,\exp\left(-i|x|\cos\theta\right),$$

we obtain

$$\begin{aligned}&\iiint d^2x\, d^2x_1\, d^2x_2\, |x|^2\, K_0^3(|x_2-x_1|)K_0(|x_1|)K_0(|x_2-x|)K_0(|x|)\\&\quad = 4(2\pi)^3 \int dr \int dp\, \frac{r\,p\,\left(1-p^2\right) K_0^3(r)J_0(pr)}{\left(p^2+1\right)^5}.\end{aligned}$$

Using NIntegrate in the program Mathematica for the double integral in this expression one obtains the value 0.04874. Similar manipulations with the last contribution to $A(\alpha,\lambda)$ led us to the following numerical results:

$$\int dr \int dp \int dk \frac{r\,p\,k\,\left(1-p^2\right) K_0^3(r)J_0(pr)J_0(kr)}{\left(p^2+1\right)^4\left(k^2+1\right)^2} = 0.0169622,$$

and

$$\int dr\, r^5\, K_0^2(r)K_1^2(r) = 0.08783.$$

Putting all this together we have

$$\begin{aligned}A(\alpha,\lambda) = {}& \frac{1}{12\pi\alpha^2}\left(1+\frac{\lambda}{4\pi}\right) + \frac{\lambda^2}{\alpha^2(2\pi)^6}\left[\frac{1}{8}\, 2\pi^3 + \frac{1}{6}\, 4\,(2\pi)^3\, 0.04874\right.\\&\left. +\frac{1}{4}\,(2\pi)^3\left(8\times 0.0169622 - \frac{1}{8}\, 0.08783\right)\right].\end{aligned}$$

Working along the same lines with $B(\alpha,\lambda)$ and $C(\alpha,\lambda)$ we find

$$B(\alpha,\lambda) = \frac{-32\lambda^2}{\alpha^2(2\pi)^3 4!}\int dr \int dp\, \frac{r\,p\,\left(1-p^2\right) K_0^3(r)J_0(p\,r)}{\left(p^2+1\right)^4} = \frac{-32\lambda^2}{\alpha^2(2\pi)^3 4!}\times 0.0501125$$

and

$$C(\alpha,\lambda) = \frac{\lambda^2}{\alpha^2(2\pi)^3 4!}\int dr\, r^3\, K_0^4(r) = \frac{\lambda^2}{\alpha^2(2\pi)^3 4!}\times 0.0754499.$$

All these numerical values were confirmed by using Fortran.

Finally, inserting these results in the left hand side of (3) we get

$$F(\alpha,\lambda) = \frac{1}{12\pi\alpha^2} + \frac{\lambda}{48\pi^2\alpha^2} + \frac{\lambda^2}{4!(2\pi)^3\alpha^2}\left(\frac{3}{4} + 5.4\times 10^{-6}\right).$$

Comparing this expression with the expansion in λ of the right hand side of (3) one sees that they are equal up to first order and differ in a small quantity $\left(O\left(10^{-6}\right)\right)$ up to second order in λ.

In summary, we have verified the validity of a quantitative prediction of CFT in the context of the SG model. Taking into account the well-known bosonization identity between the SG theory and the massive Thirring model (characterized by a coupling g^2) [9] which takes place for $\beta^2/(4\pi) = \left(1+g^2/\pi\right)^{-1}$, it becomes apparent that our result implies that (1) also holds in the strong coupling limit of the Thirring model. This, in turn, allows us to improve the proof of the validity of (1) for the Baxter and Ashkin–Teller models which, up to now, was restricted to the weak coupling limit [7].

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A Symmetric Treatment of Damped Harmonic Oscillator in Extended Phase Space

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Extended phase space (EPS) formulation of quantum statistical mechanics treats the ordinary phase space coordinates on the same footing and thereby permits the definition of the canonical momenta conjugate to these coordinates. The extended Lagrangian and extended Hamiltonian are defined in EPS by the same procedure as one does for ordinary Lagrangian and Hamiltonian. The combination of ordinary phase space and their conjugate momenta exhibits the evolution of particles and their mirror images in the same manner. As an example the resultant evolution equation in EPS for a damped harmonic oscillator DHO, is such that the energy dissipated by the actual oscillator is absorbed in the same rate by the image oscillator leaving the whole system as a conservative system. We use the EPS formalism to obtain the dual Hamiltonian of a damped harmonic oscillator, first proposed by Bateman, by a simple extended canonical transformations. The extended canonical transformations are capable of converting the damped system of actual and image oscillators to an undamped one, and transform the evolution equation into a simple form. The resultant equation is solved and the eigenvalues and eigenfunctions for damped oscillator and its mirror image are obtained. The results are in agreement with those obtained by Bateman. At last, the uncertainty relation are examined for above system.

1 Introduction

Although the formulation of dissipative systems from the first principles are cumbersome and little transparent, however, it is not so difficult to account for dissipative forces in classical mechanics in a phenomenological manner. Stokes' linear frictional force proportional to the velocity $\boldsymbol{v}$, Coulomb's friction $\sim \boldsymbol{v}/v$, Dirac's radiation damping $\sim \ddot{\boldsymbol{v}}$ and the viscous force $\sim \nabla^2\boldsymbol{v}$ are noteworthy examples in this respect. Unfortunately, the situation is much more complicated in quantum level (see Dekker [1], and the references there in). In his review article on classical and quantum mechanics of the damped harmonic oscillator, Dekker outlines that: "Although completeness is certainly not claimed, it is felt that the present text covers a substantial portion of the relevant work done during the last half century. All models agree on the classical dynamics … however, the actual quantum mechanics of the various models reveals a considerable variety in fluctuation behavior. … close inspection further shows that none of them … are completely satisfactory in all respects". As an example of the dissipative systems, the DHOs is investigated through different approaches by different people. Caldirola [2] and Kanai [3] using the familiar canonical quantization procedure, obtained the Schrödinger equation which gives the eigenvalue and eignfunctions for damped oscillator. However the difficulty with this approach is that it violates the Heisenberg uncertainty relation in the long time limit. Another approach is the Schrödinger–Langevin method, which introduces a nonlinear wave equation for the evolution of the damped oscillator [4]. In this method the superposition principle is obviously violated. Using the Wigner equation, Dodonov and Manko [5] introduced the loss energy state for DHO as consequence of the Bateman dissipation, by introducing a dual Hamiltonian considered the evolution of the DHO in parallel with its mirror image [6]. In this method the energy dissipated by the actual oscillator of interest is absorbed at the same rate by the image oscillator. The image oscillator, in fact, plays the role of the physical reservoir. Therefore, the energy of the total system, as a closed one, is a constant of motion.

Here we use the EPS method [7] to investigate the evolution of the DHO. The method looks like the Bateman approach, however, the uncertainty principle, when looked upon from a different point of view, is not violated. That is, the extended uncertainty relation is satisfied for combination of actual and image oscillators, while reducing into ordinary uncertainty relations for actual and image oscillators, separately, in zero dissipation limit.

This paper is organized as follows. In Section 2, a review of the EPS formulation is given. In Section 3, we investigate the quantization procedure for the DHO. In Section 4, we use the path integral technique directly to calculate the exact propagators, and then the uncertainties of position and momentum for the actual and image oscillator system. Section 5 is devoted to concluding remarks.

2 A review of the EPS formulation

A direct approach to quantum statistical mechanics is proposed by Sobouti and Nasiri [7], by extending the conventional phase space and applying the canonical quantization procedure to extended quantities in this space. Assuming the phase space coordinates q and p to be independent variables on the virtual trajectories allows one to define momenta π_q and π_p, conjugate to q and p, respectively. This is done by introducing the extended Lagrangian

$$\mathcal{L}(q,p,\dot q,\dot p) = -\dot q p - \dot p q + \mathcal{L}^q(q,\dot q) + \mathcal{L}^p(p,\dot p), \tag{1}$$

where $\mathcal{L}^q$ and $\mathcal{L}^p$ are the q and p space Lagrangians of the given system. Using equation (1) one may define the momenta, conjugate to q and p, respectively, as follows

$$\pi_q = \frac{\partial \mathcal{L}}{\partial \dot q} = \frac{\partial \mathcal{L}^q}{\partial \dot q} - p, \tag{2}$$

$$\pi_p = \frac{\partial \mathcal{L}}{\partial \dot p} = \frac{\partial \mathcal{L}^p}{\partial \dot p} - q. \tag{3}$$

In the EPS defined by the set of variables $\{q,p,\pi_q,\pi_p\}$, one may define the extended Hamiltonian

$$\begin{aligned}\mathcal{H}(q,p,\pi_q,\pi_p) &= \dot q\pi_q + \dot p\pi_p - \mathcal{L} = H(p+\pi_q,q) - H(p,q+\pi_p)\\ &= \sum \frac{1}{n}\left\{\frac{\partial^n H}{\partial p^n}\pi_q^n - \frac{\partial^n H}{\partial q^n}\pi_p^n\right\},\end{aligned} \tag{4}$$

where $H(q,p)$ is the Hamiltonian of the system. Using the canonical quantization rule, the following postulates are outlined:

a) Let q, p, π_q and π_p be operators in Hilbert space X, of all square integrable complex functions, satisfying the following commutation relations

$$[\pi_q, q] = -i\hbar, \qquad \pi_q = -i\hbar\frac{\partial}{\partial q}, \tag{5}$$

$$[\pi_p, p] = -i\hbar, \qquad \pi_p = -i\hbar\frac{\partial}{\partial p}, \tag{6}$$

$$[q,p] = [\pi_q,\pi_p] = 0. \tag{7}$$

By virtue of equations (5)–(7), the extended Hamiltonian $\mathcal{H}$, will be an operator in X.

b) A state function $\chi(q,p,t) \in X$ is assumed to satisfy the following dynamical equation

$$i\hbar\frac{\partial\chi}{\partial t} = \mathcal{H}\chi = \left[H\left(p - i\hbar\frac{\partial}{\partial q}, q\right) - H\left(p, q - i\hbar\frac{\partial}{\partial p}\right)\right]\chi$$
$$= \sum \frac{1}{n}\left\{\frac{\partial^n H}{\partial p^n}\pi_q^n - \frac{\partial^n H}{\partial q^n}\pi_p^n\right\}\chi. \tag{8}$$

The general solution for this equation is

$$\chi(q,p,t) = \psi(q)\phi^*(p)e^{-\frac{i}{\hbar}qp}, \tag{9}$$

where $\psi(q)$ and $\phi(p)$ are the solutions of the Schrödinger equation in q and p space, respectively.

c) the averaging rule for an observable $O(q,p)$, a c-number operator in this formalism, is given as

$$\langle O(q,p)\rangle = \int O(q,p)\chi^*(q,p,t)dpdq. \tag{10}$$

For details of selection procedure of the admissible state functions, see Sobouti and Nasiri [7].

3 Damped harmonic oscillator in EPS

Extended Hamiltonian of equation (4) for undamped harmonic oscillator is given by

$$\mathcal{H} = \frac{1}{2}{\pi_q}^2 + p\pi_q - \frac{1}{2}{\pi_p}^2 - q\pi_p. \tag{11}$$

By a canonical transformation of the form

$$q_1 = q, \qquad \pi_{q_1} = -\pi_q - p, \qquad p_1 = p, \qquad \pi_{p_1} = -\pi_p - q,$$

equation (10) yields

$$\mathcal{H} = \frac{1}{2}\pi_{q_1}^2 + q_1^2 - \frac{1}{2}\pi_{p_1}^2 - \frac{1}{2}p_1^2. \tag{12}$$

This extended Hamiltonian evidently represents the subtraction of Hamiltonians of two independent identical oscillators, which is called actual and image oscillators [5]. The position q and momentum π_q denote the actual oscillator, while p and π_p denote the image oscillator. The minus sign has its origin in equation (4) and has an important role in this theory [7]. The following canonical transformation

$$q_2 = q_1, \qquad \pi_{q_2} = \pi_{q_1} - \lambda q_1, \qquad p_2 = p_1, \qquad \pi_{p_2} = \pi_{p_1} + \lambda p_1. \tag{13}$$

changes the extended Hamiltonian of an undamped harmonic oscillator into that of the damped one, i.e.

$$\mathcal{H}_2 = \frac{1}{2}\left\{\pi_{q_2}^2 + 2\lambda q_2\pi_{q_2} + \omega^2 q_2^2\right\} - \frac{1}{2}\left\{\pi_{p_2}^2 - 2\lambda p_2\pi_{p_2} + \omega^2 p_2^2\right\}, \tag{14}$$

where $\omega = 1 + i\lambda$. One further transformation generated by

$$F_2(q_2, p_2, \pi_{q_3}, \pi_{p_3}) = q_2\pi_{q_3}e^{-\lambda,t} + p_2\pi_{p_3}e^{\lambda t}, \tag{15}$$

finally leads to

$$\mathcal{H}_3 = \frac{1}{2}\left\{\pi_{q3}^2 e^{-2\lambda t} + \omega^2 q_3^2 e^{2\lambda t}\right\} - \frac{1}{2}\left\{\pi_{p3}e^{2\lambda t} + \omega^2 p_3^2 e^{-2\lambda t}\right\}. \tag{16}$$

The first part of the extended Hamiltonian in equation (16) is Caldirola–Kanai Hamiltonian, which is widely used to study the dissipation in quantum mechanics [3]. Using equation (16), the extended Hamilton equations [7] gives the following classical evolution equations for actual and image oscillators, respectively

$$\ddot{q_3} + 2\lambda\dot{q_3} + \omega^2 q_3 = 0, \tag{17}$$

and

$$\ddot{p_3} - 2\lambda\dot{p_3} + \omega^2 p_3 = 0. \tag{18}$$

Almost trivially, the energy dissipated by actual oscillator, with phase space coordinates (q_3, π_{q_3}) is completely absorbed at the same pace by the image oscillator with phase space coordinates (p_3, π_{p_3}).

To quantize the above system as usual, the dynamical variables (q_3, π_{q_3}) and (p_3, π_{p_3}) are considered as operators in a linear space. They obey the commutation relations in equations (5)–(7). The dynamical equation (8), now becomes

$$i\hbar\frac{\partial\chi}{\partial t} = \mathcal{H}\chi = \left(\frac{1}{2}\left\{\pi_{q_3}^2 e^{-2\lambda t} + \omega^2 q_3^2 e^{2\lambda t}\right\} - \frac{1}{2}\left\{\pi_{p_3}^2 e^{2\lambda t} + \omega^2 p_3^2 e^{-2\lambda t}\right\}\right)\chi. \tag{19}$$

By an infinitesimal canonical transformation which in quantum level corresponds to the following unitary transformation

$$U = \exp\left(\frac{i\lambda}{2\hbar}\left\{e^{2\lambda t}{q_4}^2 + e^{-2\lambda t}p_4^2\right\} + \frac{i\lambda t}{\hbar}\left\{q_4\pi_{q_4} - p_4\pi_{p_4}\right\}\right), \tag{20}$$

equation (19) may be written as

$$i\hbar\frac{\partial\chi}{\partial t} = \mathcal{H}\chi = \left(\frac{1}{2}\left\{-\hbar^2\frac{\partial^2}{\partial q_4^2} + \omega'^2{q_4}^2\right\} - \frac{1}{2}\left\{-\hbar^2\frac{\partial^2}{\partial p_4^2} + \omega'^2{p_4}^2\right\}\right)\chi. \tag{21}$$

where $\omega' = \omega + i\lambda$. The eigenvalues of equation (21) may be obtained as follows [7],

$$\mathcal{E}_{mn} = E_n - E_m = (n-m)\hbar\omega'. \tag{22}$$

The corresponding eigenfunctions are,

$$\begin{aligned}\chi_{mn}(q_4, p_4, t) &= U\chi_{mn}(q_3, p_3, t)\\ &= \exp\left(\frac{i\lambda}{2\hbar}\left\{e^{2\lambda t}{q_4}^2 + e^{-2\lambda t}{p_4}^2\right\}\right)\psi_m\left(e^{\lambda t}q_4\right)\phi_n^\star\left(e^{-\lambda t}p_4\right)e^{-\frac{i}{\hbar}p_4 q_4},\end{aligned} \tag{23}$$

where $\psi_m(q)$ and $\phi_n(p)$ eigenfunctions of the harmonic oscillator in configuration and momentum space (Hermit functions). The result obtained above are in agreement with those obtained by Bateman [6]. However, here in contrast to the Bateman approach, the Heisenberg uncertainty relation is looked upon from a different point of view and is not violated. This is discussed in the next section using the eigenfunctions in equation (23).

4 Uncertainty relations for actual and image oscillators

In this section we calculate the uncertainties in position and momentum for the actual and the image oscillators. We calculate the extended propagator [8] for the combined actual and the image oscillators as follows

$$
\begin{aligned}
K(q,p,t,q_i,p_i,t_i) &= \left(\frac{1}{2\pi i\hbar}\right)\left[\frac{\omega'}{\sin\omega'(t-t_i)}\right] \\
&\times \exp\left[\frac{1}{2}\left(\frac{\omega' e^{\lambda(t+t_i)}}{\sin\omega'(t-t_i)}\right)\left\{e^{\lambda(t-t_i)}q^2\left(\cos\omega'(t-t_i)-\frac{\lambda}{\omega'}\sin(\omega'(t-t_i))\right)\right.\right. \\
&\left.\left.+ e^{-\lambda(t-t_i)}q_i^2\left(\cos\omega'(t-t_i)+\frac{\lambda}{\omega'}\sin(\omega'(t-t_i))\right)-2qq_i\right\}\right] \\
&\times \exp\left[\frac{1}{2}\left(\frac{\omega' e^{-\lambda(t+t_i)}}{\sin\omega'(t-t_i)}\right)\left\{e^{-\lambda(t-t_i)}p^2\left(\cos\omega'(t-t_i)+\frac{\lambda}{\omega'}\sin(\omega'(t-t_i))\right)\right.\right. \\
&\left.\left.+ e^{\lambda(t-t_i)}p_i^2\left(\cos\omega'(t-t_i)-\frac{\lambda}{\omega'}\sin(\omega'(t-t_i))\right)-2pp_i\right\}\right]. \qquad (24)
\end{aligned}
$$

When $\lambda \to 0$, then equation (24) reduces to the familiar form of the undamped extended harmonic oscillator propagator [8]. We assume that the initial state function for combined system in ground state is $\chi_{00}(q,p,0) = (\pi\delta^2)^{-\frac{1}{2}}\exp\left(-\frac{q^2+p^2}{2\delta^2}\right)$, where δ is the width of the extended wave packet. Then one gets using equation (9)

$$
\begin{aligned}
\chi_{00}(q,p,t) &= \int\int dq_i dp_i K(q,p,t,q_i,p_i,0)\chi_{00}(q_i,p_i,0) \\
&= \left(\frac{\pi}{\delta^2}\right)\left[\frac{1}{2\delta^2}-\frac{i}{2}\frac{\omega'}{\hbar}\left(\frac{\cos\omega' t}{\sin\omega' t}+\frac{\lambda}{\omega'}\right)\right]^{-\frac{1}{2}} \\
&\times \left(\frac{\omega' e^{\lambda t}}{2\pi i\hbar\sin\omega' t}\right)^{\frac{1}{2}}\exp\left[-\frac{q^2}{2}\left\{\frac{1}{\delta^2}e^{2\lambda t}\left(1+\left[\frac{1}{\delta^4}\left(\frac{\hbar}{\omega'}\right)^2+2\left(\frac{\lambda}{\omega'}\right)^2-1\right]\sin^2\omega'\right.\right.\right. \\
&\left.+\frac{\lambda}{\omega'}\sin 2\omega' t\right)^{-1} - i\left\{\frac{\omega'}{\hbar}\frac{e^{2\lambda t}}{\sin\omega' t}\left[\cos\omega' t-\frac{\lambda}{\omega'}\sin\omega' t-\left(\cos\omega' t+\frac{\lambda}{\omega'}\sin\omega' t\right)\right]\right. \\
&\left.\left.\left.\times\left[1+\frac{1}{\delta^4}\left(\frac{\hbar}{\omega'}\right)^2+2\left(\frac{\lambda}{\omega'}\right)^2-1\right]\sin 2\omega' t+\frac{\lambda}{\omega'}\sin 2\omega' t\right\}\right\}^{-1}\right] \\
&\times \left[\frac{1}{2\delta^2}+\frac{i}{2}\frac{\omega'}{\hbar}\left(\frac{\cos\omega' t}{\sin\omega' t}+\frac{-\lambda}{\omega'}\right)\right]^{-\frac{1}{2}}\frac{-\omega' e^{-\lambda t}}{2\pi i\hbar\sin\omega' t}^{\frac{1}{2}}\exp\left[-\frac{p^2}{2}\left\{\frac{1}{\delta^2}e^{-2\lambda t}\right.\right. \\
&\times\left(1+\left[\frac{1}{\delta^4}\left(\frac{\hbar}{\omega'}\right)^2+2\left(\frac{\lambda}{\omega'}\right)^2-1\right]\sin^2\omega'-\frac{\lambda}{\omega'}\sin 2\omega' t\right)^{-1} \\
&+i\left\{\frac{\omega'}{\hbar}\frac{e^{-2\lambda t}}{\sin\omega' t}\left[\cos\omega' t+\frac{\lambda}{\omega'}\sin\omega' t-\left(\cos\omega' t-\frac{\lambda}{\omega'}\sin\omega' t\right)\right]\right. \\
&\left.\left.\left.\times\left[1+\frac{1}{\delta^4}\left(\frac{\hbar}{\omega'}\right)^2+2\left(\frac{\lambda}{\omega'}\right)^2-1\right]\sin 2\omega' t-\frac{\lambda}{\omega'}\sin 2\omega' t\right\}\right\}^{-1}\right]e^{-\frac{ipq}{\hbar}}. \qquad (25)
\end{aligned}
$$

Using equations (10) and (25) the uncertainties of positions and momenta we can calculate for the actual and the image oscillators as follows

$$
\langle\Delta q\rangle = \frac{\delta}{\sqrt{2}}e^{-\lambda t}\left\{1+\left[\left(\frac{\sqrt{\hbar}}{\delta}\right)^4\left(\frac{1}{\omega'}\right)^2+\left(\frac{\lambda}{\omega'}\right)^2-1\right]\sin^2\omega' t+\frac{\lambda}{\omega'}\sin 2\omega' t\right\}^{\frac{1}{2}}, \qquad (26)
$$

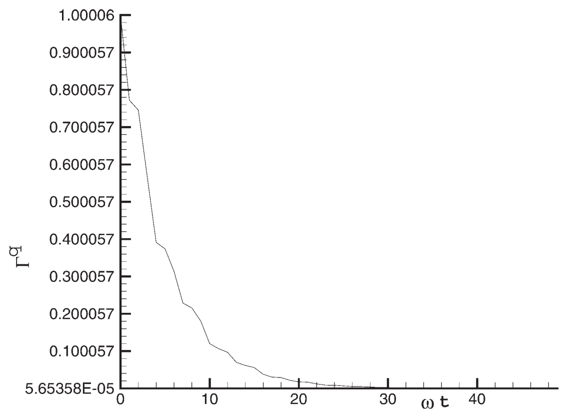

Figure 1. Uncertainty relation for actual oscillator as a function of time, for $\lambda = 0.1\omega$.

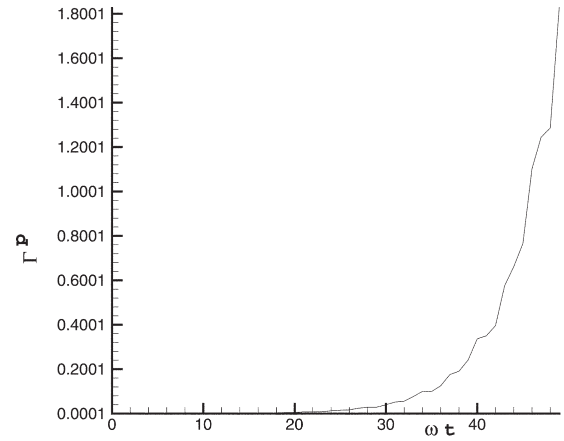

Figure 2. Uncertainty relation for image oscillator as a function of time, for $\lambda = 0.1\omega$.

$$\langle \Delta \pi_q \rangle = \frac{\delta}{\sqrt{2}} e^{-\lambda t} \left\{ 1 + \left[\left(\frac{\sqrt{\hbar}}{\delta} \right)^4 \left(\frac{1}{\omega'} \right)^2 + \left(\frac{\lambda}{\omega'} \right)^2 - 1 \right] \sin^2 \omega' t - \frac{\lambda}{\omega'} \sin 2\omega' t \right\}^{\frac{1}{2}}, \tag{27}$$

$$\langle \Delta p \rangle = \frac{\delta}{\sqrt{2}} e^{\lambda t} \left\{ 1 + \left[\left(\frac{\sqrt{\hbar}}{\delta} \right)^4 \left(\frac{1}{\omega'} \right)^2 + \left(\frac{\lambda}{\omega'} \right)^2 - 1 \right] \sin^2 \omega' t - \frac{\lambda}{\omega'} \sin 2\omega' t \right\}^{\frac{1}{2}}, \tag{28}$$

and

$$\langle \Delta \pi_p \rangle = \frac{\delta}{\sqrt{2}} e^{\lambda t} \left\{ 1 + \left[\left(\frac{\sqrt{\hbar}}{\delta} \right)^4 \left(\frac{1}{\omega'} \right)^2 + \left(\frac{\lambda}{\omega'} \right)^2 - 1 \right] \sin^2 \omega' t + \frac{\lambda}{\omega'} \sin 2\omega' t \right\}^{\frac{1}{2}}. \tag{29}$$

The above results for actual and image oscillators, in separate form, are in agreement with those obtained by Bateman. It is clear that the Heisenberg uncertainty relation is not valid for each oscillator independently. In fact for $\lambda \neq 0$ it is not possible to separate the oscillators, and the Heisenberg uncertainty relations would not hold for them separately, as shown in Figs. 1 and 2. In the presence of dissipation, i.e. $\lambda \neq 0$, the actual and image oscillators are coupled with each other and the area which is preserved during the evolution is $\Gamma(t) = \Delta\pi_q \Delta\pi_p \Delta q \Delta p$ in EPS. In contrast to the case of undamped harmonic oscillator, neither $\Gamma^q(t) = \Delta\pi_q \Delta q$ nor $\Gamma^p(t) = \Delta\pi_p \Delta p$ are preserved for DHO in q and p representation of quantum mechanics. This is shown in Fig. 3, where $\Gamma(t)$ is plotted versus time. It is clear that $\Gamma(t)$ never goes the zero. In other words, $\Gamma^q(t)$ and $\Gamma^p(t)$ which goes to zero and infinity in the long time limit, respectively, behave in such a manner that their product $\Gamma(t)$, always keeps a positive and finite value.

5 Concluding remarks

The EPS formulation of quantum mechanics seems to be a suitable method to handle the dissipative systems. Introducing the notion of mirror image oscillator beside the actual oscillator is a possibility that the extension of the ordinary phase space allows one to consider. This possibility introduces a conservative system of combined actual and image oscillators evolving together in the course of time. The eigenvalues and eigenfunctions obtained in this way is in agreement with those obtained by Bateman by introducing a dual Hamiltonian. However, the uncertainty principle, as one of the major problems on the way of the different approaches to the dissipative systems, including the Bateman approach, is valid in the extended form. This means

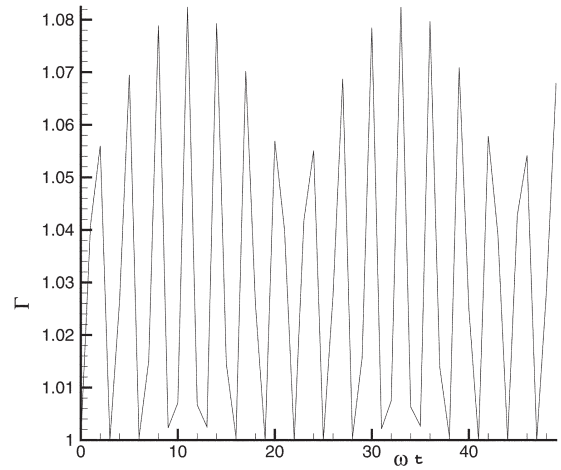

Figure 3. Uncertainty relation for combined system (actual and image oscillator) as a function of time, for $\lambda = 0.1\omega$.

that the dissipative systems can not be considered as isolated systems and it really interacts with its surrounding medium. The effect of the medium must be included as well. The mirror image oscillator plays the role of the interacting medium for the total conservative system, and the uncertainty relation is still valid.

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Canonical Realization of Poincaré Algebra: from Field Theory to Direct-Interaction Theory

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The canonical realization of the Poincaré group for the systems of the pointlike particles coupled with the electromagnetic, massive vector and scalar fields is constructed. The reduction of the canonical field degrees of freedom is done in the linear approximation in the coupling constant. The Poincaré generators in terms of particle variables are found. The relation between covariant and physical particle variables in the Hamiltonian description is written. The approximation up to c^{-2} is examined.

1 Introduction

So many field-theoretical models in the classical relativistic mechanics are based on the Lagrangian formalism due to its conceptual simplicity [1, 2]. However, the transition from Lagrangian description, when the fields are eliminated by means of substitution of the formal solutions of the field equations, to Hamiltonian one is not simple and demands the use of various approximations. For this reason, it is natural to construct the Hamiltonian description of the "particle plus field" systems, and *then* to exclude field degrees of freedom. Such a program is discussed in the series of papers by Lusanna with collaborators (see [3]).

Here at the beginning we apply simpler approach of the use of the geometrical forms of dynamics [2] fixing chronometrical invariance of the action integral. We construct the Hamiltonian description of charged particles with electromagnetic field, and perform the canonical transformation which isolates nonphysical (gauge) degrees of freedom of the electromagnetic field. We also consider the massive scalar and vector interactions and obtain generators of time evolution and Lorentz transformations on the physical phase space. In Section 3 the procedure of the exclusion of the field degrees of freedom is described within the linear approximation in the coupling constant. We obtain the canonical generators of the Poincaré group (the direct-interaction theory) for considered interactions. We demonstrate that the approximation up to c^{-2} agrees with the well known results of various approaches.

2 Hamiltonian formulation of the "field+particle" systems

Let particles be described by their world lines in the Minkowski space-time[1] $\gamma_a : \tau \mapsto x_a^\mu(\tau)$. The electromagnetic interaction between charges is mediated by the field $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$ with the electromagnetic potential $A_\mu(x)$; $\partial_\nu \equiv \partial/\partial x^\nu$. An action for the system of N charges

[1]The Minkowski space-time is endowed with a metric $\|\eta_{\mu\nu}\| = \mathrm{diag}(1, -1, -1, -1)$. The Greek indices $\mu, \nu, \ldots$ run from 0 to 3; the Roman indices from the middle of alphabet, $i, j, k, \ldots$ run from 1 to 3 and both types of indices are subject of the summation convention. The Roman indices from the beginning of alphabet, a, b, label the particles and run from 1 to N. The sum over such indices is indicated explicitly.

is

$$S = -\sum_{a=1}^{N} \int d\tau_a \left\{ m_a \sqrt{u_a^2(\tau_a)} + e_a u_a^\nu(\tau_a) A_\nu[x_a(\tau_a)] \right\} - \frac{1}{4} \int F_{\lambda\sigma}(x) F^{\lambda\sigma}(x) d^4x, \tag{1}$$

where m_a and e_a are the mass and the charge of particle a, respectively, and $u_a^\mu(\tau_a) = dx_a^\mu(\tau_a)/d\tau_a$. The action is manifestly invariant under reparametrization of the particle world lines and ordinary gauge transformation of the electromagnetic potential:

$$\tau_a \mapsto \phi(\tau_a), \qquad \phi' > 0, \tag{2}$$

$$A_\mu \mapsto A_\mu + \partial_\mu \Lambda. \tag{3}$$

Moreover, action (1) is invariant under (global) transformations of the Poincaré group; this invariance results in the conservation of the symmetric energy-momentum tensor [4]:

$$\theta^{\mu\nu}(x) = \sum_{a=1}^{N} \int m_a \frac{u_a^\mu(\tau_a) u_a^\nu(\tau_a)}{\sqrt{u_a^2(\tau_a)}} \delta^4(x - x_a(\tau_a)) d\tau_a - F^{\mu\lambda} F^\nu{}_\lambda + \frac{\eta^{\mu\nu}}{4} F_{\lambda\sigma} F^{\lambda\sigma}, \tag{4}$$

$$\theta^{\mu\nu}(x) = \theta^{\nu\mu}(x), \qquad \partial_\nu \theta^{\mu\nu}(x) = 0. \tag{5}$$

We fix the freedom in the parametrization of particle world lines by means of gauge condition:

$$x^0 = f(t, \boldsymbol{x}), \qquad \boldsymbol{x} = \left(x^1, x^2, x^3\right), \tag{6}$$

which defines the form of relativistic dynamics. Then, the Minkowski space-time is foliated by the family of space-like or isotropic hypersurfaces Σ_t parametrized by t. The functions $x^i = x_a^i(t)$, $i = 1, 2, 3$, completely determine the parametric equations of the particle world lines in a given form of dynamics:

$$x^0 = f(t, \boldsymbol{x}_a(t)), \qquad x^i = x_a^i(t). \tag{7}$$

The variable t serves as a common evolution parameter of the system.

Accounting (6), we come to a single-time form of the action [5]

$$S = \int dt L \tag{8}$$

with Lagrangian $L(t)$ depending on the functions $\boldsymbol{x}_a(t)$, $A^\mu(t, \boldsymbol{x})$ and their first order derivatives with respect to evolution parameter, $\dot{\boldsymbol{x}}_a(t) = d\boldsymbol{x}_a(t)/dt$ and $\dot{A}^\mu(t, \boldsymbol{x})$.

The conservation of the energy-momentum tensor (4) gives us ten conserved quantities in a given form of dynamics:

$$P^\mu = \int_{\Sigma_t} \theta^{\mu\nu} d\sigma_\nu, \qquad M^{\mu\nu} = \int_{\Sigma_t} \left(x^\mu \theta^{\nu\rho} - x^\nu \theta^{\mu\rho}\right) d\sigma_\rho. \tag{9}$$

However, the Lagrangian L still remains invariant under gauge transformation (3) and leads to the constrained Hamiltonian description. It is demonstrated in [5] that the form of dynamics determines the structure of the corresponding constraints. In the following we confine ourselves by the most common case of the instant form of dynamics ($x^0 = t$). The Lagrangian function in this form of dynamics is represented by

$$L = -\sum_{a=1}^{N} \left\{ m_a \sqrt{1 - \dot{\boldsymbol{x}}_a^2} + e_a \left[A_0(t, \boldsymbol{x}_a) + \dot{x}_a^i A_i(t, \boldsymbol{x}_a)\right] \right\} - \frac{1}{4} \int \left(2E_i E^i + F_{ij} F^{ij}\right) d^3x, \tag{10}$$

where $F_{ij} = \partial_i A_j - \partial_j A_i$ and $E_i = \partial_i A_0 - \dot{A}_i$.

In the Hamiltonian formulation of our system we start with canonical variables $x_a^i(t)$, $A_\mu(t,\boldsymbol{x})$ and conjugated momenta $p_{ai}(t)$, $E^\mu(t,\boldsymbol{x})$ which are subject of the first class constraints [6]

$$E^0 \approx 0, \qquad \Gamma \equiv \varrho - \partial_i E^i \approx 0, \tag{11}$$

where $\approx$ means "weak equality" in the sense of Dirac and $\varrho(t,\boldsymbol{x}) = \sum_{a=1}^{N} e_a \delta^3(\boldsymbol{x} - \boldsymbol{x}_a(t))$ is a charge density.

Now we break the field phase space by means of canonical transformation so that the physical part is described by the gauge invariant variables $a_\alpha = \left(\delta_\alpha^i - \delta_3^i \partial_\alpha/\partial_3\right) A_i$, E^α; $\alpha = 1,2$, and unphysical part is parametrized by the canonical pairs (Q,Γ) and (A_0, E^0).

The time evolution of the physical degrees of freedom is generated by the Hamiltonian

$$H = \sum_{a=1}^{N} \sqrt{m_a^2 + [\boldsymbol{p}_a - e_a \boldsymbol{A}_\perp(\boldsymbol{x}_a)]^2} - \frac{1}{2}\int \left(A_i^\perp \Delta A_i^\perp - E_\perp^i E_\perp^i + \varrho \Delta^{-1} \varrho\right) d^3x, \tag{12}$$

where

$$E_\perp^i = \left(\delta_\alpha^i - \delta_3^i \partial_\alpha/\partial_3\right) E^\alpha, \qquad A_i^\perp = \left(\delta_i^\alpha + \partial_i \Delta^{-1} \partial^\alpha\right) a_\alpha. \tag{13}$$

Inverse differential operators are defined so that

$$1/\partial_3 \delta^3(\boldsymbol{x}) = (1/2)\delta\left(x^1\right)\delta\left(x^2\right)\operatorname{sgn}\left(x^3\right), \qquad \Delta^{-1}\delta^3(\boldsymbol{x}) = -1/(4\pi|\boldsymbol{x}|). \tag{14}$$

Reexpression of the conserved quantities (9) in the terms of canonical variables leads to the canonical realization of the Poincaré group. On the physical subspace the generator P^0 coincides with the Hamiltonian (12), and the generator of the Lorentz transformation is given by

$$\begin{aligned} M^{k0} = &\sum_{a=1}^{N} \left\{ x_a^k \sqrt{m_a^2 + [\boldsymbol{p}_a - e_a \boldsymbol{A}_\perp(\boldsymbol{x}_a)]^2} - t p_a^k \right\} - \frac{1}{2}\int x^k \varrho \Delta^{-1} \varrho d^3x \\ &+ \int x^k \left(\frac{1}{4} F_{ij}^\perp F_{ij}^\perp + \frac{1}{2} E_\perp^i E_\perp^i + E_\perp^l \partial_l \Delta^{-1} \varrho \right) d^3x - t \int E_\perp^l \partial^k A_l^\perp d^3x. \end{aligned} \tag{15}$$

where $F_{ij}^\perp = \partial_i A_j^\perp - \partial_j A_i^\perp$.

Let us consider in a similar manner the Hamiltonian description of the system of particles with massive vector and scalar interactions. In the first case a system is described by action that differs from (1) by the massive term $\frac{1}{2}\mu^2 A^\nu A_\nu$. The instant form Hamiltonian description of the system is based on the canonical variables $x_a^i(t)$, $A_\mu(t,\boldsymbol{x})$ and $p_{ai}(t)$, $E^\mu(t,\boldsymbol{x})$. Moreover, there is a pair of the second class constraints:

$$E^0 \approx 0, \qquad \Gamma - \mu^2 A_0 \approx 0, \tag{16}$$

which can be excluded by means of the Dirac bracket. The canonical Hamiltonian is

$$\begin{aligned} H = &\sum_{a=1}^{N} \sqrt{m_a^2 + [\boldsymbol{p}_a - e_a \boldsymbol{A}(t,\boldsymbol{x}_a)]^2} \\ &+ \int \left[\frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} E^i E^i - \frac{1}{2}\mu^2 A_i A^i + A_0 \left(\Gamma - \frac{1}{2}\mu^2 A_0 \right) \right] d^3x. \end{aligned} \tag{17}$$

After exclusion of the constraints (16) one obtains for the boost generator

$$
\begin{aligned}
M^{k0} = &\sum_{a=1}^{N} \left\{ x_a^k \sqrt{m_a^2 + [\boldsymbol{p}_a - e_a \boldsymbol{A}(t, \boldsymbol{x}_a)]^2} - t p_a^k \right\} \\
&+ \int x^k \left[\frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} E^i E^i - \frac{1}{2} \mu^2 A_i A^i + \frac{1}{2\mu^2} \Gamma^2 \right] d^3 x \\
&- t \int \left[E^j \partial^k A_j - \frac{1}{2} \mu^2 A^k \Gamma \right] d^3 x. \qquad (18)
\end{aligned}
$$

In the case of a system of particles interacting by means of the scalar field $\varphi(x)$ we construct the standard Hamiltonian formalism without constraints with the Hamiltonian

$$
H = \sum_{a=1}^{N} \sqrt{\boldsymbol{p}_a^2 + [m_a - e_a \varphi(t, \boldsymbol{x}_a)]^2} + \frac{1}{2} \int \left[\pi^2 + (\boldsymbol{\nabla} \varphi)^2 + \mu^2 \varphi^2 \right] d^3 x, \qquad (19)
$$

and the boost generator

$$
\begin{aligned}
M^{k0} = &\sum_{a=1}^{N} \left\{ x_a^k \sqrt{\boldsymbol{p}_a^2 + [m_a - e_a \varphi(t, \boldsymbol{x}_a)]^2} - t p_a^k \right\} \\
&+ \frac{1}{2} \int x^k \left[\pi^2 + (\boldsymbol{\nabla} \varphi)^2 + \mu^2 \varphi^2 \right] d^3 x - t \int \pi \partial^k \varphi d^3 x. \qquad (20)
\end{aligned}
$$

In the next section we will see that elimination of the field degrees of freedom into the three considered cases gives us the canonical generators of a similar structure.

3 Elimination of the field degrees of freedom

In the systems, where the free radiation is not essential, the physical field degrees of freedom can be excluded. As a result, we obtain the description of our systems in the terms of particle variables only.

Let us perform the field reduction by three steps [7]. First, we must find a solution of the field equations of motion. Here, using coupling constant expansion, we solve the linearized equations. However, we touch the problem of choice of Green's function. Fortunately, in the first-order (linear) approximation in the coupling constant the advanced, retarded, or symmetric solutions coincide. We use here the time-symmetric Green's function $G(x^2) = G\left(x_0^2 - \boldsymbol{x}^2\right)$. It is well known [1] that the Green's function determines the nonrelativistic potential $u(r)$:

$$
u(r) = \int d\alpha G\left(\alpha^2 - r^2\right). \qquad (21)
$$

The general solution of the field equations is a sum of the source free field A_s^{rad} (s is the number of the physical field components), which satisfies the homogeneous equation, and the solution of the inhomogeneous equation $\mathcal{A}_s$ in the terms of canonical particle variables.

Second, we perform a canonical transformation [7]:

$$
A_s = A_s^{\mathrm{rad}} + \mathcal{A}_s, \qquad E^s = E_{\mathrm{rad}}^s + \mathcal{E}^s, \qquad (22)
$$

$$
x_a^i = q_a^i + \int \left[\left(A_s^{\mathrm{rad}} + \frac{1}{2} \mathcal{A}_s \right) \frac{\partial \mathcal{E}^s}{\partial k_{ai}} - \left(E_{\mathrm{rad}}^s + \frac{1}{2} \mathcal{E}^s \right) \frac{\partial \mathcal{A}_s}{\partial k_{ai}} \right] d^3 x, \qquad (23)
$$

$$
p_{ai} = k_{ai} - \int \left[\left(A_s^{\mathrm{rad}} + \frac{1}{2} \mathcal{A}_s \right) \frac{\partial \mathcal{E}^s}{\partial q_a^i} - \left(E_{\mathrm{rad}}^s + \frac{1}{2} \mathcal{E}^s \right) \frac{\partial \mathcal{A}_k^{\perp}}{\partial q_a^i} \right] d^3 x, \qquad (24)
$$

here the free field terms $(A_s^{\mathrm{rad}}, E_{\mathrm{rad}}^s)$ are treated as the new canonical variables.

Third step consists in elimination of the field variables by means of constraints

$$A_s^{\rm rad} \approx 0, \qquad E^s_{\rm rad} \approx 0. \tag{25}$$

The Dirac bracket for the systems with additional canonical constraints (25) coincides with the particle Poisson bracket $\{q^i_a, k_{bj}\} = -\delta_{ab}\delta^i_j$.

It is true, in order to simplify the form of the Poincaré generators for the system with vector interaction, we need to canonically transform the particle variables. Finally, the canonical generators of the Poincaré group for the considered interactions in the linear approximation are

$$H = c\sum_{a=1}^{N} k_a^0 + \frac{c}{2}\sum_{a,b=1}^{N}{}' e_a e_b \frac{f(\omega_{ab})}{k_a^0} u(\rho_{ab}), \qquad k_a^0 = \sqrt{m_a^2c^2 + \boldsymbol{k}_a^2}, \tag{26}$$

$$P^k = \sum_{a=1}^{N} k_a^k, \qquad M^{ij} = \sum_{a=1}^{N}\left(q_a^i k_a^j - q_a^j k_a^i\right), \tag{27}$$

$$M^{k0} = \sum_{a=1}^{N}\left(\frac{q_a^k}{c}k_a^0 - tk_a^k\right) + \frac{1}{2c}\sum_{a,b=1}^{N}{}' e_a e_b q_b^k \frac{f(\omega_{ab})}{k_a^0} u(\rho_{ab}), \tag{28}$$

where the prime over sum denotes that $a \neq b$ ($a = b$ terms is excluded by means of mass renormalization); $\rho^2_{ab} = q^2_{ab} + \left(\boldsymbol{k}_a \boldsymbol{q}_{ab}/k_a^0\right)^2$, $\boldsymbol{q}_{ab} = \boldsymbol{q}_a - \boldsymbol{q}_b$, $q_{ab} = |\boldsymbol{q}_{ab}|$, $\omega_{ab} = k_a^\mu k_{b\mu}/m_a m_b c^2$, and $f(\omega) = 1$ for the scalar interaction and $f(\omega) = \omega$ for the vector interaction. It can easy be demonstrated that the expressions (26)–(28) satisfy the commutation relations of the Poincaré group in a given approximation with arbitrary functions $u(r)$ and $f(\omega)$.

According to (23), the covariant particle positions x_a^i are connected with the canonical variables as

$$x_a^i = q_a^i + \frac{1}{2}\int\left[\mathcal{A}_s\frac{\partial\mathcal{E}^s}{\partial k_{ai}} - \mathcal{E}^s\frac{\partial\mathcal{A}_s}{\partial k_{ai}}\right]d^3x. \tag{29}$$

It can be verified directly that in a given approximation the expression (29) satisfies the world line condition

$$\left\{x_a^i, M^{k0}\right\} = x_a^k\left\{x_a^i, H\right\} - t\delta^{ik}. \tag{30}$$

The Poisson brackets between particle positions do not vanish,

$$\left\{x_a^i, x_b^j\right\} = \int\left(\frac{\partial\mathcal{A}_s}{\partial k_{bj}}\frac{\partial\mathcal{E}^s}{\partial k_{ai}} - \frac{\partial\mathcal{E}^s}{\partial k_{bj}}\frac{\partial\mathcal{A}_s}{\partial k_{ai}}\right)d^3x, \tag{31}$$

in a full agreement with the famous no-interaction theorem [8].

Similarly, the direct-interaction theory can be obtained in the different forms of relativistic dynamics. They are physically equivalent. So, the Poincaré generators in the front form ($x^0 = t+x^3$), which corresponds to foliation of the Minkowski space-time by the isotropic hypersurfaces, are connected with the instant form generators ("in") by means of the following canonical transformation:

$$q_a^i \to q_a^i - q_a^3\frac{k_a^i + \delta_3^i h_a}{h_a}, \qquad k_a^i \to k_a^i + \delta_a^i h_a, \tag{32}$$

$$G_{\rm in} - G_{\rm fr} = \{F, G_{\rm in}\}, \tag{33}$$

$$h_a = \frac{\boldsymbol{k}_a^2 + m_a^2}{2k_{a3}}, \qquad F = \int\left(\exp\left(-x^3\partial_t\right) - 1\right)\mathcal{F}d^3x, \tag{34}$$

where $\partial_t\mathcal{F}$ is equal to the spatial density of the instant form interaction term.

Now let us examine the generators (26), (28) up to c^{-2} approximation. We immediately find that

$$u(\rho_{ab}) = u(q_{ab}) + \frac{(\boldsymbol{q}_{ab}\boldsymbol{k}_a)^2}{2q_{ab}m_a^2c^2}\frac{du(q_{ab})}{dq_{ab}}, \qquad f(\omega_{ab}) = 1 + \frac{f'(0)}{2c^2}\left(\frac{\boldsymbol{k}_a}{m_a} - \frac{\boldsymbol{k}_b}{m_b}\right)^2. \tag{35}$$

Performing the canonical transformation generated by the function

$$\Lambda = \frac{1}{4c^2}\sum_{a<b}^{N} e_a e_b u(q_{ab})\left[\boldsymbol{q}_{ab}\left(\frac{\boldsymbol{k}_a}{m_a} - \frac{\boldsymbol{k}_b}{m_b}\right)\right], \tag{36}$$

finally, we obtain the expressions

$$H = H^{(0)} + H^{(1)}, \tag{37}$$

$$M^{k0} = \sum_{a=1}^{N}(q_a^k m_a - t k_a^k) + \frac{1}{2c^2}\sum_{a,b=1}^{N}{}' e_a e_b q_b^k u(q_{ab}). \tag{38}$$

where

$$H^{(0)} = \sum_{a=1}^{N}\left(m_a c^2 + \frac{\boldsymbol{k}_a^2}{2m_a}\right) + U^{(0)}, \qquad U^{(0)} = \sum_{a<b}^{N} e_a e_b u(q_{ab}), \tag{39}$$

$$H^{(1)} = -\sum_{a=1}^{N}\frac{\boldsymbol{k}_a^4}{8m_a^3c^2} - \sum_{a<b}^{N} e_a e_b\left\{\frac{1}{2c^2 m_a m_b}\left[\boldsymbol{k}_a\boldsymbol{k}_b u(q_{ab})\right.\right.$$
$$\left.\left. +(\boldsymbol{k}_a\boldsymbol{q}_{ab})(\boldsymbol{k}_b\boldsymbol{q}_{ab})\frac{du(q_{ab})}{q_{ab}dq_{ab}}\right] - \frac{A}{2c^2}\left(\frac{\boldsymbol{k}_a}{m_a} - \frac{\boldsymbol{k}_b}{m_b}\right)^2 u(q_{ab})\right\}, \tag{40}$$

and $A = f'(0) - 1$. Specifically, $A = -1$ for the scalar and $A = 0$ for the vector interactions. The latter in the massless case produces by the Darwin's Lagrangian for electromagnetic interaction. Expression (40) agrees with the post-Newtonian Hamiltonians obtained within various approaches [1].

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Towards Uniform T-Duality Rules

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In this contribution based on the talk given at the SUSY'01, Dubna, Russia, I discuss the reasons of appearing the different sets of fields entering into the T-duality transformations and a way to construct the uniform T-duality rules.

1 Introduction

Discovery in the past decade of the new class of non-compact symmetries allowed to revise one of the main problems of Superstrings: The puzzle of having too much "fundamental" theories. The resolution was in the conjecture on the M-theory in framework of which it turned out possible to unify all the five different superstring theories. The useful tool for establishing the M-theory evidence are dualities connected to the new symmetries of perturbative or non-perturbative sectors of Superstrings.

In this Contribution I would like to discuss some questions related to the so-called T-duality (see [1] for review). Namely, I will focus my attention on the following:

Q1: What are the different ways of T-duality rules derivation?

Q2: Do the T-duality rules coincide?

Q3: How uniform T-duality rules could be derived?

For the sake of simplicity I restrict myself in what follows to the pure classical frames, that means neglecting the dilaton field, which receives its corrections from quantum effects, and will consider backgrounds with single isometry direction. However, these restrictions are not so crucial, and more general case can be considered more or less straightforwardly in the same manner.

2 How one can derive the T-duality rules?

There are at least three different ways for getting the T-duality rules. The first one is based on the consideration of fundamental string dynamics in special kind of backgrounds allowing for existence of isometry direction [2]. Such consideration gives a possibility to derive the T-duality rules for Neveu–Schwarz (NS) sectors of the original and dual theories. The second way appeals to the unification of D=10 type IIA and D=10 type IIB supergravity (SUGRA) theories, which are the low energy effective actions for superstrings, after dimensional reduction down to nine dimensions. Requirement of having the same spectra in nine dimensions leads to the T-duality rules relating both the NS and Ramond (RR) sectors of the original (say, type IIA) and dual (type IIB) SUGRAs [3, 4]. Finally, one can derive the T-duality rules from the consideration of Dirichlet (D)-branes [5, 6, 7]. Again, the requirement of having the same physics after direct dimensional reduction of action for the Dp-brane, propagating in the background of type IIA theory, and double dimensional reduction of action for the D(p+1)-brane in the type

IIB background gives a chance to arrive at the T-duality transformations which relate the NS and the RR fields of the type IIA and type IIB theories to each other.

3 Do the T-duality rules coincide?

Let us consider the picture we have dealt with again. From the point of view of the fundamental string dynamics in the background with isometry no any *additional* redefinitions of the original fields which enter into the T-duality transformations are required. It should be pointed out since derivation of the T-duality rules is accompanied by dimensional reduction, target-space gauge fields can always be redefined to pick up the term which is proportional to the Kaluza–Klein vector field. This is due to the so-called transgression of the field strengths which is well-known in Kaluza–Klein literature. In such a situation *additional* means other possible redefinitions which do not relate to the transgression.

However, remind that in the case of fundamental string we are dealing with the NS sector only. If we extend our consideration to involve the RR sector into the game and to consider the T-duality rules from the point of view of the SUGRAs, we observe that the RR fields entering into the T-duality transformations are not always the same as the original ones [3, 4]. In other words some of the RR fields *additional* redefinition is required. The reason of this phenomenon is easy to understand. One of the features of the D=10 type IIB SUGRA is S-duality symmetry under the global $SL(2,\mathbb{R})$ transformations. This symmetry is absent in the case of the D=10 type IIA SUGRA. Since we can establish the connection between these theories only after dimensional reduction, we expect the trace of the higher dimensional $SL(2,\mathbb{R})$ transformations to be in lower dimensions. To make this symmetry manifest it is necessary to redefine the fields and to recollect them into the $SL(2,\mathbb{R})$ multiplets [3, 4]. On the other hand there always exists a freedom in the field (re)definition due to Bianchi identities (BIs) for the RR fields. But nevertheless, having the same BIs for two different sets of fields one describes the same physics [8]. Important point, which is worth to note, is that the additional redefinitions of the RR fields just fall into the class of admissible from the point of view of the BIs ones.

To complete the picture we need to know what happens in the D-brane case. The action for any D-brane (modulo instanton in type IIB) has the following structure (see, e.g., [9] for details)

$$S^{Dp} = -\int d\xi^{p+1}\sqrt{-\det(g_{mn}+\mathcal{F}_{mn})} + \int_{M^{p+1}} C\wedge e^{\mathcal{F}_2}, \qquad C = \sum_{n=0}^{d/2(-1)} C^{IIB(IIA)}_{2n(+1)}. \tag{1}$$

The first term is the kinetic term represented by generalization of Dirac action for relativistic membrane and the action for nonlinear electrodynamics proposed by Born and Infeld and called therefore by Dirac–Born–Infeld (DBI) term, and the second term is the Wess–Zumino (WZ) term. The DBI term is constructed out the NS target-space fields and worldvolume gauge field a_1 entering by means of $\mathcal{F}_2 = da_1 - B_2$, while both the NS 2-form B_2 and the RR target-space fields enter in the WZ term. In T-duality business two parts of the action play their roles independently giving the T-duality rules for the NS and for the RR sectors respectively [5, 7]. One can demonstrate that in such a case no original fields redefinition is required. The reason for this lies, in particular, in the natural restriction on the RR fields when they are fixed by the canonical form of the WZ term (cf. (1)), where C is treated as formal sum over the RR potentials in d-spacetime dimensions.

Therefore we are able to conclude the T-duality rules which we can read off from the Fundamental String, the SUGRA and the D-brane considerations coincide in the NS sector. However, even if the RR T-duality rules which follow from the SUGRA and the D-brane considerations coincide, the fields entering into the rules are not always the same.

4 Can we derive the uniform T-duality rules?

The key point here is that the analysis based mainly on the two-dimensional CFT and the vertex operators technique shows that *T-duality is the exact symmetry of string theory* [10, 11] (and Refs. therein). Therefore, in the language of low-energy classical effective dynamics of quantum string theory, it should be the symmetry of the following action

$$S = S_{\text{SUGRA (NS+RR)}} + S_{\text{NS SOURCES}} + S_{\text{RR SOURCES}}, \tag{2}$$

where the first term denotes schematically the action for the SUGRA with the NS and the RR fields (strictly speaking with their field strengths) and the last terms denote dynamical sources for the NS and the RR fields which are the actions for fundamental string and D-branes. This approach leads to the complete set of classical dynamical equations of motion describing dynamics of the SUGRA in presence of matter-type sources as well as dynamics of sources in the dynamical-type background, and gives therefore enclosed interaction picture.

In such a consideration it is naturally to expect that one can derive the uniform T-duality rules for both the NS and RR sectors since all the possible additional redefinitions coming from the SUGRA consideration will be under control and shall be in accordance with T-duality rules for the sources. One can observe some details in favour of this claim in [12]. There the model with action [13, 14]

$$S = S_{\text{SUGRA (NS)}} + S_{\text{Fundamental String SOURCE}} \tag{3}$$

was investigated from the point of view of invariance under the T-duality transformations.

The result on the T-duality invariance of such a system is predictable in view of the statement in the beginning of this Section. However, as a by-product, it became clear that as it can be expected from the discussion in the Section before the NS T-duality rules which come from equation (3) are in accordance with the rules derived from the fundamental string and the SUGRA considerations and no additional fields redefinition is required.

This observation simplifies explicit derivation of T-duality transformations, because, roughly speaking, having fundamental string as a source in effective action one can "forget" about the DBI part of D-brane sources and consider the contributions coming only from the WZ part of the D-branes action.

5 Discussion and conclusions

To summarize, I have discussed the reasons of appearing different sets of fields in the T-duality transformations and have sketched a way of the T-duality rules derivation in uniform basis of fields. Actually, the main problem is not even in derivation of T-duality rules in the uniform basis of fields, but rather in verification at the dynamical level the statement on T-duality as the exact symmetry of (perturbative and non-perturbative) string theory.

Vice versa, as we believe in the result achieved in the framework of quantum approach of the CFT and vertex operators, this gives confidence that it should be correct for effective theory describing the low-energy classical dynamics of quantum string theory. Hence, as a by-product, we can derive the T-duality transformations in the uniform basis of fields.

Beside the questions discussed above, consideration of T-duality in the SUGRA+SOURCES type interacting systems plays important role in String Cosmology with the pre-big-band scenario [15] where it is supposed that one deals with the string theory effective action in the background with isometries (see, e.g., the discussion in [16]). But this model should include actions for matter fields (strings and branes) and they have to be invariant under duality transformations.

Finally, in view of the recent paper [17] where an example of supersymmetric interacting system of the SUGRA+SOURCES type has been proposed, it looks very attractive to derive the supersymmetric T-duality rules and to compare the result with that of [18].

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First Order Equations of Motion from Breaking of Super Self-Duality

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First order differential equations, which satisfy second order equations of motion for $N=2$ Super Yang–Mills theory, are obtained with help of breaking of super self-duality.

First order equations of motion, by definition, are differential equations of first order, which satisfy second order equations of motion of the theory. For example, in the $N=1$ supersymmetric $SU(2)$ Yang–Mills theory

$$L = \mathrm{Tr}\left\{-\frac{1}{4}F_{mn}F^{mn} - i\bar{\lambda}\bar{\sigma}^m\mathcal{D}_m\lambda + \frac{1}{2}D^2\right\}, \tag{1}$$

where

$$F_{mn} = \partial_m V_n - \partial_n V_m + ig[V_m, V_n],$$
$$\mathcal{D}_m = \partial_m + ig[V_m, \cdot\,], \qquad \eta_{mn} = \mathrm{diag}(-1,1,1,1),$$

the super self-duality equations in component fields are first order equations of motion. The Yang–Mills strength in spinor indices has the following form

$$F_{\alpha\dot{\alpha},\beta\dot{\beta}} \equiv \sigma^m{}_{\alpha\dot{\alpha}}\sigma^n{}_{\beta\dot{\beta}}F_{mn} = \frac{1}{2}\varepsilon_{\alpha\beta}f_{\dot{\alpha}\dot{\beta}} + \frac{1}{2}\varepsilon_{\dot{\alpha}\dot{\beta}}f_{\alpha\beta},$$

where

$$f_{\dot{\alpha}\dot{\beta}} \equiv \varepsilon^{\alpha\gamma}F_{\gamma\dot{\alpha},\alpha\dot{\beta}}, \qquad f_{\alpha\beta} \equiv \varepsilon^{\dot{\alpha}\dot{\gamma}}F_{\alpha\dot{\gamma},\beta\dot{\alpha}}.$$

The super self-duality equations of the theory (1) look as follows [1]

$$f_{\alpha\beta} = 0, \qquad D = 0, \qquad \lambda_\alpha = 0, \qquad \mathcal{D}_{\alpha\dot{\beta}}\bar{\lambda}^{\dot{\beta}} = 0. \tag{2}$$

The system (2) is invariant under the following $N=1$ supersymmetric transformations

$$\delta_\xi V_{\alpha\dot{\alpha}} = -2i(\xi_\alpha\bar{\lambda}_{\dot{\alpha}} + \bar{\xi}_{\dot{\alpha}}\lambda_\alpha), \qquad \delta_\xi D = -\xi^\alpha\mathcal{D}_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} + \bar{\xi}_{\dot{\alpha}}\mathcal{D}^{\alpha\dot{\alpha}}\lambda_\alpha,$$
$$\delta_\xi\lambda_\alpha = \frac{1}{2}\xi^\beta f_{\alpha\beta} + i\xi_\alpha D, \qquad \delta_\xi\bar{\lambda}_{\dot{\alpha}} = \frac{1}{2}\bar{\xi}^{\dot{\beta}}f_{\dot{\alpha}\dot{\beta}} - i\bar{\xi}_{\dot{\alpha}}D, \tag{3}$$

where ξ_α, $\bar{\xi}_{\dot{\alpha}}$ are the parameters of $N=1$ supersymmetric transformations. Invariance of (2) under transformations (3) means that the system of transformed equations

$$\delta_\xi f_{\alpha\beta} = 0, \qquad \delta_\xi D = 0, \qquad \delta_\xi\lambda_\alpha = 0, \qquad \delta_\xi\left(\mathcal{D}_{\alpha\dot{\beta}}\bar{\lambda}^{\dot{\beta}}\right) = 0$$

is satisfied on the system (2).

It easy to verify that the system of super self-duality equations (2) can be derived from the only equation

$$\lambda_\alpha = 0 \tag{4}$$

by applying sypersymmetric transformations to this equation twice. In other words, the system (2) can be written in the following form

$$\lambda_\alpha = 0, \qquad \delta_\xi \lambda_\alpha = 0, \qquad \delta_\eta \delta_\xi \lambda_\alpha = 0.$$

Adding to (4) one more equation (which breaks super self-duality)

$$\lambda_\alpha = 0, \qquad \bar{\lambda}_{\dot{1}} = k\bar{\lambda}_{\dot{2}}, \tag{5}$$

where k is complex number, and by applying twice the transformations (3) to (5), we obtain another system of first order equations of motion

$$f_{\alpha\beta} = 0, \qquad f_{1\dot\beta} = kf_{2\dot\beta}, \qquad D = 0, \qquad \lambda_\alpha = 0, \qquad \bar{\lambda}_{\dot{1}} = k\bar{\lambda}_{\dot{2}}, \qquad \mathcal{D}_{\alpha\dot\beta}\bar{\lambda}^{\dot\beta} = 0. \tag{6}$$

Though this system is overdetermined, it is invariant under supersymmetric transformations.

This example prompts the procedure for obtaining of first order equations of motion in supersymmetric theories.

In this paper we present some systems of first order equations in the $N = 2$ supersymmetric Yang–Mills theory, which are obtained by breaking super self-duality. By definition, the system of super-self-duality equations has the following properties: i) it includes the self-duality equation for pure Yang–Mills theory $f_{\alpha\beta} = 0$; ii) it satisfies the equations of motion of the corresponding supersymmetric theory; iii) it is invariant under supersymmetric transformations.

The $SU(2)$ Yang–Mills theory with extended $N = 2$ supersymmetry, given by the Lagrangian [2]

$$\begin{aligned} L = \mathrm{Tr}\Big(&-\frac{1}{4}F_{mn}F^{mn} - i\bar{\lambda}_{\dot\alpha i}\bar{\sigma}^{m\dot\alpha\beta}\mathcal{D}_m\lambda^i_\beta - 2\mathcal{D}_m C\mathcal{D}^m C^* - \frac{1}{2}\vec{C}^2 \\ &+ igC\{\bar{\lambda}_{\dot\alpha i}, \bar{\lambda}^{\dot\alpha i}\} + igC^*\{\lambda^i_\alpha, \lambda^\alpha_i\} + 4g^2 C[C, C^*]C^*\Big), \end{aligned} \tag{7}$$

is invariant under $N = 2$ supersymmetric transformations [3]:

$$\begin{aligned} &\delta_\xi C = -\xi^\alpha_i \lambda^i_\alpha, \\ &\delta_\xi C^* = -\bar{\xi}_{\dot\alpha i}\bar{\lambda}^{\dot\alpha i}, \\ &\delta_\xi V_{\alpha\dot\alpha} = 2i\left(\xi^i_\alpha \bar{\lambda}_{\dot\alpha i} + \bar{\xi}_{\dot\alpha i}\lambda^i_\alpha\right), \\ &\delta_\xi \lambda^i_\alpha = -\frac{1}{2}\xi^{\beta i} f_{\alpha\beta} + 2ig\xi^i_\alpha[C, C^*] - \xi_{\alpha j}\vec{C}\vec{\tau}^{\,ij} + 2i\bar{\xi}^{\dot\alpha i}\mathcal{D}_{\alpha\dot\alpha}C, \\ &\delta_\xi \bar{\lambda}_{\dot\alpha i} = -\frac{1}{2}\bar{\xi}^{\dot\beta}_i f_{\dot\alpha\dot\beta} - 2ig\bar{\xi}_{\dot\alpha i}[C, C^*] + \bar{\xi}^j_{\dot\alpha}\vec{C}\vec{\tau}_{ij} + 2i\xi^\alpha_i \mathcal{D}_{\alpha\dot\alpha}C^*, \\ &\delta_\xi \vec{C} = -i\xi^{\alpha i}\left(\mathcal{D}_{\alpha\dot\beta}\bar{\lambda}^{\dot\beta j} + 2g[\lambda^j_\alpha, C^*]\right)\vec{\tau}_{ij} + i\bar{\xi}^i_{\dot\alpha}\left(\mathcal{D}^{\alpha\dot\beta}\lambda^j_\alpha - 2g[\bar{\lambda}^{\dot\beta j}, C]\right)\vec{\tau}_{ij}, \end{aligned} \tag{8}$$

where ξ^i_α, $\bar{\xi}_{\dot\alpha i}$ are the parameters of $N = 2$ supersymmetric transformations, and $\vec{\tau}^{\,j}_i$ are Pauli matrices.

The $N = 2$ super self-dual system

$$\begin{aligned} &f_{\alpha\beta} = 0, \qquad C = 0, \qquad \mathcal{D}_{\alpha\dot\beta}\mathcal{D}^{\alpha\dot\beta}C^* - ig\{\bar{\lambda}_{\dot\alpha i}, \bar{\lambda}^{\dot\alpha i}\} = 0, \\ &\vec{C} = 0, \qquad \lambda^i_\alpha = 0, \qquad \mathcal{D}_{\alpha\dot\beta}\bar{\lambda}^{\dot\beta i} = 0 \end{aligned} \tag{9}$$

includes one second order equation. The system (9) can be written as

$$\lambda^i_\alpha = 0, \qquad \delta_\xi \lambda^i_\alpha = 0, \qquad \delta_\eta \delta_\xi \lambda^i_\alpha = 0.$$

In order to obtain the systems of first order equations, we can break super self-duality in two ways: i) adding to the equation $\lambda^i_\alpha = 0$ other conditions for spinor fields; ii) imposing some conditions on supersymmetric parameters. In the case of proper choice of the above-mentioned conditions, after applying twice the supersymmetric transformations (8) to the equations for spinor fields we will obtain the system of first order equations.

The first example is (we have two systems, which correspond to $i = 1$ and $i = 2$)

$$\begin{aligned}
&\lambda^i_\alpha = 0, \qquad \delta_\xi \lambda^i_\alpha = 0, \qquad \delta_\eta\delta_\xi\lambda^i_\alpha = 0,\\
&\bar\lambda_{\dot 1 i} - k\bar\lambda_{\dot 2 i} = 0, \qquad \delta_\xi\left(\bar\lambda_{\dot 1 i} - k\bar\lambda_{\dot 2 i}\right) = 0, \qquad \delta_\eta\delta_\xi\left(\bar\lambda_{\dot 1 i} - k\bar\lambda_{\dot 2 i}\right) = 0.
\end{aligned} \tag{10}$$

After transformations, we obtain from (10)

$$\begin{aligned}
&f_{\alpha\beta} = 0, \qquad f_{\dot 1\dot\beta} = k f_{\dot 2\dot\beta}, \qquad C = 0, \qquad \mathcal{D}_{\alpha\dot 1} C^* = k\mathcal{D}_{\alpha\dot 2} C^*,\\
&\vec C = 0, \qquad \lambda^i_\alpha = 0, \qquad \mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta i} = 0, \qquad \bar\lambda_{\dot 1 i} = \bar\lambda_{\dot 2 i}.
\end{aligned} \tag{11}$$

The system (11) is invariant under $N = 2$ supersymmetric transformations and satisfies the equations of motion of the theory (7). It is not overdtermined.

The following system of first order equations of motion looks as follows

$$\begin{aligned}
&\bar\xi_{\dot\alpha i} = 0,\\
&\lambda_{\alpha i} = 0, \qquad \delta_\xi\lambda_{\alpha i} = 0, \qquad \delta_\eta\delta_\xi\lambda_{\alpha i} = 0,\\
&\bar\lambda_{\dot\alpha i} = 0, \qquad \delta_\xi\bar\lambda_{\dot\alpha i} = 0, \quad \delta_\eta\delta_\xi\bar\lambda_{\dot\alpha i} = 0,\\
&\lambda^i_1 - k\lambda^i_2 = 0, \qquad \delta_\xi(\lambda^i_1 - k\lambda^i_2) = 0, \qquad \delta_\eta\delta_\xi(\lambda^i_1 - k\lambda^i_2) = 0,
\end{aligned} \tag{12}$$

or, in the equivalent form,

$$\begin{aligned}
&f_{\alpha\beta} = 0, \qquad \mathcal{D}_{1\dot\alpha} C = k\mathcal{D}_{2\dot\alpha} C, \qquad C^* = \vec C = 0,\\
&\mathcal{D}^{\alpha\dot\beta}\lambda^i_\alpha - 2g[\bar\lambda^{\dot\beta i}, C] = 0, \qquad \mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta i} = 0,\\
&\lambda_{\alpha i} = 0, \qquad \bar\lambda_{\dot\alpha i} = 0, \qquad \lambda^i_1 = k\lambda^i_2, \qquad \bar\xi_{\dot\alpha i} = 0.
\end{aligned} \tag{13}$$

In this case we have put a constraint on supersymmetric parameters and included it into the system of first order equations of motion. This underlines that the system (13) is invariant under $N = 2$ supersymmetric transformations on the condition that $\bar\xi_{\dot\alpha 1} = 0$ or $\bar\xi_{\dot\alpha 2} = 0$ correspondingly.

Another system of first order equations of motion

$$\begin{aligned}
&\xi_{\alpha i} = 0, \qquad \bar\xi_{\dot\alpha i} = 0,\\
&\lambda_{\alpha i} = 0, \qquad \delta_\xi\lambda_{\alpha i} = 0, \qquad \delta_\eta\delta_\xi\lambda_{\alpha i} = 0,\\
&\bar\lambda_{\dot\alpha i} = 0, \qquad \delta_\xi\bar\lambda_{\dot\alpha i} = 0, \qquad \delta_\eta\delta_\xi\bar\lambda_{\dot\alpha i} = 0,
\end{aligned} \tag{14}$$

or, in the explicit form,

$$\begin{aligned}
&f_{\alpha\beta} = 0, \qquad C = 0, \qquad \mathcal{D}_{1\dot\alpha} C^* = k\mathcal{D}_{2\dot\alpha} C^*, \qquad \vec C = 0,\\
&\mathcal{D}^{\alpha\dot\beta}\lambda^i_\alpha = 0, \qquad \mathcal{D}_{\alpha\dot\beta}\bar\lambda^{\dot\beta i} + 2g[\lambda^i_\alpha, C^*] = 0,\\
&\lambda_{\alpha i} = 0, \qquad \bar\lambda_{\dot\alpha i} = 0, \qquad \xi_{\alpha i} = 0, \qquad \bar\xi_{\dot\alpha i} = 0.
\end{aligned} \tag{15}$$

In such way one can find some more systems of first order equations of motion.

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The Maxwell–Dirac Equations, Some Non-Perturbative Results

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In this talk I will review some recent work on the Maxwell–Dirac equations. This system of equations can be thought of as the classical equations for electronic matter, the quantisation of which yields that most successful of physical theories, QED. The talk will focus on qualitative, non-perturbative properties of this highly non-linear system of equations. We will be particularly interested in properties which might be used to describe a single isolated electron.

1 Introduction

The Maxwell–Dirac system consists of the Dirac equation

$$\gamma^\alpha(\partial_\alpha - i\,e\,A_\alpha)\psi + im\psi = 0, \tag{1}$$

with electromagnetic interaction given by the potential A_α; and the Maxwell equations (sourced by the Dirac current, j^α),

$$\begin{aligned} &F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha,\\ &\partial^\alpha F_{\alpha\beta} = -4\pi e j_\beta = -4\pi e\bar{\psi}\gamma_\beta\psi. \end{aligned} \tag{2}$$

Most studies of the Dirac equation treat the electromagnetic field as given and ignore the Dirac current as a source for the Maxwell equations, i.e. these treatments ignore the electron "self-field". A comprehensive survey of these results can be found in the book by Thaller [1]. This is not surprising, inclusion of the electron self-field via the Dirac current leads to a very difficult, highly non-linear set of partial differential equations. So difficult in fact that the existence theory and solution of the Cauchy problem for small initial data was only solved in 1997 (Gross [2], Chadam [3], Georgiev [4], Esteban *et al* [5], Bournaveas [6], and Flato, Simon and Taflin [7]) – seventy years after Dirac first wrote down his equation!

There are no known non-trivial, exact solutions to the Maxwell–Dirac equations in $1 + 3$ dimensions – all known solutions involve some numerical work. These solutions do, however, exhibit interesting non-linear behaviour which would not have been apparent through perturbation expansions. The particular solutions found in [8] and [9] exhibit just this sort of behaviour – localisation and charge screening. See also Das [10] and the recent work of Finster, Smoller and Yau [11].

Finster, Smoller and Yau also point out in [12] that solving the system (Einstein–Maxwell–Dirac system in their case) gives, in effect, all the Feynman diagrams of the quantum field theory, with the exception of the fermionic loop diagrams. Study of the Maxwell–Dirac system should provide an interesting insight into non-perturbative QED.

In the discussion which follows we will focus on two broad reductions of the equations, the static case (including the spherically symmetric sub-case) and the stationary case. Precise statements of theorems will be given, however only brief indications as to the methods of proof are supplied – details can be found in the original papers cited in the bibliography.

2 The Maxwell–Dirac equations

In [8] the 2-spinor form of the Dirac equations was employed to solve (1) for the electromagnetic potential, under the non-degeneracy condition $j^\alpha j_\alpha \neq 0$. Requiring A^α to be a real four-vector then gave a set of partial differential equations in the Dirac field alone, *the reality conditions.*

For 2-spinors u_A and v^B (see [13] for an exposition of the 2-spinor formalism) we have

$$\psi = \begin{pmatrix} u_A \\ \overline{v}^{\dot{B}} \end{pmatrix}, \qquad \text{with} \qquad u_C v^C \neq 0 \quad \text{(non-degeneracy)},$$

where $A, B = 0, 1,\ \dot{A}, \dot{B} = \dot{0}, \dot{1}$ are two-spinor indices. The *Dirac equations* are

$$\begin{aligned} &\left(\partial^{A\dot{A}} - i\,e\,A^{A\dot{A}}\right) u_A + \frac{im}{\sqrt{2}} \overline{v}^{\dot{A}} = 0, \\ &\left(\partial^{A\dot{A}} + i\,e\,A^{A\dot{A}}\right) v_A + \frac{im}{\sqrt{2}} \overline{u}^{\dot{A}} = 0, \end{aligned} \tag{3}$$

where $\partial^{A\dot{A}} \equiv \sigma^{\alpha A\dot{A}}\partial_\alpha$, $A^{A\dot{A}} = \sigma^{\alpha A\dot{A}} A_\alpha$; here $\sigma^{\alpha A\dot{A}} A_\alpha$ are the Infeld-van der Waerden symbols. The *electromagnetic potential* is (see [8] for details),

$$A^{A\dot{A}} = \frac{i}{e(u^c v_c)} \left\{ v^A \partial^{B\dot{A}} u_B + u^A \partial^{B\dot{A}} v_B + \frac{im}{\sqrt{2}} \left(u^A \overline{u}^{\dot{A}} + v^A \overline{v}^{\dot{A}} \right) \right\}. \tag{4}$$

The *reality conditions* are,

$$\begin{aligned} &\partial^{A\dot{A}} \left(u_A \overline{u}_{\dot{A}} \right) = -\frac{im}{\sqrt{2}} \left(u^C v_C - \overline{u}^{\dot{C}} \overline{v}_{\dot{C}} \right), \\ &\partial^{A\dot{A}} \left(v_A \overline{v}_A \right) = \frac{im}{\sqrt{2}} \left(u^C v_C - \overline{u}^{\dot{C}} \overline{v}_{\dot{C}} \right), \\ &u_A \partial^{A\dot{A}} \overline{v}_{\dot{A}} - \overline{v}_{\dot{A}} \partial^{A\dot{A}} u_A = 0. \end{aligned} \tag{5}$$

The Maxwell equations are,

$$\partial^\alpha F_{\alpha\beta} = -4\pi e\, j_\beta = -4\pi e\, \sqrt{2} \sigma_\beta^{A\dot{A}} \left(u_A \overline{u}_{\dot{A}} + v_A \overline{v}_{\dot{A}} \right). \tag{6}$$

The equations (4), (5) and (6) are entirely equivalent to the original Maxwell–Dirac equations, (1) and (2).

3 The static Maxwell–Dirac equations

A Maxwell–Dirac system is said to be *static* if there exists a Lorentz frame in which the Dirac current vector is purely timelike, i.e. $j^\alpha = j^0 \delta_0^\alpha$, in this Lorentz frame there is no current flow.

As noted in [8] this definition implies,

$$v^A = e^{i\chi} \sqrt{2} \sigma^{0A\dot{A}} \overline{u}_{\dot{A}}, \quad \text{with } \chi \text{ a real function.}$$

The gauge may be fixed (see [8]) by the choice,

$$u^0 = X\, e^{\frac{i}{2}(\chi + \eta)}, \qquad u^1 = Y\, e^{\frac{i}{2}(\chi - \eta)},$$

with X, Y, and η real functions on $\mathbb{R}^4$.

Defining the null vector L,

$$L = \left(\sigma^{\alpha}_{A\dot{A}} u^A \bar{u}^{\dot{A}}\right) = \left(L^0, \frac{1}{\sqrt{2}}\boldsymbol{V}\right), \quad \text{with} \quad L^0 = \frac{1}{\sqrt{2}}\left(X^2+Y^2\right) \quad \text{and}$$
$$\boldsymbol{V} = \left(2XY\cos\eta,\ 2XY\sin\eta,\ X^2-Y^2\right),$$

our equations become,

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(X^2+Y^2\right) = 0,\\
&\boldsymbol{\nabla}.\boldsymbol{V} = -2m\left(X^2+Y^2\right)\sin\chi,\\
&\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{\nabla}\chi)\times\boldsymbol{V} = \boldsymbol{0}.
\end{aligned} \tag{7}$$

With electromagnetic potential

$$\begin{aligned}
&A^0 = \frac{m}{e}\cos\chi + \frac{\left(X^2-Y^2\right)}{2\,e\left(X^2+Y^2\right)}\frac{\partial\eta}{\partial t} + \frac{(\boldsymbol{\nabla}\chi)\,.\boldsymbol{V}}{2e\left(X^2+Y^2\right)},\\
&\boldsymbol{A} = \frac{1}{2e\left(X^2+Y^2\right)}\left[\frac{\partial\chi}{\partial t}\boldsymbol{V} + \left(X^2-Y^2\right)\boldsymbol{\nabla}\eta - \boldsymbol{\nabla}\times\boldsymbol{V}\right], \quad \text{where} \quad \boldsymbol{A} = \left(A^1, A^2, A^3\right).
\end{aligned} \tag{8}$$

The full system is given by the above two sets of equations and the Maxwell equations.

Further simplification can be made to the system by imposing the stationary condition: A Maxwell–Dirac system is said to be *stationary* if there is a gauge in which $\psi = e^{i\omega t}\phi$, with the bi-spinor ϕ independent of t. Such a gauge will be referred to as a stationary gauge. We will be examining isolated, stationary, static systems in Section 3.2. A stationary gauge is not unique.

3.1 Spherical symmetry

Spherical symmetry of the stationary and static Maxwell–Dirac system is imposed (in a gauge independent way) by demanding that the null vector L, defined above, is spherically symmetric. This has the following consequences, in terms of spherical polar coordinates,

$$X = \sqrt{R}\cos(\theta/2), \qquad Y = \sqrt{R}\sin(\theta/2), \qquad \text{and} \qquad \eta = \phi.$$

The equations are

$$\begin{aligned}
&\boldsymbol{A} = \frac{1}{2\,e}\frac{\cot\theta}{r}\hat{\boldsymbol{\phi}}, \qquad A^0 = \frac{m}{e}\cos\chi + \frac{1}{2\,e}\frac{d\chi}{dr},\\
&\frac{d}{dr}\left(r^2 R\right) = -2mr^2 R\sin\chi, \qquad \frac{d}{dr}\left(r^2\frac{dA_0}{dr}\right) = 8\pi e r^2 R,
\end{aligned}$$

with χ and R functions of r only. The Dirac field is

$$\psi = \sqrt{R}\begin{pmatrix} -e^{\frac{i}{2}(\chi-\phi)}\sin\left(\frac{\theta}{2}\right)\\ e^{\frac{i}{2}(\chi+\phi)}\cos\left(\frac{\theta}{2}\right)\\ -e^{\frac{-i}{2}(\chi+\phi)}\sin\left(\frac{\theta}{2}\right)\\ e^{\frac{-i}{2}(\chi-\phi)}\cos\left(\frac{\theta}{2}\right)\end{pmatrix}.$$

The first thing one notices is that there is a central magnetic monopole, with Dirac magnetic charge $\frac{1}{2e}$. In fact, we can obtain a reasonably complete characterisation of these solutions [8]. Briefly, under quite weak (physically reasonable) assumptions, we find that the solutions can be thought of as a central magnetically and electrically charged point source (external to the Dirac field) surrounded by an electrically (oppositely) charged Dirac field. Near ∞ the electrostatic potential behaves as $A^0 \sim -\frac{m}{e} + \frac{1/(me)}{r^2}$ and near $r = 0$ the potential behaves as $A^0 \sim -\frac{m}{e} + \frac{\gamma/e}{r}$ (for some constant γ).

The object is highly compact, with a radius of about $1/m$ a (reduced) Compton wavelength. Inside this radius it has an onion like structure consisting of an infinite series of spherical shells. The system is electrically neutral, with the central Coulomb point source effectively screened by the Dirac field for $r > 1/m$.

3.2 Isolated systems

In most physical processes that we would wish to model using the Maxwell–Dirac system we would be interested in isolated systems – systems where the fields and sources are largely confined to a compact region of $\mathbb{R}^3$. This requires that the fields 'die-off' sufficiently quickly as $|x| \to \infty$.

The best language for the discussion of such decay conditions and other regularity issues is the language of weighted function spaces; specifically weighted classical and Sobolev spaces. In [14] the weighted Sobolev spaces $W^{k,p}_\delta$ are used following the definitions of [15]. These definitions have the advantage that the decay rate is explicit: under appropriate circumstances a function in $W^{k,p}_\delta$ behaves as $|x|^\delta$ with $|x| \to \infty$. An element f of $W^{k,p}_\delta$ has $\sigma^{-\delta+|\alpha|-\frac{3}{p}}\partial^{|\alpha|}f$ in L^p for each multi-index α for which $0 \le |\alpha| \le k$; here $\sigma = \sqrt{1+|x|^2}$ and we are working on $\mathbb{R}^3$ (or some appropriate subset thereof) – see [15] or [16] and [17] (the later papers use a different indexing of the Sobolev spaces).

We will be interested in the asymptotic region (spatially) of the Maxwell-Dirac system, which we denote by $E_\rho = \mathbb{R}^3 \backslash B_\rho$, where B_ρ is the ball of radius ρ. A minimal condition that one may impose on the Dirac field is that it have finite total charge in the region E_ρ, this amounts to

$$\int_{E_\rho} j^0\, dx = \int_{E_\rho} \left(|u_o|^2 + |u_1|^2 + |v^0|^2 + |v^1|^2\right)\, dx \; < \infty.$$

This, of course, simply means that u_A and v^A are in L^2.

Suppose we have a stationary system and we are in a stationary gauge for which $A^\alpha \to 0$ as $|x| \to \infty$. Write, $u_A = e^{-iEt}U_A$ and $\bar{v}^{\dot{A}} = e^{-iEt}\bar{V}^{\dot{A}}$ with U_A, V_A and A^α all independent of time t. Then U_A and V_A must be in $L^2(E_\rho)$ if the total charge due to the Dirac field is finite. So U and V must have L^2 decay as $|x| \to \infty$, roughly U and V must decay faster than $|x|^{-\frac{3}{2}}$. We also note that A^α is given by equation (4) in terms of U and V and their first derivatives. If we substitute this expression for the electromagnetic potential into the Maxwell equations then we have equations that are of third order for U and V. For these equations to make sense we require that U and V are three times differentiable (in the weak sense at least). This suggests that U and V should be in $W^{3,2}_{-\tau}(E_\rho)$, where $\tau > \frac{3}{2}$.

To make this all a little more precise we introduce some more notation. Note that $u_C v^C = U_C V^C$ is a gauge and Lorentz invariant complex scalar function, this means we can introduce a (unique up to sign) "spinor dyad" $\{o_A, \iota_B\}$, with $\iota^A o_A = 1$. The dyad is defined as follows, let $U_C V^C = Re^{i\chi}$ – where R and χ are real functions – then write,

$$U_A = \sqrt{R}e^{i\frac{\chi}{2}}o_A \qquad \text{and} \qquad V_A = \sqrt{R}e^{i\frac{\chi}{2}}\iota_A.$$

Definition 1. A stationary Maxwell–Dirac system will be said to be isolated if, in some stationary gauge, we have

$$\psi = e^{-iEt}\sqrt{R}\begin{pmatrix} e^{\frac{i\chi}{2}}o_A \\ e^{-\frac{i\chi}{2}}\bar{\iota}_{\dot{A}} \end{pmatrix},$$

with E constant and $\sqrt{R}e^{i\frac{\chi}{2}} \in W^{3,2}_{-\tau}(E_\rho)$; $o_A, \iota_A \in W^{3,2}_{\epsilon}(E_\rho)$ and $A^\alpha \in W^{2,2}_{-1+\epsilon}(E_\rho)$, for some $\tau > \frac{3}{2}$ and some $\rho > 0$ and any $\epsilon > 0$.

Remark 1. This definition ensures, after use of the Sobolev inequality and the multiplication lemma, that $\psi = o(r^{-\tau})$ and $A^\alpha = o(r^{-1+\epsilon})$.

Remark 2. Notice our condition places regularity restrictions on the fields in the region E_ρ only. In the "interior" B_ρ there are no regularity assumptions.

The spherically symmetric solution in fact provides an excellent example of an *isolated*, *stationary* and *static* Maxwell–Dirac system.

The main theorem proved in [14] shows that the electric neutrality of the spherically symmetric solution is generic for these isolated, static systems.

Theorem 1. *An isolated, stationary, static Maxwell–Dirac system is electrically neutral.*

The theorem is remarkable in that it depends only on asymptotic regularity and decay – almost anything can happen in B_ρ! Another theorem of [14] shows that the association of a magnetic monopole with the central, external Coulomb field, in the spherically symmetric case, is also generic (at least for axial symmetry). That is, associated to each external Coulomb point charge in a stationary, static Maxwell–Dirac system there is a magnetic monopole with magnetic charge of Dirac value $\frac{1}{2e}$.

4 Stationary isolated systems

To close this brief overview of the Maxwell–Dirac system we will take a quick look at some very recent results [18].

The first observation one makes is that under the regularity and decay conditions assumed (those of an isolated system) we can always perform a gauge transformation to the Lorenz gauge. The Maxwell equation for A^α now becomes an elliptic equation (remember the system is stationary)

$$\triangle\, A^\alpha = 4\pi e\sqrt{2}R\sigma^{A\dot{A}}_\beta(o_A\bar{o}_{\dot{A}} + \iota_A\bar{\iota}_{\dot{A}}). \tag{9}$$

Writing $a^\alpha = E\delta^\alpha_0 + A^\alpha$ the Dirac equations (3) are,

$$\begin{aligned}
&\frac{o_A}{2}\left(\frac{\partial^{A\dot{A}}R}{R} + i\partial^{A\dot{A}}\chi\right) + \partial^{A\dot{A}}o_A - iea^{A\dot{A}}o_A + \frac{im}{\sqrt{2}}\,\bar{\iota}^{\dot{A}}e^{-i\chi} = 0,\\
&\frac{\iota_A}{2}\left(\frac{\partial^{A\dot{A}}R}{R} + i\partial^{A\dot{A}}\chi\right) + \partial^{A\dot{A}}\iota_A + iea^{A\dot{A}}\iota_A + \frac{im}{\sqrt{2}}\,\bar{\iota}^{\dot{A}}e^{-i\chi} = 0.
\end{aligned} \tag{10}$$

A straightforward "bootstrap" argument (based on elliptic regularity) can be made to show that A and U and V must in fact be C^∞ if U and V are taken to be in $L^2(E_\rho)$ and A is in L^1_{loc}.

One can show (using an argument based on Thaller, [1]) that the essential spectrum of the Dirac operator in this case is the same as that for the free Dirac operator, i.e. $(-\infty, -m]\cup[m, \infty)$. So we would expect to get bound states for $E \in (-m, m)$ – cf. [5]. In fact under very weak assumptions we can show that there are *no embedded eigenvalues*, i.e. $E \in [-m, m]$ – cf. [1].

Under more restrictive assumptions there is also a version of the "electric neutrality" theorem. The interested reader may find details of these and other results in the forthcoming paper.

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Asymptotic Expansions of the Potential Curves of the Relativistic Quantum-Mechanical Two-Coulomb-Centre Problem

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The asymptotic expansions (at small and large internuclear distances R) of the eigenvalues (potential curves) $E(R)$ of the two-Coulomb-centre problem by the perturbation theory are obtained.

1 Introduction

In the present time a severe asymmetry exists in development of the theories of nonrelativistic and relativistic quantum-mechanical problems of two Coulomb centres (the so-called Z_1eZ_2 problem). Numerous effective asymptotic and numerical methods of solving the two-Coulomb-centre problem for the Schrödinger equation (see, for instance, [1] and references therein) can be opposed only by seldom examples of the consideration of same problem for the Dirac equation within various approximations [2, 3, 4, 5] (the Galerkin method, diagonalization, variational method, perturbation theory, Furry–Sommerfeld–Maue approximation). Such situation is a surprising example of passivity of the theory at the deficiency of experimental data for heavy and superheavy quasi-molecular systems due to the difficulties in construction of sources of multiply charged ions and formation of beams of rather slow particles.

Besides, with the recent erection of powerful accelerators of highly charged ions in many laboratories [6, 7] the need of the consistent Dirac theory of the quantum mechanical problem is more and more urgent in different fields of physics. Previously, this problem was applied, basically, in the theory of supercritical atoms for the description of effects of spontaneous and enforced creation of positrons in a supercritical field of a quasi-atom formed at slow collisions of heavy ions with a total atomic number $Z_1 + Z_2 > 173$ [3, 8, 9]. Rather recently [10], Z_1eZ_2 problem was used as a model approximation in the investigations of elementary processes of collisions (excitation, charge exchange, ionization) of multiply charged ions. Other application of the relativistic problem in theory of collisions is more traditional, and is reduced to using the model functions of a continuous spectrum for the analysis of scattering of relativistic electrons on heavy diatomic molecules [10].

The difficulty in considering the problem consists in the fact that the Dirac equation with the potential of two Coulomb centres does not permit complete separation of variables in any orthogonal system of coordinates and, thus, one has to deal with first-order partial differential equations. This highly complicates the whole specific problem of finding the electron wave function and potential curves. Unfortunately, numerical solving this system of differential equations is rather complicated and cumbersome problem [4, 5] requiring complicated calculations for each specific system Z_1eZ_2. This causes the necessity of creating and investigating approximative methods of solving this problem, which are based on clear physical ideas and well elaborated mathematical device and have a clear area of application.

In the present paper we determine the energy of an electron for two asymptotic cases, when the distance R between the Coulomb centres is rather small or rather large. For this we use the scheme of the perturbation theory which does not require the separation of variables. As a result of the performed calculations, the asymptotic expressions for levels of energy of system Z_1eZ_2 are obtained at $R \to 0$ ($R \to \infty$) up to the terms $O(R^3)$ ($O(R^{-3})$).

2 Asymptotic expansions of the solutions of the problem at $R \to 0$

When the total charge of Coulomb centres $Z = Z_1 + Z_2$ is positive and internuclear distance R tends to zero, it is possible to consider the relativistic problem within the perturbation theory. The Dirac Hamiltonian of the problem Z_1eZ_2 is of the form ($m_e = e = \hbar = 1$):

$$\widehat{H} = c\vec{\alpha} \cdot \widehat{\vec{p}} + c^2\beta + V, \qquad V = -\frac{Z_1}{r_1} - \frac{Z_2}{r_2}, \tag{1}$$

where $r_{1,2}$ is the distance between the electron and the corresponding nucleus, $\widehat{\vec{p}} = -i\hbar$ is the momentum operator, and c is the velocity of light. In standard representation [11],

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

Here $\vec{\sigma}$ are Pauli matrices, and 0 and I are, respectively, 2×2 zero and identity matrices. Let us represent the complete Hamiltonian of the two-Coulomb-centre problem $\widehat{H}$ by the Hamiltonian of zero approximation $\widehat{H}^{UA}$ and perturbation $\widehat{W}$:

$$\widehat{H} = \widehat{H}^{UA} + \widehat{W}.$$

As $\widehat{H}^{UA}$ the Dirac Hamiltonian of the united relativistic atom

$$\widehat{H}^{UA} = c\vec{\alpha} \cdot \widehat{\vec{p}} + c^2\beta - \frac{Z}{r_0}$$

is taken, the atom being placed on the axis z, directed from centre Z_1 to centre Z_2, in the point $z = z_0$ that is the centre of electric charges and divides the internuclear distance into two segments:

$$R_1 = \frac{Z_2}{Z}R, \qquad R_2 = \frac{Z_1}{Z}R.$$

We consider a spherical system of coordinates r_0, θ_0, φ_0: the origin is in the point $(0, 0, z_0)$ and the angle θ_0 is measured from the axis z.

Now we construct the unperturbed wave function of an united atom. The eigenvalues of the operator are characterized by spherical quantum numbers n, j, l, m, where n is the principal quantum number, j and l are the total electron and orbital angular moments, respectively, is the projection of j onto the internuclear axis z. The explicit form of the eigenfunctions of the operator $\widehat{H}^{UA}$ can be found in [11]. Expanding the perturbation operator $\widehat{W}$ in the Legendre polynomials and calculating the matrix elements of the matrix $\left\| W_{njlm}^{njl'm'} \right\|$ to the first (within the terms $O\left(R^3\right)$) nonzero term we see that at $R \to 0$ the matrix $\left\| W_{njlm}^{njl'm'} \right\|$ is diagonal with respect to each group of mutually degenerated (on l and m) states. The residual result for energy of Z_1eZ_2 system at is $R \to 0$

$$E_{njlm}(R) = \varepsilon c^2 + \frac{Z_1Z_2}{2N^3} \cdot \frac{3m^2 - j(j+1)}{j(j+1)} \cdot \frac{\left[3\varepsilon\aleph(\varepsilon\aleph - 1) - \gamma^2 + 1\right] \cdot (ZR)^2}{\gamma(\gamma^2 - 1)(4\gamma^2 - 1)} + O\left(R^3\right), \tag{2}$$

where

$$n_r = n - j - 1/2, \qquad \aleph = (-1)^{k-l}k, \qquad k = j + 1/2, \qquad l = j \pm 1/2, \tag{3}$$

$$N = \sqrt{n^2 - 2n_r(k-\gamma)}, \qquad \gamma = \sqrt{k^2 - (Z\alpha_0)^2},$$
$$\varepsilon = \left[1 + \left(\frac{Z\alpha_0}{n_r + \gamma}\right)^2\right]^{-1/2}, \qquad \alpha_0 = \frac{1}{c}. \tag{4}$$

We have compared (see Fig. 1) the binding energies of some bound states of the Pb-Pb system calculated by asymptotic formula (2) with results of paper [5]. The difference $\approx 5\%$ is connected with the finite extension of the Pb nuclei in [5].

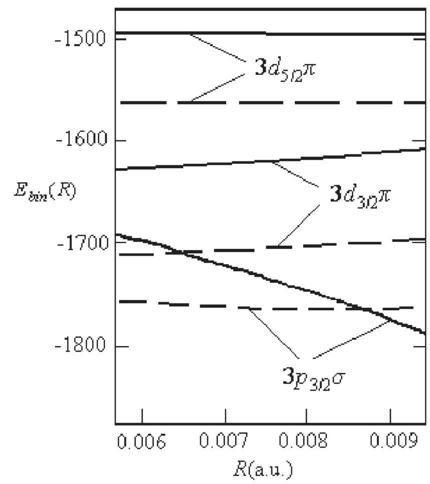

Figure 1.

3 Asymptotic expansions of the solutions of the problem at $R \to \infty$

Now we shall determine the energy $E(R)$ and the wave functions $\Psi(\vec{r}; R)$ of an electron in the asymptotic region, when the distance R between the Coulomb centres is large. This distance should be so large that the quantum penetrability of the potential barrier separating atomic particles is much smaller than unity. When atoms 1 and 2 are different, the eigenvalues (potential curves) $E(R)$ of the two-Coulomb-centre problem, dependent on the internuclear distance R as a parameter, are divided into two classes in the asymptotic limit $R \to \infty$: E_I- and E_{II}-potential curves that, for $R \to \infty$, transform into the energy levels of isolated atoms 1 and 2, respectively.

Having placed the origin at the position of the hydrogen-like ion eZ_1 with nuclear charge Z_1 and run the polar axis along the R axis, we represent a complete Hamiltonian of the two-Coulomb-centre problem (1) by a Hamiltonian of zero-approximation $\widehat{H}^{SA}$ and perturbation $\widehat{V}$:

$$\widehat{H} = \widehat{H}^{SA} + \widehat{V}.$$

As $\widehat{H}^{SA}$ the Hamiltonian of the relativistic hydrogen-like atom with charge Z_1

$$\widehat{H}^{SA} = c\vec{\alpha} \cdot \widehat{\vec{p}} + c^2\beta - \frac{Z_1}{r_1}$$

is taken. In a spherical coordinate system wave functions $\Psi^{SA}_{n_1 j_1 l_1 m_1}(\vec{r}_1)$ of eZ_1 atom, belonging to a discrete energy spectrum, are characterized by the set quantum numbers n_1, j_1, l_1, m_1. At large internuclear distances the operator of the interaction between the electron and the Z_2 nucleus $\widehat{V} = -Z_2/\left|\vec{R} - \vec{r}\right|$ can be considered as a small perturbation of the Hamiltonian $\widehat{H}^{SA}$. As in previous case we expand the perturbation operator $\widehat{V}$ in the Legendre polynomials and calculate the matrix $\left\| V^{n_1 j_1 l_1' m_1'}_{n_1 j_1 l_1 m_1} \right\|$ of the perturbation operator to the first non-zero diagonal term.

Diagonalizing the complete matrix of energy with respect to each group of mutually degenerate states we obtain the analytical expression for E_I-potential curves in the first order of the perturbation theory

$$E_I(R) = \varepsilon_1 c^2 - \frac{Z_2}{R} \pm \frac{3}{4}\sqrt{N_1^2 - \aleph_1^2}\,\frac{(n_{r1} + \gamma_1)\, m_1}{j_1(j_1+1)}\,\frac{Z_2}{Z_1 R^2} + O\left(R^{-3}\right), \tag{5}$$

where the quantities n_{r1}, $\aleph_1$, k_1, l_1, N_1, γ_1, ε_1 are obtained from (3), (4) by adding index 1. The third term in (5) coincides with the Stark shift of level in the weak electric field with the intensity $-Z_2/R^2$ [12].

The asymptotic expansion of the potential curve E_{II} is obtained from E_I by the substitutions $\varepsilon_1 \to \varepsilon_2$, $Z_{1,2} \to Z_{2,1}$, $n_1, \aleph_1, j_1, m_1 \to n_2, \aleph_2, j_2, m_2$.

4 Conclusions

Here we briefly summarize the results obtained in this paper. By means of the perturbation theory we have calculated the asymptotic expansion of the eigenvalues (potential curves) $E(R)$ of the two-Coulomb-centre problem in the limits of united ($R \to 0$) and separated ($R \to \infty$) atoms with the precision to $O\left(R^3\right)$ and $O\left(R^{-3}\right)$, respectively. Note that asymptotic expressions of the potential curves obtained here are applicable under the condition that quantities γ, $\gamma_{1,2}$ are real only, which corresponds to the range of applicability of the Dirac equation solutions for the point-charge.

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WKB Method for the Dirac Equation with the Central-Symmetrical Potential and Its Application to the Theory of Two Dimensional Supercritical Atoms

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Solutions of the Dirac equation in a strong external field are obtained in the WKB approximation. A field is considered strong if the electron binding energy exceeds $2mc^2$ and the discrete spectrum levels may be lowered into the lower continuum. The wave functions in the classically allowed and forbidden regions are found and the conditions for matching them on transition through the turning point are obtained. The WKB method is applied to the following problems: 1) generalization of the Bohr–Sommerfeld quantization conditions with allowance for relativistic effects and the spin in $2+1$ dimensions; 2) energy and width of the quasistationary level in the lower continuum.

1 Introduction

It is known [1, 2] that in three spatial dimensions the expression for the electron ground state energy in the Coulomb field of a point-charge $Z|e|$ becomes purely imaginary when $Z > 137$, and that its interpretation as electron energy no longer has a physical meaning. To determine the electron energy spectrum in the Coulomb field with such a charge we need to eliminate the singularity of the Coulomb potential of a point-charge at $r = 0$ by cutting off the Coulomb potential at small distances. This is equivalent to taking into account of the nucleus size. In three space dimensions the electron energy spectrum in the Coulomb field regulated at small distances was first considered by Pomeranchuk and Smorodinsky (see, for instance, [3]). With increasing Z in the region $Z > 137$, the electron energy levels in such a field were found to decrease, become negative, and may cross the boundary of the lower energy continuum, $E = -mc^2$. The value of $Z|e| = Z_{\rm cr}|e|$ at which the lowest electron energy level cross the boundary of the lower energy continuum is called the critical charge for the electron ground state [2, 4]. If Z continues to grow and enters the transcritical region with $Z > Z_{\rm cr}$, the lowest electron energy level "sinks" into the lower energy continuum, which result in a rearrangement of the vacuum of the QED. This rearrangement is constrained by Pauli exclusion principle. If the electron ground state at $Z < Z_{\rm cr}$ is vacant, two electron-positron pairs are created; if it is half-occupied, one pair is created; and if it is occupied, no pairs are created. The Coulomb potential is repulsive for the created positrons, so they go to infinity. Hence at $Z > Z_{\rm cr}$ a quasistationary state appears in the lower energy continuum and the new vacuum of QED, which corresponds to the filling of all the electron states with $E < -mc^2$, has the total electric charge $2e$ [2, 4]. Indeed, all the electron states with $E < -mc^2$ (the Dirac sea) were filled at $Z < Z_{\rm cr}$, so electrons created by the strong Coulomb field with $Z > Z_{\rm cr}$ cannot be described by means of a convenient wave function, and the notion of charged vacuum was introduced to describe these states [4, 5, 6, 7]. In terms of the new vacuum, the density of electric charge $\rho(r)$ is classical. It is a function characterising the spatial distribution of the real electric charge appearing in the new (charged) vacuum, while in terms of the old (uncharged) vacuum this function should be interpreted as the probability of two electrons (with charge $2e$) being present at a given point in space.

We would like to see how the same system behaves in two dimensions. With this aim we shall apply the WKB method to the Dirac equation in a strong Coulomb field. Such approach works rather well for states with energy both $0 < E < mc^2$ and $E < -mc^2$. The obtained by this way quasiclassical formulae for the energy of quasistationary levels of the Dirac equation solutions in the lower continuum in $(2+1)$ dimensions allow to consider a wide range of problems in the theory of supercritical atoms.

2 The Dirac equation in an external Coulomb field in 2 + 1 dimensions

Since [8] in $2+1$ dimensions the Dirac algebra may be represented in terms of the Pauli matrices as $\gamma^0 = \sigma^3$, $\gamma^k = i\sigma^k$, the Dirac equation for an electron minimally coupled to an external electromagnetic field has the form ($\hbar = c = m_e = 1$)

$$\left(i\frac{\partial}{\partial t} - H_D\right)\Psi = 0, \tag{1}$$

where

$$H_D = \vec{\alpha}\widehat{\vec{p}} + \beta - eA^0\widehat{I} = \sigma^1 p_2 - \sigma^2 p_1 + \sigma^3 - eA^0\widehat{I} \tag{2}$$

is the Dirac Hamiltonian, $p_\mu = i\partial_\mu + eA_\mu$ is the operator of generalized momentum of the electron, A_μ is the vector potential of the external electromagnetic field, $-e < 0$ ($e > 0$) is electric charge of the electron, and the conserved total angular momentum has only a single component, namely, $J_z = L_z + S_z$, where $L_z = -i\partial/\partial\varphi$ and $S_z = \sigma^3/2$.

Let us apply the Dirac equation (1), (2) to study two-dimensional hydrogen-like ion with nuclear charge $eZ \gg 1$. Consider the problem neglecting the nucleus size and assuming the vector potential to be Coulomb

$$A^0(r) = -\frac{Ze}{r}, \qquad A^x = A^y = 0 \tag{3}$$

for $0 \le r < \infty$.

We seek the solutions of the Dirac equation (1) in the field (3) in the polar coordinates in the form

$$\Psi(t, \vec{x}) = \frac{1}{\sqrt{2\pi}} \exp(-i\varepsilon t + il\varphi)\psi(r, \varphi), \tag{4}$$

where ε is the energy, l is an integer number and

$$\psi(r, \varphi) = \frac{1}{\sqrt{r}} \begin{pmatrix} F(r) \\ G(r)e^{i\varphi} \end{pmatrix}. \tag{5}$$

Note that the function (4) is an eigenfunction of the the Dirac Hamiltonian H_D and the total angular momentum J_z with eigenvalues ε and $l + 1/2$, respectively.

Substituting (4) and (5) into (1), and taking into account of the equations

$$p_x \pm p_y = -ie^{\pm i\varphi}\left(\frac{\partial}{\partial r} \pm \frac{i}{r}\frac{\partial}{\partial\varphi}\right),$$

we obtain

$$\frac{dF}{dr} - \frac{l+1/2}{r} F + (\varepsilon + 1 - V(r))\, G = 0, \tag{6}$$

$$\frac{dG}{dr} + \frac{l+1/2}{r} G - (\varepsilon - 1 - V(r))\, F = 0, \tag{7}$$

where $V(r) = -Z\alpha/r$, $\alpha = e^2 \approx 1/137$ is the fine structure constant.

The exact solutions and the energy eigenvalues with $\varepsilon < 1$ corresponding to stationary states of the Dirac equation may be found in full analogy with the case of three space dimensions [1]. Let us look for functions F and G in the form

$$F = \sqrt{1+\varepsilon} \cdot e^{-\rho/2} \rho^{\gamma} \left(Q_1 + Q_2\right), \qquad G = \sqrt{1-\varepsilon} \cdot e^{-\rho/2} \rho^{\gamma} \left(Q_1 - Q_2\right), \tag{8}$$

where

$$\rho = 2\lambda r, \qquad \lambda = \sqrt{1-\varepsilon^2}, \qquad \gamma = \sqrt{(l+1/2)^2 - (Z\alpha)^2}.$$

The value of γ is to be found by studying the behavior of the wave function at small r. The functions Q_1 and Q_2 which rendered the solutions of (6), (7) finite at $\rho = 0$ are given in terms of the confluent hypergeometric function $F(a, b; z)$ as:

$$Q_1 = AF(\gamma - \varepsilon Z\alpha/\lambda, 2\gamma + 1; \rho), \qquad Q_2 = BF(\gamma - \varepsilon Z\alpha/\lambda + 1, 2\gamma + 1; \rho).$$

The constants A and B are related by

$$B = \frac{\gamma - \varepsilon Z\alpha/\lambda}{l + 1/2 + Z\alpha/\lambda} A,$$

and the energy eigenvalues are defined by

$$\gamma - \varepsilon Z\alpha/\lambda = -n_r. \tag{9}$$

It is easy to show that the following values of the quantum number n_r are allowed: $n_r = 0, 1, 2, \ldots$, if $l \geqslant 0$, and $n_r = 1, 2, 3, \ldots$ if $l < 0$.

From the normalization condition for the wave function $\Psi(t, \vec{x})$ one can obtain the expression for the constant A:

$$A = \frac{1}{\Gamma(2\gamma+1)} \left\{ \frac{\lambda \left[Z\alpha + \lambda(l+1/2)\right] \Gamma(2\gamma + n_r + 1)}{2Z\alpha \cdot n_r!} \right\}^{1/2}.$$

From (9) we find the electron energy spectrum in the Coulomb field (3):

$$\varepsilon = \left[1 + \frac{(Z\alpha)^2}{\left(n_r + \sqrt{(l+1/2)^2 - (Z\alpha)^2}\right)^2} \right]^{-1/2}.$$

It is seen that

$$\varepsilon_0 = \sqrt{1 - (2Z\alpha)^2}$$

for $l = n_r = 0$, and ε_0 becomes zero at $Z\alpha = 1/2$, whereas in three spatial dimensions ε_0 equals zero at $Z\alpha = 1$. Thus, in two space dimensions the expression for the electron ground state energy in the Coulomb field of a point-charge $Z|e|$ no longer has a physical meaning at a much lower value of $Z\alpha = 1/2$, and the corresponding solution of the Dirac equation oscillates near the point $r \to 0$.

To determine the electron energy spectrum in the Coulomb field with such a charge we need to eliminate the singularity of the Coulomb potential of a point-charge at $r = 0$ by cutting off the Coulomb potential at small distance r_N. This is equivalent to taking into account of the nucleus size.

3 WKB method for the Dirac equation in the strong external field

For finding the quasiclassical solutions of the system of equations (6), (7) it is convenient to write them in the matrix form:

$$\psi' = \frac{1}{\hbar} D\psi,\ \psi = \begin{pmatrix} F \\ G \end{pmatrix}, \qquad D = \begin{pmatrix} \hbar\aleph/r & -(\varepsilon + 1 - V(r)) \\ \varepsilon - 1 - V(r) & -\hbar\aleph/r \end{pmatrix}. \tag{10}$$

Here we have restored in an obvious view the reduced Planck constant $\hbar$, the prime denotes the derivative with respect to r, $\aleph = l + 1/2$, and the external electrostatic potential is $V(r) = -eA^0(r)$. The solution of the matrix equation (10) we shall look as the formal expansion in powers of $\hbar$:

$$\psi = \varphi \exp\left(\int y dr\right), \qquad y(r) = \frac{1}{\hbar} y_{-1}(r) + y_0(r) + \hbar y_1(r) + \cdots,$$

$$\varphi(r) = \sum_{n=0}^{\infty} \hbar^n \varphi^{(n)}(r), \tag{11}$$

where the upper (lower) component $\varphi_F^{(n)}$ $\left(\varphi_G^{(n)}\right)$ of the vector $\varphi^{(n)}$ corresponds to the radial wave function F (G). By substituting (11) into (10) and equating to zero the coefficient of each power of $\hbar$, we obtain the recurrence system

$$(D - y_{-1})\,\varphi^{(0)} = 0, \tag{12}$$

$$(D - y_{-1})\,\varphi^{(n+1)} = \left(\varphi^{(n)}\right)' + \sum_{k=0}^{n} y_{n-k}\varphi^{(k)}, \qquad n = 0, 1, \ldots \tag{13}$$

Using the first two equation of system of equations (12), (13) by the left and right vectors technique we find the terms y_{-1}, y_0 and $\varphi^{(0)}$. Solving the following equations of this system by the similar procedure one can find the terms $y_2, y_3, \ldots, \varphi^{(2)}, \varphi^{(3)}, \ldots$ in the expansions (11). But formulae for them are rather cumbrous, therefore in applications one usually restricts them to only first terms. Actually the reason of this is the fact that the expansions in powers of $\hbar$ (11) in the general case do not convergent and are asymptotic series, the finite number of terms of which gives the good approximation for the wave function, if a parameter of an expansion (the reduced Planck constant $\hbar$) is rather small. So we obtained (to within a normalization constant)

$$\psi = \frac{1}{\sqrt{qQ_\mp}} \exp\left[\int \left(\pm q + \frac{V'(r)}{2qQ_\mp}\right) dr\right] \begin{pmatrix} 1 + \varepsilon - V(r) \\ \mp Q_\mp \end{pmatrix}. \tag{14}$$

Employ the obtained formula to the problem about quasistationary state that is prolongation of the discrete level into the transcritical range $Z > Z_{\mathrm{cr}}$, when $\varepsilon < -1$.

To the Dirac system of equations (6), (7) there corresponds the effective potential

$$U(r, \varepsilon) = \varepsilon V - 1/2V^2 + \aleph^2/2r^2, \tag{15}$$

which corresponds to the attraction on small distances $r < r_-$ from nuclear (at $Z\alpha > |\aleph|$) and repulsion for $r > r_-$. So $U(r, \varepsilon)$ looks like a potential with a barrier. To eliminate the singularity of the Coulomb potential of a point-charge at $r = 0$ it is necessary to cut off the Coulomb potential $V(r)$ at some small distance r_N:

$$V(r) = \begin{cases} -Z\alpha/r, & r > r_N, \\ -(Z\alpha/r)\, f(r/r_N), & r \le r_N. \end{cases} \tag{16}$$

Here $f(x)$ is cutoff function, $0 \le x = r/r_N \le 1$. Most often the following models are used: $f(x) = 1$ and $f(x) = \left(3 - x^2\right)/2$.

4 The wave function of the Dirac electron in classically allowed and forbidden regions

The wave function of quasistationary state has the various look in the various regions.

I. The region $r_0 < r < r_-$ is classically allowed; there the wave functions (14) oscillate

$$G = C_1^\pm \left(\frac{\varepsilon - V + 1}{p}\right)^{1/2} \cos\Theta_1, \qquad F = C_1^\pm \operatorname{sgn}\left(\frac{\varepsilon - V - 1}{p}\right)^{1/2} \cos\Theta_2. \tag{17}$$

Here

$$p(r) = \sqrt{(\varepsilon - V)^2 - 1 - \frac{\aleph^2}{r^2}}$$

is quasiclassical moment for the radial motion of a particle, $C_1^\pm$ is normalization constant,

$$\Theta_1 = \int_{r_-}^{r} \left(p - \frac{\aleph w}{pr}\right) dr + \frac{\pi}{4}, \qquad \Theta_2 = \int_{r_-}^{r} \left(p - \frac{\aleph \widetilde{w}}{pr}\right) dr + \frac{\pi}{4},$$
$$w = \frac{1}{2}\left(\frac{V'}{1 + \varepsilon - V} - \frac{1}{r}\right), \qquad \widetilde{w} = \frac{1}{2}\left(\frac{V'}{1 - \varepsilon + V} - \frac{1}{r}\right).$$

Signs $\pm$ correspond to values $\aleph > 0$ and $\aleph < 0$. If a width γ of a level is small (it will be shown later) the wave function of quasistationary state can be normalized on a single particle localized in the region I, neglecting its penetrability into the classically forbidden regions $r < r_0$ and $r > r_-$ [10]. Here $\cos^2\Theta_i(r)$ can be replaced with average value 1/2:

$$|C_1^\pm| = \left[\int_{r_0}^{r_-} \frac{\varepsilon - V(r)}{p(r)} dr\right]^{-1/2} = \left(\frac{2}{T}\right)^{1/2},$$

where T is the frequency period of a relativistic particle inside a potential well.

II. The below-barrier region $r_- < r < r_+$ is classically forbidden. Here $p = iq$, and quantities q, y_{-1} and y_0 are real. As known [10] the wave function should exponentially damp inside of this region. So the solutions of the Dirac system of equations (6), (7) in the below-barrier region for $\aleph < 0$ are

$$\psi = \frac{C_2^-}{\sqrt{qQ_-}} \exp\left[-\int_{r_+}^{r} \left(q + \frac{V'(r)}{2qQ_-}\right) dr\right] \begin{pmatrix} -Q_- \\ \varepsilon - 1 - V(r) \end{pmatrix}, \tag{18}$$

III. In the region $r > r_+$ the divergent wave corresponds to the quasistationary state (taking off positron); for $\aleph < 0$:

$$\psi = \frac{C_3^-}{\sqrt{pP_-}} \exp\left[\int_{r_+}^{r} \left(ip + \frac{V'(r)}{2pP_-}\right) dr\right] \begin{pmatrix} iP_- \\ \varepsilon - 1 - V(r) \end{pmatrix}, \tag{19}$$

where $P_\pm = p \pm i\aleph/r$. The formulae (17)–(19) include the whole range of values of r (except for range $r < r_0$ for which the view of a wave function here is not written out), except for neighbourhoods of turning points r_- and r_+. For bypass of these points and sewing the solutions we shall use the usual method [10]. Closely to the r_- and r_+ the system (6) reduces to the Schrödinger equation with the effective potential linearly depending on $r - r_\pm$, the solution of which expressed through the Airy function; one can sew by the more elegant Zwaan method. So the relation between the constants in various regions is of the form

$$C_2^\pm = iC_3^\pm = \sigma C_1^\pm \left[\frac{|\aleph|}{\left(r_-^2 + \aleph^2\right)^{1/2} + r_-}\right]^\sigma \exp\left[-\int_{r_-}^{r_+} \left(q + \sigma\frac{V'(r)}{qQ_\pm}\right) dr\right], \tag{20}$$

where $\sigma = \operatorname{sgn}\aleph/2$.

Though the formulae (17)–(19) essentially differ from the formulae by nonrelativistic quasiclassics and more complicated from them, their application to concrete problems does not meet difficulties, as all quantities in functions F and G express in quadratures.

5 Position and width of quasistationary levels in the lower continuum

Let us find the energy of quasistationary states that are the prolongation of the discrete spectrum levels into supercritical region $Z > Z_{\rm cr}$, $\varepsilon < -1$. Neglecting the penetrability of a barrier in the region $r_- < r < r_+$ we obtain from (15) the quantization condition:

$$\int_{r_-}^{r_+} \left(p - \frac{\aleph w}{pr}\right) dr = \pi \left(n_r + \frac{\operatorname{sgn}(l+1/2)}{2}\right). \tag{21}$$

The equation (21) determines the real part of the level energy $\varepsilon_{nl} = \varepsilon - i\gamma/2$. It is easy to show that the condition (21) reproduces the exact expression of the energy spectrum in the case $0 < \varepsilon < 1$.

Calculating the integral in (21) for the potential (14) and taking into account that $|\varepsilon| \ll Z\alpha/r_N$, we arrive at the transcendental equation ε:

$$\frac{\varepsilon Z\alpha}{2k} \ln \frac{|\varepsilon|\, Z\alpha + kg}{|\varepsilon|\, Z\alpha - kg} - g \ln \frac{r_N e\mu}{2g^2} + \sigma \arccos \frac{g^2 - \varepsilon\aleph^2}{Z\alpha\mu} + I = \pi \left(n_r + \frac{\operatorname{sgn}(l+1/2)}{2}\right), \tag{22}$$

where

$$I = Z\alpha \int_{x_0}^{1} \left[\sqrt{f^2(x) - \frac{\rho^2}{x^2}} + \frac{\aleph}{2(Z\alpha)^2}\left(\frac{f'(x)}{f(x)} + \frac{1}{x}\right)\frac{1}{\sqrt{x^2 f^2(x) - \rho^2}}\right] dx, \quad e = 2.718\ldots$$

Let now us go to determination of the level width $\gamma = -2\,\mathrm{Im}\,\varepsilon_{nl}$ that coincides with the probability of the spontaneous creation of positrons. From the equations (6), (7) we obtain the expression for γ

$$\gamma = 2\,\mathrm{Im}\,[G^*(\infty)F(\infty)].$$

By the obtained formulae for G and F γ takes the form

$$\gamma = \gamma_0 \exp\left[-2\pi Z\alpha\left(\sqrt{1+1/k^2} - \sqrt{1-\rho^2}\right)\right],$$
$$T = \frac{1}{\gamma_0} = -\frac{2}{k^2}\left[\varepsilon g + \frac{Z\alpha}{2k} \ln\left(\frac{|\varepsilon|\, Z\alpha + kg}{|\varepsilon|\, Z\alpha - kg}\right)\right].$$

6 Conclusions

In this paper we construct quasiclassical solutions of the (2+1)-dimensional Dirac equation with a strong Coulomb field. By the obtained formulae we obtain the spectrum of quasistationary levels (its position and width) in the lower energy continuum $\varepsilon < -1$ for a spherical superheavy nuclear with a charge $Z > Z_{\rm cr}$. Comparison of values of critical charge $Z_{\rm cr}$ obtained from exact solutions of the Dirac equation [9] with $Z_{\rm cr}$ obtained from the quasiclassical formula (20) shows good correlation. Note that in the ground state for the model I at $r_N = 0.03$ $Z \approx 108$ and 170 in (2 + 1)- and (3 + 1)-dimensional QED, respectively. Thus, the Dirac vacuum in two space dimensions in the presence of a strong Coulomb field is unstable against electron-positron production at significantly smaller values of the critical charge than in the case of three spatial dimensions. Another difference between these two cases results from the fact that electrons confined to a plane behave like a spinless fermion. So if the ground electron state at $Z < Z_{\rm cr}$ is vacant, one pair is created; if it is occupied, no pairs are created.

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Geometric Formulation of Berezin Quantization

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In this paper we try to formulate the Berezin quantization on projective Hilbert space $\mathbb{P}(\mathcal{H})$ and use its geometric structure. It will be shown that the star product in Berezin quantization is equivalent to the Poisson bracket on $\mathbb{P}(\mathcal{H})$ and the Berezin method to construction a correspondence between a given classical theory and a given quantum theory is used to define a classical limit for geometric quantum mechanics.

1 Introduction

In Berezin quantization one defines from a representation of C^*-algebra of quantum observables the *covariant symbols*. These symbols are expectation values of the observables in terms of *coherent states*: the holomorphic functions on classical phase space M that is assumed to be a Kähler manifold.

Berezin [2] showed that the covariant symbols form a $*_\hbar$-algebra which in limit $\hbar \to 0$ leads to the Poisson algebra between the corresponding classical observables: The functions on the phase space M.

In this paper we will see that the Berezin $*_\hbar$-algebra is in fact a Poisson algebra which is induced by the Fubini–Study 2-form on space of coherent states. This space is defined as follows: coherent states span a dense subspace $\widetilde{\mathcal{H}}$ of Hilbert space $\mathcal{H}$. $\mathbb{P}(\widetilde{\mathcal{H}})$, which is denoted by $\mathcal{M}$, is a Kähler manifold with induced symplectic structure from $\mathbb{P}(\mathcal{H})$. Therefore the covariant symbols can be considered as functions on $\mathcal{M}$. It is shown [13] that there exists an embedding mapping between the classical phase space M and $\mathcal{M}$, by which to any point $z \in M$ is associated a point $Z \in \mathcal{M}$. With this construction to all of the quantum observables are associated their covariant symbols, which form a Poisson algebra on $\mathcal{M}$ and since the corresponding classical observables form a Poisson algebra on M, the Berezin quantization is a systematic procedure to relate these two Poisson algebras. Also the relation of Berezin quantization and geometric formulation of quantum mechanics will be evident as follows. The geometric quantum mechanics is a formulation of quantum mechanics in projective Hilbert spaces. With our construction one sees that the Berezin quantization is an equivalent formulation and in addition gives a prescription as classical limit for geometric quantum mechanics [9].

2 Geometry of projective Hilbert space

Let $\mathcal{H}$ be a Hilbert space and $\mathbb{P}(\mathcal{H})$ is its projective space by the canonical projection $\pi : \mathcal{H}\backslash 0 \to \mathbb{P}(\mathcal{H})$. Any point in $\mathbb{P}(\mathcal{H})$ is shown with $[\psi]$ corresponds to the one dimensional subspace $\mathbb{C}\psi$ in $\mathcal{H}$. $\mathbb{P}(\mathcal{H})$ is a Kähler manifold, the symplectic form is given by [12]

$$\Omega^{\hbar}_{[\psi]}(T_\psi\pi(\phi_1), T_\psi\pi(\phi_2)) = -2\hbar(\langle\phi_1, \phi_2\rangle), \tag{1}$$

where $\phi \in (\mathbb{C}\psi)^{\perp}$ and $T_{\psi}\pi(\phi)$ is tangential space of $\mathbb{P}(\mathcal{H})$ in point $[\psi]$, which is isomorphic to $\mathcal{H}\backslash\mathbb{C}\psi$,

$$T_{\psi}\pi : \mathcal{H} \to T_{[\psi]}\mathbb{P}(\mathcal{H}) \simeq \mathcal{H}\backslash\mathbb{C}\psi. \tag{2}$$

and defined by

$$(T_{\psi}\pi)(\phi) = \frac{d}{dt}\pi(\psi + t\phi)\mid_{t=0}.$$

2.1 Vector fields on $\mathbb{P}(\mathcal{H})$

Let (M, ω) be a symplectic manifold. The vector field A is called *Hamiltonian* if there exists a smooth function f on M such that

$$i_A\omega = df, \tag{3}$$

where i_A is interior derivative of Ω with respect to A.

The quantum mechanical observables are self adjoint operators on Hilbert space and one can consider the expectation values of these observables as function on projective Hilbert space; in fact the expectation value of H_A is defined by

$$\langle H_A\rangle_{\psi} = \frac{\langle\psi, H_A\psi\rangle}{\langle\psi,\psi\rangle}. \tag{4}$$

By the following theorem the relation between the operators on Hilbert space and the associated Hamiltonian vector field will be evident.

Theorem 1. *Let A be a Hamiltonian vector field on $\mathbb{P}(\mathcal{H})$ and H_A the corresponding Hamiltonian operator on $\mathcal{H}$. Then the Schrödinger equation $H_A\psi(t) = i\hbar d\psi/dt$ is equivalent to the equation of motion that induced by A on $\mathbb{P}(\mathcal{H})$, such that*

$$A[\psi] = \frac{1}{i\hbar}\frac{H_A\psi}{\|\psi\|}, \tag{5}$$

where A is given in local coordinates Z on $\mathbb{P}(\mathcal{H})$ with respect to Fubini-study form as

$$A = -i\sum_{n,p}\Omega^{k,np}\left(\frac{\partial\langle H_A\rangle}{\partial\bar{Z}^k_p}\frac{\partial}{\partial Z^k_n} - \frac{\partial\langle H_A\rangle}{\partial Z^k_p}\frac{\partial}{\partial\bar{Z}^k_n}\right). \tag{6}$$

Proof. See [3, 15]. ■

As a consequence one can say that the Schrödinger equation is nothing but the classical Hamilton equations. Then it is natural to expect that there exists a Poisson structure on $\mathbb{P}(\mathcal{H})$. With the following proposition will be seen that the symplectic form on $\mathbb{P}(\mathcal{H})$ endows it with Poisson algebra. For a symplectic manifold with form Ω we have:

$$\{f, g\} = \Omega(X_f, X_g), \tag{7}$$

where X_f and X_g are Hamiltonian vector fields of f and g respectively.

Proposition 1. *Let $A, B : \mathbb{P}(\mathcal{H}) \to T\mathbb{P}(\mathcal{H})$ are two Hamiltonian vector fields corresponding to the functions $\langle H_A\rangle$ and $\langle H_B\rangle$ on $\mathbb{P}(\mathcal{H})$ respectively. Then*

$$\{\langle H_A\rangle, \langle H_B\rangle\} = \langle\frac{1}{i\hbar}[H_A, H_B]\rangle, \tag{8}$$

where the Poisson bracket is defined by (1) and the relation

$$\{\langle H_A\rangle, \langle H_B\rangle\} = \Omega_{FS}(A, B).$$

Proof. Direct calculation [12, 15]. ■

It must be pointed out that the Poisson structure is defined on quantum phase space $\mathbb{P}(\mathcal{H})$ rather than classical phase space M.

It is well known that the $\mathbb{P}(\mathcal{H})$ has a natural metric, called Fubini–Study metric g, by which the transition probability is defined [3, 9, 12]. Then the vector field A on $\mathbb{P}(\mathcal{H})$ is Hamiltonian if and only if $\mathcal{L}_A g = 0$, where $\mathcal{L}_A$ is Lie derivative along A and $A = X_{\langle H_A\rangle}$ is defined by (3). Therefore the Hamiltonian flow of the functions $\langle H_A\rangle$ preserves the geometric structures carried by $\mathbb{P}(\mathcal{H})$ and then the quantum mechanical observables generate the structural symmetries of $\mathbb{P}(\mathcal{H})$ [4].

2.2 The coherent states manifold

The generalized coherent states are elements of a G-orbit, which are generated by action of the Lie group G on a dominant weight vector ϕ_0 in the separated Hilbert space $\mathcal{H}$. This orbit $\widetilde{\mathcal{H}}$ is dense subspace of $\mathcal{H}$ [1]. If U_g is a unitary representation of $g \in G$ Then the projective space $\mathbb{P}(\widetilde{\mathcal{H}}) \equiv \mathcal{M}$ is also a dense subspace of $\mathbb{P}(\mathcal{H})$. $\mathcal{M}$ is Kählerian if G is a semi-simple group [11]. The manifold of coherent states is given by

$$\mathcal{M} = \{[U_g\psi_0] \mid g \in G\}, \tag{9}$$

where $[U_g]$ is the projective representation of G induced by U.

Let $\mathcal{K}$ denote the maximal stabilisator of G. Then there exists an isomorphism between $\mathcal{M}$ and $G/\mathcal{K}$.

With this construction there exist an embedding $\iota : \mathcal{M} \to \mathbb{P}(\mathcal{H})$ and the symplectic and other geometrical structures of projective Hilbert space are induced in $\mathcal{M}$:

$$\Omega = \iota^*\Omega_{FS} = \Omega_{FS}|_M. \tag{10}$$

2.3 The embedding of classical phase space in $\boldsymbol{\mathcal{M}}$

Let (M, ω) be a Kähler manifold as classical phase space. We define the weighted Bergman space as

$$\mathcal{H}_\hbar = \left\{ f| \int |f(z)|^2 e^{-\frac{1}{\hbar}\Psi(z,\bar{z})} d\nu(z,\bar{z}) = \|f\|^2_\hbar < \infty \right\}. \tag{11}$$

As a subspace of $L^2\left(M, e^{-\frac{1}{\hbar}\Psi}\right)$, $\mathcal{H}_\hbar$ is a Hilbert space. In fact $\mathcal{H}_\hbar$ is the space of analytic quadratic integrable functions on Kähler manifold M with measure

$$d\mu(z,\bar{z}) = e^{-\frac{1}{\hbar}\Psi(z,\bar{z})} d\nu(z,\bar{z}). \tag{12}$$

In this space the Berezin coherent states $\Phi^\hbar_{\bar{\zeta}}$ form a overcomplete set. According to definition of inner product in $\mathcal{H}_\hbar$ we have

$$\Phi^\hbar_{\bar{\zeta}}(z) = \langle \Phi^\hbar_{\bar{z}}, \Phi^\hbar_{\bar{\zeta}}\rangle_\hbar =: K_\hbar(\bar{\zeta}, z), \tag{13}$$

where $K_\hbar(\bar{\zeta}, z)$ is the Bergman kernel, which is defined uniquely for any manifold and has the reproducing property

$$f(\zeta) = \langle \Phi^\hbar_{\bar{\zeta}}, f\rangle_\hbar. \tag{14}$$

For a symmetric space the Berezin coherent states are the same as the generalized coherent states [14]. Therefore to any point of Kähler manifold M is associated a coherent state in $\mathcal{H}_\hbar$ as a kerned Hilbert space. Hence there exists a holomorphic embedding $\iota_\hbar : M \to \mathcal{M}_\hbar$, where $\mathcal{M}_\hbar$

is the projective space of $\mathcal{H}_\hbar$. This association is called the coherent states quantization [1, 13]. Two important properties of this embedding are that it is one-one and global differentiable. Then the pull-back of $\iota^*\Omega_{FS}$ of Fubini–Study form of $\mathcal{M}_\hbar$, induced from $\mathcal{M}$, is again a symplectic form. If the coherent states are generated from the representation of a Lie group G, then (M,ω) is a homogeneous symplectic manifold.

3 The Berezin quantization on the coherent states manifold

Berezin quantization [2, 6] on an arbitrary Kähler manifold is defined by the $*_\hbar$-algebra out of covariant symbols, which are the expectation values of quantum observables (self adjoint bounded operators) in terms of coherent states $\Phi^\hbar_{\bar\zeta}$

$$\widetilde{AB} = \widetilde{A} *_\hbar \widetilde{B}(z) = \int_M \widetilde{A}(\bar\zeta, z)\widetilde{B}(\bar z, \zeta)\frac{|K_\hbar(\bar\zeta, z)|^2}{K_\hbar(\bar z, z)} e^{-\frac{1}{\hbar}\Psi(\bar\zeta, z)}\, d\nu(\bar\zeta, \zeta), \tag{15}$$

where $\widetilde{A}$ is Berezin covariant symbol defined by

$$\widetilde{A}(\bar\zeta, z) = \frac{\langle K_\hbar(\bar z, \cdot), AK_\hbar(\bar\zeta, \cdot)\rangle_\hbar}{K_\hbar(\bar\zeta, z)} = \frac{\langle \Phi^\hbar_{\bar z}, A\Phi^\hbar_{\bar\zeta}\rangle_\hbar}{\langle \Phi^\hbar_{\bar z}, \Phi^\hbar_{\bar\zeta}\rangle_\hbar}. \tag{16}$$

$K_\hbar(\bar\zeta, z)$ is the Bergman kernel and $\Psi(\bar\zeta, z)$ is the Kähler function. Berezin has considered the covariant symbols as bounded functions on classical phase space (M, ω), to be a Kähler manifold, which form the $*_\hbar$-algebra $\mathcal{A}_\hbar$. The classical limit $\hbar \to 0$ results from

$$\left(\widetilde{A} *_\hbar \widetilde{B}\right)(z) = a(z)b(z) + \mathcal{O}(\hbar), \tag{17}$$

$$\frac{1}{\hbar}\left(\widetilde{A} *_\hbar \widetilde{B} - \widetilde{B} *_\hbar \widetilde{A}\right)(z) = -i\{a, b\}(z) + \mathcal{O}(\hbar). \tag{18}$$

where a, b are the $\hbar \to 0$ limits of $\widetilde{A}$, $\widetilde{B}$ respectively.

By construction in Section 2 we can also consider the covariant symbols as functions on projective Hilbert space $\mathcal{M}_\hbar$. Therefore these functions form a Poisson algebra via the induced Fubini-Study form on $\mathcal{M}_\hbar$. What we must show is that both these algebras, i.e. Poisson algebra and $*_\hbar$ algebra, are the same.

From Proposition 1 one sees clearly that for two covariant symbols $\widetilde{A}$, $\widetilde{B}$, as expectation values of the operators A, B, in terms of coherent states, we have

$$\frac{1}{i\hbar}\widetilde{[A, B]} = \{\widetilde{A}, \widetilde{B}\}_{\iota^*\Omega_{FS}}. \tag{19}$$

On other hand from equation (15) it can be easily shown

$$\frac{1}{i\hbar}\widetilde{[A, B]} = \frac{1}{i\hbar}\left(\widetilde{A} *_\hbar \widetilde{B} - \widetilde{B} *_\hbar \widetilde{A}\right). \tag{20}$$

The lhs of equations (19) and (20) are identical, so we have

$$\frac{1}{i\hbar}\left(\widetilde{A} *_\hbar \widetilde{B} - \widetilde{B} *_\hbar \widetilde{A}\right) = \left\{\widetilde{A}, \widetilde{B}\right\}_{\iota^*\Omega_{FS}}. \tag{21}$$

Hence: *The $*_\hbar$-algebra correspond to Poisson algebra on $\mathcal{M}_\hbar$.*

We emphasise again that this Poisson structure is defined on quantum phase space and preserves all of the quantum mechanical properties of the system.

The classical limit in Berezin quantization is defined now by

$$\lim_{\hbar\to 0}\left\{\widetilde{A},\widetilde{B}\right\}_{\iota^*\Omega_{FS}}(Z)=\left\{\varphi(\widetilde{A}),\varphi(\widetilde{B})\right\}(z),\qquad z\in M, Z\in\mathcal{M}_\hbar, \tag{22}$$

where φ is defined as the quantum to classical observable map:

$$\lim_{\hbar\to 0}\widetilde{A}=\varphi(\widetilde{A}). \tag{23}$$

Dynamics is also defined in Berezin quantization as follows:

The Heisenberg equation of motion for the observable A is $\frac{dA}{dt}=\frac{1}{i\hbar}[A,H]$. This equation on $\mathcal{M}_\hbar$ has the following form

$$\frac{d\widetilde{A}(Z)}{dt}=\left\{\widetilde{A}(Z),\widetilde{H}(Z)\right\}_{\iota^*\Omega_{FS}},\qquad Z\in\mathcal{M}. \tag{24}$$

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On Multi-Parameter Families of Hermitian Exactly Solvable Matrix Schrödinger Models

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Five multi-parameter families of Hermitian exactly solvable matrix Schrödinger operators in one variable was constructed.

1 Introduction

One of the principal aims of the present paper is developing a systematic algebraic procedure for constructing exactly solvable (ES) Hermitian matrix Schrödinger operators

$$\hat{H}[x] = \partial_x^2 + V(x). \tag{1}$$

Here $V(x)$ is an 2×2 matrix whose entries are smooth complex-valued functions of x. Hereafter we denote d/dx as ∂_x.

The well-known procedure of constructing a ES matrix (scalar) model is based on the concept of a Lie-algebraic Hamiltonian [1, 2] (the Turbiner–Shifman approach). We call a second-order operator in one variable Lie-algebraic if the following requirements are met:

- the Hamiltonian is a quadratic form with constant coefficients of first-order operators $Q_1, Q_2, \ldots, Q_n$ forming a Lie algebra g;
- the Lie algebra g has a finite-dimensional invariant subspace $\mathcal{I}$ of the whole representation space.

Now if a given Hamiltonian $H[x]$ is Lie-algebraic, then after being restricted to the space $\mathcal{I}$ it becomes a matrix operator $\mathcal{H}$ whose eigenvalues and eigenvectors are computed in a purely algebraic way. This means that the Hamiltonian $H[x]$ is exactly solvable.

In the paper [3] we have extended the Turbiner–Shifman approach to the construction of quasi-exactly solvable (QES) models on line for the case of matrix Hamiltonians. In this paper we suggested the method for construction of exactly solvable matrix models, which based on the idea explained in [3]. Let us remind, the method consists in supplementing a set of operators $Q_1, Q_2, \ldots, Q_n$, forming a representation of some algebra, so that the obtained set of operators left an appropriate subspace $\mathcal{I}$ invariant. However, there is a difference between the approaches suggested in this paper and in [3]. Namely, the obtained set of operators does not form a Lie algebra, in contrast to a set found in [3].

So, let us realize this method considering the set of the operators

$$Q_1 = A, \qquad Q_2 = Be^{-x}, \qquad Q_3 = c^x(\partial_x + C), \qquad Q_4 = \partial_x, \tag{2}$$

which form the representation of the algebra $L^2_{4,8}$, found in [4]. Here $A = B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$.

The operators (2) belong to the class $\mathcal{L}$ of matrix differential operators of the form

$$\mathcal{L} = \{Q : \ Q = a(x)\partial_x + A(x)\}, \tag{3}$$

where $a(x)$ is a smooth real-valued function and $A(x)$ is an 2×2 matrix whose entries are smooth complex-valued functions of x.

The corresponding finite-dimensional invariant space has the form

$$\mathcal{G} = \langle e^{-cx}\vec{e}_1, e^{-(c+1)x}\vec{e}_1, \ldots, e^{-(c+k+1)x}\vec{e}_1\rangle \oplus \langle e^{-cx}\vec{e}_2, e^{-(c+1)x}\vec{e}_2, \ldots, e^{-(c+k)x}\vec{e}_2\rangle, \tag{4}$$

where k is an arbitrary natural number.

It is easy to verify that all operators from the class (3) and acting in the space (4), are

$$\begin{aligned} &R_1 = S_0, \qquad R_2 = S_+e^x\partial_x, \qquad R_3 = S_+\partial_x, \qquad R_4 = S_0e^x\partial_x,\\ &R_5 = S_0\partial_x, \qquad R_6 = S_+e^{-x}\partial_x, \qquad R_7 = S_-e^x\partial_x, \end{aligned} \tag{5}$$

where $S_0 = \sigma_3/2$, $S_\pm = (i\sigma_2 \pm \sigma_1)/2$, σ_k are the 2×2 Pauli matrices.

Then we construct a Hamiltonian $H[x]$ of the form

$$H[x] = \xi(x)\partial_x^2 + B(x)\partial_x + C(x), \tag{6}$$

which can be obtained by using of all bilinear combinations of operators belonging to the linear span of the operators (2), (5).

Here we omitted a very cumbersome calculation and some technical methods to reduce an operator (6) to a standard Schrödinger operator

$$\hat{H}[y] = \partial_y^2 + V(y). \tag{7}$$

We give below the final results, namely, the restrictions on the choice of parameters and the explicit forms of the QES Hermitian Schrödinger operators (7). In the formulae below we denote the conjunction of two statements A and B as $[A]\bigvee[B]$.

Let complex-valued parameters $\tilde{\boldsymbol{\beta}} = (\beta_1, i\beta_2, \beta_3)$, $\tilde{\boldsymbol{\delta}} = (\delta_1, i\delta_2, \delta_3)$ and others satisfy the following conditions

$$\begin{aligned} &\left[\tilde{\boldsymbol{\beta}}^2 < 0,\ \varepsilon \neq 0,\ \beta_1 \neq \beta_2\right] \wedge \left[\{\alpha_0, \alpha_1, \alpha_2, \lambda, \gamma_0, \beta_0, \varepsilon(\beta_1-\beta_2),\ \delta_1(\beta_1-\beta_2)+\beta_3\delta_3, \delta_3\} \subset \mathbb{R}\right]\\ &\quad \wedge \left[\mu = \alpha_0 = \alpha_2\lambda - \beta_0 + 2\alpha_2\frac{\tilde{\boldsymbol{\beta}}\tilde{\boldsymbol{\delta}}}{\tilde{\boldsymbol{\beta}}^2} = -\gamma_0\lambda + 2\alpha_1\frac{\tilde{\boldsymbol{\beta}}\tilde{\boldsymbol{\varepsilon}}}{\tilde{\boldsymbol{\beta}}^2}\right.\\ &\quad \left. = \alpha_1\lambda - \beta_0\lambda - \gamma_0 + 2\alpha_1\frac{\tilde{\boldsymbol{\beta}}\tilde{\boldsymbol{\delta}}}{\tilde{\boldsymbol{\beta}}^2} + 2\alpha_2\frac{\tilde{\boldsymbol{\beta}}\tilde{\boldsymbol{\varepsilon}}}{\tilde{\boldsymbol{\beta}}^2} = 0\right]. \end{aligned}$$

Then, the following Schrödinger operator be hermitian:

$$\begin{aligned} \hat{H}[y] = \partial_y^2 &+ \frac{1}{16\left(\alpha_2e^{2x}+\alpha_1e^x\right)}\left[\left(\alpha_1^2 + 8\alpha_2\gamma_0 - 4\beta_0^2 - 4\tilde{\boldsymbol{\beta}}^2\right)e^{2x}\right.\\ &\left. + \left(8\alpha_1\gamma_0 - 8\gamma_0\beta_0 - 8\lambda\tilde{\boldsymbol{\beta}}^2\right)e^x - 4\left(\lambda^2\tilde{\boldsymbol{\beta}}^2 + \gamma_0^2\right)\right]\\ &+ \left(P\cos\left(\theta(x)\sqrt{-\tilde{\boldsymbol{\beta}}^2} + \Omega\right) + \frac{\varepsilon(\beta_1-\beta_2)}{\sqrt{-\tilde{\boldsymbol{\beta}}^2}}e^{-x}\cos\left(\theta(x)\sqrt{-\tilde{\boldsymbol{\beta}}^2}\right)\right)\sigma_1\\ &+ \left.\left(P\sin\left(\theta(x)\sqrt{-\tilde{\boldsymbol{\beta}}^2} + \Omega\right) + \frac{\varepsilon(\beta_1-\beta_2)}{\sqrt{-\tilde{\boldsymbol{\beta}}^2}}e^{-x}\sin\left(\theta(x)\sqrt{-\tilde{\boldsymbol{\beta}}^2}\right)\right)\sigma_3\right|_{x=z^{-1}(y)}, \end{aligned} \tag{8}$$

here

$$P = \sqrt{\delta_3^2 - \frac{(\tilde{\boldsymbol{\beta}}\tilde{\boldsymbol{\delta}})^2}{\tilde{\boldsymbol{\beta}}^2}}, \quad \cos\Omega = \frac{\tilde{\boldsymbol{\beta}}\tilde{\boldsymbol{\delta}}}{P\sqrt{-\tilde{\boldsymbol{\beta}}^2}}, \quad \sin\Omega = \frac{\delta_3}{P}, \quad \theta(x) = -\int\frac{e^x+\lambda}{\alpha_0+\alpha_1e^x+\alpha_2e^{2x}}dx.$$

We denote the function $z^{-1}(y)$ as the inverse of the function

$$y = z(x) \equiv \int \frac{dx}{\sqrt{\alpha_2 x^2 + \alpha_1 x + \alpha_0}}. \tag{9}$$

Furthermore, the basis elements of the corresponding transformed invariant space take the form

$$\mathcal{G} = \langle \Lambda^{-1} e^{-cz^{-1}(y)}\vec{e}_1, \Lambda^{-1} e^{-(c+1)z^{-1}(y)}\vec{e}_1, \ldots, \Lambda^{-1} e^{-(c+k+1)z^{-1}(y)}\vec{e}_1 \rangle$$
$$\oplus \langle \Lambda^{-1} e^{-cz^{-1}(y)}\vec{e}_2, \Lambda^{-1} e^{-(c+1)z^{-1}(y)}\vec{e}_2, \ldots, \Lambda^{-1} e^{-(c+k)z^{-1}(y)}\vec{e}_2 \rangle,$$

where the constant matrix

$$\Lambda = \Lambda_1 \cdot \Lambda_2 = \exp\left(\frac{\beta_3}{2\tilde{\boldsymbol{\beta}}\tilde{\boldsymbol{\epsilon}}}\tilde{\boldsymbol{\epsilon}}\boldsymbol{\sigma}\right) \cdot \exp(\nu\sigma_3), \qquad e^{2\nu} = \frac{\sqrt{-\tilde{\boldsymbol{\beta}}^2}}{\beta_1 - \beta_2}.$$

We give the particular example of a Hermitian model which has the important property. Namely, a corresponding invariant space is a Hilbert one. That is, one can define a scalar product

$$\langle f_1(y), f_2(y) \rangle = \int \vec{f_1}(y)^\dagger f_2(y) dy,$$

where $\vec{f_1}(y)^\dagger$ is a Hermitian conjugation of the vector $f_1(y)$. Let us put in the formula (8) $\alpha_2 = \beta_2 = \delta_3 = \gamma_0 = 1$, $\epsilon = \frac{1}{2}$, and the rest coefficients are equal zero. Then we have the following Hamiltonian

$$\hat{H}(y) = \partial_y^2 - \left(\sin y + \frac{1}{2} y \cos y\right)\sigma_1 + \left(\cos y - \frac{1}{2} y \sin y\right)\sigma_3 + \frac{3}{4}.$$

The corresponding invariant space for this operator $\mathcal{G}$ has $(2k+3)$-dimension and is generated by the vectors

$$\vec{f_j} = ie^{-y^2/4} y^j \exp\left(\frac{-i\sigma_2}{2} y\right)\vec{e}_1, \qquad \vec{g}_s = -ie^{-y^2/4} y^s \exp\left(\frac{-i\sigma_2}{2} y\right)\vec{e}_2,$$

where $j = 0, \ldots, k+1$, $s = 0, \ldots, k$, $\vec{e}_1 = (1, 0)^T$, $\vec{e}_2 = (0, 1)^T$, σ_i $(i = 1, 2, 3)$ are 2×2 Pauli matrices.

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Nonlinear Schrödinger Equations for Identical Particles and the Separation Property

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We investigate the separation property for hierarchies of Schrödinger operators for identical particles. We show that such hierarchies of translation invariant second order differential operators are necessarily linear. A weakened form of the separation property, related to a strong form of cluster decomposition, allows for homogeneous hierarchies of nonlinear differential operators. Some connection with field theoretic formalisms in Fock space are pointed out.

1 Introduction

In [1] we studied hierarchies of N-particle Schrödinger equations that satisfy the separation property. By this we mean that product functions evolve as product functions. The separation property was considered as a nonlinear version of the notion of non-interacting systems, as then uncorrelated states remain uncorrelated under time evolution. The motivation for studying such hierarchies came from concrete examples of nonlinear Schrödinger equations arising in problems of representations of the diffeomorphism group.

The hierarchies of Schrödinger operators that one encounters in such evolution equation satisfies a property that we called *tensor derivation* as the characteristic property is formally a derivation with respect to the tensor product of wave functions.

$$\begin{aligned} F_n(\psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_p) &= F_{n_1}(\psi_1) \otimes \psi_2 \otimes \cdots \otimes \psi_p \\ &+ \psi_1 \otimes F_{n_2}(\psi_2) \otimes \cdots \otimes \psi_p + \cdots + \psi_1 \otimes \psi_2 \otimes \cdots \otimes F_{n_p}(\psi_p), \end{aligned} \tag{1}$$

where the F_m are m-particle operators, the ψ_k are n_k-particle wave function and $n = n_1+\cdots+n_p$. Tensor derivations were fully classified in [1]. Canonical decompositions and constructions were also presented.

The analysis in [1] is incomplete in several aspects. One most apparent is that there one only considered N-particle systems in which the particles were all of different species. Thus there was no need to consider symmetric or antisymmetric wave functions. Since the world is made of bosons and fermions, one should reconsider the whole question for systems of identical particles. The tensor derivation property (1) must then be reformulated not with respect to the simple tensor product

$$\phi \otimes \psi(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) = \phi(x_1, \ldots, x_n)\psi(x_{n+1}, \ldots, x_{n+m})$$

of two wave functions, but with respect to the symmetric or anti-symmetric tensor product

$$\begin{aligned} &\phi \hat{\otimes} \psi(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \\ &\qquad = \frac{1}{n!m!} \sum_{\pi} (-1)^{fs(\pi)} \phi(x_{\pi(1)}, \ldots, x_{\pi(n)}) \psi(x_{\pi(n+1)}, \ldots, x_{\pi(n+m)}), \end{aligned}$$

where π is a permutation of $\{1, 2, \ldots, n+m\}$, $s(\pi)$ its parity, and f is the *Fermi number* equal to zero for bosons and one for fermions. The coefficient in front of the sum is conventional.

In [2] we explored the possibility of formulating a nonlinear relativistic quantum mechanics based on a nonlinear version of the consistent histories approach to quantum mechanics. A toy model led to a set of equations among which there were instances of the separation property for a symmetric tensor product. This showed once more that such a separation property is fundamental for understanding any nonlinear extension of ordinary quantum mechanics.

Given these motivations, this paper is dedicated to the beginning of a systematic exploration of the symmetric separation property.

2 Symmetric tensor derivations

A symmetric tensor derivation would be a hierarchy of operators that satisfies (1) with $\hat{\otimes}$ instead of $\otimes$. That is,

$$\begin{aligned} F_n(\psi_1\hat{\otimes}\psi_2\hat{\otimes}\cdots\hat{\otimes}\psi_p) &= F_{n_1}(\psi_1)\hat{\otimes}\psi_2\hat{\otimes}\cdots\hat{\otimes}\psi_p \\ &+ \psi_1\hat{\otimes}F_{n_2}(\psi_2)\hat{\otimes}\cdots\hat{\otimes}\psi_p + \cdots + \psi_1\hat{\otimes}\psi_2\hat{\otimes}\cdots\hat{\otimes}F_{n_p}(\psi_p). \end{aligned} \tag{2}$$

One does not have a classification of these as one has for ordinary tensor derivations as given in [1]. It seems that the conditions to be a tensor derivation in the symmetric case is rather stringent, and as we shall now see, in the case of differential operators, implies linearity under some general conditions. We only treat the case of second order operators as these are the most common kind in physical applications.

Let us consider a possibly nonlinear differential operators of second order not depending explicitly on the position coordinates (dependence on time can be construed as simply dependence on a parameter), in the case $N = 2$. Such an operator has the form

$$H\left(\phi, \frac{\partial\phi}{\partial x_i}, \frac{\partial\phi}{\partial y_j}, \frac{\partial^2\phi}{\partial x_i\partial x_j}, \frac{\partial^2\phi}{\partial x_i\partial y_j}, \frac{\partial^2\phi}{\partial y_i\partial y_j}\right).$$

Introducing variable names for the arguments of H, we write $H(a, b_i, c_j, d_{ij}, e_{ij}, f_{ij})$. When ϕ is constrained to be a symmetrized product (here f is the Fermi number)

$$\phi(x, y) = \alpha(x)\beta(y) + f\beta(x)\alpha(y)$$

then the arguments of H are constrained to take on values of the form .

$$a = \alpha_0\beta_0 + f\tilde{\beta}_0\tilde{\alpha}_0, \qquad b_i = \alpha_i\beta_0 + f\tilde{\beta}_i\tilde{\alpha}_0, \qquad c_i = \alpha_0\beta_i + f\tilde{\beta}_0\tilde{\alpha}_i, \tag{3}$$

$$d_{ij} = \alpha_{ij}\beta_0 + f\tilde{\beta}_{ij}\tilde{\alpha}_0, \qquad e_{ij} = \alpha_i\beta_j + f\tilde{\beta}_i\tilde{\alpha}_j, \qquad f_{ij} = \alpha_0\beta_{ij} + f\tilde{\beta}_0\tilde{\alpha}_{ij} \tag{4}$$

where all the quantities on the right-hand sides: α_0, β_0, α_i, β_i, α_{ij}, β_{ij}, $\tilde{\alpha}_0$, $\tilde{\beta}_0$, $\tilde{\alpha}_i$, $\tilde{\beta}_i$, $\tilde{\alpha}_{ij}$, $\tilde{\beta}_{ij}$, which we shall call the $\alpha\beta$-quantities, can be given, by Borel's lemma, arbitrary complex values by an appropriate choice of the points x and y and functions α and β. Denote the right-hand sides of the above equations by $\hat{a}$, $\hat{b}_i$, $\hat{c}_i$, $\hat{d}_{ij}$, $\hat{e}_{ij}$, and $\hat{f}_{ij}$, respectfully.

The separability condition for the symmetrized tensor product now reads:

$$\begin{aligned} F_2(\hat{a}, \hat{b}_i, \hat{c}_i, \hat{d}_{ij}, \hat{e}_{ij}, \hat{f}_{ij}) &= F_1(\alpha_0, \alpha_i, \alpha_{ij})\beta_0 \\ &+ fF_1(\tilde{\alpha}_0, \tilde{\alpha}_i, \tilde{\alpha}_{ij})\tilde{\beta}_0 + F_1(\beta_0, \beta_i, \beta_{ij})\alpha_0 + fF_1(\tilde{\beta}_0, \tilde{\beta}_i, \tilde{\beta}_{ij})\tilde{\alpha}_0. \end{aligned} \tag{5}$$

Based on the examples of separating hierarchies for the non symmetrized tensor product, we must admit that the differential operators F_1 and F_2 may be singular, so that in analyzing (5) we should avoid points in which the first argument vanishes. Aside from this we put no further restrictions the values of the $\alpha\beta$-quantities. The freedom of choice in these quantities is now such that we can give arbitrary values to a, with $a \neq 0$, b_i, c_i, d_{ij}, and f_{ij}. This is achieved by setting

$$\alpha_0 = \frac{a - f\tilde{\alpha}_0\tilde{\beta}_0}{\beta_0}, \qquad \alpha_i = \frac{b_i - f\tilde{\beta}_i\tilde{\alpha}_0}{\beta_0}, \qquad \tilde{\alpha}_i = f\frac{\beta_0 c_i - (a - f\tilde{\alpha}_0\tilde{\beta}_0)\beta_i}{\beta_0\tilde{\beta}_0},$$
$$\alpha_{ij} = \frac{d_{ij} - f\tilde{\beta}_{ij}\tilde{\alpha}_0}{\beta_0}, \qquad \tilde{\alpha}_{ij} = f\frac{\beta_0 f_{ij} - (a - f\tilde{\alpha}_0\tilde{\beta}_0)\beta_{ij}}{\beta_0\tilde{\beta}_0}$$

with these substitutions one finds

$$\hat{e}_{ij} = \frac{\tilde{\beta}_0 b_i\beta_j + \beta_0\tilde{\beta}_i c_j - a\tilde{\beta}_i\beta_j}{\beta_0\tilde{\beta}_0}.$$

Equation (8) now becomes

$$\begin{aligned} F_2(a, b_i, c_i, d_{ij}, \hat{e}_{ij}, f_{ij}) &= F_1\left(\frac{a - f\tilde{\alpha}_0\tilde{\beta}_0}{\beta_0}, \frac{b_i - f\tilde{\beta}_i\tilde{\alpha}_0}{\beta_0}, \frac{d_{ij} - f\tilde{\beta}_{ij}\tilde{\alpha}_0}{\beta_0}\right)\beta_0 \\ &+ fF_1\left(\tilde{\alpha}_0,\, f\frac{\beta_0 c_i - (a - f\tilde{\alpha}_0\tilde{\beta}_0)\beta_i}{\beta_0\tilde{\beta}_0},\, f\frac{\beta_0 f_{ij} - (a - f\tilde{\alpha}_0\tilde{\beta}_0)\beta_{ij}}{\beta_0\tilde{\beta}_0}\right)\tilde{\beta}_0 \\ &+ \frac{a - f\tilde{\alpha}_0\tilde{\beta}_0}{\beta_0}F_1\left(\beta_0,\, \beta_i,\, \beta_{ij}) + f\tilde{\alpha}_0 F_1(\tilde{\beta}_0,\, \tilde{\beta}_i,\, \tilde{\beta}_{ij}\right). \end{aligned} \tag{6}$$

The left-hand side of (6) is independent of $\tilde{\beta}_{ij}$ and the right-hand side has two terms that depend on it. Differentiating both sides with respect to β_{ij} one arrives at the following identity:

$$D_3^{ij}F_1\left(\frac{a - f\tilde{\alpha}_0\tilde{\beta}_0}{\beta_0}, \frac{b_i - f\tilde{\beta}_i\tilde{\alpha}_0}{\beta_0}, \frac{d_{ij} - f\tilde{\beta}_{ij}\tilde{\alpha}_0}{\beta_0}\right) = D_3^{ij}F_1(\tilde{\beta}_0,\, \tilde{\beta}_i,\, \tilde{\beta}_{ij}) \tag{7}$$

which must hold for all values of the variables that appear. Here D_3^{ij} stands for the partial derivative with respect to the ij component of the third argument of F_1. Choosing $\tilde{\alpha}_0 = 1$, $\beta_0 = f\tilde{\beta}_0$, $a = 2f\tilde{\beta}_0$, $b_i = f\tilde{\beta}_i$, and $d_{ij} = f\tilde{\beta}_{ij}$ one gets

$$D_3^{ij}F_1(1, 0, 0) = D_3^{ij}F_1(\tilde{\beta}_0,\, \tilde{\beta}_i,\, \tilde{\beta}_{ij})$$

which means that

$$F_1(u, v_i, w_{ij}) = G(u, v_i) + \sum_{ij} k^{ij}w_{ij},$$

where k^{ij} are constants. After substituting this into (6) and simplifying, that equation now becomes

$$\begin{aligned} &F_2(a, b_i, c_i, d_{ij}, \hat{e}_{ij}, f_{ij}) \\ &\quad = G\left(\frac{a - f\tilde{\alpha}_0\tilde{\beta}_0}{\beta_0}, \frac{b_i - f\tilde{\beta}_i\tilde{\alpha}_0}{\beta_0}\right)\beta_0 + fG\left(\tilde{\alpha}_0,\, f\frac{\beta_0 c_i - (a - f\tilde{\alpha}_0\tilde{\beta}_0)\beta_i}{\beta_0\tilde{\beta}_0}\right)\tilde{\beta}_0 \\ &\quad + \frac{a - f\tilde{\alpha}_0\tilde{\beta}_0}{\beta_0}G(\beta_0,\, \beta_i) + f\tilde{\alpha}_0 G(\tilde{\beta}_0,\, \tilde{\beta}_i) + \sum_{ij} k^{ij}(d_{ij} + f_{ij}). \end{aligned} \tag{8}$$

The linear differential operator represented by the term $\sum_{ij} k^{ij}(d_{ij}+f_{ij})$ is of the form $I\otimes L+L\otimes I$ and which is part of a $\hat{\otimes}$-separating hierarchy (in which the one-particle operator is L), so subtracting it from F_2 results in a new separating hierarchy with $k^{ij}=0$. We now note that the left-hand side of (8) is independent of $\tilde{\alpha}_0$ so differentiation both sides with respect to $\tilde{\alpha}_0$ results in the following identity:

$$\begin{aligned}
&-D_1G\left(\frac{a-f\tilde{\alpha}_0\tilde{\beta}_0}{\beta_0},\,\frac{b_i-f\tilde{\beta}_i\tilde{\alpha}_0}{\beta_0}\right)\tilde{\beta}_0\\
&\qquad-\sum_i D_2^iG\left(\frac{a-f\tilde{\alpha}_0\tilde{\beta}_0}{\beta_0},\,\frac{b_i-f\tilde{\beta}_i\tilde{\alpha}_0}{\beta_0}\right)\tilde{\beta}_i+D_1G\left(\tilde{\alpha}_0,\,f\frac{\beta_0c_i-(a-f\tilde{\alpha}_0\tilde{\beta}_0)\beta_i}{\beta_0\tilde{\beta}_0}\right)\tilde{\beta}_0\\
&\qquad+\frac{\tilde{\beta}_0}{\beta_0}\sum_i D_2^iG\left(\tilde{\alpha}_0,\,f\frac{\beta_0c_i-(a-f\tilde{\alpha}_0\tilde{\beta}_0)\beta_i}{\beta_0\tilde{\beta}_0}\right)\beta_i-\frac{\tilde{\beta}_0}{\beta_0}G(\beta_0,\beta_i)+G(\tilde{\beta}_0,\tilde{\beta}_i)=0.
\end{aligned}\tag{9}$$

Choosing now as before $\tilde{\alpha}_0=1$, $\beta_0=f\tilde{\beta}_0$, $\beta_i=0$, $a=2f\tilde{\beta}_0$, $b_i=f\tilde{\beta}_i$, and $c_i=0$ one finds

$$G(\tilde{\beta}_0,\tilde{\beta}_i)+fG(f\tilde{\beta}_0,0)+\sum_i D_2^iG(1,0)\tilde{\beta}_i=0.\tag{10}$$

This means that

$$G(u,v_i)=A(u)+\sum_i k^iv_i,\tag{11}$$

where k^i are constants. Substituting this into (8) with $k^{ij}=0$ one gets

$$\begin{aligned}
F_2(a,b_i,c_i,d_{ij},\hat{e}_{ij},f_{ij})&=A\left(\frac{a-f\tilde{\alpha}_0\tilde{\beta}_0}{\beta_0}\right)\beta_0+fA(\tilde{\alpha}_0)\tilde{\beta}_0\\
&\quad+\frac{a-f\tilde{\alpha}_0\tilde{\beta}_0}{\beta_0}A(\beta_0)+f\tilde{\alpha}_0A(\tilde{\beta}_0)+\sum_i k^i(b_i+c_i).
\end{aligned}\tag{12}$$

As before, the differential operator represented by the last term is part of a $\hat{\otimes}$-separating hierarchy, so subtracting it from F_2 results in a new separating hierarchy with $k^i=0$.

Also as before the right-hand side of (12) has to be independent of $\tilde{\alpha}_0$. Differentiating again both sides with respect to $\tilde{\alpha}_0$ one arrives at

$$-A'\left(\frac{a-f\tilde{\alpha}_0\tilde{\beta}_0}{\beta_0}\right)\tilde{\beta}_0+A'(\tilde{\alpha}_0)\tilde{\beta}_0-f\frac{\tilde{\beta}_0}{\beta_0}A(\beta_0)+fA(\tilde{\beta}_0)=0.\tag{13}$$

As the first term is the only one that depends on a, this equation can only hold if $A'(u)$ is a constant, that is $A(u)=ku+\ell$ for constants k and ℓ. Substituting this into (12) now results in

$$F_2(a,b_i,c_i,d_{ij},\hat{e}_{ij},f_{ij})=2ka+\ell\left(\beta_0+f\tilde{\beta}_0+f\tilde{\alpha}_0+\frac{a-f\tilde{\alpha}_0\tilde{\beta}_0}{\beta_0}\right)$$

which seeing that the right-hand side must be independent of $\tilde{\alpha}_0$ means that $\ell=0$, and we conclude.

Lemma 1. *A $\hat{\otimes}$-derivation of translation invariant* second order differential operators *necessarily has F_1 a* linear *operator.*

Following the procedure in [1], we define $e_{00} = a$, $e_{0j} = c_j$, $e_{i0} = b_i$, and let the upper case indices I, J, K, L range over $0, 1, \ldots, d$, then the parameterization of our variety is given by $e_{IJ} = \alpha_I \beta_J + \tilde{\alpha}_I \tilde{\beta}_J$. This is equivalent to saying that e_{IJ} is at most a rank two matrix. By standard results about determinantal ideals, the ideal of polynomials over the complex numbers vanishing on the variety of such matrices is generated by the order-three minors

$$M_{IJKABC} = \begin{vmatrix} e_{IA} & e_{IB} & e_{IC} \\ e_{JA} & e_{JB} & e_{JC} \\ e_{KA} & e_{KB} & e_{KC} \end{vmatrix}.$$

A simple rotation-invariant example would be given by jp and kq contraction of

$$\begin{vmatrix} a & c_j & c_k \\ b_p & e_{pj} & e_{pk} \\ b_q & e_{qj} & e_{qk} \end{vmatrix}$$

that is,

$$\sum_{pq} (a(e_{pp}e_{qq} - e_{pq}e_{qp}) - 2c_p b_p e_{qq} + 2c_p b_q e_{pq}).$$

For the curious, written out explicitly as a differential operator for $\phi(x, y)$, using the summation convention, this is:

$$\phi \left(\frac{\partial^2 \phi}{\partial x^p \partial y^p} \right)^2 - \phi \frac{\partial^2 \phi}{\partial x^p \partial y^q} \frac{\partial^2 \phi}{\partial x^q \partial y^p} - 2 \frac{\partial \phi}{\partial x^p} \frac{\partial \phi}{\partial y^p} \frac{\partial^2 \phi}{\partial x^q \partial y^q} + 2 \frac{\partial \phi}{\partial x^p} \frac{\partial^2 \phi}{\partial x^p \partial y^q} \frac{\partial \phi}{\partial y^q}.$$

A somewhat more concise expression results if we use the mixed Hessian

$$H_{pq} = \frac{\partial^2 \phi}{\partial x^q \partial y^p}$$

then our operator becomes

$$\phi \mathrm{Tr}(H)^2 - \phi \mathrm{Tr}\left(H^2\right) - 2\nabla_x \phi \cdot \nabla_y \phi \, \mathrm{Tr}(H) + 2\nabla_x \phi \cdot H \cdot \nabla_y \phi.$$

This is not a homogeneous operator, but dividing it by ϕ^2 turns it into one.

If we were simply interested in only the one- and two-particle equations then a separating hierarchy would consist of a linear one-particle operator, and the two particle operator would be given by the sum of the canonically lifted one-particle operator [1] and an operator that vanishes identically on symmetrized tensor products of one-particle functions. If we want a full multiparticle hierarchy with N-particle operators for all N, the story is different. An N-particle wave-function for particles in $\mathbb{R}^d$ can be viewed as a one-particle wave-function for particles (let us call these *conglomerate* particles) in $\mathbb{R}^{Nd}$. We can now consider the consequences of the separating property for the hierarchy consisting of a $2N$ particle operator on a symmetrized tensor product of two N-particle wave functions reinterpreted as one consisting of an operator for two conglomerate particles and an operator for one conglomerate particle. A wave-function of two conglomerate particles does not have the same permutation symmetry as the wave-function of $2N$ particles, but the difference is such as to impose even stronger conditions due to the separation property. Let $\phi(x_1, \ldots, x_N)$ and $\psi(y_1, \ldots, y_N)$ be two properly symmetric N-particle wave-functions. One has

$$\phi \hat{\otimes} \psi(x_1, \ldots, x_{2N}) = C \sum_I (-1)^{fp(I)} \phi(x_{i_1}, \ldots, x_{i_N}) \psi(x_{j_1}, \ldots, x_{j_N}), \tag{14}$$

where C is a combinatorial factor, $I = (i_1, \ldots, i_N)$ are N numbers from $\{1, \ldots, 2N\}$, in ascending order, $(j_1, \ldots, j_N)$ the complementary numbers, also in ascending order, and $p(I)$ is the parity (0 or 1) of the permutation $(1, \ldots, 2N) \mapsto (i_1, \ldots, i_N, j_1, \ldots, j_N)$. For (14) the possible values that one can attribute to the wave-function and its derivatives at a point is now more complicated than that given by expressions (3), (4), but by an appropriate choice of coordinates and an appeal to Borel's lemma, we can again use, as a particular case, expressions (3), (4) for two conglomerate particles. Repeating the argument presented above for the two-particle case we see that the operator for one conglomerate particle must be linear and so the N-particle operator must be linear. With this the whole hierarchy must be linear. We thus have:

Theorem 1. *A $\hat{\otimes}$-derivation of translation invariant second order differential operators is linear.*

This result of course does not rule out the physical possibility of nonlinear quantum mechanics for identical particles, but points out a further subtlety in its manifestation. The separation property cannot be used as a generalization of the idea of non-interacting systems and the notion of lack of interaction becomes more subtle.

3 Strong cluster property

Given that separation cannot hold for identical particles in the nonlinear case, one can expect on intuitive grounds that it may hold for systems in which the subsystems are distant from each other. This is usually called the cluster decomposition property. This property however holds even in the interacting case, given short range interparticle potentials. A slightly strengthened version however eliminates interaction potentials in the linear case, and can be used as a generalization that can be extended to the symmetric nonlinear case. Consider an n-fold symmetric tensor product

$$(\phi_1 \hat{\otimes} \phi_2 \hat{\otimes} \cdots \hat{\otimes} \phi_n)(x) = C \sum_{\pi \in S} \pm \phi_1(x_{(1,\pi)}) \phi_2(x_{(2,\pi)}) \cdots \phi_n(x_{(n,\pi)}), \tag{15}$$

where C is a combinatorial coefficient x is an m-tuple of space points, S is a subset of the permutation group, and each $x_{(k,\pi)}$ is a subset of the m-tuple x ordered according to its original order in x. The sum is over all permutations that distribute x into the subsets $x_{(k,\pi)}$. We say such a product is *cluster-separated* if the supports of the summand in (15) are all disjoint. We say a hierarchy of operators has the strong cluster separation property if (2) holds for cluster-separated products. A simple verification with ordinary linear Schrödinger operators shows that these satisfy the strong cluster-separation property if and only if the interparticle potentials vanish, so this is indeed a proper generalization of lack of interaction. One sees immediately that the strong cluster separation property would hold if the ordinary separation property holds and if the operators were linear on sums of functions with disjoint supports. This linearity may at first sight seem contrary to the spirit of looking for nonlinear theories, but in fact, for *differential* operators it follows from the ordinary separation property in almost all cases. As was shown in [1] tensor derivations are for the most part homogeneous. Those that are not, differ from these by a fixed canonical term. Homogeneous differential operators have the remarkable property that they are linear on spaces generated by functions with disjoint supports:

Theorem 2. *If G be a differential operator which is homogeneous of degree $k \neq 0$ then it is additive on spaces generated by functions with disjoint support and for $k = 1$ it is linear on such spaces.*

Proof. By Euler's formula $DG(\phi)\phi = kG(\phi)$ where D denotes the Frechét derivative. Let ϕ_j, $j = 1, \ldots, r$ have disjoint supports. We have

$$G\left(\sum_j \phi_j\right) = k^{-1} DG\left(\sum_j \phi_j\right)\left(\sum_\ell \phi_\ell\right) = k^{-1}\sum_\ell DG\left(\sum_j \phi_j\right)\phi_\ell.$$

Now in a neighborhood of a point where $\phi_\ell \neq 0$ one has for $j \neq \ell$ that $\phi_j = 0$. Since the value at a point of a differential operator applied to a function depends only on the values of the function in any neighborhood of the point, we can write the last term as $k^{-1}\sum_\ell DG(\phi_\ell)\phi_\ell = \sum_j G(\phi_j)$ and we have additivity. If now $k = 1$, the operator will in fact be real-linear on the subspace generated by the ϕ_j. ■

From this we deduce

Theorem 3. *Homogeneous ordinary tensor derivations of differential operators satisfy the strong cluster separation property.*

This means that we can apply all the structural theorems of [1] to symmetric tensor derivation provided that we stay within the class of homogeneous differential operators.

4 Fock space considerations

In [2] we were led to consider the problem of finding a Lorentz invariant nonlinear operator K in a relativistic scalar free field Fock space for which

$$[[K, \phi(f)], \phi(g)] = 0 \tag{16}$$

provided the supports of f and g are space-like separated. In that reference we analyzed only the simplest consequence of this equation that arising from applying it to the vacuum state. One of the conditions was a symmetric separation property for space-like separated supports. We now address (16) more systematically. We here consider only the bosonic case as the fermionic one is entirely similar.

Let $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$ be the bosonic Fock space where $\mathcal{H}_0 = \mathbb{C}$, $\mathcal{H}_1$ is the 1-particle subspace, and $\mathcal{H}_n = \mathcal{H}_1 \hat{\otimes} \cdots \hat{\otimes} \mathcal{H}_1$, the n-fold symmetric tensor product, is the n-particle subspace. We assume $\mathcal{H}_1$ has a antilinear involution $f \mapsto \bar{f}$ satisfying

$$(\bar{f}, g) = (\bar{g}, f). \tag{17}$$

For $f \in \mathcal{H}_1$ one defines the creation operator $a^+(f)$ and the annihilation operator $a(f)$ in $\mathcal{H}$ by

$$(a^+(f)\Psi)_n = \sqrt{n}\, f \hat{\otimes} \Psi_{n-1}, \tag{18}$$

$$(a(f)\Psi)_n = \sqrt{n+1}\, f \rfloor \Psi_{n+1}, \tag{19}$$

where the contraction operator $\rfloor$ is defined by

$$f \rfloor (g_1 \hat{\otimes} \cdots \hat{\otimes} g_n) = \frac{1}{n}\sum_{i=1}^{n} (f, g_i) g_1 \hat{\otimes} \cdots \hat{\otimes} \hat{g}_i \hat{\otimes} \cdots \hat{\otimes} g_n,$$

where by the hat over g_i we mean that that factor is missing. The *quantum field* is defined as

$$\phi(f) = \phi^{(+)}(f) + \phi^{(-)}(f) = a^+(f) + a(\bar{f}). \tag{20}$$

One has the famous *canonical commutation relations*

$$[\phi^{(+)}(f), \phi^{(+)}(g)] = 0,$$
$$[\phi^{(-)}(f), \phi^{(-)}(g)] = 0,$$
$$[\phi^{(-)}(f), \phi^{(+)}(g)] = (\bar{f}, g).$$

Assume that K respects particle number, that is, $(K\Psi)_n = K_n\Psi_n$ for a hierarchy of operators K_n. We analyze the equation

$$[[K, \phi(f)], \phi(g)] = 0 \tag{21}$$

by applying the left-hand side to a Fock space vector which has only an n-particle component Ψ_n. One arrives at the following three conditions

$$\begin{aligned}
&K_{n+2}\left(\sqrt{(n+2)(n+1)}\, f\hat{\otimes}g\hat{\otimes}\Psi_n\right) - \sqrt{n+2}\, f\hat{\otimes}K_{n+1}\left(\sqrt{n+1}\, g\hat{\otimes}\Psi_n\right) \\
&\quad - \sqrt{n+2}\, g\hat{\otimes}K_{n+1}\left(\sqrt{n+1}\, f\hat{\otimes}\Psi_n\right) + \sqrt{(n+2)(n+1)}\, f\hat{\otimes}g\hat{\otimes}K_n(\Psi_n) = 0, \qquad (22)\\
&K_n\left((n+1)\bar{f}\rfloor g\hat{\otimes}\Psi_n + nf\hat{\otimes}\bar{g}\rfloor\Psi_n\right) - \sqrt{n+1}\,\bar{f}\rfloor K_{n+1}\left(\sqrt{n+1}\, g\hat{\otimes}\Psi_n\right) \\
&\quad - \sqrt{n+1}\,\bar{g}\rfloor K_{n+1}\left(\sqrt{n+1}\, f\hat{\otimes}\Psi_n\right) - \sqrt{n}\, f\hat{\otimes}K_{n-1}\left(\sqrt{n}\,\bar{g}\rfloor\Psi_n\right) \\
&\quad - \sqrt{n}\, g\hat{\otimes}K_{n-1}\left(\sqrt{n}\,\bar{f}\rfloor\Psi_n\right) + (n+1)\bar{g}\rfloor f\hat{\otimes}K_n(\Psi_n) + ng\hat{\otimes}\bar{f}\rfloor K_n(\Psi_n) = 0, \qquad (23)\\
&K_{n-2}\left(\sqrt{(n-1)n}\,\bar{f}\rfloor\bar{g}\rfloor\Psi_n\right) - \sqrt{n-1}\,\bar{f}\rfloor K_{n-1}\left(\sqrt{n}\,\bar{g}\rfloor\Psi_n\right) \\
&\quad - \sqrt{n-1}\,\bar{g}\rfloor K_{n-1}\left(\sqrt{n}\,\bar{f}\rfloor\Psi_n\right) + \sqrt{(n-1)n}\,\bar{g}\rfloor\bar{f}\rfloor K_n(\Psi_n) = 0. \qquad (24)
\end{aligned}$$

In the relativistic case these conditions are to be satisfied whenever the smearing functions f and g have space-like separated supports.

We have

Theorem 4. *If K is a* linear *symmetric tensor derivation, then equations (22) and (24) are satisfied identically, while (23) is satisfied if*

$$(\bar{f}, K_1(g)) + (\bar{g}, K_1(f)) = 0. \tag{25}$$

This is a straightforward though tedious verification. It is enough to consider $\Psi_n = h_1\hat{\otimes}\cdots\hat{\otimes}h_n$ as linear operators are uniquely defined by their action on product functions.

Equation (25), imposed for all f and g says, using (17), that K_1 must be anti-symmetric, or that its exponential is unitary. This is an interesting consequence, as one of the requirements in [2] for a consistent history model is this unitarity which was states separately; here it is a consequence of the separation property and the commutation relation.

This result does not in itself provide us with an example of a nonlinear relativistic quantum mechanics, but it allows us to construct a theory, using the coherent histories approach, in which the quantum measurement process has properties similar to those we believe a nonlinear theory must have, that is, the future light-cone singular behavior pointed out in [2].

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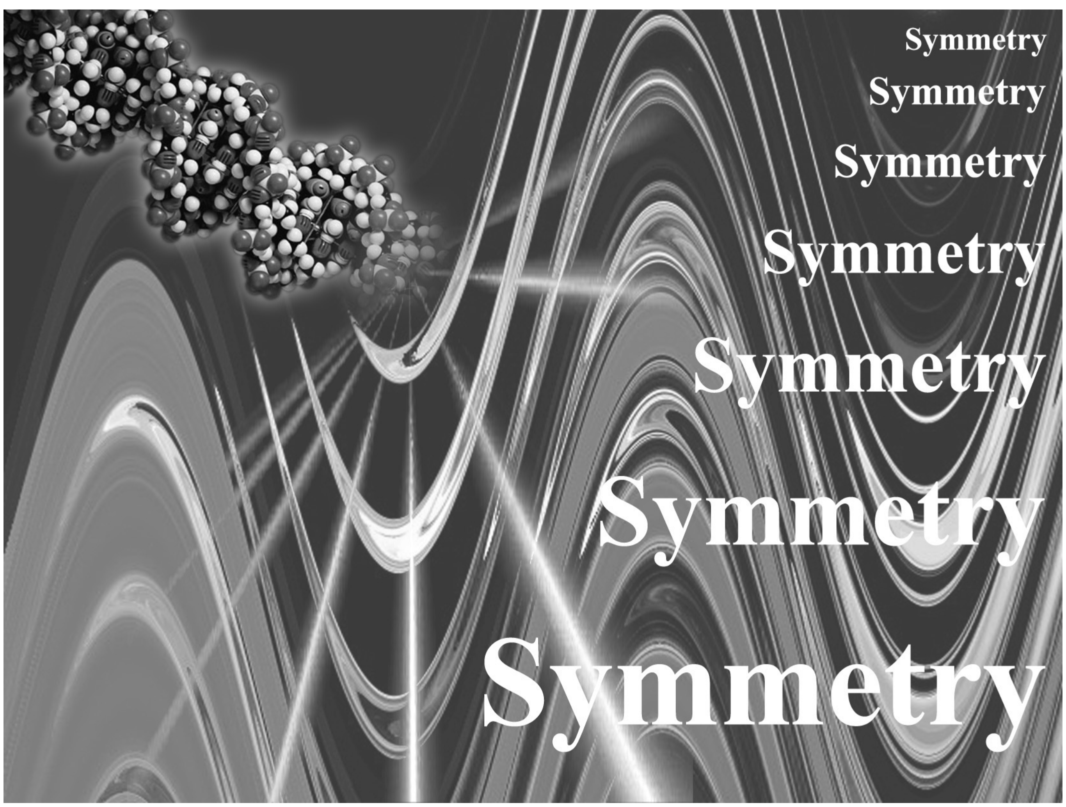

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Semiclassically Concentrates Waves for the Nonlinear Schrödinger Equation with External Field

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Classes of solutions, asymptotic in small parameter $\hbar$, $\hbar \to 0$, are constructed to the generalized nonlinear Schrödinger equation (NSE) in a multi-dimensional space with an external field in the framework of the WKB-Maslov method. Asymptotic semiclassically concentrated solutions (SCS), regarded as multi-dimensional solitary waves, are introduced for the NSE with an external field and cubic local nonlinearity. The one-dimensional soliton dynamics in an external field of a special form is discussed. Another class of asymptoitic SCS solutions is constructed for the NSE with Gaussian non-local potential and a local external field. These solutions are similar to the trajectory-coherent states or squeezed states in quantum mechanics.

1 Introduction

We study soliton-like properties of nonintegrable generalizations of the nonlinear Schrödinger equation (NSE)

$$\left\{-i\hbar\partial_t + \hat{\mathcal{H}}\left(t, |\Psi|^2\right)\right\}\Psi = 0 \tag{1}$$

within the framework of the semiclasical WKB-Maslov method [1]. Here, $\Psi = \Psi(\vec{x}, t)$ is a complex smooth function, $\vec{x} \in \mathbb{R}^n$, $t \in \mathbb{R}^1$; $|\Psi|^2 = \Psi^*\Psi$, Ψ^* is the function complex conjugate of Ψ; $\hat{\mathcal{H}}\left(t, |\Psi|^2\right)$ is a nonlinear operator, $\partial_t\Psi = \partial\Psi/\partial t$. The Planck's constant $\hbar$ plays the role of an asymptotic parameter.

Equation (1) arises in the statistical physics and quantum theory of condensed matter [2]. The evolution of bosons is described in terms of the secondary quantized Schrödinger equation. In Hartree's approximation it leads to the classical multi-dimensional Schrödinger equation with a non-local nonlinearity for one-particle functions, i.e. a Hartree type equation. The special case of equation (1), the NSE with local cubic nonlinearity

$$i\hbar\Psi_{,t} + \frac{\hbar^2}{2}\Psi_{,xx} + 2g|\Psi|^2\Psi = 0, \tag{2}$$

is used, in particular, in nonlinear optics (see, for example, [3, 4]). Here $\Psi = \Psi(x, t)$, $x \in \mathbb{R}^1$, g is a real nonlinearity parameter, $\Psi_{,t} = \partial\Psi/\partial t$, $\Psi_{,x} = \partial\Psi/\partial x$.

Equation (2) is integrated by the Inverse Scattering Transform (IST) method and has soliton solutions [5]. Solitons are localized wave packets propagating without distortion and interacting elastically in mutual collisions. The soliton conception is of commonly used in various fields of nonlinear physics and mathematics (see [6, 7, 8] and Refs. herein).

A fairly wide class of nonlinear equations, nonintegrable via the IST method, was found to possess soliton-like solutions. They are concentrated in a sense and conserve this property in the course of evolution. These solutions are referred to as solitary waves (SWs), quasi-solitons, etc. There is a large number of papers studying SWs. For example, so called squeezed (compressed) light states and the important problem of the correspondence between the stressed states describing the quantum properties of a radiation and the optical solitons are analyzed in [11] in terms of NLS-solitons. Systematic study of soliton excitations in molecular systems was carried out by Davidov [9] and was continued in subsequent works [10].

Note that in the optical pulse propagation theory the function Ψ is an envelope of the electromagnetic field that is quite different from the quantum mechanical meaning of Ψ. Though, in both cases Ψ is a square-integrable function which norm is conserved. This can be considered as a ground to apply quantum mechanical ideas and methods to the pulse propagation theory. The semiclassical approach in this case implies that we deal with narrow wave packets, and the asymptotic small parameter $\hbar$ is a characteristic of the packet width.

Soliton properties in nonintegrable systems can be investigated either using computer simulations or by approximate methods.

We construct asymptotic semiclassically concentrated solutions, regarded as multi-dimensional solitary waves, for the NSE with cubic local nonlinearity in the presence of an external field. The one-dimensional soliton dynamics in the external field of a special form is discussed in terms of the asymptotic SCS as an illustration.

Another class of the SCS is introduced and studied for the NSE with non-local unitary nonlinearity, the Hartree type equation. This class of solutions is similar to the trajectory-coherent states or squeezed states in quantum mechanics. A class of such solutions, asymptotic in small parameter $\hbar$ ($\hbar \to 0$), is constructed for the one-dimensional Hartree type equation with Gaussian non-local potential.

2 The nonlinear Schrödinger equation with external field

The generalized NSE with cubic local nonlinearity is written as follows [2, 7, 9]:

$$\left\{-i\hbar\partial/\partial t + \frac{1}{2}(-i\hbar\nabla - \vec{\mathcal{A}}(\vec{x},t))^2 + u(\vec{x},t) - 2g|\Psi(\vec{x},t)|^2\right\}\Psi(\vec{x},t) = 0. \tag{3}$$

Here $u(\vec{x},t)$, $\vec{\mathcal{A}}(\vec{x},t)$ are given functions determining an external field; g is a real parameter of nonlinearity.

The key moment of the asymptotic method is choice of a class of functions singularly depending on the asymptotic parameter in which asymptotic solutions are constructed.

To define soliton-like asymptotic solutions to (3) we need some auxiliary notions. Let $\hat{\vec{x}}(=\vec{x})$ and $\hat{\vec{p}}(=-i\hbar\nabla)$ are the position and momentum operators, respectively, with the commutators

$$[\hat{x}_k, \hat{p}_s] = i\hbar\delta_{k,s}, \qquad [\hat{x}_k, \hat{x}_s] = [\hat{p}_k, \hat{p}_s] = 0, \qquad k, s = \overline{1, n}.$$

A smooth function $A(t, \vec{x}, \vec{p})$ of t and of real vector variables $\vec{x}$ and $\vec{p}$ is a symbol of the (Weyl) operator $\hat{A}(t, \vec{x}, \hat{\vec{p}})$.

The mean value of the operator $\hat{A}$ by a function $\Psi(\vec{x}, t, \hbar)$ is defined as

$$\langle A\rangle = \langle\Psi|\hat{A}|\Psi\rangle/\|\Psi\|^2, \qquad \|\Psi\|^2 = \langle\Psi|\Psi\rangle = \int_{\mathbb{R}^n} |\Psi|^2 d\vec{x}, \tag{4}$$

$$\langle\Psi|\hat{A}(t)|\Psi\rangle = \int_{\mathbb{R}^n} \Psi^*(\vec{x},t)\hat{A}(t)\Psi(\vec{x},t)d\vec{x}.$$

For the operators $\vec{x}$, $\hat{\vec{p}}$ we have $\langle\vec{x}\rangle = \vec{X}(t,\hbar)$, $\langle\vec{p}\rangle = \vec{P}(t,\hbar)$. We assume that there exist the limits $\lim\limits_{\hbar\to 0} \vec{X}(t,\hbar) = \vec{X}(t)$, $\lim\limits_{\hbar\to 0} \vec{P}(t,\hbar) = \vec{P}(t)$. The $2n$-vector function $Z(t) = \{\vec{X}(t), \vec{P}(t),\ 0 \le t \le T\}$ is referred to as the phase orbit corresponding to the function $\Psi(\vec{x}, t)$.

Let $\mathbb{CS}^\hbar(Z(t)) \equiv \mathbb{CS}^\hbar$ be the class of semiclassically concentrated functions associated with an arbitrary phase orbit $Z(t)$ as follows.

Definition 1. A function $\Psi(\vec{x},t)$ belongs to the class $\mathbb{CS}^\hbar$

(i) if there exists the limit

$$\lim_{\hbar\to 0} |\Psi(\vec{x},t,\hbar)|^2/\|\Psi\|^2 = \delta(\vec{x}-\vec{X}(t)),$$

(ii) there exist the centered moments of arbitrary order with respect to $\vec{X}(t)$, $\vec{P}(t)$.

A solution $\Psi(\vec{x},t,\hbar)$ of (3), $\Psi\in\mathbb{CS}^\hbar$, is called the semiclassically concentrated solution (SCS).

It was proved in Ref. [12] that if $\Psi(\vec{x},t,\hbar)$ is a semiclassically concentrated solution of (3), then $Z(t) = \{\vec{X}(t),\vec{P}(t)\}$ is a solution of the classical Hamilton system with the Hamiltonian $\mathcal{H}_{\rm cl}(\vec{p},\vec{x},t) = \frac{1}{2}(\vec{p}-\vec{\mathcal{A}}(\vec{x},t))^2 + u(\vec{x},t)$.

Let us denote by $\mathcal{Q}^t_\hbar$ a class of semiclassically concentrated functions $\Psi(\vec{x},t,\hbar)$ singularly depending on the asymptotic parameter $\hbar$, $\hbar\to 0$,

$$\mathcal{Q}^t_\hbar = \left\{\Psi(\vec{x},t,\hbar) : \Psi(\vec{x},t,\hbar) = \rho(\theta,\vec{x},t,\hbar)\exp\left[\frac{i}{\hbar}S(\vec{x},t,\hbar)\right]\right\}. \tag{5}$$

Here $\theta = \hbar^{-1}\sigma(\vec{x},t,\hbar)$ is a "fast" variable; $\sigma(\vec{x},t,\hbar)$, $\rho(\theta,\vec{x},t,\hbar)$, and $S(\vec{x},t,\hbar)$ are real functions regular in $\hbar$, that is $S(\vec{x},t,\hbar) = S^{(0)}(\vec{x},t) + \hbar S^{(1)}(\vec{x},t) + \cdots$.

The class $\mathcal{Q}^t_\hbar$ can be considered as a generalization of the solitary wave since the one-soliton solution for the NSE (2) belongs to the $\mathcal{Q}^t_\hbar$. Note that the derivative operators $\partial/\partial t$ and ∇ are extended in acting on the functions of the class (5):

$$-i\hbar\partial/\partial t = -i\hbar\partial/\partial t\Big|_{\theta=\rm const} - i\sigma_{,t}\partial/\partial\theta, \qquad -i\hbar\nabla = -i\hbar\nabla|_{\theta=\rm const} - i(\nabla\sigma)\partial/\partial\theta,$$

where $\sigma_{,t} = \partial\sigma/\partial t$. In what follows we put

$$\partial/\partial t|_{\theta=\rm const} \equiv \partial_t, \qquad \nabla|_{\theta=\rm const}\equiv\nabla, \qquad \partial/\partial\theta\equiv\partial_\theta. \tag{6}$$

Let us set estimates for these operators.

Definition 2. An operator $\hat{A}$ has the asymptotic estimate $\hat{O}(\hbar^\alpha)$ on the class $\mathcal{Q}^t_\hbar$, $\hat{A} = \hat{O}\left(\hbar^\alpha\right)$, if $\forall\ \Psi\in\mathcal{Q}^t_\hbar$ the asymptotic estimate

$$\|\hat{A}\Psi\|/\|\Psi\| = O\left(\hbar^\alpha\right), \qquad \hbar\to 0, \tag{7}$$

is valid.

Note that similar estimates are also valid for mean values of operators,

$$|\langle\Psi|\hat{A}|\Psi\rangle|/\|\Psi\| = O\left(\hbar^\alpha\right), \qquad \hbar\to 0.$$

For the derivative operators (6) we have

$$i\hbar\partial_t + S_{,t} = \hat{O}(\hbar), \qquad i\hbar\nabla + \nabla S = O(\hbar), \qquad \vec{x} = O(1), \qquad \partial_\theta = O(1). \tag{8}$$

These estimates permits us to construct a solution of equation (3) in the form of asymptotic series in $\hbar$.

When studying the asymptotic solution, the leading term is of primary interest. So, we construct the asymptotic SCS to equation (3) in the class $\mathcal{Q}^t_\hbar$ with an accuracy of $O\left(\hbar^2\right)$.

To this end we substitute the function $\Psi(\vec{x},t)$ of the form (5) into (3), gather and sum both $\hbar$–free terms and terms of the power $\hbar^1$, and put every of these sums to zero. Note that the residual has the estimates $O\left(\hbar^2\right)$. Next, we separate the equations for the function ρ with the "fast" variable θ from the other equations and solve them under the constraint $\lim\limits_{\theta\to\infty}\rho(\theta,\vec{x},t,\hbar) = \lim\limits_{\theta\to\infty}\rho_{,\theta}(\theta,\vec{x},t,\hbar) = 0$.

As a result the asymptotic solution taken with the accuracy of $O\left(\hbar^2\right)$ is of the form

$$\Psi = \Psi_0(\theta,\vec{x},t,\hbar)[1+\hbar(w(\theta,\vec{x},t)+iv(\theta,\vec{x},t))] + O\left(\hbar^2\right), \tag{9}$$

where

$$\Psi_0 = \rho(\theta,\vec{x},t,\hbar)\exp\left[\frac{i}{\hbar}(S^{(0)}(\vec{x},t)+\hbar S^{(1)}(\vec{x},t))\right], \tag{10}$$

$$\theta = \frac{1}{\hbar}\sigma^{(0)}(\vec{x},t)+\sigma^{(1)}(\vec{x},t), \tag{11}$$

$$\rho = \sqrt{\frac{\left(\nabla\sigma^{(0)}\right)^2}{2g}}\cosh^{-1}\theta, \quad g>0. \tag{12}$$

Here, $S^{(0)}$, $S^{(1)}$, $\sigma^{(0)}$, $\sigma^{(1)}$ are real functions of $\vec{x}$ and t independent from $\hbar$ which are determined by the following system:

$$S^{(0)}_{,t} + u + \frac{1}{2}\left(\nabla S^{(0)}-\vec{\mathcal{A}}\right)^2 = \frac{1}{2}\left(\nabla\sigma^{(0)}\right)^2, \tag{13}$$

$$\sigma^{(0)}_{,t} + \langle\left(\nabla S^{(0)}-\vec{\mathcal{A}}\right), \nabla\sigma^{(0)}\rangle = 0, \tag{14}$$

$$\begin{aligned} &S^{(1)}_{,t} + \langle\left(\nabla S^{(0)}-\vec{\mathcal{A}}\right), \nabla S^{(1)}\rangle - \langle\nabla\sigma^{(0)},\nabla\sigma^{(1)}\rangle \\ &\quad + \frac{\nu}{2}\langle\nabla\sigma^{(0)},\nabla\rangle\ln\frac{\left(\nabla\sigma^{(0)}\right)^2}{g} + \frac{\nu}{2}\Delta\sigma^{(0)} = 0, \end{aligned} \tag{15}$$

$$\begin{aligned} &\sigma^{(1)}_{,t} + \langle\left(\nabla S^{(0)}-\vec{\mathcal{A}}\right), \nabla\sigma^{(1)}\rangle + \langle\nabla\sigma^{(0)},\nabla S^{(1)}\rangle - \frac{\nu}{2}\left[\left(\ln\frac{\left(\nabla\sigma^{(0)}\right)^2}{g}\right)_{,t}\right. \\ &\quad \left. + \langle\left(\nabla S^{(0)}-\vec{\mathcal{A}}\right), \nabla\ln\frac{\left(\nabla\sigma^{(0)}\right)^2}{g}\rangle + \langle\nabla,\left(\nabla S^{(0)}-\vec{\mathcal{A}}\right)\rangle\right] = 0. \end{aligned} \tag{16}$$

Here, $\nu = \mathrm{sign}(\theta)$ and $\langle\vec{a},\vec{b}\rangle$ denotes the Euclidean scalar product of the vectors: $\sum\limits_{j=1}^{n} a_j b_j$.

The functions $w(\theta,\vec{x},t)$, $v(\theta,\vec{x},t)$ are written as

$$\begin{aligned} \rho(\theta,\vec{x},t)w(\theta,\vec{x},t) = &\sqrt{\frac{2}{g\left(\nabla\sigma^{(0)}\right)^2}}\frac{1}{\cosh\theta}\left\{c_1(\vec{x},t)\tanh\theta + \frac{1}{2}\langle\nabla\sigma^{(0)},\nabla\sigma^{(1)}\rangle\right. \\ &\left. + \frac{1}{12}\left[\Delta\sigma^{(0)} + \langle\nabla\sigma^{(0)},\nabla\ln\frac{\left(\nabla\sigma^{(0)}\right)^2}{g}\rangle\right]\left(\sinh\theta\cosh\theta - \nu\cosh^2\theta\right)\right\}, \end{aligned} \tag{17}$$

$$\begin{aligned} \rho(\theta,\vec{x},t)v(\theta,\vec{x},t) = &\sqrt{\frac{2}{g^2\left(\nabla\sigma^{(0)}\right)^2}}\left\{\frac{c_1(\vec{x},t)}{\cosh\theta} + \frac{1}{4}\left[\langle\nabla,\nabla S^{(0)}-\vec{\mathcal{A}}\rangle\right.\right. \\ &\left.\left. + \left(\partial_t + \langle\left(\nabla S^{(0)}-\vec{\mathcal{A}}\right),\nabla\rangle\right)\ln\frac{(\nabla\sigma^{(0)})^2}{g}\right](\nu\sinh\theta - \cosh\theta)\right\}. \end{aligned} \tag{18}$$

Here, a function $c_1(\vec{x},t)$ is determined by successive approximations.

3 One-dimensional NSE-soliton in external field

To assess an efficacy of the asymptotic approach it is of interest to compare the asymptotic results with a well known problem. To this end let us apply the above asymptotic solution to the one-dimensional nonlinear Schrödinger equation with an external field $u(x,t)$ that is read as

$$i\hbar\Psi_{,t} + \frac{\hbar^2}{2}\Psi_{,xx} + 2g|\Psi|^2\Psi - u\Psi = 0. \tag{19}$$

In accordance with (9)–(12) soliton-like asymptotic solution for equation (19) is

$$\Psi = \sqrt{\frac{(\sigma^{(0)}_{,x})^2}{2g}}\ \exp\left[\frac{i}{\hbar}\left(S^{(0)}(x,t) + \hbar S^{(1)}(x,t)\right)\right]\cosh^{-1}\theta. \tag{20}$$

Here, $S^{(0)}$, $\sigma^{(0)}$, $S^{(1)}$, $\sigma^{(1)}$ are functions of x and t, independent of $\hbar$. Equations (13)–(16) takes the form

$$S^{(0)}_{,t} + \frac{1}{2}\left(S^{(0)}_{,x}\right)^2 + u = \frac{1}{2}\left(\sigma^{(0)}_{,x}\right)^2, \qquad \sigma^{(0)}_{,t} + S^{(0)}_{,x}\sigma^{(0)}_{,x} = 0, \tag{21}$$

$$S^{(1)}_{,t} + S^{(0)}_{,x}S^{(1)}_{,x} - \sigma^{(0)}_{,x}\sigma^{(1)}_{,x} + \frac{\nu}{2}\sigma^{(0)}_{,x}\left(\ln\frac{(\sigma^{(0)}_{,x})^2}{g}\right)_{,x} + \frac{\nu}{2}\sigma^{(0)}_{,xx} = 0, \tag{22}$$

$$\sigma^{(1)}_{,t} + S^{(0)}_{,x}\sigma^{(1)}_{,x} + \sigma^{(0)}_{,x}S^{(1)}_{,x} = \frac{\nu}{2}\left(\ln\frac{(\sigma^{(0)}_{,x})^2}{g}\right)_{,t} + \frac{\nu}{2}S^{(0)}_{,x}\left(\ln\frac{(\sigma^{(0)}_{,x})^2}{g}\right)_{,x} + \frac{\nu}{2}\left(S^{(0)}\right)_{,xx}. \tag{23}$$

At $u = 0$ the functions

$$S^{(0)} = 2(\eta^2 - \xi^2)t + 2\xi x + \varphi_0, \tag{24}$$

$$\sigma^{(0)} = -4\xi\eta t + 2\eta(x - x_0), \tag{25}$$

$$S^{(1)} = \sigma^{(1)} = 0. \tag{26}$$

satisfy the system (21)–(23) and determine the *exact* one-soliton solution to the nonlinear Schrödinger equation (19) in the form (20). Here, constants ξ, η, φ_0, x_0 are soliton parameters: 2ξ is a velocity, η is related to an amplitude, φ_0 is an initial phase, x_0 is an initial soliton position.

Let us construct the asymptotic solution of the form (20) so that it turns into the exact one-soliton solution at $u \to 0$. We will refer to this asymptotic solution as *asymptotic soliton* for equation (19).

In accordance with (24)–(26) we take the solutions of equations (21) as

$$S^{(0)} = 2\left(\eta^2 - \xi^2\right)t + 2\xi x + \varphi_0 + h(x,t), \tag{27}$$
$$\sigma^{(0)} = -4\xi\eta t + 2\eta(x - x_0) + f(x,t).$$

Then for functions h and f we have

$$h_{,t} + \frac{1}{2}\left(4\xi h_{,x} + h^2_{,x}\right) + u = \frac{1}{2}\left(4\eta f_{,x} + f^2_{,x}\right),$$
$$f_{,t} + (2\xi + h_{,x})f_{,x} + 2\eta h_{,x} = 0. \tag{28}$$

Taking f as $f(x,t) = -2\eta x + 4\eta\xi t + w(x,t)$, we obtain

$$\sigma^{(0)} = -2\eta x_0 + w(x,t). \tag{29}$$

Equations (21), (28) result in the following equations for the functions h and w:

$$h_{,t} + \frac{1}{2}\left(4\xi h_{,x} + h_{,x}^2\right) + u = \frac{1}{2}\left(-4\eta^2 + w_{,x}^2\right), \tag{30}$$

$$w_{,t} + (2\xi + h_{,x})w_{,x} = 0. \tag{31}$$

For $h_{,x} = h_{,x}(x)$ the characteristic equation of (31), $dx/dt = 2\xi + h_{,x}$, has a special solution as an arbitrary function $w = w(z)$ of the variable $z = t - \int(2\xi + h_{,x})^{-1}dx$. Then with the change of variables $(x,t) \to (x,z)$ (22), (23) are simplified as

$$\frac{w'(z)}{2\xi + h_{,x}}\left(\sigma^{(1)}_{,x} - \frac{1}{2\xi + h_{,x}}\sigma^{(1)}_{,z}\right) + (2\xi + h_{,x})S^{(1)}_{,x} + \frac{3\nu}{2}\frac{w''(z) + w'(z)h_{,xx}}{(2\xi + h_{,x})^2} = 0, \tag{32}$$

$$\frac{w'(z)}{2\xi + h_{,x}}\left(S^{(1)}_{,x} - \frac{1}{2\xi + h_{,x}}S^{(1)}_{,z}\right) - (2\xi + h_{,x})\sigma^{(1)}_{,x} = \frac{\nu}{2}h_{,xx}. \tag{33}$$

Had we chosen a special solution of equations (32), (33) in the form $w(z) = \alpha z$, $\alpha = \text{const}$, then the functions $\sigma^{(1)} \equiv m(x)$ and $S^{(1)} \equiv n(x)$ are dependent on x only and are determined by the equations

$$\frac{\alpha}{2\xi + h_{,x}}m'(x) + (2\xi + h_{,x})n'(x) + \frac{3\nu}{2}\frac{\alpha h_{,xx}}{(2\xi + h_{,x})^2} = 0,$$

$$\frac{\alpha}{2\xi + h_{,x}}n'(x) - (2\xi + h_{,x})m'(x) = \frac{\nu}{2}h_{,xx}.$$

The potential u according to (30) reads

$$u = \frac{1}{2}\cdot\frac{\alpha^2}{(2\xi + h_{,x})^2} - \frac{1}{2}(2\xi + h_{,x})^2 + 2\left(\xi^2 - \eta^2\right). \tag{34}$$

Let us take into account that the velocity V of the exact one-soliton solution of the NSE (19) at $u = 0$ is equal to $V = 2\xi$. In terms of the "fast" variable $\theta = (2\eta/\hbar)(x - x_0 - 2\xi t)$ it will be

$$V = -\frac{\partial\theta}{\partial t}\Big/\frac{\partial\theta}{\partial x}. \tag{35}$$

For the considered asymptotic solution

$$\theta = \frac{1}{\hbar}\sigma^{(0)} + \sigma^{(1)} = \frac{\alpha}{\hbar}\left(t - \int\frac{dx}{2\xi + h_{,x}}\right) - \frac{2\eta x_0}{\hbar} + m(x), \tag{36}$$

and, with respect to (35), we have

$$V = V(x) = (2\xi + h_{,x})\left[1 - \frac{\hbar}{\alpha}m'(x)(2\xi + h_{,x})\right]^{-1}. \tag{37}$$

Note that at $\hbar \to 0$ we obtain $V \to (2\xi + h_{,x})$. The function

$$V_0(x) = 2\xi + h_{,x} \tag{38}$$

has the meaning of the velocity (at $\hbar \to 0$) of the asymptotic soliton moving in the external field $u(x)$. From (34) it follows

$$\lim_{h(x)\to 0} u(x) = \frac{\alpha^2}{8\xi^2} - 2\eta^2.$$

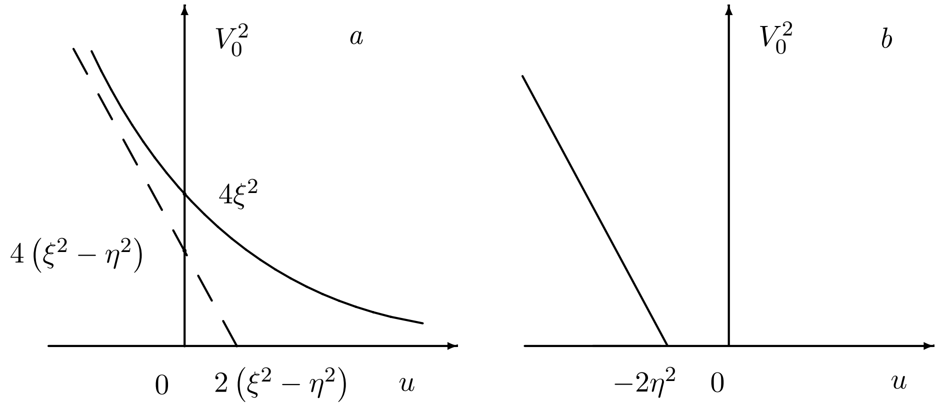

Figure 1.

For $u(x) \to 0$ at $h_{,x} \to 0$ one needs to put $\alpha = \pm 4\xi\eta$. If we take $\alpha = -4\xi\eta$ then the potential u according to (34) and (38) becomes

$$u(x) = \frac{8\xi^2\eta^2}{V_0(x)^2} - \frac{1}{2}V_0(x)^2 + 2\left(\xi^2 - \eta^2\right). \tag{39}$$

Solving (39) with respect to V_0^2 we obtain

$$V_0(x)^2 = -u(x) - 2(\eta^2 - \xi^2) + \sqrt{\left[u(x) + 2\left(\eta^2 - \xi^2\right)\right]^2 + 16\eta^2\xi^2}. \tag{40}$$

Note that in (40) we are to take the positive value of the square root and $V_0^2 \to 4\xi^2$ at $u \to 0$.

The general form of the function $V_0^2(u)$ is shown as in Fig. 1 at $\xi \neq 0$ (a) and at $\xi = 0$ (b). It can be seen that the potential well ($u \leq 0$) increases the soliton velocity and the potential barrier ($u \geq 0$) monotonously decreases it with respect to the free soliton velocity equal to 2ξ without a barrier reflection. The last feature is the nonlinearity effect.

Let us collect the expressions determining the asymptotic one-soliton solution (20) for equation (19) with the external field $u(x)$.

Equations (27), (29) for the functions $S^{(0)}$, $\sigma^{(0)}$ are written as

$$S^{(0)} = 2\left(\eta^2 - \xi^2\right)t + \int_{-\infty}^{x} (V_0(y) - 2\xi)dy + 2\xi x + \varphi_0,$$

$$\sigma^{(0)} = 4\xi\eta\left(\int_{-\infty}^{x}\left(\frac{1}{V_0(y)} - \frac{1}{2\xi}\right)dy - t\right) + 2\eta(x - x_0).$$

The functions $S^{(1)} \equiv n(x)$, $\sigma^{(1)} \equiv m(x)$ are given by

$$\sigma^{(1)'} = m'(x) = -\frac{\nu V_{0,xx}(x)}{2D}\left(\frac{48\xi^2\eta^2}{V_0(x)^3} + V_0(x)\right),$$

$$S^{(1)'}(x) = n'(x) = \frac{4\nu\xi\eta V_{0,xx}(x)}{V_0(x)D}, \qquad D = \frac{16\xi^2\eta^2}{V_0^2(x)} + V_0^2(x).$$

The "fast" variable θ (29) takes the form

$$\theta = \frac{4\xi\eta}{\hbar}\left(\int_{-\infty}^{x}\left(\frac{1}{V_0(y)} - \frac{1}{2\xi}\right)dy - t\right) + 2\frac{\eta}{\hbar}(x - x_0) + m(x),$$

the phase

$$\Phi = \frac{1}{\hbar}S^{(0)} + S^{(1)} = \frac{2}{\hbar}\left(\eta^2 - \xi^2\right)t + \frac{1}{\hbar}\int_{-\infty}^{x}(V_0(y) - 2\xi)dy + \frac{2\xi x + \varphi_0}{\hbar} + n(x).$$

The velocity $V(x)$ of the asymptotic soliton in the external field $u(x)$ with respect to (37) is

$$V(x) = V_0(x)\left[1 + \frac{\hbar}{4\xi\eta}m'(x)V_0(x)\right]^{-1}.$$

4 The Hartree type equation

The asymptotic approach appears to be more effective for the NSE with non-local nonlinearity, the Hartree type equation (HTE). A construction of asymptotic solution to the multi-dimensional HTE with external field and unitary non-local nonlinearity in terms of the WKB-Maslov method is developed in [13]. Here we consider the one-dimensional HTE with Gaussian non-local potential

$$\left\{-i\hbar\partial_t + \mathcal{H}(\hat{p}, x, t) + \hat{g}V_0\int_{-\infty}^{+\infty} dy\,\exp\left[\frac{-(x-y)^2}{2\gamma^2}\right]\frac{|\Psi(y,t)|^2}{\|\Psi\|^2}\right\}\Psi = 0, \tag{41}$$

where $\mathcal{H}(p,x,t) = \frac{p^2}{2m} + u(x,t)$, $u(x,t) = \frac{1}{2}kx^2 + lx$ is the Hamiltonian of an effective particle in the external field that is the sum of an oscillator field and a stationary homogeneous field. Note that $\hat{g} = g\|\Psi\|^2$ is assumed to be $O(1)$ and k, V_0, l are real parameters.

The HTE is not solvable by the IST method even in one-dimensional case. To define a class of semiclassically concentrated functions similar to (5) we turn to the quantum mechanics where functions of this type are well known coherent and “squeezed” states (see, for example, [14, 15]).

Following to these ideas, consider a class of functions $\mathcal{P}_\hbar^t$ in which we will find asymptotic solutions of equation (41), it as

$$\mathcal{P}_\hbar^t = \left\{\Psi : \Psi(x,t,\hbar) = \varphi\left(\frac{\Delta x}{\sqrt{\hbar}}, t, \hbar\right)\exp\left[\frac{i}{\hbar}(S(t) + P(t)\Delta x)\right]\right\}. \tag{42}$$

Here the function $\varphi(\xi,t,\hbar)$ belongs to the Schwartz space $\mathbb{S}$ in variable $\xi \in \mathbb{R}^1$ and depends smoothly on t and regularly on $\sqrt{\hbar}$ for $\hbar \to 0$. We assume here that $\Delta x = x - X(t)$; the real function $S(t)$ and the 2-dimensional vector function $Z(t) = (P(t), X(t))$, which characterize the class $\mathcal{P}_\hbar^t$, are indepent of $\hbar$ and are to be determined. More general case when S, P, X are regular functions of $\sqrt{\hbar}$ is considered in [13]. The functions of the class $\mathcal{P}_\hbar^t$ are normalized to $\|\Psi(t)\|^2 = \langle\Psi(t)|\Psi(t)\rangle$ in the space $L_2(\mathbb{R}_x^1)$ with the norm (4).

In addition, let us define the following class of functions

$$\mathcal{C}_\hbar^t = \left\{\Psi : \Psi(x,t,\hbar) = \varphi\left(\frac{\Delta x}{\sqrt{\hbar}}, t\right)\exp\left[\frac{i}{\hbar}(S(t,\hbar) + \langle P(t,\hbar), \Delta x\rangle)\right]\right\}, \tag{43}$$

where the functions $\varphi(\xi,t)$, as distinct from (42), are independent of $\hbar$.

At any fixed point in time $t \in \mathbb{R}^1$, the functions of the class $\mathcal{P}_\hbar^t$ *are concentrated*, in the limit of $\hbar \to 0$, in a neighborhood of a point lying on the phase curve $z = Z(t,0)$, $t \in \mathbb{R}^1$ [13] and are referred to as *trajectory-concentrated functions* (TCF).

In definition of the class of the TCF the phase trajectory $Z(t,\hbar)$ and the scalar function $S(t,\hbar)$ are free “parameters”. Note that for a linear Schrödinger equation, $g = 0$, the class $\mathcal{P}_\hbar^t$ includes the well-known dynamic (compressed) coherent states of quantum systems with quadratic Hamiltonians (see for details [16]).

Let us consider principal moments of the asymptotic solution construction for equation (41) in the class $\mathcal{P}_\hbar^t$ (see for details [12]).

Consider functions Φ of the class $\hat{\mathcal{P}}^t_\hbar$ that is defined by the functions $(Z(t), \hat{S}(t))$,

$$\Phi(x,t,\hbar) = \varphi\left(\frac{\Delta x}{\sqrt{\hbar}}, t, \hbar\right) \exp\left[\frac{i}{\hbar}(\hat{S}(t) + P(t)\Delta x)\right], \tag{44}$$

$$\hat{S} = S + \int_0^t \left[\frac{P(t)^2}{2m} + \frac{k}{2}X(t)^2 + lX(t) - \dot{X}(t)P(t) + \hat{g}V_0 - \hat{g}\frac{V_0}{2\gamma^2}\alpha_\Phi^{(2)}\right]. \tag{45}$$

The following estimates are valid for the functions $\Phi \in \hat{\mathcal{P}}^t_\hbar$ (44) in terms of Definition 2:

$$\Delta x = \hat{O}\left(\sqrt{\hbar}\right), \qquad \hat{\Delta p} = \hat{O}\left(\sqrt{\hbar}\right), \qquad -i\hbar\partial_t - \dot{\hat{S}}(t) + \dot{X}(t)\hat{p} - \dot{P}(t)\Delta x = \hat{O}(\hbar), \tag{46}$$

$$\Delta x = x - X(t), \qquad \Delta p = p - P(t), \qquad \hat{p} = -i\hbar\partial_x. \tag{47}$$

Let us expand the exponential in equation (41) in a Taylor series of $\Delta x = x - X(t)$, $\Delta y = y - X(t)$ and restrict ourselves to the terms of the order of not above four in Δx and Δy. In view of the estimates (46), (47) equation (41) takes the form

$$\begin{aligned}\Big\{\hat{L}_0 + \dot{\hat{S}} - \dot{X}(t)\hat{p} + \dot{P}(t)\Delta x + \dot{X}(t)P(t) + \frac{1}{m}P(t)\hat{\Delta p} + kX(t)\Delta x + l\Delta x \\ + \frac{\hat{g}V_0}{\gamma^2}\Delta x\alpha_\Phi^{(1)} + \hat{L}_1\Big\}\Phi = \hat{O}\left(\hbar^{5/2}\right),\end{aligned} \tag{48}$$

where

$$\hat{L}_0 = -i\hbar\partial_t - \dot{\hat{S}}(t) + \dot{X}(t)\hat{p} - \dot{P}(t)\Delta x + \frac{1}{2m}\hat{\Delta p}^2 + \frac{1}{2}\left(k - \frac{\hat{g}V_0}{\gamma^2}\right)\Delta x^2 = \hat{O}(\hbar), \tag{49}$$

$$L_1 = \frac{\hat{g}V_0}{8\gamma^4}\left(\Delta x^4 - 4\Delta x^3\alpha_\Phi^{(1)} + 6\Delta x^2\alpha_\Phi^{(2)} - 4\Delta x\alpha_\Phi^{(3)} + \alpha_\Phi^{(4)}\right) = \hat{O}\left(\hbar^2\right), \tag{50}$$

$$\alpha_\Phi^{(k)}(t,\hbar) = \frac{1}{\|\Phi\|^2}\int_{-\infty}^{\infty}(\Delta y)^k|\Phi(y,t)|^2 dy, \quad k = 0, 1, \ldots, \qquad \alpha_\Phi^{(k)}(t,\hbar) = O\left(\hbar^{k/2}\right). \tag{51}$$

Let us expand $\varphi(\xi, t, \hbar)$ in $\sqrt{\hbar}$ then

$$\Phi = \Phi^{(0)} + \sqrt{\hbar}\Phi^{(1)} + \hbar\Phi^{(2)} + \cdots, \qquad \Phi^{(k)} \in \mathcal{C}^t_\hbar, \tag{52}$$

$$\begin{aligned}\alpha_\Phi^{(1)} &= \alpha_{\Phi^{(0)}}^{(1)} + \sqrt{\hbar}\frac{2}{\|\Phi^{(0)}\|^2}\mathrm{Re}\langle\Phi^{(0)}|\Delta x|\Phi^{(1)}\rangle \\ &\quad + \hbar\frac{1}{\|\Phi^{(0)}\|^2}(\langle\Phi^{(1)}|\Delta x|\Phi^{(1)}\rangle + 2\mathrm{Re}\langle\Phi^{(0)}|\Delta x|\Phi^{(2)}\rangle).\end{aligned} \tag{53}$$

From (4) and (49)–(53) we have

$$\dot{\hat{S}} = 0, \qquad \dot{P}(t) = -kX(t) - l, \qquad \dot{X}(t) = \frac{1}{m}P(t), \tag{54}$$

$$\left(L_0 + \frac{\hat{g}V_0}{\gamma^2}\Delta x\alpha_{\Phi^{(0)}}^{(1)}\right)\Phi^{(0)} = 0, \tag{55}$$

$$\left(L_0 + \frac{\hat{g}V_0}{\gamma^2}\Delta x\alpha_{\Phi^{(0)}}^{(1)}\right)\Phi^{(1)} = -\frac{2}{\|\Phi^{(0)}\|^2}\frac{\hat{g}V_0}{\gamma^2}\Delta x\,\mathrm{Re}\langle\Phi^{(0)}|\Delta x|\Phi^{(1)}\rangle\Phi^{(0)}. \tag{56}$$

The function $\Phi^{(0)}$ is governed by (49), (55). It is defined as a linear Schrödinger equation with quadratic Hamiltonian that has the special solution (see, for example, [14, 15, 16]) in the form of Gaussian wave packet

$$\Phi_0^{(0)} = N(t)\exp\left\{\frac{i}{\hbar}\left[a(t) + a_1(t)\Delta x + \frac{1}{2}f(t)\Delta x^2\right]\right\}, \qquad \mathrm{Im}\, f(t) > 0. \tag{57}$$

Here, the functions $a(t)$, $a_1(t)$, $f(t)$ are to be determined. With (49), (54), equation (55) takes the form

$$\Big\{-i\hbar\partial_t + \frac{1}{m}P(t)\hat{p} - \dot{P}(t)\Delta x + \frac{1}{2m}\hat{\Delta p}^2 + \frac{1}{2}\left(k - \frac{\hat{g}V_0}{\gamma^2}\right)\Delta x^2 + \frac{\hat{g}V_0}{\gamma^2}\Delta x \alpha^{(1)}_{\Phi_0^{(0)}}\Big\}\Phi_0^{(0)} = 0. \tag{58}$$

Note that for the Gaussian packet of general form we have

$$\alpha^{(1)}_{\Phi_0^{(0)}} = 0. \tag{59}$$

From (57)–(59) it follows that $a(t) = \text{const}$, $a_1(t) = P(t)$, $f(t) = \dot{C}(t)/C(t)$, $N(t) = C(t)^{-1/2}$, and (57) becomes

$$\Phi_0^{(0)} = \frac{1}{C^{1/2}}\exp\left\{\frac{i}{\hbar}\left[a + P(t)\Delta x + \frac{m}{2}\frac{\dot{C(t)}}{C(t)}\Delta x^2\right]\right\}. \tag{60}$$

With the initial conditions $C(0) = 1$, $B(0) = mb$, $\operatorname{Im} b < 0$, the function $C(t)$ can be found as follows:

$$1)\ \frac{1}{m}\left(k - \frac{\hat{g}V_0}{\gamma^2}\right) = \Omega^2 \geq 0, \qquad C(t) = \cos\Omega t + \frac{b}{\Omega}\sin\Omega t, \tag{61}$$

$$2)\ \frac{1}{m}\left(k - \frac{\hat{g}V_0}{\gamma^2}\right) = -\Omega^2 \leq 0, \qquad C(t) = \cosh\Omega t + \frac{b}{\Omega}\sinh\Omega t. \tag{62}$$

The variance of the coordinate x with respect to (60) will be

$$\alpha^{(2)}_{\Phi_0^{(0)}} = \frac{1}{\|\Phi_0^{(0)}\|^2}\int_{-\infty}^{\infty}\Delta x^2|\Phi_0^{(0)}(x,t)|dx = \frac{\hbar|C(t)|^2}{2m\operatorname{Im}\left(\frac{\dot{C}}{C}\right)}. \tag{63}$$

It can be seen that for $\hat{g}V_0 < 0$ the variance $\alpha^{(2)}_{\Phi_0^{(0)}}(t,\hbar)$ is limited in t, i.e. $|\alpha^{(2)}_{\Phi_0^{(0)}}(t,\hbar)| \leq M$, $M = \text{const}$, while for $\hat{g}V_0 > 0$ it increases exponentially. In the limit of $\gamma \to 0$ and with $V_0 = (2\pi\gamma)^{-1/2}$, equation (4) becomes a nonlinear Schrödinger equation with the local nonlinearity, while in the case where $\hat{g}V_0 < 0$ ($\hat{g}V_0 > 0$) it corresponds to the condition of existence (nonexistence) of solitons.

Consider (60) as the vacuum solution of (58) regarded as the linear Schrödinger equation with quadratic Hamiltonian. Then the Fock basis of solutions of equation (58) yields a class of asymptotic solutions to the HTE. Due to the condition (59) the superposition principle is not fulfilled for these solutions. The last ones can be modified so that $\alpha^{(1)}_{\Phi_0^{(0)}} \neq 0$ and the superposition principle becomes valid.

Acknowledgements

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Arnold Diffusion: a Functional Analysis Approach

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We present, in the context of nearly integrable Hamiltonian systems, a functional analysis approach to study the "splitting of the whiskers" and the "shadowing problem" developed in collaboration with P. Bolle in the recent papers [1] and [2] . This method is applied to the problem of Arnold diffusion for nearly integrable partially isochronous systems improving known results.

1 Introduction

Topological instability of action variables in multidimensional nearly integrable Hamiltonian systems is known as Arnold diffusion. This phenomenon was pointed out in 1964 by Arnold himself in his famous paper [3]. For autonomous Hamiltonian systems with two degrees of freedom KAM theory generically implies topological stability of the action variables (i.e. the time-evolution of the action variables for the perturbed system stay close to their initial values for all times). On the contrary, for systems with more than two degrees of freedom, outside a wide range of initial conditions (the so-called "Kolmogorov set" provided by KAM theory), the action variables may undergo a drift of order one in a very long, but finite time called the "diffusion time". After thirty years from Arnold's seminal work [3], attention to Arnold diffusion has been renewed by [4], followed by several papers (see e.g. [5, 6] and references therein).

The Hamiltonian models which are usually studied (as suggested by normal form theory near simple resonances) have the form

$$H(I,\varphi,p,q) = \frac{1}{2}I_1^2 + \omega \cdot I_2 + \frac{1}{2}p^2 + \varepsilon(\cos q - 1) + \varepsilon\mu f(I,\varphi,p,q), \tag{1}$$

where ε and μ are small parameters (the "natural" order for μ being ε^d for some positive d); (I_1, I_2, p) and (φ, q) are standard symplectic action-angle variables ($I_i \in \mathbb{R}^{n_i}$, $n_1 + n_2 = n$, $\varphi = (\varphi_1, \varphi_2) \in \mathbb{T}^n$, $(p,q) \in \mathbb{R} \times \mathbb{T}$, $\mathbb{T}$ being the standard torus $\mathbb{R}/2\pi\mathbb{Z}$). In Arnold's model $I_1, I_2 \in \mathbb{R}$, $\omega = 1$, $f(I,\varphi,p,q) = (\cos q - 1)(\sin\varphi_1 + \cos\varphi_2)$ and in [3] diffusion is proved for μ exponentially small w.r.t. $\sqrt{\varepsilon}$. Physically (1) describes a system of n_1 "rotators" and n_2 harmonic oscillators weakly coupled with a pendulum through a perturbation term.

The existence of Arnold diffusion is usually proved following the mechanism proposed in [3]. For $\mu = 0$, Hamiltonian H admits a continuous family of n-dimensional partially hyperbolic invariant tori $\mathcal{T}_I$ possessing stable and unstable manifolds $W_0^s(\mathcal{T}_I) = W_0^u(\mathcal{T}_I)$, called "whiskers" by Arnold. Arnold's mechanism is then based on the following three main steps.

- *(i)* For $\mu \neq 0$ small enough, the perturbed stable and unstable whiskers $W_\mu^s(\mathcal{T}_I^\mu)$ and $W_\mu^u(\mathcal{T}_I^\mu)$ split and intersect transversally ("splitting of the whiskers");
- *(ii)* Prove the existence of a chain of "transition" tori connected by heteroclinic orbits ("transition chain");
- *(iii)* Prove the existence of an orbit, "shadowing" the transition chain, for which the action variables I undergo a variation of $O(1)$ in a certain time T_d called the *diffusion time.*

The shadowing problem (iii) has been extensively studied in the last years by geometrical (see e.g. [4, 7, 8, 9, 10, 11]) and by variational methods (see e.g. [5, 12]). A rich literature is also available for the splitting problem see e.g. [4, 13, 14, 15, 16, 17, 18] and references therein.

The aim of this note is to summarize the functional analysis approach developed in the recent papers [1, 2] (see also [19]), apt to deal with Arnold diffusion, especially with "**splitting**" (i) and "**shadowing**" (iii) problems. The method is illustrated on a relatively simple class of models, namely harmonic oscillators weakly coupled with a pendulum through purely spatial perturbations. Precisely we consider nearly integrable partially *isochronous* Hamiltonian systems given by

$$\mathcal{H}_\mu = \omega \cdot I + \frac{p^2}{2} + (\cos q - 1) + \mu f(\varphi, q), \tag{2}$$

where $(\varphi, q) \in \mathbb{T}^n \times \mathbb{T}^1$ and $(I, p) \in \mathbb{R}^n \times \mathbb{R}^1$. When $\mu = 0$ the energy $\omega_i I_i$ of each oscillator is a constant of the motion. The unperturbed Hamiltonian possesses n-dimensional invariant tori $\mathcal{T}_{I_0} = \{(\varphi, I, q, p) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T}^1 \times \mathbb{R}^1 \mid I = I_0,\ q = p = 0\}$ with stable and unstable manifolds $W^s(\mathcal{T}_{I_0}) = W^u(\mathcal{T}_{I_0}) = \{(\varphi, I, q, p) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{T}^1 \times \mathbb{R}^1 \mid I = I_0,\ p^2/2 + (\cos q - 1) = 0\}$. The problem of *Arnold diffusion* in this context is whether, for $\mu \neq 0$, there exist motions whose net effect is to transfer $O(1)$-energy from one oscillator to the others. In order to exclude trivial drifts of the actions due to resonance phenomena, it is standard to assume a Diophantine condition for the frequency vector ω. Precisely we will always suppose that ω is (γ, τ)-Diophantine, i.e.

- $(H1)$ $\exists\, \gamma > 0$, $\tau \geq n - 1$ such that $|\omega \cdot k| \geq \gamma / |k|^\tau$, $\forall\, k \in \mathbb{Z}^n$, $k \neq 0$.

Under assuption $(H1)$ *all* the invariant tori are preserved by the perturbation, being just slightly deformed (in [1] an elementary proof, not based on any KAM technique, is given). As a consequence, the existence of a transition chain (ii) follows immediately once the "splitting of the whiskers" (i) is proved.

As applications of our shadowing theorems and our splitting estimates, we will consider the following two particular cases

(a) the frequencies of the harmonic oscillators form a Diophantine vector ω of order 1 ("a priori-unstable case");

(b) the frequencies of the harmonic oscillators form a Diophantine vector $\omega_\varepsilon = (1/\sqrt{\varepsilon}, \beta\varepsilon^a)$ with $a \geq 0$, $\mu\varepsilon^{-3/2}$ small and the perturbation $f(\varphi, q) = (1 - \cos q) f(\varphi)$ ("three-time-scales problem" with perturbations preserving all the unperturbed invariant tori). This corresponds, after a time rescaling, to $\omega = (1, \beta\varepsilon^a\sqrt{\varepsilon})$ in (1). Hamiltonian systems with three time scales have been introduced in [4] as a description of the d'Alembert problem in celestial mechanics.

2 The functional analysis approach

We now describe the **functional analysis approach** developed to prove both the results on the shadowing theorem and on the "splitting of the whiskers". It is based on a finite dimensional reduction of Lyapunov–Schmidt type, variational in nature, introduced in [20] and in [21], and later extended in [22, 23] in order to construct shadowing orbits of "multibump" type. For simplicity we describe our approach when the perturbation term $f(\varphi, q) = (1 - \cos q) f(\varphi)$ so that the tori $\mathcal{T}_{I_0}$ are still invariant for $\mu \neq 0$ (we underline however that in [1] the shadowing analysis is carried out also for a general perturbation term $f(\varphi, q)$).

The equations of motion derived by Hamiltonian $\mathcal{H}_\mu$ are

$$\dot{\varphi} = \omega, \qquad \dot{I} = -\mu(1 - \cos q)\, \nabla f(\varphi), \qquad \dot{q} = p, \qquad \dot{p} = \sin q - \mu \sin q\ f(\varphi). \tag{3}$$

The dynamics on the angles φ is given by $\varphi(t) = \omega t + A$ so that (3) are reduced to the quasi-periodically forced pendulum equation

$$-\ddot{q} + \sin q = \mu \ \sin q \ f(\omega t + A), \tag{4}$$

corresponding to the Lagrangian

$$\mathcal{L}_{\mu,A}(q, \dot{q}, t) = \frac{\dot{q}^2}{2} + (1 - \cos q) + \mu(\cos q - 1) f(\omega t + A). \tag{5}$$

For each solution $q(t)$ of (4) one recovers the dynamics of the actions $I(t)$ by quadratures in (3).

For $\mu = 0$ equation (4) is autonomous and possesses the one parameter family of homoclinic solutions (mod 2π) $q_\theta(t) = 4 \arctan(\exp(t - \theta))$, $\theta \in \mathbb{R}$. Consider the Lagrangian action functional $\Phi_{\mu,A} : q_0 + H^1(\mathbb{R}) \to \mathbb{R}$ associated to the quasi-periodically forced pendulum (4)

$$\Phi_{\mu,A}(q) := \int_{\mathbb{R}} \mathcal{L}_{\mu,A}(q(t), \dot{q}(t), t) \, dt. \tag{6}$$

$\Phi_{\mu,A}$ is smooth on $q_0 + H^1(\mathbb{R})$ and critical points q of $\Phi_{\mu,A}$ are homoclinic solutions to 0, mod 2π, of (4). These critical points q are in fact smooth functions of the time t and are exponentially decaying to 0, mod 2π, as $|t| \to +\infty$.

The unperturbed functional $\Phi_0 := \Phi_{0,A}$ does not depend on A and possesses the 1-dimensional manifold of critical points $Z := \{q_\theta \mid \theta \in \mathbb{R}\}$ with tangent space at q_θ spanned by $\dot{q}_\theta$. All the unperturbed critical points q_θ are degenerate since $d^2\Phi_0(q_\theta)[\dot{q}_\theta] = 0$. However q_θ are non-degenerate critical points of the restriction $\Phi_{0|E_\theta}$ for any subspace E_θ supplementary to $\langle \dot{q}_\theta \rangle$. It is then possible to apply a Lyapunov–Schmidt type reduction, based on the Implicit Function Theorem, to find near q_θ, for μ small, critical points $q^\mu_{A,\theta}$ of $\Phi_{\mu,A}$ restricted to E_θ; more precisely $q^\mu_{A,\theta} = q_\theta + w^\mu_{A,\theta}$ with $w^\mu_{A,\theta} \in E_\theta$, $||w^\mu_{A,\theta}|| = O(\mu)$ and $d\Phi_{\mu,A}(q^\mu_{A,\theta})_{|E_\theta} = 0$. We call the functions $q^\mu_{A,\theta}$ "**1-bump pseudo-homoclinic solutions**" of the quasi-periodically forced pendulum (4).

It turns out that the 1-dimensional manifold $Z_\mu = \{q^\mu_{A,\theta} \mid \theta \in \mathbb{R}\}$ is a "natural constraint" for the action functional $\Phi_{\mu,A}$, namely any critical point of $\Phi_{\mu,A|Z_\mu}$ is a critical point of $\Phi_{\mu,A}$, and hence a true solution of equation (4) homoclinic to 0 (mod 2π).

In [1] the above finite dimensional reduction is performed using two different supplementary spaces to $\langle \dot{q}_\theta \rangle$: one is better suited for the shadowing arguments, the other is better suited for studying the splitting problem in presence of "high frequencies".

Shadowing. For dealing with the shadowing problem, we choose as supplementary space

$$E_\theta = \big\{ w : \mathbb{R} \to \mathbb{R} \; \big| \; w(\theta) = 0 \big\}. \tag{7}$$

E_θ and $\langle \dot{q}_\theta \rangle$ are supplementary since $\dot{q}_0(0) \neq 0$. The choice of the supplementary space E_θ is very well suited to perform the shadowing theorem because the corresponding "1-bump pseudo solutions" $q^\mu_{A,\theta}(t)$ are true solutions of (4) except at the instant $t = \theta$ where $\dot{q}^\mu_{A,\theta}(t)$ may have a jump, even though $q^\mu_{A,\theta}(t)$ is continuous at $t = \theta$ and assumes the value $q^\mu_{A,\theta}(\theta) = q_\theta(\theta) + w^\mu_{A,\theta}(\theta) = q_0(0) = \pi$. The corresponding "reduced action functional" turns out to be

$$F_\mu(A, \theta) := \Phi_{\mu,A}(q^\mu_{A,\theta}) = \int_{-\infty}^{\theta} \mathcal{L}_{\mu,A}(q^\mu_{A,\theta}(t), \dot{q}^\mu_{A,\theta}(t), t) \, dt + \int_{\theta}^{+\infty} \mathcal{L}_{\mu,A}(q^\mu_{A,\theta}(t), \dot{q}^\mu_{A,\theta}(t), t) \, dt.$$

By the autonomy of the system $F_\mu(A, \theta) = G_\mu(A + \omega\theta)$ where $G_\mu(A) := F_\mu(A, 0)$. The function $G_\mu : \mathbb{T}^n \to \mathbb{R}$, called the **homoclinic function**, has a neat geometric meaning: it is the difference between the generating functions of the exact Lagrangian stable and unstable manifolds $W^{s,u}(\mathcal{T}_{I_0})$ at section $\{q = \pi\}$. Hence, from a geometrical point of view, the choice of the supplementary space E_θ means to study $W^s(\mathcal{T}_{I_0})$ and $W^u(\mathcal{T}_{I_0})$ at the fixed Poincaré section $\{q = \pi\}$.

If $\partial_\theta F_\mu(A,\theta) = 0$ then $q^\mu_{A,\theta}$ is a true homoclinic orbit of the quasi periodically forced pendulum equation (4). Moreover it results that $\partial_A F_\mu(A,\theta) = \int_{-\infty}^{+\infty} \dot I_\mu(t)\, dt =$ *"heteroclinic jump"* and hence critical points of $\mathcal{F}_\mu(A,\theta) := F_\mu(A,\theta) - (I'_0 - I_0)\cdot A$ give rise to true heteroclinic orbits between the tori $\mathcal{T}_{I_0}$ and $\mathcal{T}_{I'_0}$. By a Taylor expansion in μ it results that $\mathcal{F}_\mu(A,\theta) = \Phi_0(q_0) + \mu\Gamma(A+\omega\theta) + O(\mu^2) - (I'_0 - I_0)\cdot A$ where $\Gamma(B) = \int_{\mathbb{R}}(\cos q_0(t) - 1) f(\omega t + B)$ is nothing but the Poincaré-Melnikov primitive. Hence, roughly speaking, critical points of Γ give rise, for μ small, $I'_0 - I_0 = O(\mu)$ and $(I'_0 - I_0)\cdot\omega = 0$, to heteroclinic orbits between $\mathcal{T}_{I_0}$ and $\mathcal{T}_{I'_0}$.

Following [22, 23] the above finite dimensional reduction is generalized in [1] in order to find a natural constraint for "k-bump pseudo homoclinic solutions" turning k times near the unperturbed separatrices of the pendulum. The search for "diffusion orbits" is then reduced to find critical points of a finite dimensional functional, which is the natural generalitation of the previous one: the Lagrangian action functional evaluated on the k-bump pseudo homoclinic solutions. In this way under a suitable "splitting condition", satisfied for instance if $G_\mu(A)$ possesses a proper minimum we can prove a general shadowing theorem with explicit estimates on the diffusion time T_d. Denoting by $B_\alpha(A_0)$ the open ball centered at $A_0 \in \mathbb{R}^n$ and of radius α, let assume

"Splitting condition". There exist $A_0 \in \mathbb{T}^n$, $\alpha > 0$, a bounded open set $U \subset \mathbb{R}^n$ (the covering space of $\mathbb{T}^n$) such that $B_\alpha(A_0) \subset U$ and a positive constant $\delta > 0$ such that

(i) $\inf_{\partial U} G_\mu \geq \inf_U G_\mu + \delta$;

(ii) $\sup_{B_\alpha(A_0)} G_\mu \leq \frac{\delta}{4} + \inf_U G_\mu$;

(iii) $d(\{A \in U \mid G_\mu(A) \leq \delta/2 + \inf_U G_\mu\}, \{A \in U \mid G_\mu(A) \geq 3\delta/4 + \inf_U G_\mu\}) \geq 2\alpha$.

The following shadowing type theorem holds, where $\rho_U := \mathrm{diam}(\Pi_\omega(U))$ and $\Pi_\omega : \mathbb{R}^n \to \mathbb{R}^n$ denotes the orthogonal projection onto $\omega^\perp$.

Theorem 1. *Assume $(H1)$ and the "splitting condition". Then $\forall\, I_0, I'_0$ with $\omega\cdot I_0 = \omega\cdot I'_0$, there is a heteroclinic orbit connecting the invariant tori $\mathcal{T}_{I_0}$ and $\mathcal{T}_{I'_0}$. Moreover there exists $C_3 > 0$ such that $\forall\, \eta > 0$ small enough the "diffusion time" T_d needed to go from a η-neighbourhood of $\mathcal{T}_{I_0}$ to a η-neighbourhood of $\mathcal{T}_{I'_0}$ is bounded by*

$$T_d \leq C_3 \frac{|I_0 - I'_0|}{\delta} \cdot \rho_U \cdot \max\left(|\log\delta|, \frac{1}{\gamma\alpha^\tau}\right) + C_3|\log(\eta)|. \tag{8}$$

The meaning of the previous estimate (8) is roughly the following: T_d is estimated by the product of the number of heteroclinic transitions k (= number of tori forming the transition chain = heteroclinic jump/splitting) and of the time T_s required for a single transition, namely $T_d = kT_s$. The time for a single transition T_s is bounded by the maximum time between the time needed to "shadow" homoclinic orbits for the quasi-periodically forced pendulum and the "ergodization time" T_e of the torus $\mathbb{T}^n$ run by the linear flow ωt, defined as the time needed for the flow $\{\omega t\}$ to make an α-net of the torus. By a well known result this time can be estimated by $T_e = O(1/\alpha^\tau)$.

The a-priori unstable systems (Case (a)) highlight the improvement of our estimates on diffusion times. In this case it is easy to show that the splitting of the whiskers is $O(\mu)$ using the classical Poincaré–Melnikov function, which for a general perturbation turns out to be $M(A) = \int_{\mathbb{R}}[f(\omega t + A, 0) - f(\omega t + A, q_0(t))]\, dt$ (note that when the perturbation is $f(\varphi, q) = (1 - \cos q) f(\varphi)$ then $M(A)$ reduces to the Poincaré–Melnikov function Γ previously defined). Then our shadowing method yields

Theorem 2. *Assume (H1) and let $M(A)$ possess a proper minimum (or maximum) A_0. Then, for μ small enough, there exist orbits whose action variables undergo a drift of order one, with diffusion time $T_d = O((1/\mu)\log(1/\mu))$.*

Theorem 2 answers a question raised in [24, Section 7] proving that, at least for isochronous systems, it is possible to reach the maximal speed of diffusion $\mu/|\log\mu|$. The estimate on the diffusion time obtained in [4] is $T_d >> O(\exp(1/\mu))$ and that in [8] it is improved to be $T_d = O(\exp(1/\mu))$; recently in [12], by means of Mather theory, the estimate on the diffusion time has been improved to be $T_d = O(1/\mu^{2\tau+1})$; in [10] it is obtained via geometric methods that $T_d = O(1/\mu^{\tau+1})$. It is worth pointing out that the estimates given in [12] and [10], while providing a diffusion time polynomial in the splitting, depend on the diophantine exponent τ and hence on the number of rotators n. Instead our estimate (as well as that discussed in [11]) does not depend upon the number of degrees of freedom.

The main reason for which Theorem 2 improves the polynomial estimates $T_d = O(1/\mu^{2\tau+1})$ and $T_d = O(1/\mu^{\tau+1})$, obtained respectively in [12] and [10], is that our shadowing orbit can be chosen, at each transition, to approach the homoclinic point only up to a distance $O(1)$ and not $O(\mu)$ like in [12] and [10]. This implies that the time spent by our diffusion orbit at each transition is $T_s = O(\log(1/\mu))$. Since the number of tori forming the transition chain is equal to $O(1/\text{splitting}) = O(1/\mu)$ the diffusion time is finally estimated by $T_d = O((1/\mu)\log(1/\mu))$.

As mentioned in the introduction variational methods in the context of Arnold diffusion have been used also in [5] and [12]. One possible advantage of our approach is that it may be used to consider more general critical points of the reduced functional, not only minima. Another advantage is that the same shadowing arguments can be used also when the hyperbolic part is a general Hamiltonian in $\mathbb{R}^{2m}$, $m \geq 1$, possessing one hyperbolic equilibrium and a transversal homoclinic orbit.

Splitting. Detecting and measuring the splitting of the whiskers is a difficult problem when the frequency vector $\omega = \omega_\varepsilon$ depends on some small parameter ε and contains some "fast frequencies" $\omega_i = O(1/\varepsilon^b)$, $b > 0$. Indeed, in this case, the variations of the Melnikov function along some directions turn out to be exponentially small with respect to ε and then the naive Poincaré-Melnikov expansion provides a valid measure of the splitting only for μ exponentially small with respect to some power of ε.

The typical argument to estimate exponentially small splittings, used virtually in all papers dealing with this problem is based on Fourier analysis on complex domains.

For this reason we would like to extend analytically the "reduced action functional" $F_\mu(A,\theta) = \Phi_{\mu,A}(q^\mu_{A,\theta})$ in a complex strip sufficiently wide in the θ variable. However $F_\mu(A,\theta)$ can not be easily analytically extended. Indeed, for θ complex, the supplementary space $E_\theta = \{w : \mathbb{R} \to \mathbb{C} \mid w(\text{Re }\theta) = 0\}$, appearing naturally when we try to extend the definition of $q^\mu_{A,\theta}$ to $\theta \in \mathbb{C}$, does not depend analytically on θ. This breakdown of analyticity, arising when measuring the "splitting of the whiskers" at the fixed Poincaré section $\{q = \pi\}$, is a well known difficulty and has been compensated in [4, 14, 15] via the introduction of tree techniques which enable to prove cancellations in the power series expansions.

Our method to overcome this "loss of analyticity" is different and relies on the introduction of another supplementary space $\widetilde{E}_\theta$, which depends analytically on θ. This "trick" was yet used in [20] to study the exponentially small splitting in rapidly periodically forced systems. Our supplementary space $\widetilde{E}_\theta$ is defined by

$$\widetilde{E}_\theta = \left\{ w : \mathbb{R} \to \mathbb{R} \;\Big|\; \int_{\mathbb{R}} \psi_\theta(t) w(t)\, dt = 0 \right\},$$

where $\psi_0(t) = \cosh^2(t)/(1+\cosh t)^3$. $\widetilde{E}_\theta$ and $\langle \dot q_\theta \rangle$ are supplementary spaces since $\int_{\mathbb{R}} \psi_0(t)\dot q_0(t)dt \neq 0$ (the above choice of ψ_0 is motivated by the fact that $\psi_0(t)$ decays at zero as $\dot q_0(t)$ for $|t| \to$

$\pm\infty$ and its singularities are located at $\pm i\pi$ while those of $\dot{q}_0$ stay at $i\pi/2$). The corresponding reduced functional $\widetilde{F}_\mu(A,\theta) := \Phi_{\mu,A}(Q^\mu_{A,\theta})$, where $Q^\mu_{A,\theta}$ are the corresponding "1-bump pseudo-homoclinics solutions", can be analytically extended in a sufficiently large complex strip. More precisely let $f(\varphi,q) = (1-\cos q)f(\varphi)$ with $f(\varphi)$ analytic in $D := \prod_{j=1}^n(\mathbb{R}+i[-a_j,a_j])$ for some $a_i \geq 0$ and define $||f|| := \sup_{A\in D}|f(A)|$. Since $q_0(t)$ has a analytic extension up to the streep $|\text{Im } t| < \pi/2$ we manage to extend, using the contraction mapping theorem, $\widetilde{F}_\mu(A,\theta)$ in $D \times \{|\text{Im }\theta| < ((\pi/2)-\sigma)\}$, for $\mu\sigma^{-3}||f||$ small enough.

By an estimate of $\widetilde{F}_\mu(A,\theta) - \mu\Gamma(A+\omega\theta)$ over its complex domain and a standard lemma on Fourier coefficients of analytical functions we easily obtain an exponentially small bounds for the Fourier coefficients of the splitting function $\widetilde{G}_\mu := \widetilde{F}_\mu(A,0)$. Setting $\widetilde{G}_\mu(A) = \sum_{k\in\mathbb{Z}^n}\widetilde{G}_k\exp^{ik\cdot A}$, $\Gamma(A) = \sum_{k\in\mathbb{Z}^n}\Gamma_k\exp^{ik\cdot A}$ and $f(A) = \sum_{k\in\mathbb{Z}^n} f_k\exp(ik\cdot A)$, the following theorem holds

Theorem 3. *For $\mu||f||\sigma^{-3}$ small enough, $\forall\ k\neq 0,\ k\in\mathbb{Z}^n,\ \forall\ \sigma\in\left(0,\frac{\pi}{2}\right)$,*

$$|\widetilde{G}_k - \mu\Gamma_k| = O\left(\frac{\mu^2||f||^2}{\sigma^4}\exp\left(-\sum_{i=1}^n a_i|k_i|\right)\exp\left(-|k\cdot\omega|\left(\frac{\pi}{2}-\sigma\right)\right)\right). \tag{9}$$

Γ_k are explicitely given by $\Gamma_k = f_k 2\pi(k\cdot\omega)/\sinh\left(k\cdot\omega\frac{\pi}{2}\right)$.

The **crucial point** is now to observe that "reduced action functionals" corresponding to different choices of the supplementary space are equivalent: it turns out that the reduced functionals F_μ and $\widetilde{F}_\mu$ are simply the same up to a change of variables close to the identity,

$$F_\mu(A,\theta) = \widetilde{F}_\mu(A,\theta+h_\mu(A,\theta)), \qquad h_\mu = O(\mu). \tag{10}$$

This fact enables to transpose the informations on $\widetilde{F}_\mu$ to F_μ and viceversa. The introduction of $\widetilde{F}_\mu$ (which recover the analiticity) may then be interpreted simply as measuring the splitting with a non fixed Poincaré section. Setting $\theta = 0$ in equation (10) we get

$$G_\mu(A) = \widetilde{G}_\mu(A+g_\mu(A)\omega), \qquad \text{where} \quad g_\mu(A) := h_\mu(A,0), \tag{11}$$

namely the splitting function G_μ and $\widetilde{G}_\mu$ are the same up to the change of variables of the torus $\psi_\mu := Id + g_\mu$ close to identity, i.e. $G_\mu = \widetilde{G}_\mu\circ\psi_\mu$.

3 Systems with three time scales

We consider now Hamiltonians with three time scales

$$\mathcal{H} = \frac{I_1}{\sqrt{\varepsilon}} + \varepsilon^a\beta\cdot I_2 + \frac{p^2}{2} + (\cos q - 1) + \mu(\cos q - 1)f(\varphi_1,\varphi_2),\ I_1\in\mathbb{R}, \beta, I_2\in\mathbb{R}^{n-1},\ n\geq 2,$$

namely $\mathcal{H}_\mu$ with $\omega_\varepsilon = \left(\frac{1}{\sqrt{\varepsilon}},\varepsilon^a\beta\right)$. In this case, as an application of the previous estimate (9), it follows easily lower estimates for the splitting and hence for the diffusion time. Assume that f is analytical w.r.t φ_2. Set $\widetilde{G}_\mu(A) = \sum_{k_1\in\mathbb{Z}}\widetilde{g}_{k_1}(A_2)\exp^{ik_1\cdot A_1}$ and $\Gamma(\varepsilon,A) = \sum_{k_1\in\mathbb{Z}}\Gamma_{k_1}(\varepsilon,A_2)\exp^{ik_1\cdot A_1}$. In [1] it is proved

Theorem 4. *For $\mu||f||\varepsilon^{-3/2}$ small there holds*

$$\begin{aligned}\widetilde{G}_\mu(A_1,A_2) &= \widetilde{g}_0(A_2) + 2\text{Re}\left[\widetilde{g}_1(A_2)e^{iA_1}\right] + \sum_{|k_1|\geq 2}\widetilde{g}_{k_1}(A_2)\exp(ik_1A_1)\\ &= b + (\mu\Gamma_0(\varepsilon,A_2) + R_0(\varepsilon,\mu,A_2)) + 2\text{Re}\left[(\mu\Gamma_1(\varepsilon,A_2) + R_1(\varepsilon,\mu,A_2))\,e^{iA_1}\right]\\ &\quad + O\left(\mu\varepsilon^{-1/2}||f||\exp\left(-\frac{\pi}{\sqrt{\varepsilon}}\right)\right),\end{aligned}$$

where $b := \Phi_0(q_0)$

$$R_0(\varepsilon,\mu,A_2) = O\left(\mu^2||f||^2\right) \quad \textit{and} \quad R_1(\varepsilon,\mu,A_2) = O\left(\frac{\mu^2||f||^2}{\varepsilon^2}\exp\left(-\frac{\pi}{2\sqrt{\varepsilon}}\right)\right).$$

This result improves the main Theorem I in [18] which holds for $\mu = \varepsilon^p$, $p > 2+a$; w.r.t. [14] (which deals with more general systems) we remark that our results hold in any dimension, while the results of [14], based on tree techniques and cancellations, are proved for $n = 2$.

Using Theorem 4, in [1] simple conditions on the perturbation f which imply the "splitting condition", are given. For example, setting $f(A_1,A_2) = \sum_{k_1\in\mathbb{Z}} f_{k_1}(A_2)\exp(ik_1A_1)$, the "splitting condition" is satisfied if (i) $a > 0$, $f_0(A_2)$ admits a strict local minimum at the point $\overline{A}_2$ and $f_1(\overline{A}_2) \neq 0$. (ii) $a = 0$, $f_0(A_2)$ admits a strict local minimum at the point $\overline{A}_2$ and $f_1(\overline{A}_2 + i(\pi/2)\beta) \neq 0$.

In systems with three time scales it appears that the splitting is not uniform in all the directions. Since for larger splitting one would expect a faster speed of diffusion, one could guess the existence of diffusion orbits that drift along the "fast" directions $I_2 \in \mathbb{R}^{n-1}$, where the splitting is just polynomially small w.r.t. $1/\varepsilon$, in a polynomially long diffusion time $T_d = O(1/\varepsilon^q)$. In [2] we prove that, for $n \geq 3$, this is indeed the case (note that Arnold diffusion can take place in the direction I_2 only for $n \geq 3$ because of the conservation of the energy along the orbits). For example we can prove

Theorem 5. *Let* $f(\varphi) = \sum_{j=1}^{n}\cos\varphi_j$, $n \geq 3$, *and* ω_ε *be a* $(\gamma_\varepsilon,\tau)$*-diophantine vector. Assume* ε, $\mu\varepsilon^{-3/2}$ *and* $\mu\varepsilon^{-2a-1}$ *to be sufficiently small. Then, for all* I_0, I_0' *with* $\omega_\varepsilon\cdot I_0 = \omega_\varepsilon\cdot I_0'$ *and* $(I_0)_1 = (I_0')_1$ *there exists a heteroclinic orbit connecting the invariant tori* $\mathcal{T}_{I_0}$ *and* $\mathcal{T}_{I_0'}$ *with a diffusion time*

$$T_d \leq C\frac{|I_0'-I_0|}{\mu\varepsilon^{a+(1/2)}} \times \max\left\{\frac{1}{\gamma_\varepsilon(\varepsilon^{a+(1/2)})^\tau}, |\ln(\mu)|\right\}. \tag{12}$$

The previous phenomenon can not be seen by the splitting estimates given in [14] and [18] where the size of the splitting is measured by the "determinant of the splitting matrix" which turns out to be exponentially small.

In order to prove Theorem 5 we refine the shadowing Theorem 1. The reasons for which we are able to move in polynomial time w.r.t. $1/\varepsilon$ along the fast I_2 directions are the following three ones. (i) As in Theorem 1, since the homoclinic orbit decays exponentially fast to 0, the time needed to "shadow" homoclinic orbits for the quasi-periodically forced pendulum (4) is only polynomial. (ii) Since the splitting is polynomially small in the directions I_2, we can choose just a polynomially large number of tori forming the transition chain $k = O(1/\varepsilon^p)$ to get a $O(1)$-drift of I_2. (iii) Finally, the most difficult task is to get a polynomial estimate for the "ergodization time" T_e. The crucial improvement of the shadowing Theorem 5 allows the shadowing orbit to approach the homoclinic point only up to a polynomially small distance $\alpha = O(\varepsilon^p)$, $p > 0$, (and not exponentially small as it would be required applying the shadowing Theorem 1). Since the ergodization time is estimated by $T_e = O(1/\alpha^\tau)$, it results that the minimum time after which the homoclinic trajectory can "jump" to another torus is only polynomially long w.r.t. $1/\varepsilon$.

These results are the first steps to prove the existence of this phenomenon also for more general systems (with non isochronous terms and more general perturbations).

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Melnikov Analysis for Multi-Symplectic PDEs

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In this work the Melnikov method for perturbed Hamiltonian wave equations is considered in order to determine possible chaotic behaviour in the systems. The backbone of the analysis is the multi-symplectic formulation of the unperturbed PDE and its further reduction to travelling waves. In the multi-symplectic approach two separate symplectic operators are introduced for the spatial and temporal variables, which allow one to generalise the usual symplectic structure. The systems under consideration include perturbations of generalised KdV equation, nonlinear wave equation, Boussinesq equation. These equations are equivariant with respect to Abelian subgroups of Euclidean group. It is assumed that the external perturbation preserves this symmetry. Travelling wave reduction for the above-mentioned systems results in a four-dimensional system of ODEs, which is considered for Melnikov type chaos. As a preliminary for the calculation of a Melnikov function, we prove the persistence of a fixed point for the perturbed Poincaré map by using Lyapunov–Schmidt reduction. The framework sketched will be applied to the analysis of possible chaotic behaviour of travelling wave solutions for the above-mentioned PDEs within the multi-symplectic approach.

1 Introduction

Recently it was shown how many nonlinear PDEs can be formulated in a multi-symplectic form [3, 4, 5]. This formulation assigns distinct symplectic structures to the spatial and the temporal coordinates, thereby generalising the usual Hamiltonian formulation. By means of the multi-symplectic approach questions concerning stability of solitary waves, existence of generalised basic state at infinity, equivariant properties of the solutions etc. can be considered in a more general setting, yielding new results on these issues.

In order to analyse chaotic behaviour of travelling wave solutions to Hamiltonian PDEs, Melnikov's method can be used. The problem with its direct application is due to the symmetry in a multi-symplectic formulation of these PDEs, which results in the presence of unit eigenvalue among the spectrum of the Poincaré map. This complication is solved by means of Lyapunov–Schmidt reduction.

Finally, we illustrate the application of the Melnikov method to the study of chaotic behaviour in a perturbed Korteweg-de Vries equation.

2 General setting

We start to consider a perturbed multi-symplectic PDE of the form [5]:

$$\mathbf{M}Z_t + \mathbf{K}Z_x = \nabla S(Z) + \epsilon S_1(Z, x-ct), \qquad Z = \begin{pmatrix} U \\ V \\ W \\ \Phi \end{pmatrix} \in \mathbb{R}^4, \qquad x \in \mathbb{R}, \tag{1}$$

where $\mathbf{M}$ and $\mathbf{K}$ are constant skew-symmetric matrices on $\mathbb{R}^4$ and $S : \mathbb{R}^4 \to \mathbb{R}$ is sufficiently smooth (at least twice continuously differentiable). The perturbation S_1 is assumed to be a periodic function of its last argument: $S_1(\cdot, x) = S_1(\cdot, x+T)$ and also C^r, $r \geq 2$.

We suppose that the system (1) is equivariant with respect to a one-parameter Lie group, whose algebra is spanned by the generator ξ. For the unperturbed case ($\epsilon = 0$), multi-symplectic Noether theory provides the existence of the two functionals $P(Z)$ and $Q(Z)$ such that [3]

$$\mathbf{M}\xi(Z) = \nabla P(Z), \qquad \mathbf{K}\xi(Z) = \nabla Q(Z). \tag{2}$$

The state at infinity should satisfy [4, 5]:

$$\nabla S(Z_0) = a\nabla P(Z_0) + b\nabla Q(Z_0) \tag{3}$$

with $P(Z_0) = \mathcal{P}$ and $Q(Z_0) = \mathcal{Q}$ specified real parameters, $a, b \in \mathbb{R}$.

A shape of an unperturbed solitary wave travelling at speed c, $Z(x,t) = Z(x - ct)$, which is biasymptotic to this state should satisfy the equation

$$Z_x = \mathbf{J}_c^{-1}\nabla H_0(Z), \tag{4}$$

where $H_0(Z) = S(Z) - aP(Z) - bQ(Z)$, and $\mathbf{J}_c = \mathbf{K} - c\mathbf{M}$ [4, 5].

To study the existence of travelling waves and their chaotic behaviour, we consider the dynamical system (similar consideration can be found in [7])

$$\frac{d}{dx}Z = f_0(Z) + \epsilon f_1(Z,x), \qquad Z \in \mathbb{R}^4, \qquad 0 < \epsilon \ll 1, \qquad 0 < x < \infty, \tag{5}$$

where $f_0(Z) = \mathbf{J}_c^{-1}\nabla H_0(Z)$, and $f_1(Z,x) = \mathbf{J}_c^{-1}S_1(Z,x)$.

The following hypotheses are imposed on the system:

(H1) **a)** $f_0 : \mathbb{R}^4 \to \mathbb{R}^4$ is C^r $(r \geq 2)$;

b) $f_1 : \mathbb{R}^4 \times S^1 \to \mathbb{R}^4$ is C^r $(r \geq 2)$.

The system (5) can be rewritten as a following suspended system:

$$\frac{dZ}{dx} = f_0(Z) + \epsilon f_1(Z,\theta), \qquad \frac{d\theta}{dx} = \omega, \tag{6}$$

with the frequency $\omega = 2\pi/T$. Its flow $\Phi_t^\epsilon : \mathbb{R}^4 \times S^1 \to \mathbb{R}^4 \times S^1$ is defined for all $t \in \mathbb{R}$.

(H2) a) *The unperturbed system*

$$\frac{d}{dx}Z = f_0(Z) = \mathbf{J}_c^{-1}\nabla H_0(Z) \tag{7}$$

is Hamiltonian with energy $H_0 : \mathbb{R}^4 \to \mathbb{R}$.

So, we have the corresponding symplectic form

$$\Omega(Z_1, Z_2) = \langle \mathbf{J}_c Z_1, Z_2 \rangle. \tag{8}$$

b) *The system (7) is equivariant with respect to a one-parameter symmetry group $\mathcal{G}$ spanned by the generator $\xi(Z)$. This group $\mathcal{G}$ is assumed to be either compact or a subgroup of affine translations. We also suppose that the perturbation preserves this symmetry.*

c) *The system (7) has a family of fixed points ϕp_0, where $p_0 = 0$ and $\phi \in \mathcal{G}$, and corresponding (heteroclinic) orbits $Z_0(x)$ such that*

$$\frac{d}{dx}Z_0(x) = f_0\left(Z_0(x)\right), \tag{9}$$

and $\lim_{x\to-\infty} Z_0(x) = \phi_1 p_0$ as well as $\lim_{x\to+\infty} Z_0(x) = \phi_2 p_0$, for some $\phi_1, \phi_2 \in \mathcal{G}$. Correspondingly, the unperturbed suspended system (6) $\epsilon = 0$ has a family of periodic orbits $\phi\gamma_0(x) = (\phi p_0, \omega x)$.

(H3) *$f_1(Z,x) = A_1 Z + f(x) + g(Z,x)$, where A_1 is a linear operator, $f(x) = f(x+T)$, $g(Z,x)$ is time-periodic with period T and also satisfies $g(0,x) = 0$, $Dg(0,x) = 0$.*

(H4) a) *For $\epsilon = 0$ the spectrum $\sigma[\exp(TA)] = \{1, 1, e^{\pm\lambda T}\}$, $\lambda > 0$, where $A = \mathbf{J}_c^{-1} D^2 H_0(p_0)$,*
b) i. (Hamiltonian case) *For $\epsilon > 0$ $\sigma[\exp[T(A + \epsilon A_1)]] = \left\{1, 1, e^{T\lambda_\epsilon^\pm}\right\}$,*

ii. (Dissipative case) *For $\epsilon > 0$ $\sigma[\exp[T(A+\epsilon A_1)]] = \left\{1, \lambda^d, e^{T\lambda_\epsilon^\pm}\right\}$, where $C_1\epsilon \leqslant \mathrm{dist}(\lambda^d, |z| = 1) \leqslant C_2\epsilon$, $C_1 > 0$, $C_2 > 0$.*

Next, one can define the Poincaré map $P^\epsilon : \mathbb{R}^4 \to \mathbb{R}^4$ as

$$P^\epsilon(Z) = \pi_1 \Phi_T^\epsilon(Z, 0), \tag{10}$$

where $\pi_1 : \mathbb{R}^4 \times S^1 \to \mathbb{R}^4$ denotes the projection onto the first factor. Equivalently, one can define

$$P_{x_0}^\epsilon(Z) = \pi_1 \Phi_T^\epsilon(Z, x_0). \tag{11}$$

We rewrite the fixed point equation $P^\epsilon(p_\epsilon) = p_\epsilon$ in the form:

$$\mathcal{P}^\epsilon(p_\epsilon) = 0, \tag{12}$$

where $\mathcal{P}^\epsilon(Z) = P^\epsilon(Z) - Z$, and the operator $L = D\mathcal{P}^0(0)$ is introduced.

3 Main results

Lemma 1. *Let* **(H1)**–**(H4)** *hold. For ϵ small, there exists a unique group orbit ϕp_ϵ of fixed points of the perturbed Poincaré map near the group orbit ϕp_0 such that*

$$\min_{\phi_1, \phi_2 \in \mathcal{G}} \{\phi_1 p_\epsilon - \phi_2 p_0\} = \mathcal{O}(\epsilon).$$

Equivalently, there is a family of periodic orbits $\phi\gamma_\epsilon(x) = (\phi p_\epsilon, \omega x)$ of the perturbed system (6) near $\phi\gamma_0(x)$ for $\phi \in \mathcal{G}$.

Lemma 2. *For $\epsilon > 0$ sufficiently small, we have $\sigma\left[DP^\epsilon(p_\epsilon)\right] = \left\{1, 1, e^{T\lambda_\epsilon^\pm}\right\}$ for the case* **(H4b i)** *and $\sigma\left[DP^\epsilon(p_\epsilon)\right] = \left\{1, \lambda^d, e^{T\lambda_\epsilon^\pm}\right\}$ for the case* **(H4b ii)** *respectively.*

Conjecture 3. Corresponding to the eigenvalues $e^{T\lambda_\epsilon^\pm}, 1$ there exist invariant manifolds: $W^{ss}(\gamma_\epsilon(x))$ (the strong stable manifold), $W^u(\gamma_\epsilon(x))$ (the unstable manifold), and $W^c(\gamma_\epsilon(x))$ (the centre manifold) of p_ϵ for the Poincaré map $P^\epsilon(Z)$ such that

i) $W^u_{\mathrm{loc}}(\gamma_\epsilon(x))$ and $W^{ss}_{\mathrm{loc}}(\gamma_\epsilon(x))$ are tangent to the eigenspaces of $e^{T\lambda_\epsilon^\pm}$ respectively at γ_ϵ, while $W^c_{\mathrm{loc}}(\gamma_\epsilon(x))$ is at the same point tangent to the eigenspace corresponding to unity eigenvalue (double in the case of Hamiltonian perturbations). Their global analogues are obtained in the usual way:

$$W^{ss}(\gamma_\epsilon(x)) = \bigcup_{x \leq 0} \Phi_x^\epsilon W^{ss}_{\mathrm{loc}}(\gamma_\epsilon(x)), \qquad W^u(\gamma_\epsilon(x)) = \bigcup_{x \geq 0} \Phi_x^\epsilon W^u_{\mathrm{loc}}(\gamma_\epsilon(x)); \tag{13}$$

ii) they are invariant under $P^\epsilon(\cdot)$;

iii) $W^{ss}(\gamma_\epsilon(x))$ and $W^u(\gamma_\epsilon(x))$ are C^r $\mathcal{O}(\epsilon)$-close to $W^s(\gamma_0(x))$ and $W^u(\gamma_0(x))$ respectively.

Lemma 4. *Let $(Z_\epsilon^{s,u}(x, x_0), \omega x)$ be orbits lying in $W^{ss,u}(\gamma_\epsilon(x))$ and originating in an $\mathcal{O}(\epsilon)$-neighbourhood of $(Z_0(-x_0), 0)$. Then the following expressions hold with uniform validity in the indicated time intervals:*

$$
\begin{aligned}
Z_\epsilon^s(x, x_0) &= Z_0(x - x_0) + \epsilon y_\epsilon^s(x, x_0) + \mathcal{O}\left(\epsilon^2\right), \qquad x \in [x_0, \infty), \\
Z_\epsilon^u(x, x_0) &= Z_0(x - x_0) + \epsilon y_\epsilon^u(x, x_0) + \mathcal{O}\left(\epsilon^2\right), \qquad x \in (-\infty, x_0],
\end{aligned} \tag{14}
$$

where $y_\epsilon^{s,u}$ satisfy the first variational equation:

$$
\frac{dy}{dx} = \mathbf{J}_c^{-1} D^2 H_0(Z_0(x - x_0))y + \epsilon f_1(Z_0(x - x_0), \omega x). \tag{15}
$$

We introduce Melnikov function as:

$$
\begin{aligned}
M(x_0) &= \int_{-\infty}^{\infty} DH_0(Z_0(x)) \cdot f_1(Z_0(x), x + x_0) dx \\
&= \int_{-\infty}^{\infty} \Omega\left(f_0(Z_0(x)), f_1(Z_0(x), x + x_0\right) dx.
\end{aligned} \tag{16}
$$

Theorem 1. *Suppose $M(x_0)$ has simple zeros. Then for $\epsilon > 0$ sufficiently small $W^{ss}(\gamma_\epsilon)$ and $W^u(\gamma_\epsilon)$ intersect transversely.*

This result implies via the Smale–Birkhoff theorem [6] the appearance of a horseshoe near the saddle-centre point of the perturbed Poincaré map, what results in a chaotic dynamics in the corresponding region of the phase space.

4 Example

We consider the perturbed Korteweg-de Vries equation [1, 2]:

$$
u_t + \Delta u_x + \alpha u u_x + u_{xxx} = \epsilon f_x(u, u_x, u_t, x - ct), \tag{17}
$$

where the perturbation is assumed to be T-periodic in its last argument. The unperturbed equation can be rewritten in a multi-symplectic form as

$$
\mathbf{M}Z_t + \mathbf{K}Z_x = \nabla S, \qquad Z = \begin{pmatrix} u \\ v \\ w \\ \Phi \end{pmatrix} \in \mathbb{R}^4, \qquad x \in \mathbb{R} \tag{18}
$$

with

$$
\mathbf{M} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{K} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{19}
$$

and

$$
S = \frac{1}{2}v^2 - \frac{1}{2}uw + u\left(\frac{1}{2}\Delta u + \frac{\alpha}{6}u^2\right). \tag{20}
$$

The unperturbed solution is defined as:

$$\begin{aligned}
u_0(x) &= 2b + \frac{3}{\alpha}K^2 \mathrm{sech}^2\left(\frac{Kx}{2}\right),\\
v_0(x) &= -\frac{3}{\alpha}K^3 \sinh\left(\frac{Kx}{2}\right)\mathrm{sech}^3\left(\frac{Kx}{2}\right),\\
w_0(x) &= 2a + 4b(\Delta + \alpha b) + \frac{3}{\alpha}cK^2 \mathrm{sech}^2\left(\frac{Kx}{2}\right),\\
\Phi_0(x) &= \frac{3}{\alpha}K\left[1 + \tanh\left(\frac{Kx}{2}\right)\right],
\end{aligned} \tag{21}$$

with $K = \sqrt{c - \Delta - 2\alpha b}$. Perturbation can be represented in a multi-symplectic setting as:

$$\epsilon f_1(Z, x) = \begin{pmatrix} 0 \\ \epsilon \tilde{f}(Z, x) \\ 0 \\ 0 \end{pmatrix}. \tag{22}$$

Here we expressed the perturbation $\tilde{f}(Z, x) = f(u, v, -cv, x)$ in terms of the components of Z. Therefore Melnikov function (16) for the system (17) yields

$$M(x_0) = \int_{-\infty}^{\infty} v_0(x)\tilde{f}(Z_0(x), x + x_0)dx. \tag{23}$$

For the one-harmonic dissipative driving force this Melnikov function will have simple zeros [2], and therefore one can conclude the chaotic dynamics of u.

5 Conclusions

We have considered a modification of Melnikov's method, which can be used for the analysis of chaotic behaviour of travelling wave solutions to multi-symplectic PDEs. The results are illustrated with the example of the perturbed KdV equation.

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Smoothness Properties of Green's–Samoilenko Operator-Function the Invariant Torus of an Exponentially Dichotomous Bilinear Matrix Differential System

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In this paper the smoothness properties of Green's operator-function an exponentially dichotomous bilinear matrix system and the smoothness properties the invariant torus of nonhomogeneous matrix system of equations have been considered. It hHave been proved that if some conditions, concerning the properties of coefficient of the system hold this operator-function has smoothness index which depends on both the smoothness of matrix coefficients of the system and their spectral properties.

We consider the system of equations

$$\frac{d\phi}{dt} = a\,(\phi)\,, \qquad \frac{dX}{dt} = A\,(\phi)\,X - XB\,(\phi) + F\,(\phi)\,, \tag{1}$$

where $a^T(\phi) = (a_1(\phi), a_2(\phi), \ldots, a_m(\phi))$, $\phi^T = (\phi_1, \phi_2, \ldots, \phi_m)$, $\phi_i \in [0, 2\pi)$, $i = \overline{1, m}$, are vectors, $A(\phi) = A_{n_1\times n_1}$, $B(\phi) = B_{n_2\times n_2}$, $F(\phi) = F_{n_1\times n_2}$, $X = X_{n_1\times n_2}$ are matrix functions defined and continuous with respect to $\phi \in T_m$, where $T_m = T_1 \times T_1 \times \cdots \times T_1$ is m-dimensional torus, $\phi_i \in T_1 = [0, 2\pi)$, $i = \overline{0, m}$. We shall call the system (1) a matrix bilinear non-homogeneous system of equations defined on a direct product of m-dimensional torus T_m and the space of matrices $M_{n_1\times n_2}$, under assumption that spectral sets of matrices $A(\phi)$ and $B(\phi)$ satisfy the condition $\sigma(A(\phi)) \bigcap \sigma(B(\phi)) = \varnothing$, and the system (1) is exponentially dichotomous. We define the norm of matrix in the space $M_{n_1\times n_2}$ as Frobenius or trace-norm $\|X\|^2 = \operatorname{tr}(X^*X)$. The Green's operator-function for the system of equations (1) defined by relation

$$[G_t(\tau,\phi)]F(\phi_\tau(\phi)) = \begin{cases} [\Omega^t_0(\phi)][P_1(\phi)][\Omega^0_\tau(\phi)]F(\phi_\tau(\phi)), & t \geq \tau, \\ -[\Omega^t_0(\phi)][P_2(\phi)][\Omega^0_\tau(\phi)]F(\phi_\tau(\phi)), & t < \tau. \end{cases} \tag{2}$$

where $[\Omega^{\,t}_{\,\tau}(\phi)]Z = \underset{A}{\Omega}{}^{t}_{\tau}(\phi) Z \underset{B}{\Omega}{}^{\tau}_{t}(\phi)$, $[P_k(\phi)]Z = \sum P_{i_k}(\phi) Z Q_{j_k}(\phi)$, (k=1,2), $\underset{A}{\Omega}{}^{t}_{\tau}(\phi)$, $\underset{B}{\Omega}{}^{t}_{\tau}(\phi)$ are matricents of matrix differential equation associated with matrix A and B accordingly. $P_i(\phi)$, $Q_j(\phi)$ are projection operators to proper subspace in Euclidean space E_{n_1} and E_{n_2}. $[P_1]$, $[P_2]$ are projection operators in the space of matrices $M_{n_1\times n_2}$, $[P_1(\phi)] + [P_2(\phi)] = [I]$, $[I]$ is the identity operator in the space $M_{n_1\times n_2}$, $[I]Z = I_{n_1} Z I_{n_2} = Z$, $\eta_{i_k,j_k}(A(\phi), B(\phi)) = \lambda_{i_k}(A(\phi)) - \mu_{j_k}(B(\phi))$ is an eigenvalue of operator $\Phi(\phi)X = A(\phi)X - XB(\phi)$, $k = 1$ when $\eta_{i_k,j_k}(A(\phi), B(\phi)) < 0$ and $k = 2$ when $\eta_{i_k,j_k}\ (A(\phi), B(\phi)) > 0$, $\lambda_{i_k}(A(\phi))$ and $\mu_{j_k}(B(\phi))$ are eigenvalues of matrices $A(\phi)$ and $B(\phi)$ accordingly. The solution of the second homogeneous matrix equation (1) has the form [1] $X_t(\phi, X) = \underset{A}{\Omega}{}^{t}_{\tau}(\phi) X \underset{B}{\Omega}{}^{\tau}_{t}(\phi) = [\Omega^{\,t}_{\,\tau}(\phi)]X$. The operator $[\Omega^{\,t}_{\,\tau}(\phi)]$ in the space $M_{n_1\times n_2}$ has the group property $[\Omega^{\,t}_{\,\tau}(\phi_\theta(\phi))] = [\Omega^{\,t+\theta}_{\,\tau+\theta}(\phi)]$ that follows from the properties of matricents $\underset{A}{\Omega}{}^{t}_{\tau}(\phi)$, $\underset{B}{\Omega}{}^{t}_{\tau}(\phi)$ [2].

We call $[G_t(\tau,\phi)]$ a Green's operator-function for system of equations (1) in the case when integral

$$\int_{-\infty}^{\infty} \|[G_0(\tau,\phi)]\|\, d\tau \le K < \infty$$

is uniformly bounded with respect to ϕ. We give the simplest properties of the Green's operator-function. It follows from its definition that $[G_0(\tau,\phi)] \in C(T_m)$ for $\forall\ \tau$ and $[G_0(-0,\phi)] - [G_0(+0,\phi)] = [P_1(\phi)] + [P_2(\phi)] = [I]$. Suppose that matricents $\underset{A}{\Omega}{}^t_\tau(\phi)$, $\underset{B}{\Omega}{}^t_\tau(\phi)$ satisfy inequalities

$$\|\underset{A}{\Omega}{}^t_\tau(\phi)\| \le K_1 \exp(-\gamma_1(t-\tau)), \qquad \|\underset{B}{\Omega}{}^t_\tau(\phi)\| \le K_2 \exp(-\gamma_2(\tau-t)), \qquad t>\tau, \tag{3}$$

for all $\phi \in T_m$, and some positive K_i, γ_i, $(i=1,2)$ independent of ϕ. From (3) the estimate follows $(t>\tau)$

$$|[G_t(\tau,\phi)]F(\phi_\tau)\| \le \|\underset{A}{\Omega}{}^t_\tau(\phi)\|\, \|F(\phi_\tau)\|\, \|\underset{B}{\Omega}{}^\tau_t(\phi)\| \le Ke^{-(\gamma_1-\gamma_2)(t-\tau)}\, \|F(\phi_\tau(\phi)\|\,. \tag{4}$$

We suppose that homogeneous system of equations (1) is exponentially dichotomous, then a Green's operator-function satisfies the estimate [3, 4]

$$\|[G_t(\tau,\phi)]\| \le Ke^{-(\gamma_1-\gamma_2)|t-\tau|}, \qquad t,\tau\in\mathbb{R}, \quad \phi\in T_m, \tag{5}$$

where $K>0$, $\gamma=\gamma_1-\gamma_2>0$ are positiv constants independent of ϕ.

From estimate (5) the existence of invariant matrix torus of the system (1) follows, which is given by the relation

$$U(\phi) = \int_{-\infty}^{0} [\Omega^t_\tau(\phi)][P_1(\phi_\tau(\phi))]F_\tau(\phi)d\tau - \int_{0}^{\infty} [\Omega^t_\tau(\phi)][P_2(\phi_\tau(\phi))]F_\tau(\phi)d\tau. \tag{6}$$

The smoothness of invariant torus (6) of the system (1) depends essentially on the properties of the Green's function $[G_0(\tau,\phi)]$ and the solution $\phi_t(\phi)$ of the first equation of the system, which defines a trajectory flow for system (1) on the torus $U(\phi)$ [2]. We need to have the estimate of derivative $\partial\phi_t(\phi)/\partial\phi_j$ which is equal j-th column Jacobi matrix for vector-function $\phi_t(\phi)$, which is satisfying the system of equations $d\theta/dt = a'(\phi)\theta$, where $a'(\phi) = D\phi_t(\phi)/D\phi$ is the matrix of partial derivative of the function $\phi_t(\phi)$ or Jacobi matrix. We denote $\underset{a}{\Omega}{}^t_0(\phi)$ matricient of this system, it is characterized the stability of solutions of a nonperturbed system on a torus [2]. For obtaining of estimate of derivatives of operator-function $[G_t(\tau,\phi)]$ in ϕ_i we use the estimate of derivatives $\partial^s\phi_t(\phi)/\partial^s\phi_i = D^s_{\phi_i}\phi_t(\phi)$ which was obtained in [2].

$$\|D^s_{\phi_i}\phi_t(\phi)\| \le Ke^{(s\alpha+\varepsilon)|t|}, \qquad t\in\mathbb{R}, \quad \phi\in T_m, \tag{7}$$

Taking $s=1$ we obtain the estimate $\|\underset{a}{\Omega}{}^t_0(\phi)\| \le Ke^{(\alpha+\varepsilon)|t|}$. The estimate of derivatives of operator-function $[G_t(\tau,\phi)]$ is essentially defined by smoothness properties of invariant torus of the nonhomogeneous system of equations and depends on smoothness of coefficients of the system (1) $a(\phi)$, $A(\phi)$, $B(\phi)$ and spectral properties of matrices $A(\phi)$, $B(\phi)$.

Theorem 1. *Assume that for some integer positive number $l \ge 0$ the following conditions holds: $A(\phi)\in C^l_{\mathrm{Lip}}(T_m)$, $B(\phi)\in C^l_{\mathrm{Lip}}(T_m)$, $a(\phi)\in C^l_{\mathrm{Lip}}(T_m)$ and $\tilde\gamma=\gamma_1-\gamma_2-\varepsilon \ge l\alpha$, where $\alpha>0$, $\varepsilon>0$ is an arbitrary small positive number. Then*

$$\|D^s_\phi[G_0(\tau,\phi)]\| \le Ke^{-(\gamma_1-\gamma_2-\varepsilon-s\alpha)|\tau|}, \tag{8}$$

where $0\le s\le l$, $K=K(\varepsilon)$ is a positive constant independent of $\phi\in T_m$.

Proof. Because $e^{\varepsilon|\tau|} > 1$ for $\tau \neq 0$, then $e^{-\gamma|\tau|} < e^{-\gamma|\tau|+\varepsilon|\tau|} = e^{-\tilde{\gamma}|\tau|}$, $|\tau| < (1/\varepsilon)e^{\varepsilon|\tau|}$. If $l = 0$ the estimate (8) followed from (5) and the operator $[G_0(\tau,\phi)]$ belongs to the space $C(T_m)$, we therefore suppose that $l > 0$. Consider the difference $[Z_t(\tau,\bar{\phi},\phi)] = [G_t(\tau,\bar{\phi})] - [G_t(\tau,\phi)]$, where $\bar{\phi} = \phi + \Delta\phi_i e_i$, $e_i^T = (0,\ldots,0,1,0,\ldots,0)$ is unit vector and $\Delta\phi_i$ is a scalar constant. $[Z_t(\tau,\bar{\phi},\phi)]F$ satisfies the matrix equation $(t \neq \tau)$

$$d([Z_t(\tau,\bar{\phi},\phi)]F)/dt = \Phi_{A,B}\left\{[Z_t(\tau,\bar{\phi},\phi)]F\right\} + \Psi_{A,B}(t,\tau,\phi,\bar{\phi}), \tag{9}$$

where

$$\begin{aligned}
&\Phi_{A,B}\left\{X_t\right\} = A_t(\bar{\phi})X_t - X_tB_t(\bar{\phi}),\\
&\Psi_{A,B}(t,\tau,\phi,\bar{\phi}) = \Phi_{\Delta A,\Delta B}\left\{[G_t(\tau,\phi)]F\right\} = \Delta A(\phi_t)\left([G_t(\tau,\phi)]F\right) - \left([G_t(\tau,\phi)]F\right)\Delta B(\phi_t),\\
&\Delta A(\phi_t) = A(\phi_t(\bar{\phi})) - A(\phi_t(\phi)), \qquad \Delta B(\phi_t) = B(\phi_t(\bar{\phi})) - B(\phi_t(\phi)).
\end{aligned}$$

It has a unique bounded solution on R given by the expression

$$[Z_t(\tau,\bar{\phi},\phi)]F_t(\phi) = \int_{-\infty}^{\infty}[G_t(s,\bar{\phi})]\Psi_{A,B}(s,\tau,\phi,\bar{\phi})ds. \tag{10}$$

Inequality (5) ensures that the operator-function $[Z_t(\tau,\bar{\phi},\phi)]$ is bounded on $(-\infty,\infty)$. If we devide the expression (10) on $\Delta\phi_i$ and equal it to zero, we obtain $\lim\limits_{\Delta\phi_i\to 0}(\Delta A(\phi_t))/\Delta\phi_i = D_{\phi_i}A(\phi_t)$, $\lim\limits_{\Delta\phi_i=\to 0}(\Delta B\phi_t)/\Delta\phi_i = D_{\phi_i}B(\phi_t)$, $\lim\limits_{\Delta\phi_i\to 0}([Z_t(\tau,\bar{\phi},\phi)]F)/\Delta\phi_i = D_{\phi_i}([G_t(\tau,\phi)]F)$. We will be use notations $A(\phi_t(\phi)) = A_{1,t}(\phi)$, $B(\phi_t(\phi)) = A_{2,t}(\phi)$, $\partial/\partial\phi_i = D_{\phi_i}$. Since $\lim\limits_{\Delta\phi\to 0}\Psi_{A,B}(s,\tau,\phi,\bar{\phi})$ uniformly with respect to $\phi \in T_m$ and $\tau, s \in D_2$, it follows that

$$\lim_{\Delta\phi_i\to 0}[Z_t(\tau,\bar{\phi},\phi)]F/\Delta\phi_i = D_{\phi_i}[G_t(\tau,\phi)]F = \int_{-\infty}^{\infty}J_t(s,\tau,\phi,F)ds, \tag{11}$$

where

$$\begin{aligned}
&J_t(s,\tau,\phi,F) = [G_t(s,\phi)]\Phi_{D_\phi A,D_\phi B}\left\{[G_s(\tau,\phi)]F\right\},\\
&\Phi_{D_{\phi_i}A,D_{\phi_i}B}\left\{[G_s(\tau,\phi)]F\right\} = D_{\phi_i}A(\phi_s)([G_s(\tau,\phi)]F) - ([G_s(\tau,\phi)]F)D_{\phi_i}B(\phi_s),\\
&D_{\phi_i}A_k(\phi_s) = \sum_{\nu=1}^{m}(\partial A_k(\phi_s(\phi))/\partial(\phi_s)_\nu)(\partial(\phi_s)_\nu/\partial\phi_i), \qquad (k = 1,2).
\end{aligned}$$

Here D_2 is any bounded domain of the τ, s plane. The value $\lim\limits_{\Delta\phi_i\to 0}[Z_t(\tau,\bar{\phi},\phi)]$ equal derivative of operator-function, when integral is uniformly convergent.

For the following estimates we will be use the formulas of Faa de Bruno [5]

$$D_\phi^r f(\phi_t(\phi)) = \sum_{q=1}^{r}D_{\phi_t}^q f(\phi_t(\phi))\sum_p c_{qp}(D_\phi\phi_t(\phi))^{p_1}\left(D_\phi^2\phi_t(\phi)\right)^{p_2}\cdots\left(D_\phi^r\phi_t(\phi)\right)^{p_r}, \tag{12}$$

where $p_1 + p_2 + \cdots + p_r = q$, $p_1 + 2p_2 + \cdots + rp_r = r$. For obtaining the estimate of function $J_t(s,\tau,\phi,F)$ we need to have the estimate of only the first summand, because the estimate for the second one is different from the first summand by a constant multiplier

$$\|[G_t(s,\phi)]D_{\phi_i}A(\phi_s)[G_s(\tau,\phi)]F\| \le K\exp(-\gamma|s-t| - \gamma|s-\tau| + \tilde{\alpha}|s|)\|F\|. \tag{13}$$

Therefore summarizing the estimates (13) from both summands, we obtain

$$\|J_t(s,\tau,\phi,F)\| \le K\exp(-\gamma|t-s| + \tilde{\alpha}|s| - \gamma|s-\tau|)\|F\|,$$

where $\varepsilon > 0$, $\tilde{\alpha} = \alpha + \varepsilon$, $K = K(\varepsilon)$ and independent of ϕ. Taking $t = 0$, we obtain

$$\|J_0(s,\tau,\phi,F)\| \leq K\exp(-(\gamma-\tilde{\alpha})|s| - \gamma|s-\tau|)\|F\|. \tag{14}$$

For obtaining the estimate of derivative $\|D_{\phi_i}[G_0(\tau,\phi)]\|$ it is necessary to have the estimate of the integral of function $J_0(s,\tau,\phi,F)$, we consider the case $\tau > 0$. We represent the integral as sum of three integrals $(-\infty,\infty) = (-\infty,0)\bigcup(0,\tau)\bigcup(\tau,\infty)$, after simple transformation we obtain the estimate

$$\int_{-\infty}^{\infty}\|J_0(s,\tau,\phi,F)\|ds \leq K(\varepsilon)e^{-(\gamma-\tilde{\alpha})\tau+\varepsilon\tau}\|F\|,$$

where $K(\varepsilon) = (2/(2\gamma-\tilde{\alpha}) + \tau)$, from which follow estimate

$$||D_{\phi_i}[G_0(\tau,\phi)]|| \leq K(\varepsilon_1)e^{-(\tilde{\gamma}-\alpha)|\tau|}$$

for $\forall\tau \in R$, $\tilde{\gamma} = \gamma - \varepsilon_1$, $\varepsilon_1 = 2\varepsilon$, $K(\varepsilon_1) = K(2/(2\gamma-\tilde{\alpha}) + |\tau|)$. The estimate for the second derivative we obtain from relation

$$D^2_{\phi_i}[G_0(\tau,\phi)]F = \int_{-\infty}^{\infty} D_{\phi_i}J_0(s,\tau,\phi,F)ds. \tag{15}$$

The estimate of both summands of function $D_{\phi_i}J_0(s,\tau,\phi,F)$ will be similar, therefore we need only one of this estimate

$$\begin{aligned} D_{\phi_i}\left([G_0(s,\phi)]D_{\phi_i}A(\phi_s)[G_s(\tau,\phi)]F\right) &= D_{\phi_i}[G_0(s,\phi)]D_{\phi_i}A(\phi_s)[G_s(\tau,\phi)]F \\ &+ [G_0(s,\phi)]D^2_{\phi_i}A(\phi_s)[G_s(\tau,\phi)]F + [G_0(s,\phi)]D_{\phi_i}A(\phi_s)D_{\phi_i}[G_s(\tau,\phi)]F. \end{aligned} \tag{16}$$

For obtaining the estimate of last summand of (16) transform $[G_s(\tau,\phi)]$ to the form $[G_0(\tau - s,\phi_s(\phi))]$, and use the formulas Faa de Bruno

$$D^k_{\phi_i}[G_s(\tau,\phi)] = \sum_{j=1}^{k} D^j_{\phi_i}[G_0(\tau-s,\phi_s(\phi))]\sum_m c_{jm}(D_{\phi_i}\phi_s(\phi))^{m_1}\cdots(D^k_{\phi}\phi_s(\phi))^{m_k}$$

where $m_1 + m_2 + \cdots + m_k = j$, $m_1 + 2m_2 + \cdots + km_k = k$. For $k = 1$ we obtain an estimate

$$\|D_{\phi_i}J_0(s,\tau,\phi,F)\| \leq K(\varepsilon)(e^{-(\tilde{\gamma}-2\alpha)|s|-\tilde{\gamma}|s-\tau|} + e^{-(\tilde{\gamma}-2\alpha)|s|-(\tilde{\gamma}-\alpha)|s-\tau|})\|F\|. \tag{17}$$

Taking the integral from expression on right hand side, we obtain estimate

$$\|D^2_{\phi_i}[G_0(\tau,\phi)]F\| \leq K(\varepsilon)e^{-(\tilde{\gamma}-2\alpha)|\tau|}\|F\|. \tag{18}$$

We carry out the proof by induction. Suppose that inequality (8) holds for $s = k$, we will prove that it then holds for $s = k+1$. To prove this we differentiate the identity (10) k times, $(t = 0)$

$$D^{k+1}_{\phi_i}[G_0(\tau,\phi)]F = \int_{-\infty}^{\infty} D^k_{\phi_i}J_0(s,\tau,\phi,F)ds. \tag{19}$$

Consider one of summands of function $D^k_{\phi_i}J_0(s,\tau,\phi,F)$, it has the form

$$\sum_{j=0}^{k} C_k^{k-j}([G_0(s,\phi)]D_{\phi_i}A(\phi_s))^{(k-j)}([G_0(\tau-s,\phi_s)])^{(j)}F.$$

For the first multiplier, under sign of the sum, an estimate has the form

$$\|D^{k-j}_{\phi_i}([G_0(s,\phi)D_{\phi_i}A(\phi_s))\| \leq \tilde{K}e^{-(\tilde{\gamma}-(k-j+1)\alpha-\varepsilon)|s|}, \tag{20}$$

where $\tilde{K} = K(\varepsilon)\sum\limits_{p=1}^{k-j} C_{k-j}^{k-j-p}$. The Faa de Bruno formulas allow one to obtain an estimate for $[G_0(\tau - s, \phi_s(\phi))]^{(j)}$ of the form

$$\|D_{\phi_i}^j[G_0(\tau - s, \phi_s(\phi))]\| \leq jK(\varepsilon)e^{-(\tilde{\gamma}-j\alpha)|s-\tau|+(j\alpha+\varepsilon)|s|}. \tag{21}$$

Using estimate (20), (21) one can obtain estimate

$$\|D_{\phi_i}^k([G_0(s,\phi)]D_{\phi_i}A_q(\phi_s)[G_s(\tau,\phi)])F\| \leq \bar{K}_q e^{-(\tilde{\gamma}-(k+1)\alpha)|s|-(\tilde{\gamma}-k\alpha)|s-\tau|}\|F\|,$$

where $\bar{K}_q = K_q(\varepsilon)\sum\limits_{j=1}^{k} jC_k^{k-j}$, $(q = 1, 2)$ independent of ϕ. Summarizing all estimates we have the inequality

$$\|D_{\phi_i}^{k+1}[G_0(\tau,\phi)]F\| \leq K(\varepsilon)e^{-(\tilde{\gamma}-(k+1)\alpha)|\tau|}\|F\|$$

and the proof of the Theorem 1 is complete. ■

Theorem 1 allows one to prove the theorem about smoothness of invariant torus of the dichotomous matrix bilinear equation.

Theorem 2. *Let the following conditions be satisfied:* $A(\phi) \in C^l_{\mathrm{Lip}}(T_m)$, $B(\phi) \in C^l_{\mathrm{Lip}}(T_m)$, $a(\phi) \in C^l(T_m)$ *and* $F(\phi) \in C^l(T_m)$, *then the invariant matrix torus (6) of system (1) belongs to the space* $C^l(T_m)$ *and admits the estimate*

$$|U(\phi)|_l \leq K|F(\phi)|_l.$$

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Systems of Linear Differential Equations of Rational Rank with Multiple Root of Characteristic Equation

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A method of the reduction of linear differential equations with multiple root of the characteristic equation to which some multiple elementary divisors correspond to the system, the perturbed characteristic equation of which has the simple roots as well as asymptotic estimation of solutions obtained are presented.

Consider the system of linear differential equations of the following type

$$\varepsilon^{p/q}\frac{dx}{dt} = A(t,\varepsilon)x, \tag{1}$$

where x is an n-dimensional vector, $A(t,\varepsilon) = \sum\limits_{s=0}^{\infty} \varepsilon^s A_s(t)$ is real square $(n\times n)$ dimension matrix, whose elements are infinitely differentiable by t on the segment $[0; L]$, $\varepsilon > 0$ is a small parameter, p and q are natural relatively prime numbers. Besides let the inequality $p < n \le q$ take place.

Let us denote $\varepsilon^{1/q} = \mu$. Then the system (1) reduces to the form

$$\mu^p\frac{dx}{dt} = \left(A_0(t) + \mu^q A_1(t) + \mu^{2q}A_2(t) + \cdots\right)x, \tag{2}$$

where $\varepsilon = \mu^q$, $\varepsilon^{\frac{p}{q}} = \mu^p$.

The systems for which small parameter has a fractional power were studied by V.K. Grigorenko in [1]. The case of the simple roots of the characteristic equations and the case of the equation having only one multiple n root were studied separately. Let us construct the asymptotic solution of the system (1) by the method of perturbed characteristic equation [2] for the case when the matrix $A_0(t)$ is such that the characteristic equation has one multiple root λ_0, to which $m \geqslant 1$ multiple elementary divisors correspond.

It means that there is non-degenerate matrix $T(t)$ which leads matrix $A_0(t)$ to the matrix with the simplest structure of quasi-diagonal type

$$W(t) = \{H_1(\lambda_0(t)), H_2(\lambda_0(t)), \ldots, H_m(\lambda_0(t))\},$$

where $H_i(\lambda_0(t))$ is Jordan cells, and the length of a cell is equal a multiplicity of elementary divisor, $i = 1, 2, \ldots, m$, m is the number of elementary divisors. Let us put that the elementary divisors with every value of $t \in [0; L]$ have the same multiplicity. Let a set of elementary divisors be $k_1, k_2, \ldots, k_m$ and $k_1 \ge k_2 \ge \cdots \ge k_m$. The substitution $x = T(t)y$ reduces the system (2) to system

$$\mu^p\frac{dy}{dt} = D(t,\mu)y, \tag{3}$$

where

$$D(t,\mu) = D_0(t,\mu) + \sum_{s=1}^{\infty}\mu^{qs}D_s(t),$$

$$D_0(t,\mu) = W(t) - \mu^p T^{-1}(t)T'(t), \qquad D_s(t) = T^{-1}(t)A_s(t)T(t),$$

$T'(t)$ is a derivative of matrix $T(t)$.

Let us consider perturbed equation

$$\det \|D_0(t,\mu) - \lambda E\| = 0. \tag{4}$$

Or opening the determinant (4),

$$(\lambda_0 - \lambda)^n + (\lambda_0 - \lambda)^{n-1} c_{n-1}(t,\mu) + \cdots + c_1(t,\mu)(\lambda_0 - \lambda) + c_0(t,\mu) = 0. \tag{5}$$

It is known that coefficients $c_i(t,\mu)$ in expansion (5) will be equal to the sum of all principal minors $n-i$ order of the matrix

$$D_0(t,\mu) - \lambda_0(t)E = W(t) - \mu^p T^{-1}(t)T'(t) - \lambda_0(t)E$$
$$= \begin{pmatrix} \mu^p\overline{t}_{11} & 1+\mu^p\overline{t}_{12} & \dots & \mu^p\overline{t}_{1k_1} & \mu^p\overline{t}_{1k_1+1} & \mu^p\overline{t}_{1k_1+2} & \dots & \mu^p\overline{t}_{1n} \\ \mu^p\overline{t}_{21} & \mu^p\overline{t}_{22} & \dots & \mu^p\overline{t}_{2k_1} & \mu^p\overline{t}_{2k_1+1} & \mu^p\overline{t}_{2k_1+2} & \dots & \mu^p\overline{t}_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \mu^p\overline{t}_{k_1,1} & \mu^p\overline{t}_{k_1,2} & \dots & \mu^p\overline{t}_{k_1,k_1} & \mu^p\overline{t}_{k_1,k_1+1} & \mu^p\overline{t}_{k_1,k_1+2} & \dots & \mu^p\overline{t}_{k_1,n} \\ \mu^p\overline{t}_{k_1+1,1} & \mu^p\overline{t}_{k_1+1,2} & \dots & \mu^p\overline{t}_{k_1+1,k_1} & \mu^p\overline{t}_{k_1+1,k_1+1} & 1+\mu^p\overline{t}_{k_1+1,k_1+2} & \dots & \mu^p\overline{t}_{k_1+1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \mu^p\overline{t}_{n1} & \mu^p\overline{t}_{n2} & \dots & \mu^p\overline{t}_{n,k_1} & \mu^p\overline{t}_{n,k_1+1} & \mu^p\overline{t}_{n,k_1+2} & \dots & \mu^p\overline{t}_{nn} \end{pmatrix},$$

where $\overline{t}_{ij}$ is a matrix element $-T^{-1}(t)T'(t)$, $i,j = \overline{1,n}$.

If m multiple elementary divisors correspond to multiple root, it means that all elements of the given matrix will be of $O(\mu^p)$ order, but it $n-m$ of the first over-diagonal elements will be $1 + \mu^p\overline{t}_{ij}$. Proceeding from this, for estimation $\lambda - \lambda_0$ let us draw the first diagram of equation (5).

As ρ_{n-1} corresponds to polynomial power $c_{n-1}(t,\mu) = spD(t,\mu)$, and $\rho_{n-1} = p$. It's easy to see that all main minors, the order of which will be less or equal to k_1 will be of $O(\mu^p)$ order. So, $\rho_{n-k_1} = \rho_{n-k_1+1} = \cdots = \rho_{n-1} = p$. The order of the next k_2 polynomials $\rho_{n-k_1-1}, \rho_{n-k_1-2}, \dots, \rho_{n-k_1-k_2}$ will be $O(\mu^{2p})$, because 1 more line of $O(\mu^p)$ order is added, and further $k_2 - 1$ the lines of $O(\mu^0)$ will be added. Estimating further the main minors of matrix we will come to conclusion that the main minors of $n, n-1, \dots, n-k_m+1$ order will be of $O(\mu^{mp})$ order. Thus, minimal power by μ of polynomials $c_i(t,\mu)$ may have the next values:

$$\begin{aligned}
&\rho_n = 0, \qquad \rho_{n-1} = p, \qquad p \le \rho_{n-2} \le 2p, \qquad p \le \rho_{n-3} \le 3p, \quad \dots, \\
&p \le \rho_{n-k_1} \le k_1 p, \qquad 2p \le \rho_{n-k_1-1} \le (k_1+1)p, \quad \dots, \\
&2p \le \rho_{n-k_1-k_2} = \rho_{k_m+k_{m-1}+\cdots+k_3} \le (k_1+k_2)p, \quad \dots, \\
&(m-2)p \le \rho_{k_m+k_{m-1}} = \rho_{n-k_1-\cdots-k_{m-2}} \le (k_1+\cdots+k_{m-2})p = (n-k_m-k_{m-1})p, \\
&(m-1)p \le \rho_{k_m+k_{m-1}-1} \le (n-k_m-k_{m-1}+1)p, \quad \dots, \\
&l(m-1)p \le \rho_{k_m} = \rho_{n-k_1-k_2-\cdots-k_{m-1}} \le (n-k_m)p, \\
&mp \le \rho_{k_m-1} \le (n-k_m+1)p, \quad \dots, \\
&mp \le \rho_1 \le (n-1)p, \qquad mp \le \rho_0 \le np.
\end{aligned}$$

Let us draw the obtained results (see Fig. 1).

Here $*$ denotes values meanings of ρ_i if coefficients with the smaller theoretically possible powers μ are equal zero. Figure shows that k_i of solving the equation (5) will be $O\left(\mu^{p/k_i}\right)$ order, $i = 1, 2, \dots, m$, moreover, they all will be different. So, for the case of several multiple elementary divisors the following theorem takes place.

Theorem 1. *If matrices $A_s(t)$ $(s = 0, 1, \dots)$ on the segment $[0; L]$ are infinitely differentiable and proper meanings of matrix $D_0(t,\mu)$ on the given segment are simple when $0 < \mu \le \mu_0$:*

$$\lambda_i(t,\mu) \ne \lambda_j(t,\mu), \qquad i,j = 1,\dots,n, \qquad i \ne j, \qquad \forall\, t \in [0; L],$$

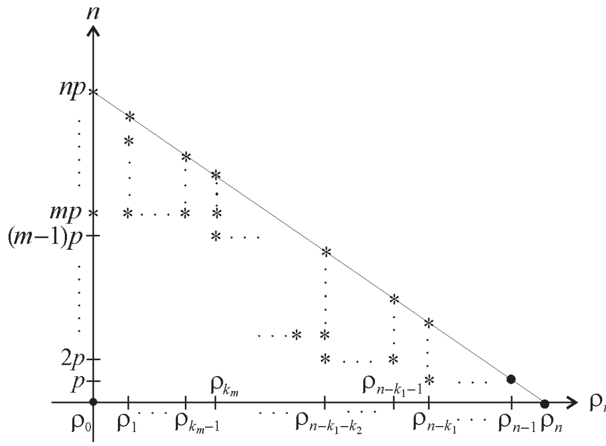

Figure 1.

then the system of differential equations (3) has a formal matrix-solution

$$Y(t,\mu) = U(t,\mu,\mu)\exp\left(\frac{1}{\mu^p}\int_0^t \Lambda(\tau,\mu,\mu)\,d\tau\right),$$

where $U(t,\mu,\mu)$ is a square matrix of n order, $\Lambda(t,\mu,\mu)$ is a diagonal matrix of n order, they are represented by formal series

$$U(t,\mu,\mu) = \sum_{s=0}^{\infty}\mu^s U_s(t,\mu), \qquad \Lambda(t,\mu,\mu) = \sum_{s=0}^{\infty}\mu^s \Lambda_s(t,\mu), \tag{6}$$

where $\mu = \sqrt[q]{\varepsilon}$.

This theorem is proved by the method from [3], as a result we have

$$U_0(t,\mu) = B(t,\mu), \qquad \Lambda_0(t,\mu) = W^*(t,\mu),$$

where $B(t,\mu)$ is the transforming matrix, which leads the matrix $D_0(t,\mu)$ to the diagonal matrix $W^*(t,\mu) = \{\lambda_1(t,\mu), \lambda_2(t,\mu), \ldots, \lambda_n(t,\mu)\}$,

$$\Lambda_s(t,\mu) = G_{1s}(t,\mu), \qquad s = 1,2,\ldots, \tag{7}$$

$G_{1s}(t,\mu)$ is obtained from the diagonal elements of matrix

$$G_s(t,\mu) = B^{-1}(t,\mu)H_s(t,\mu), \tag{8}$$

$$H_s(t,\mu) = \sum_{j=1}^{\left[\frac{s}{q}\right]} D_j(t)U_{s-jq}(t,\mu) - \sum_{i=1}^{s-1} U_i(t,\mu)\Lambda_{s-i}(t,\mu) - U'_{s-p}(t,\mu), \tag{9}$$

$$U_s(t,\mu) = B(t,\mu)Q_s(t,\mu), \tag{10}$$

where $Q_s(t,\mu)$ is the matrix the elements of which are found from the formulas

$$q_{sij}(t,\mu) = \frac{g_{sij}(t,\mu)}{\lambda_j(t,\mu) - \lambda_i(t,\mu)}, \qquad i \neq j, \qquad i,j = \overline{1,n}. \tag{11}$$

The diagonal elements of the matrix $Q_s(t,\mu)$ vanish.

Consider the matrix (9). It is easily seen that for $s < p$ $H_s(t,\mu) \equiv 0$, and from (7)–(11) $U_s(t,\mu) \equiv \Lambda_s(t,\mu) \equiv 0$, $s = 1, 2, \ldots, p-1$ is produced. As in expansion (6) these elements will follow $U_0(t,\mu), \Lambda_0(t,\mu)$, then we will write down the series (6) in the following way

$$U(t,\mu,\mu) = B(t,\mu) + \sum_{s=p}^{\infty} \mu^s U_s(t,\mu) = B(t,\varepsilon) + \sum_{s=p}^{\infty} \varepsilon^{\frac{s}{q}} U_s(t,\varepsilon),$$
$$\Lambda(t,\mu,\mu) = W^*(t,\mu) + \sum_{s=p}^{\infty} \mu^s \Lambda_s(t,\mu) = W^*(t,\varepsilon) + \sum_{s=p}^{\infty} \varepsilon^{\frac{s}{q}} \Lambda_s(t,\mu). \tag{12}$$

The following theorem is true.

Lemma 1. *Let the conditions of Theorem 1 be satisfied be* $\bar{t}_{k_1,1}(t) \neq 0$. *Then the coefficients of the formal series (12) are given by*

$$U_s(t,\mu) = B(t,\varepsilon) + \varepsilon^{-\frac{p}{qk_1}(s-p+1)} U_s^a(t,\varepsilon),$$
$$\Lambda_s(t,\mu) = W^*(t,\varepsilon) + \varepsilon^{-\frac{p}{qk_1}(s-p)} \Lambda_s^a(t,\varepsilon), \qquad s = p, p+1, \ldots, \tag{13}$$

where $U_s^a(t,\mu)$, $\Lambda_s^a(t,\mu)$ *are matrices which do not have asingularity in point* $\mu = 0$.

This Lemma is proved by immediate analysis of the matrixes elements (7)–(11).

Let us substitute (13) for (12). We will have

$$U(t,\mu,\mu) = U_0(t,\varepsilon) + \sum_{s=p}^{\infty} \varepsilon^{\frac{s}{q}} \varepsilon^{-\frac{p}{qk_1}(s-p+1)} U_s^a(t,\varepsilon),$$
$$\Lambda(t,\mu,\mu) = \Lambda_0(t,\varepsilon) + \sum_{s=p}^{\infty} \varepsilon^{\frac{s}{q}} \varepsilon^{-\frac{p(s-p)}{qk_1}} \Lambda_s^a(t,\varepsilon).$$

Lemma 2. *Let the condition of Theorem 1, Lemma 1,*

$$\operatorname{Re}(\lambda_i(t,\mu)) \leq 0$$

be satisfied on the set $\{K : t \in [0;L], 0 < \mu \leq \mu_0\}$, *then on the segment* $[0;L]$ *m-th approximation satisfies the differential system* (1) *up to the order of magnitude* $O\left(\varepsilon^{\frac{1}{q}\left((m+1-p)\left(1-\frac{p}{k_1}\right)+p\right)}\right)$.

Theorem 2. *Let the condition of Theorem 1, Lemma 2 be satisfied and for* $t = 0$

$$y(t,\mu) = y_m(t,\mu),$$

where $y(t,\mu)$ *is the exact solution of the system (3), then for any* $L > 0$ *there is* $c > 0$, *which does not depend upon* μ *and is such that for all* $t \in [0;L]$, $\mu \in (0;\mu_0]$ *the inequality is satisfied*

$$\|y(t,\mu) - y_m(t,\mu)\| \leq \mu^{(m+1-p)\left(1-\frac{p}{k_1}\right)-p+1} c.$$

Lemma 2 and Theorem 2 are proved by the methods from [3].

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On the Spectral Problem for the Finite-Gap Schrödinger Operator

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Solving of the spectral problem for the finite-gap Schrödinger operator in terms of hyperelliptic Weierstrass functions is proposed. Corresponding solutions with help of unknown coefficients are expressed through the Weierstrass functions which also contain unknown parameters. These unknown quantities are determined by corresponding band equations and polynomial solutions of the inverse Jacobi problem. Corresponding equations can be reduced to simple algebraic equations. The elliptic finite-gap case is considered in the framework of the proposed approach.

1 Introduction

The spectral problem for finite-gap linear differential operator is interesting both of its own and as an auxiliary problem in the finite-gap theory of integrable partial differential equations. Furthermore it may be applied to electron spectra theory.

The spectral problem is reduced to building of finite-gap eigenfunctions and finding of their parameters from the spectral linear differential equation. Symmetrized products of the functional part of these parameters (so-called μ-functions) are expressed through functional coefficients (so-called potentials) of a linear differential operator by the fundamental system of finite-gap equations (see [1, 2]). This system follows from the comparison of asymptotic series developments of general and finite-gap eigenfunctions.

Usually, solving of the spectral linear differential equation in μ-functions is realized with help of the known Abelian transformation with a subsequent introducing of the corresponding Riemann surface. In so doing, μ-functions are considered as points of this surface and its symmetrized degrees are solutions of the Jacobi inversion problem (see [3, 4]). Corresponding solutions are expressed in terms of the Riemann theta functions.

Thus, the system of the finite-gap equations and the Abelian transformation leads to solving of the finite-gap spectral problem for linear differential operators through the Riemann theta functions. The above mentioned solution of the Jacobi problem is connected with the complicated analysis of properties of theta functions on the Riemann surface. At the same time utilization of the known (see [5, 6, 7]) relations for 2-differential of second kind can lead to essential simplification of the latter problem.

Taking above mentioned circumstances we suggest simplification for solving the spectral problem for finite-gap linear differential operators in the case of the Schrödinger operator in the class of hyperelliptic finite-gap functions. Consideration will be based on known relations for fundamental 2-differential on the hyperelliptic Riemann curves and the system of finite-gap equations connecting the hyperelliptic Weierstrass functions and its derivatives.

The paper is organized as follows. In Section 2 the building of the hyperelliptic finite-gap eigenfunction and finite-gap equations of the spectral problem for the Schrödinger operator are formulated. In Section 3 solving of the Jacobi inversion problem with the help of the known relations for the fundamental 2-differential on the Riemann hyperelliptic curves is considered.

In Section 4 on the basis of the finite-gap equation relation for the hyperelliptic Weierstrass functions are obtained.

2 The finite-gap function and finite gap equations for the Schrödinger operator

The general form of eigenfunctions for one-dimensional Schrödinger operator $H = -\partial_x^2 + U(x)$ (where $\partial_x^n \equiv d^n/dx^n$) is determined by the symmetry of the Schrödinger equation $H\Psi(x,E) = E\Psi(x,E)$, where $\Psi(x,E)$ means the eigenfunction, x and E are space and spectral variables respectively. Such symmetry is expressed in each specific case by corresponding integrals of motion. In the case under consideration when the differential equation has two solutions $\Psi_1(x,E)$ and $\Psi_2(x,E)$ this integral of motion has the form

$$\Psi_2(x,E)\partial_x\Psi_1(x,E) - \Psi_1(x,E)\partial_x\Psi_2(x,E) = 2G, \tag{1}$$

where G means a constant.

Introducing the variable $X(x,E) = \Psi_1(x,E)\Psi_2(x,E)$ we can write the evident relation

$$\Psi_2(x,E)\partial_x\Psi_1(x,E) + \Psi_1(x,E)\partial_x\Psi_2(x,E) = \partial_x X(x,E). \tag{2}$$

The system of two equations (1) and (2) result in the equation

$$\partial_x \ln\Psi_1(x,E) = \frac{12}{\partial\ x} \ln X(x,E) + \frac{G}{X(x,E)}. \tag{3}$$

The solutions of this equation

$$\Psi_{1,2}(x,E) = \sqrt{X(x,E)}\exp\left(\pm\int_{x_0}^{x} dx\frac{G}{X(x,E)}\right) \tag{4}$$

determine the general form of the Schrödinger eigenfunctions taking into account the symmetry of the system (see [8]).

The finite-gap case imposes on the X-function the polynomial dependence on E. Then the differentiation (3) with taking into account this circumstance results in the equality (see [8])

$$U(x) = \frac{1}{2X}\partial_x^2 - \frac{1}{2X^2}(\partial_x X)^2 + \left(\frac{G}{X}\right).$$

Multiplication of the last on X^2 at zero points $x = a_i$ $(X(a)_i) = 0)$ yields the relation

$$\partial_x X|_{x=a_i} = 4G. \tag{5}$$

This relation is used for computation of the finite-gap eigenfunctions.

The above mentioned general Schrödinger eigenfunction in terms of the function $\chi = G/X$ can be written in the form

$$\Psi(x,E) = \sqrt{\chi(x,E)}\exp\left(\int^{x} dx\chi(x,E)\right), \tag{6}$$

where χ is real function with the asymptotic series

$$\chi(x,E) = \sqrt{E}\left(1 + \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+1}}\chi_{2n+1}(x)E^{-(n+1)}\right) \tag{7}$$

(further we shall omit the argument of the coefficient functions χ_n). Coefficients of (7) satisfy the known [1] recurrence relation

$$\chi_{n+1} = \frac{d}{dx}\chi_n + \sum_{k=1}^{n-1} \chi_k \chi_{n-k},\ \chi_1 = -U(x), \tag{8}$$

from which follow that χ_n-functions are polynominal in the potential U ant its derivatives.

Thus power series (7) and the recurrence relation (8) determine the power series of χ-function in the expression (4) for the general eigenfunction through the Schrödinger potential and its derivatives.

The finite-gap spectrum of the Schrödinger operator imposes the condition of a polynomial form of the X-function in the expression (4). In the case of g-gap spectra (which have g gaps and $2g+1$ boundaries $\{E_i\}$) the quantities G and X are described by the expressions [1]

$$G = \sqrt{P(E)} = \sqrt{\prod_{n=1}^{2g+1}(E - E_n)}, \qquad X = Q(E,x) = \prod_{n=1}^{g}(E - \mu_n(x)). \tag{9}$$

Then the χ-function transforms to the form

$$\chi_R(x,E) = \frac{\sqrt{\sum\limits_{n=0}^{2g+1} a_n E^{-n}}}{\sum\limits_{n=0}^{g} b_n E^{-n}}, \qquad a_0 = 1, \quad b_0 = 1. \tag{10}$$

Here a_n and b_n are symmetrized products of spectral boundaries E_j and μ-functions of the nth order, respectively;

$$a_n = (-1)^n \sum_{j_1,\neq j_2,\ldots,\neq j_n}^{2g+1} \prod_{i=1}^{n} E_{j_i}, \qquad b_n = (-1)^n \sum_{j_1,\neq j_2,\ldots,\neq j_n}^{g} \prod_{i=1}^{n} \mu(x)_{j_i},$$

The expression (10) as in the case (7) can be represented in the form of the asymptotic series

$$\chi_R \sim \sqrt{E}\left(1 + \sum_{n=1}^{\infty} A_n E^{-n}\right), \tag{11}$$

with coefficients

$$A_n = \frac{1}{n!}\frac{d^n}{dz^n} \left.\frac{\sqrt{\sum\limits_{n=0}^{2g+1} a_n z^n}}{\sum\limits_{n=0}^{g} b_n z^n}\right|_{(z=0)}. \tag{12}$$

Comparing coefficients at the same power of E^{-1} in the expressions (7) and (11) we obtain finite-gap equations

$$\frac{(-1)^n}{2^{2n+1}}\chi_{2n+1} = \frac{1}{(n+1)!}\partial_z^{n+1}\left.\left(\frac{\sqrt{\sum\limits_{n=0}^{2g+1} a_n z^n}}{\sum\limits_{n=0}^{g} b_n z^n}\right)\right|_{z=0}, \qquad b_0 = 0. \tag{13}$$

The system (13) in accord with the definition (12) and (8) determines relations between coefficient functions b_n (symmetrized products of μ_i-functions), a_n and polynomials of nth power in the potential $U(x)$ and its derivatives. The first g equations of this (13) are a system of algebraic equations which is solvable in respect to $(b_1, \ldots, b_g)$. This system at $n \geq g+1$ determines relations between the Schrödinger potential and its derivatives.

Thus finite-gap Schrödinger eigenfunctions and potentials are determined by the finite-band equations (13) presenting by relations between symmetrized products of μ-functions (b_i-coefficient functions) and the Schrödinger potential (U) with its derivatives. But complete solution of the spectral problem assumes computation of symmetrized products of μ-functions.

The general hyperelliptic finite-gap Schrödinger operator U in accord with the finite-band equation (13) is described linear symmetrized combination of μ-functions by the expression

$$U(x) = 2\sum_{j=1}^{g} \mu_j(x) - \sum_{j=1}^{2g+1} E_j. \tag{14}$$

The linear combination of μ-functions in (14) can be obtained by substitution the above mentioned finite-gap function in the Schrödinger equation and using the Abelian change of variables. In so doing, the solution is reduced to the Jacobi inverse problem. The latter is solving with help of a theorem about theta function zeros.

3 Calculating the symmetrized products of μ-functions

Substitution of the finite-gap Schrödinger eigenfunction (4) taking into account (9) in the Schrödinger equation results in the differential equation with respect to μ_i-functions. In the class of the Abelian hyperelliptic functions it can be integrated by the Abelian transformation of the form

$$\boldsymbol{v} = \sum_{j=1}^{g} \int_{P_0}^{\mu_j(z)} d\boldsymbol{v}, \qquad d\boldsymbol{v} = (2\omega)^{-1} d\boldsymbol{u}, \qquad du_j = \frac{z^{j-1}}{y(z)}, \tag{15}$$

Here $y^2(z) = \sum\limits_{j=1}^{2g+1} \lambda_j z^j$ means a hyperelliptic Riemann curve Γ; du_j means holomorphic differential of the first kind on Γ. Moreover, 2ω means a matrix of periods on the canonical basis cycles a_j (see [9]) on the Riemann surface Γ,

$$(2\omega)_{i,j=1,\ldots,g} = \left(\oint_{\mathfrak{a}_i} \mathrm{d}u_j \right),$$

which exists together with a matrix $2\omega'$ of periods on the canonical basis cycles b_j,

$$(2\omega')_{i,j=1,\ldots,g} = \left(\oint_{\mathfrak{b}_i} \mathrm{d}u_j \right).$$

The first matrix of periods on the Riemann surface is obtained from the condition reducing the canonical basis of holomorphic differentials du_j to the normal form dv_j.

Thus the finite-gap spectral problem for the Schrödinger operator was reduced to the problem of the Abelian integral (15) conversion with respect to the symmetrized products of μ_i-functions, i.e. the Jacobi inversion problem.

Taking into account (14) and using the known Riemann vanishing theorem for theta Riemann function $\theta(z|\tau)$ (which will be defined below) we can obtain the expression

$$U(x) = 2\sum_{i,j} \alpha_i \alpha_j \partial_{\alpha_i,\alpha_j} \ln \theta(\alpha x - K|\tau), \tag{16}$$

(where $\alpha_i = (2\omega)^{-1}_{gi}$) for hyperelliptic U-potentials.

Calculation of symmetrized products of higher degree can be realized with help of the so-called fundamental 2-differential of the second kind which is defined through the function of the form (see [7, 9])

$$F(z_1,z_2)=2y_2^2+2(z_1-z_2)y_2\partial_z y_2+(z_1-z_2)^2\sum_{j=1}^{g} z_1^{j-1}\sum_{k=j}^{2g+1-j}(k-j+1)\lambda_{k+j+1}z_2^k, \quad (17)$$

$$F(z_1,z_2)=2\lambda_{2g+2}z_1^{g+1}z_2^{g+1}+\sum_{i=0}^{g} z_1^i z_2^i(2\lambda_{2i}+\lambda_{2i+1}(z_1+z_2)). \quad (18)$$

Here any pair of points $(y_1,z_1),(y_2,z_2)\in\Gamma$.

Then the fundamental Abelian 2-differential of the second kind with the unique pole of the second order along $z_1=z_2$ can be written in the form

$$\mathrm{d}\widehat{\omega}(z_1,z_2)=\frac{2y_1y_2+F(z_1,z_2)}{4(z_1-z_2)^2}\frac{\mathrm{d}z_1}{y_1}\frac{\mathrm{d}z_2}{y_2}. \quad (19)$$

Taking into account (17) the expression (19) can be rewritten in the form

$$\mathrm{d}\widehat{\omega}(z_1,z_2)=\frac{\partial}{\partial z_2}\left(\frac{y_1+y_2}{2y_1(z_1-z_2)}\right)\mathrm{d}z_1\mathrm{d}z_2+\mathrm{d}\boldsymbol{u}^T(x_1)\mathrm{d}\boldsymbol{r}(x_2), \quad (20)$$

where

$$\mathrm{d}r_j=\sum_{k=j}^{2g+1-j}(k+1-j)\lambda_{k+1+j}\frac{z^k\mathrm{d}z}{4y}, \qquad j=1,\dots,g \quad (21)$$

is a canonical Abelian differential of the second kind.

Solution of the Jacobi inversion problem (15) is based on the known relation of the fundamental 2-differential (19) which can be written as

$$\int_{\mu}^{z}\sum_{i=1}^{g}\int_{\mu_i}^{z_i}\frac{2yy_i+F(z,z_i)}{4(z-z_i)^2}\frac{\mathrm{d}z}{y}\frac{\mathrm{d}z_i}{y_i}$$
$$=\ln\left\{\frac{\theta\left(\int_{a_0}^{z}\mathrm{d}\boldsymbol{v}-\sum\limits_{i=1}^{g}\int_{a_i}^{z_i}\mathrm{d}\boldsymbol{v}\right)}{\theta\left(\int_{a_0}^{z}\mathrm{d}\boldsymbol{v}-\sum\limits_{i=1}^{g}\int_{a_i}^{\mu_i}\mathrm{d}\boldsymbol{v}\right)}\right\}-\ln\left\{\frac{\theta\left(\int_{a_0}^{\mu}\mathrm{d}\boldsymbol{v}-\sum\limits_{i=1}^{g}\int_{a_i}^{z_i}\mathrm{d}\boldsymbol{v}\right)}{\theta\left(\int_{a_0}^{\mu}\mathrm{d}\boldsymbol{v}-\sum\limits_{i=1}^{g}\int_{a_i}^{\mu_i}\mathrm{d}\boldsymbol{v}\right)}\right\}. \quad (22)$$

The right hand of this equation contains the known Riemann theta function (see [1])

$$\mathcal{R}(z)=\theta(\boldsymbol{w}(z)|\tau)=\sum_{\boldsymbol{m}\in Z^n}\exp\left\{\pi\imath\left(\boldsymbol{m}^T\tau\boldsymbol{m}\right)+2\pi\imath\left(\boldsymbol{w}(z)^T\boldsymbol{m}\right)\right\}. \quad (23)$$

Here $(a\cdot b)$ means a scalar product,

$$\boldsymbol{w}(z)=\int_{a_0}^{z}d\boldsymbol{v}+\sum_{k=1}^{g}\int_{z_0}^{z_i}\mathrm{d}\boldsymbol{v}-\boldsymbol{K}_{z_0},$$

where components of $\boldsymbol{K}$ defined as

$$K_j=\frac{1+\tau_{jj}}{2}-\sum_{l\neq j}\oint_{\mathfrak{a}_l}\mathrm{d}v_l(x)\int_{z_0}^{z}\mathrm{d}v_j, \qquad j=1,\dots,g$$

is the vector of Riemann constants with respect to the base point z_0. In the considered case with the base point a the vector of Riemann constants has the form $\boldsymbol{K}_a = \sum\limits_{k=1}^{g} \int_a^{a_i} \mathrm{d}\boldsymbol{v}$. Taking into account definition of the hyperelliptic Weierstrass function through theta function (23) as

$$\wp_{ij} = \partial^2_{v_i, v_j} \ln \theta(v|\tau)$$

and differentiating (22) on variables z and z_r we can obtain the relation

$$\sum_{i=1}^{g} \wp_{ij} \left(\int_{a_0}^{z} d\boldsymbol{v} + \sum_{k=1}^{g} \int_{a_k}^{z_k} d\boldsymbol{v} + \boldsymbol{K}_a \right) z^{i-1} z_r^{j-1} = \frac{F(z, z_r) - 2yy_r}{4(z - z_r)^2}, \tag{24}$$

which expresses a second kind 2-differential through the linear combination of the hyperelliptic Weierstrass functions. In the limit case $z \to \infty$ from (24) the relation follows

$$\mathcal{P}(z; \boldsymbol{v}) = z^g - \sum_{j=1}^{g-1} \left(\sum_{l,m} \wp_{l,m}(\boldsymbol{v}) \alpha_l \alpha_j \right) z^{j-1}, \tag{25}$$

in which points $\{z_i\}$ of the Riemann surface are presented as roots of the polynomial $\mathcal{P}$ $(z_j \equiv \mu_j)$. In so doing, symmetrized products of these points are expressed by linear combinations hyperelliptic Weierstrass functions through the period matrix of holomorphic differentials 2ω. The α-coefficients are obtained from algebraic equations which can be obtained by substitution (16) in the finite-band equation (13) taking into account (25). One will be demonstrated in next section in the case elliptic finite-gap Schrödinger potentials.

The matrix 2ω can be calculated with help the known Thomae formulae of the form

$$\theta^4[\varepsilon(\boldsymbol{I}_0)] = \pm \det 2\omega \prod_{i,j \in \boldsymbol{J}_0} (E_i - E_j) \prod_{n,m \in \tilde{\boldsymbol{J}}_0} (E_n - E_m),$$

$$\theta_j^4[\varepsilon(\boldsymbol{I}_1)] = \pm \frac{\det(2\omega)^{-2}}{16} \prod_{i,j \in \boldsymbol{J}_1} (E_i - E_j) \prod_{n,m \in \tilde{\boldsymbol{J}}_1} (E_n - E_m) \sum_{i=1}^{g} \oint_{\mathfrak{a}_i} \frac{z^{j-1} \mathrm{d}z}{y} S_{i-1}(\boldsymbol{I}_1).$$

Here expressions

$$S_0(\boldsymbol{I}_1) = 1, \quad S_1(\boldsymbol{I}_1) = \sum_{j \in \boldsymbol{I}_1} E_j, \quad \ldots, \quad S_{g-1}(\boldsymbol{I}_1) = \prod_{j \in \boldsymbol{I}_1} E_j$$

denotes symmetrized products of the branching points of the Riemann surface.

Thus finite-gap Schrödinger eigenfunctions and potentials can be expressed through hyperelliptic Weierstrass functions containing theta-constants instead unknown elements of a 2ω-matrix.

4 The finite-gap relations for elliptic Weierstrass functions

The system of finite-gap equations (10) is solvable with respect to symmetrized products μ_i-functions expressed as coefficient functions b_i. Therefore excluding b_i-functions at $n > g$ we can obtain the system of algebraic equations with respect to the Schrödinger potential and its derivatives. These equations determine relations for the hyperelliptic Weierstrass functions.

We consider above mentioned relations in the case of the Riemann curves of low genus g which correspond to small number of gaps in the eigenvalue spectrum of the Schrödinger operator.

One-gap spectrum. The Schrödinger potential $U(z)$ is determined by the system of three finite-gap equations of the form (13) at $n = \overline{0, 2}$. Substitution into these equations of the explicit expressions (12) for A_n and polynomial in U expressions for χ_n-functions which follow from (8) yields the system

$$\frac{1}{2}a_1 - b_1 = -\frac{1}{2}U, \qquad \frac{1}{2}\left\{\left(a_2 - \frac{1}{4}a_1^2\right) + 2\left(b_1^2 - b_2\right) - a_1 b_1\right\} = -\frac{1}{2^3}\left\{U^2 - U^{(2)}\right\},$$
$$\frac{1}{3!}\left\{\left(\frac{3}{8}a_1^3 - \frac{3}{2}a_1 a_2 + \frac{3}{2}a_3\right) + 3\left(\frac{1}{4}a_1^2 - a_2\right)b_1 + 3a_1\left(b_1^2 - b_2\right) + \left(12b_1b_2 - 4!b_3 - 6b_1^3\right)\right\}$$
$$= \frac{1}{2^5}\left\{U^{(4)} - 5U^{(1)^2} + 6UU^{(2)} - 2U^3\right\}$$

in which $b_n|_{n\geq 2} = 0$ (in view of the relation $b_n|_{n\geq g+1} = 0$, where g is the number of gaps in the eigenvalue spectrum of the Schrödinger equation. Excluding b_n from the last system we can obtain the equations

$$b_2 = 0 = \frac{1}{8}\left(3U^2 - U^{(2)}\right) + \frac{1}{4}a_1 U + \frac{1}{2}a_2 - \frac{1}{8}a_1^2,$$
$$b_3 = 0 = -\frac{1}{32}\left(U^{(4)} + 10U^3 - 5U^{(1)^2} - 10UU^{(2)}\right) - \frac{1}{16}a_1\left(3U^2 - U^{(2)}\right)$$
$$+ \frac{1}{16}U\left(a_1^2 - 4a_2\right) + \frac{1}{2}a_3 + \frac{1}{4}a_1 a_2 - \frac{1}{16}a_1^3.$$

Inserting into the latter system the expression $U = 2\alpha_1^2\wp_{1,1} - a_1$ (in accord with (25)) we can obtain unknown parameters for 1-gap Schrödinger potentials.

Two-gap spectrum. The 2-gap Schrödinger potential is determined by the system of the four finite-gap equations of the form (13) at $n = \overline{0,3}$. Analogically to the one-gap case their explicit form can be obtained by the substitution of the expressions (12) for A_n and expressions for χ_n (following from (8)) into (13). In so doing, the first two equations are solvable with respect to b_1 and b_2. Excluding the latter from the fourth and fifth equation and taking into account the equality $b_n|_{n\geq 3} = 0$ we can obtain the finite-gap system

$$b_3 = 0 = \frac{1}{2^5}\left(16a_3 + 8a_2U + 10U^3 - 5U'^2 - 2a_1U'' - 10UU'' + U^{(4)}\right),$$
$$b_4 = 0 = \frac{1}{2^7}\left(-16a_2^2 + 64 * a_4 + 32a_3U + 24a_2U^2 + 35U^4\right.$$
$$\left. - 70UU'^2 - 8a_2U'' - 70U^2U'' + 21U''^2 + 28U'U^{(3)} + 14UU^{(4)} - U^{(6)}\right).$$

Inserting into the latter system the expression $U = 2\sum_{i,j}\alpha_i\alpha_j\wp_{i,j} - a_1$, $i,j = 1,2$ (in accord with (25)) we can obtain unknown parameters for 2-gap Schrödinger potentials.

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A Covering Second-Order Lagrangian for the Relativistic Top without Forces

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A parameter-homogeneous manifestly covariant Lagrangian of second order is considered, which covers the case of the free relativistic top at constraint manifold of constant acceleration. Relation to other models is discussed in brief.

1 Introduction

The interest to the description of quasi-classical physical particle by the means of some higher-order equations of motion and the methods of generalized Ostrohrads'kyj mechanics arose some 60 years ago and since then has been continuous [1–7]. Recently renewed attention was paid to such the models, which basically involve the notions of the first and higher curvatures of the particle's world line [8–12]. In most cases, people start with an *a priori* given higher order Lagrangian, and then try to interpret the dynamical system thus obtained as one describing the motion of quasi-classical spin (the relativistic top). Technical misunderstanding of two kinds happens to arise. First, certain nonholonomic constraints sometimes are imposed from the very beginning. These constraints are chosen in such a way as to ensure that the Lagrangian is in fact written in terms of the moving frame components [13]. But, as shown in [14], non-holonomic constraints require a more subtle approach. In particular, the constraint system does not retain the property of variationality any more. Second, sometimes the very tempting assumption of unit four-velocity vector is imposed *after* the variation procedure has already been carried out (cf. [15]). Such approach was quite justifiably criticized by several authors [16, 17]. On the other hand, there exist the established equations of Mathisson & Papapetrou [18] and of Dixon [19], which are believed to be well based from the point of view of physics. In 1945 Weyssenhoff [2] asserted, referring to one paper of Mathisson [20], "Even for a free particle in Galileian domains the equations of motion of a material particle endowed with spin do not coincide with the Newtonian laws of motion; there remains an additional term depending on the internal angular momentum or spin of the particle, which *raises the order of these differential equations to three*". We add to this that the procedure of *complete elimination* of spin variables in fact raises the order of the differential equations to *four*. In the present note this fourth order differential equation will be shown to follow from Dixon's form of the relativistic top equation of motion and in case of flat space-time a Lagrange function will be proposed which produces the world lines of thus governed spinning particle without any preliminary constraints being imposed *before* the variation procedure in undertaken. A constraint of *constant curvature* must be imposed *after* the variation, and this is why we call the corresponding Lagrange function a *covering* Lagrangian.

2 Relativistic top

To start from the lowest possible order let us recall the Dixon equations of the quasi-classical spinning particle in the gravitational field:

$$\dot{\mathcal{P}}^\alpha = \frac{1}{2} R^\alpha{}_{\beta\gamma\delta} u^\beta \mathcal{S}^{\gamma\delta}, \qquad \dot{\mathcal{S}}^{\alpha\beta} = \mathcal{P}^\alpha u^\beta - \mathcal{P}^\beta u^\alpha. \tag{1}$$

This system (1) does not prescribe any preferable way of parametrization along the world line of the particle.

It was proved in [21] and announced in [22] that under the so-called auxiliary condition of Pirani

$$u_\beta \mathcal{S}^{\alpha\beta} = 0, \tag{2}$$

equations (1), (2) are equivalent to the following system of equations (3), (4), and (5)

$$\varepsilon_{\alpha\beta\gamma\delta}\ddot{u}^\beta u^\gamma s^\delta - 3\frac{\dot{u}_\beta u^\beta}{\|\boldsymbol{u}\|^2}\varepsilon_{\alpha\beta\gamma\delta}\dot{u}^\beta u^\gamma s^\delta - m\left(\|\boldsymbol{u}\|^2\dot{u}_\alpha - \dot{u}_\beta u^\beta u_\alpha\right) = \frac{\|\boldsymbol{u}\|^2}{2}\varepsilon_{\mu\nu\gamma\delta}R_{\alpha\beta}{}^{\mu\nu}u^\beta u^\gamma s^\delta, \tag{3}$$

$$\|\boldsymbol{u}\|^2\dot{s}^\alpha + s_\beta\dot{u}^\beta\, u^\alpha = 0, \tag{4}$$

$$s_\alpha u^\alpha = 0. \tag{5}$$

The correspondence between the skewsymmetric spin tensor $\mathcal{S}^{\alpha\beta}$ and spin four-vector s^α under the assumption that we recognize Pirani's condition is given by

$$s_\alpha = \frac{1}{2\|\boldsymbol{u}\|}\,\varepsilon_{\alpha\beta\gamma\delta}u^\beta S^{\gamma\delta}, \qquad S_{\alpha\beta} = \frac{1}{\|\boldsymbol{u}\|}\,\varepsilon_{\alpha\beta\gamma\delta}u^\gamma s^\delta.$$

Equation (3) in flat space-time was considered from variational point of view in [21] and some Lagrange functions for it were offered in [23].

As promised, from now on we put $R_{\alpha\beta}{}^{\mu\nu} = 0$ and proceed to eliminate the variable s^α (in fact, a four-vector constant quantity). To facilitate the calculations, it is appropriate to chose the would line parametrization in the usual way: $\|\boldsymbol{u}\| = 1$. Then we get immediately that (3) takes on the shape ("$*$" denotes the dual tensor)

$$*\,\ddot{\boldsymbol{u}}\wedge\boldsymbol{u}\wedge\boldsymbol{s} + m\dot{\boldsymbol{u}} = \boldsymbol{0} \tag{6}$$

and possesses the first integral $k^2 = \dot{\boldsymbol{u}}^2$, which is nothing but the squared first curvature of the world line.

Now contract the above vector equation with the tensor $*\,\boldsymbol{u}\wedge\boldsymbol{s}$ and remember of (5) to obtain after some algebraic manipulations

$$\boldsymbol{s}^2\left(\ddot{\boldsymbol{u}} + k^2\boldsymbol{u}\right) = -m\,*\,\dot{\boldsymbol{u}}\wedge\boldsymbol{u}\wedge\boldsymbol{s}.$$

Differentiating and then substituting the right hand side from (6), we finally obtain

$$\dddot{\boldsymbol{u}} + \left(k^2 - \frac{m^2}{\boldsymbol{s}^2}\right)\dot{\boldsymbol{u}} = 0. \tag{7}$$

Now let us return to equations (1) and recall the standard fact that under Pirani's condition (2) the particle's momentum $\boldsymbol{\mathcal{P}}$ may be expressed in terms of spin tensor $\mathcal{S}^{\alpha\beta}$, or, equivalently, in terms of spin for-vector $\boldsymbol{s}$

$$\boldsymbol{\mathcal{P}} = \frac{m}{\|\boldsymbol{u}\|}\,\boldsymbol{u} + \frac{1}{\|\boldsymbol{u}\|^3}\,*\,\dot{\boldsymbol{u}}\wedge\boldsymbol{u}\wedge\boldsymbol{s},$$

where $m = \frac{\boldsymbol{\mathcal{P}}\cdot\boldsymbol{u}}{\|\boldsymbol{u}\|}$ is a constant of motion, and that the square momentum

$$\boldsymbol{\mathcal{P}}^2 = m^2 - k^2\boldsymbol{s}^2 + \frac{1}{\|\boldsymbol{u}\|^6}\left[(\dot{\boldsymbol{u}}\cdot\boldsymbol{s})\,\boldsymbol{u} - (\boldsymbol{u}\cdot\boldsymbol{s})\,\dot{\boldsymbol{u}}\right]^2 = m^2 - k^2\boldsymbol{s}^2$$

by virtue of (2) is a constant of motion too. Thus denoting $\omega^2 = -\frac{\boldsymbol{\mathcal{P}}^2}{\boldsymbol{s}^2}$, we finally obtain the desired fourth-order equation for the free relativistic top:

$$\dddot{\boldsymbol{u}} + \omega^2\dot{\boldsymbol{u}} = 0 \tag{8}$$

3 Hamilton–Ostrohrads'kyj approach

Let us again notify that we tend to set a parameter-invariant variational problem in order to get the world lines without any additional parametrization. Recall the general formula for the first curvature of the world line in arbitrary parametrization

$$k = \frac{\|\boldsymbol{u} \wedge \dot{\boldsymbol{u}}\|}{\|\boldsymbol{u}\|^3} \tag{9}$$

and consider the following Lagrange function:

$$\mathcal{L} = \frac{1}{2}\|\boldsymbol{u}\| \left(k^2 + A\right). \tag{10}$$

This Lagrange function (10) constitutes a parameter-homogeneous variational problem because it satisfies the Zermelo conditions:

$$\left(\boldsymbol{u}\boldsymbol{.}\frac{\partial}{\partial \boldsymbol{u}} + 2\,\dot{\boldsymbol{u}}\boldsymbol{.}\frac{\partial}{\partial \dot{\boldsymbol{u}}}\right)\mathcal{L} = \mathcal{L}, \qquad \boldsymbol{u}\boldsymbol{.}\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{u}}} = 0. \tag{11}$$

Variational equations are given by

$$-\dot{\boldsymbol{\wp}} = \boldsymbol{0}, \tag{12}$$

where

$$\boldsymbol{\wp} = \frac{\partial \mathcal{L}}{\partial \boldsymbol{u}} - \left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{u}}}\right)^{\boldsymbol{\cdot}}.$$

Now, one can calculate the quantity $\dot{\boldsymbol{\wp}}$ and *afterwards* set $\|\boldsymbol{u}\| = 1$, thus benefiting from the parameter homogeneity of equation (12). We get for (12):

$$\ddot{\boldsymbol{u}} + \left(\frac{3}{2}\,\dot{\boldsymbol{u}}^2 - A\right)\dot{\boldsymbol{u}} + 3\,(\ddot{\boldsymbol{u}}\boldsymbol{\cdot}\dot{\boldsymbol{u}})\,\boldsymbol{u} = \boldsymbol{0}. \tag{13}$$

Now, on the surface $k = k_0$ equation (13) will coincide with (8) if we put

$$A = \frac{3}{2}\,{k_0}^2 - \omega^2.$$

This completes the proof, as asserted in [24].

To pass to the canonical formalism, it is necessary to introduce the parametrization by time, setting $x^0 = t$, $u^0 = 1$, and denoting $\frac{dx^i}{dt} = v^i$. In this coordinates formula (10) suggests the following expression for the Lagrange function:

$$L = \frac{1}{2}\sqrt{1 + \boldsymbol{v}^2}\left(k^2 + A\right), \qquad k^2 = \frac{\boldsymbol{v}'^2 + (\boldsymbol{v}' \times \boldsymbol{v})^2}{(1 + \boldsymbol{v}^2)^3}. \tag{14}$$

Generalized Hamilton function H is expressed in terms of $\boldsymbol{v}$ and the couple of momenta

$$\boldsymbol{p}' = \frac{\partial L}{\partial \boldsymbol{v}'}, \qquad \boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} - \frac{d}{dt}\,\boldsymbol{p}', \tag{15}$$

namely,

$$H = \boldsymbol{p}\boldsymbol{.}\boldsymbol{v} + \boldsymbol{p}'\boldsymbol{.}\boldsymbol{v}' - L.$$

It is possible to find the inverse of the generalized Legendre transformation (15) and after some laborious calculating efforts the generalized Hamiltonian reads:

$$H = \boldsymbol{p}\boldsymbol{.}\boldsymbol{v} + \frac{1}{2}\left(1 + \boldsymbol{v}^2\right)^{3/2}\left(\boldsymbol{p}'^{\boldsymbol{2}} + (\boldsymbol{p}'\boldsymbol{.}\boldsymbol{v})^2\right) - \frac{A}{\sqrt{1 + \boldsymbol{v}^2}}.$$

4 Concluding notes

1. Equation (8) was known to Riewe [15], but its deduction directly from (1) or from the Mathisson–Papapetrou equations [18] apparently was not obvious.

2. By means of the formula $kk_2k_3 = \|\boldsymbol{u}\wedge\dot{\boldsymbol{u}}\wedge\ddot{\boldsymbol{u}}\wedge\dddot{\boldsymbol{u}}\|$, which presents the relationship between the successive curvatures of a curve (in natural parametrization), we see immediately, that all the extremals of (10) have zero third curvature, and in terms of the space-like world line it means that the particle evolves in a plane.

3. In [21] we proved by means of generalized Ostrohrads'kyj momenta approach, that every one of the successive curvatures of a curve, taken as the Lagrange function, produces the extremals with this same curvature being the constant of motion. This was also observed by Arodź for the first curvature [9]. But the problem of the simultaneous conservation of all the curvatures, i.e. the variational description of helices, remains open (cf. [25]).

4. Surprisingly enough, the Lagrange function (10) in fact coincides with one, considered by Bopp in [1] for the motion of a charged particle in electromagnetic field (in part, not including the external four-potential itself). That equations (1) in their differential prolongation cover both the Mathisson–Papapetrou equations of spinning particle and the Lorentz–Dirac equations of self-radiating particle, was already noted in [23] in relation to the prediction of Barut [26]. This gives still more grounds to call (10) the *covering* Lagrangian.

5. Following the ideas of [6] we considered in [27] some non-local transformations which leave invariant the exact form of the action integral

$$\int \sqrt{\epsilon^2 d\tau^2 - d\alpha^2} = \int \mathcal{L}_\epsilon d\tau, \tag{16}$$

where $d\alpha$ measures the rotation of the tangent to the world line during the increment $d\tau$ of the proper time along it, so the curvature $k = \frac{d\alpha}{d\tau}$. There was an attempt to interpret these non-local transformations (linear in α and τ) as such that explain the transition between the uniformly accelerated frames of reference in special relativity. Treating in quite formal way the variables α and τ as independent, one may stay hoping that the variation of (16) will produce the world lines of constant curvature (i.e. constant acceleration). On the other hand, looking more closely at the Lagrange function

$$\mathcal{L}_\epsilon = \sqrt{\epsilon^2 - k^2}, \tag{17}$$

immediately leads to the concept of maximal acceleration $\epsilon = c^{7/2}G^{(-1/2)}\hbar^{-1/2} = 3/5{\cdot}10^{52}\mathrm{m/sec}^2$ [28].

Two shortcomings spring up. First, the Lagrange function (17), viewed as a higher-order Lagrangian, does not correspond to constant curvature world lines. Second, the variational problem is not parameter-independent, at least because $\mathcal{L}_\epsilon$, with k given by (9), does not satisfy the Zermelo conditions (11). The Lagrangian (10) is free of these shortcomings.

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General Even and Odd Coherent States as Solutions of Discrete Cauchy Problems

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The explicit and exact solutions of the linear homogeneous difference equation with initial conditions (Cauchy problem) are constructed. The approach is quite general and relies on a novel and successful treatment of the linear recursion appropriately cast in matrix form. Our approach is exploited to solve the eigenvalues problem of a special set of non-Hermitian operators. A new class of generalized even and odd coherent states of a quantum harmonic oscillator are defined.

The occurrence of linear or nonlinear difference equations is ubiquitous in applied sciences. The exact treatment of many important problem in physics, chemistry, biology, economy, psychology and so on, depend on our ability to solve recursive relations of various kind. The importance of this particular chapter of mathematics may be for instance appreciated taking into account the close relation existing between difference and differential equations. Systematic methods for approximating intractable ordinary or partial differential equations by easier-to manage appropriate difference equations, are currently and successfully used in many contexts of applied sciences [1, 2, 3]. By definition a nth-order linear discrete Cauchy problem consists of a linear normal nth-order difference equation associated to given initial conditions. It is well known that when the vectors of the initial conditions defining n different discrete Cauchy problems relative to the same nth-order recursive equation are independent, then the n corresponding solutions constitute a fundamental set of solutions. In this paper we construct the explicit and exact solution of the following discrete Cauchy problems

$$\begin{aligned}&y_{k+n} = f_1(k)y_{k+n-1} + \cdots + f_{n-1}(k)y_{k+1} + f_n(k)y_k,\\ &y_0 = y_1 = y_2 = \cdots = y_{k-1} = 0, \quad y_k = 1, \quad y_{k+1} = y_{k+2} = \cdots = y_{n-1} = 0,\end{aligned} \tag{1}$$

where $f_i : S \to \mathbb{R}\ \forall\ i = 1, 2, \ldots, n$ and $f_n(k) \neq 0\ \forall\ k \in S = \{0, 1, 2, \ldots\}$. Our treatment is new and leads to a resolutive formula whose usefulness is vividly illustrated by an application to the physics of the quantum harmonic oscillator. To this end we transform equation (1) into the following homogeneous, linear, matrix, first order equation

$$Z_{k+1} = A^{(k)}Z_k, \qquad k = 0, 1, 2, \ldots \tag{2}$$

with $A^{(k)}$ $n \times n$ matrix defined by $A_{1j} = \delta_{jn}$, $A^{(k)}_{21} = f_n(k)$, $A^{(k)}_{2j} = f_{j-1}(k)$ $(j = 2, \ldots, n)$, $A_{rj} = \delta_{r-1,j}$ for $r = 3, \ldots, n$ and

$$Z^T_k = (y_k\ \ y_{k+n-1}\ \ \cdots\ \ y_{k+1}), \tag{3}$$

where superscript T denotes the transposition of column vector Z_k. Its formal solution has the form

$$Z_k = P^{(k)}Z_0, \tag{4}$$

where $P^{(0)} = I$ and $P^{(k)} = A^{(k-1)}A^{(k-2)} \cdots A^{(0)}$, $k \geq 1$. It is easy to verify that $P^{(0)}_{1j} = \delta_{1j}$, $P^{(1)}_{1j} = \delta_{nj}$ and, for any $k > 1$

$$P^{(k)}_{1j} = \sum_{h_0,h_1,\ldots,h_{k-2}} A^{(k-1)}_{1h_{k-2}} A^{(k-2)}_{h_{k-2}h_{k-1}} \cdots A^{(1)}_{h_1 h_0} A^{(0)}_{h_0 j}, \tag{5}$$

where h_i $(i = 0, 1, \ldots, k-2)$ runs from 1 to n. Consider first $1 \leq k \leq n-1$. In this case, in view of equation (5), $P^{(k)}_{1j}$ does not vanish only if $k = n-j+1$. In fact only this condition ensures the existence of a not vanishing contribution to $P^{(k)}_{1j}$ in the form of the following product of matrix elements

$$A^{(k-1)}_{1n} A^{(k-2)}_{nn-1} \cdots A^{(k+j-n-1)}_{j+1j} = 1. \tag{6}$$

Thus we arrive at the conclusion that $P^{(k)}_{1j} = \delta_{n+1-k,j}$. Moreover $P^{(n)}_{1j} = A^{(0)}_{2j}$ and $P^{(n+1)}_{1j} = A^{(0)}_{2j} A^{(1)}_{22} + (1 - \delta_{1,j}) A^{(1)}_{2j+1}$ provided we consistently put $A^{(k)}_{2\,n+1} \equiv A^{(k)}_{21}$. Observe that

$$\sum_{h_0,h_1,\ldots,h_{k-2}} A^{(k-1)}_{1h_{k-2}} A^{(k-2)}_{h_{k-2}h_{k-1}} \cdots A^{(1)}_{h_1 h_0} A^{(0)}_{h_0 j} = \sum_{h_0,h_1,\ldots,h_{k-n-1}} A^{(k-n)}_{2h_{k-n-1}} \cdots A^{(1)}_{h_1 h_0} A^{(0)}_{h_0 j}, \tag{7}$$

where the $n-1$ indices $h_{k-2}, h_{k-1}, \ldots, h_{k-n}$ have been eliminated with the help of relations essentially similar to that expressed by equation (6). The expression (5) for $P^{(k)}_{1j}$ with $k > n+1$ and $j = 1, 2, \ldots, n$, may be put in the following form

$$P^{(k)}_{1j} = \sum_{r=j}^{r^*} (\delta_{1,r} + \vartheta(j-1)) A^{(r-j)}_{2r} \sum_{h_{r-j+1},\ldots,h_{k-n-1}} A^{(k-n)}_{2h_{k-n-1}} \cdots A^{(r-j+1)}_{h_{r-j+1}2} + f_n(k,j), \tag{8}$$

where

$$\begin{aligned} f_n(k,j) &= (1-\delta_{1,j})(1-\vartheta(k-2n+j-2)) \\ &\quad \times \left[(1-\delta_{k-n,n-j+2}) A^{(k-n)}_{2j+k-n} + A^{(k-n)}_{22} A^{(k-n-1)}_{21} \delta_{k-n,n-j+2} \right], \end{aligned} \tag{9}$$

$\vartheta(x)$ is the Heaviside step function such that $\vartheta(0) = 0$ and $r^* = \min\{(n+1), k+j-n-2\}$. Equation (8) expresses $P^{(k)}_{1j}$ in terms of finite sums like

$$\sum_{h_{r-j+1},\ldots,h_{k-n-1}} A^{(k-n)}_{2h_{k-n-1}} \cdots A^{(r-j+1)}_{h_{r-j+1}2} \tag{10}$$

which may be further simplified exploiting the structural presence of "1" and "0" in the characteristic matrices $\{A^{(k)}\}$. To this end we note that there are only n not vanishing products of matrix elements beginning in the second row and ending in the second column:

$$\begin{aligned} & A^{(k-n)}_{21} A^{(k-n-1)}_{1n} \cdots A^{(k-2n+1)}_{32} = A^{(k-n)}_{21}, \\ & A^{(k-n)}_{22} = A^{(k-n)}_{22}, \\ & A^{(k-n)}_{23} A^{(k-n-1)}_{32} = A^{(k-n)}_{23}, \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & A^{(k-n)}_{2j} A^{(k-n-1)}_{jj-1} \cdots A^{(k-n)}_{32} = A^{(k-n)}_{2j}, \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & A^{(k-n)}_{2n} A^{(k-n-1)}_{nn-1} \cdots A^{(k-n)}_{32} = A^{(k-n)}_{2n}. \end{aligned} \tag{11}$$

To take advantage from equation (11) let us introduce the function $g : \{1, 2, \ldots, n\} \to \mathbb{N} \times \mathbb{N}$ defined putting $g(1) = (2, 2)$; $g(j) = (2, j + 1)$, $2 \le j \le n - 1$, $(n > 2)$ and $g(n) = (2, 1)$ which helps in writing down $P_{1j}^{(k)}$ as given by equation (8) in a conveniently irreducible form. In what follows we shall use the following symbols

$$A_{g(1)}^{(p)} \equiv A_{22}^{(p)}, \quad \ldots, \quad A_{g(j)}^{(p)} \equiv A_{21}^{(p)}. \tag{12}$$

Looking at equation (11) we see that the length of the sequence beginning with $A_{21}^{(k-n)}$ $\left(A_{2j}^{(k-n)}\right)$ (that is the number of factors) is $n(j-1)$. It is easy to convince oneself that each not vanishing contribution to the sum expressed by equation (10) may be subdivided into products of sequences of different length explicitly written down in equation (11). This circumstance provides the key for defining a useful algorithm to cast $P_{1j}^{(k)}$ into an explicit form where only elements of the second row of the matrices $\{A^{(k)},\ k = 1, 2, \ldots\}$ are present. For this purpose we find convenient to put the following definitions. Let h, n and p be positive integers. We say that a sequence of integers has order h and high n if it is constructed by h eventually repeated positive integers not exceeding n. A generic sequence of order h and high n is denoted by $(r_1, \ldots, r_h)_n$ and the set of all such sequences by $I_n(h)$. For each prefixed integer p such that $h \le p \le nh$, we say that $(r_1, \ldots, r_h)_n \in I_n(h)$ represents a $p-$sequence when it satisfies the additional condition to be also a partition of the integer p, that is $\sum_{\nu=1}^{h} r_\nu = p$. A generic $p-$sequence of order h and high n is denoted by $(r_1, \ldots, r_h)_n^p$ and the certainly not empty subset of all such p-sequences by $I_n^p(h)$. Finally we put $I_n^p = \bigcup_{h=1}^{p} I_n^p(h)$, that is I_n^p is the set of all the p-sequences of high n in correspondence with all the possible orders.

For instance if $n = 5$ and $p = 3$, I_5^3 has 4 elements: $1+1+1 = 2+1 = 1+2 = 3$, and for $p = 6$ I_5^6 has 31 elements: $1+1+1+1+1+1 = 2+1+1+1+1 = 1+2+1+1+1 = \cdots = 1+5 = 5+1 = 6$. It is possible to convince oneself that for any $k > n+1$ and $1 \le j \le n$ the expression (10) appearing in equation (8) may be cast in the following form

$$\sum_{(r_1,\ldots,r_h)_n^p \in I_n^p} A_{g(r_1)}^{(k-n-r_0)} A_{g(r_2)}^{\left(k-n-\sum_{t=0}^{1} r_t\right)} \cdots A_{g(r_h)}^{\left(k-n-\sum_{t=0}^{h-1} r_t\right)}, \tag{13}$$

where $r_0 = 0 \le h \le p$, $p = (k - n) - (r - j)$.

The important difference between the two expression (10) and (13) is of course that the latter equation contains only matrix elements of the second rows of the (at most) $(k - n)$ matrices $A^{(1)}, A^{(2)}, \ldots, A^{(k-n)}$ and therefore is irreducible. We wish to point out that $k - n - \sum_{t=1}^{h-1} r_t > r - j + 1$ as it should be. Inserting equation (13) into equation (8) we are now in position to write down the definite expression of $P_{1j}^{(k)}$ as

$$\begin{aligned}
&\delta_{1,j}, \quad k = 0, \qquad \delta_{n+1-k,j}, \quad 1 \le k \le n, \qquad A_{21}^{(0)}, \quad k = n, \\
&A_{2j}^{(0)} A_{2j}^{(1)} + (1 - \delta_{1j}) A_{2j}^{(1)}, \quad k = n + 1, \\
&\sum_{r=j}^{r^*} [\delta_{1,r} + \vartheta(j-1)] A_{2r}^{(r-j)} \\
&\qquad \times \sum_{(r_1,\ldots,r_h)_n^p \in I_n^p} A_{g(r_1)}^{(k-n-r_0)} \cdots A_{g(r_h)}^{\left(k-n-\sum_{t=0}^{h-1} r_t\right)} + f_n(k, j), \quad k > n + 1.
\end{aligned} \tag{14}$$

When j runs from 1 to n, $P_{1j}^{(k)}$ defines n independent solutions of equation (1). Introduce the non-hermitian operators

$$C_n = a^n e^{i\frac{2\pi}{n}a^\dagger a}, \qquad n = 1, 2, \dots \tag{15}$$

The eigenstates of C_1 are the coherent states and the eigenstates of C_2 pertaining to a generic not null eigenvalue are the even and odd coherent states. The eigenvalue problem for C_n, formulated to construct generalizations of the even and odd coherent states,

$$C_n \sum_{k=0}^{\infty} b_k|k\rangle = \lambda \sum_{k=0}^{\infty} b_k|k\rangle, \qquad \sum_{k=0}^{\infty} |b_k|^2 < \infty, \qquad \lambda \in \mathbb{C} \setminus \{0\} \tag{16}$$

may be easily reduced to the resolution of the following linear discrete Cauchy problem:

$$\begin{aligned} &b_{k+n} = \lambda \sqrt{\frac{k!}{(k+n)!}} e^{-i\frac{2\pi}{n}k} b_k, \\ &b_k = 1, \qquad b_0 = b_1 = \cdots = b_{k-1} = b_{k+1} = \cdots = b_{n-1}, \end{aligned} \tag{17}$$

where k runs from 0 to $n-1$. The Fock states $|0\rangle, |1\rangle, \dots, |n-1\rangle$ are eigenstates of C_n, with eigenvalue 0. If normalizable solutions of the linear difference equation (17) of order n and with variable coefficients exist in correspondence to such initial conditions, then, in view of equation (17), the relative eigensolutions of C_n satisfy the property that the distance between two successive Fock states of their number representations is fixed and equal to n. Thus the n different eigenstates of C_n correspond to the n initial conditions, if normalizable, provide possible generalizations of the even and odd coherent states. We now solve equation (17) exploiting the formula (14) deduced in this paper. A comparison between equation (17) and (1) yields

$$A_{21}^{(k)} = \lambda \sqrt{\frac{k!}{(k+n)!}} e^{-i\frac{2\pi}{n}k}, \qquad A_{2j}^{(k)} = 0, \qquad j = 2, \dots, n. \tag{18}$$

As a consequence, we immediately deduce that only when the choice $g(r_1) = g(r_2) = \cdots = g(r_h) = g(n)$ is compatible with a prefixed value of p, that is $p = nh$, the expression (12) does not vanish. This fact implies that the sum over r appearing in equation (16), contributes for $j = 1$ with the $(r = 1)$-term only and with the $(r = n+1)$-term only, if $j > 1$ and $k > 2n - j + 2$. Thus exploiting equation (16) the general expression of $P_{11}^{(k)}$ in our case may be cast as follows:

$$\begin{aligned} P_{11}^{(k)} &= A_{21}^{(h-1)n} A_{21}^{(h-2)n} \cdots A_{21}^{(n)} A_{21}^{(0)} = \frac{\lambda^h}{\sqrt{(hn)!}}, \qquad \forall\, k = hn, \quad h = 1, 2, \dots, \\ &0, \qquad \text{otherwise.} \end{aligned} \tag{19}$$

$P_{11}^{(k)}$ is the solution of equation (17) in correspondence to the initial condition $b_0 = 1$, $b_1 = b_2 = \cdots = b_{n-1} = 0$. The corresponding normalized eigenstate of C_n may be written down as

$$|\psi_0^{(n)}\rangle = N_0 \sum_{h=0}^{\infty} \frac{\lambda^h}{\sqrt{(hn)!}} |hn\rangle, \tag{20}$$

where

$$N_0 = n e^{-\frac{1}{2}|\beta|^2} \left\{ n + 2 \sum_{\nu=1}^{n-1} (n-\nu) e^{-2|\beta|^2 \sin^2\left(\frac{\pi}{n}\nu\right)} \cos\left(|\beta|^2 \sin\left(\frac{2\pi}{n}\nu\right)\right)\right\}^{-\frac{1}{2}} \tag{21}$$

and the relative eigenvalue is of course λ. The solution of equation (17) relative to the initial condition $b_{n-j+1} = 1$, $b_0 = b_1 = \cdots = b_{n-j} = b_{n-j+2} = \cdots = b_{n-1} = 0$ for $j > 1$ is $P_{1j}^{(k)}$ and corresponds to the following eigenstate of C_n

$$|\psi_k^{(n)}\rangle = N_k \sum_{h=0}^{\infty} \frac{\lambda^h}{\sqrt{(hn+k)!}} |hn+k\rangle, \qquad k = 0, \ldots, n-2, \tag{22}$$

where N_k is an appropriate normalization constant explicitly calculable.

It is possible to demonstrate that $|\psi_k^{(n)}\rangle$ can be expressed as linear combination of n equal-amplitude coherent states. In particular,

(a) $|\psi_0^{(n)}\rangle$ may be represented as the equal right linear combination of all the eigenstates of a^n pertaining to the same eigenvalue λ. It therefore generalizes the even coherent state;

(b) $|\psi_{\frac{n}{2}}^{(n)}\rangle$ (n are *even*) can also be expanded in terms of the same set of coherent states as before, with the difference that now the ratio between successive coefficients is -1. Appropriately adjusting its global phase, we may therefore state that it generalizes the odd coherent state.

In this paper we have derived a new way of representing the general solution of an arbitrary homogeneous linear difference equation. Our resolutive keys of this problem are two. The first one is the choice of the fundamental set of solutions used. The second one is the algorithm by which we succeed to express in the (best possible) closed form the first row of the product of an arbitrary number of the noncommutating matrices. Our resolutive formula (14) has been applied to solve the eigenvalue problem of a particular non-Hermitian operator building up a new class of states of a quantum harmonic oscillator. These states should attract interest in quantum optician community, for instance, in view of the fact that these generalized even and odd coherent states might exhibit remarkable non-classical features.

Concluding we wish to emphasize that the material presented in this paper provides a concrete stimulus toward other interesting applicable developments both in physics and in mathematics.

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Knot Manifolds of Double-Covariant Systems of Elliptic Equations and Preferred Orthonormal Three-Frames

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We show that the problem of existence of preferred orthonormal frame in general relativity, which is formulated as a problem of solvability for the nonlinear system of elliptic equations, can be reduced to the linear problem. For the obtained system of equations and for more general one we find the necessary and sufficient conditions of existence and uniqueness of the solution for Dirichlet problem and the conditions of zeros absence for the solution, taking into account the availability of double symmetry. This allows in particular to prove existence of wide class of hypersurfaces on which the Sen–Witten equation and the Nester gauge are equivalent (up to the sign).

Let (M, g) be $M = \Sigma \times R$ with spacelike $\Sigma_t \times \{t\}$ and metric g of signature $(+, -, -, -)$[1]. We assume that on Σ_t the constraints of general relativity are satisfied:

$$-R^{(3)} - \mathcal{K}_{\mu\nu}\mathcal{K}^{\mu\nu} + \mathcal{K}^2 = 2\mu, \tag{1}$$

$$D_\mu \left(\mathcal{K}^{\mu\nu} - \mathcal{K}h^{\mu\nu}\right) = \mathcal{J}^\nu, \tag{2}$$

where $R^{(3)}$ is scalar curvature of Σ_t, $h = g - n \otimes n$ is induced metric on Σ_t. D_μ is induced by connection ∇_μ on M connection on Σ_t, $\mathcal{K}_{\mu\nu}$ is extrinsic curvature of Σ_t, $\mathcal{K} = \mathcal{K}^\nu_\nu$. μ and $\mathcal{J}^\nu$ are the energy density and momentum density, respectively, of the matter in the frame of reference of an observer, whose one-form of 4-velocity is $\xi = dt$. μ and $\mathcal{J}^\nu$ satisfy the dominant energy condition

$$\mu \geq \mid \mathcal{J}^\nu \mathcal{J}_\nu \mid^{1/2} .$$

There are three globally defined on Σ_t linearly independent one-forms θ^a that may thus be used as a coframe basis. Vector basis will be denoted by e_a. The connection one-forms coefficients ${\omega^a}_{bc}$ are determined as usually: ${\omega^a}_{bc} = \langle \theta^a, \nabla_{e_b} e_c \rangle$.

Definition 1. A set of N $(0 < N \leq 10)$ equations for the components of orthonormal vector basis ${e_m}^\mu$ (tetrad, vierbein)

$$\Phi_N \left({e_{m'}}^{\mu'}, \ \partial_{\nu'} {e_{m'}}^{\nu'}, \ \partial^2_{\nu'\rho'} {e_{p'}}^{\pi'} \right) = 0, \tag{3}$$

which are not covariant under the local Lorentz transformations and (or) coordinate basis transformations, is said to be auxiliary conditions.

Definition 2. The auxiliary conditions (3) are said to be gauge fixing conditions in some domain Ω, if in this domain there exists the solution $x^{\mu'}(x^\nu)$, $L^{m'}_n(x)$ of the system of equations

$$\Phi_N \left({e_n}^\nu \frac{\partial x^{\mu'}}{\partial x^\nu} {L^n}_{m'}, \ldots, \ldots \right) = 0 \tag{4}$$

with arbitrary coefficients ${e_n}^\nu$.

[1]Greek indices α to λ run through $1, 2, 3$; indices κ to ω run through $0, 1, 2, 3$. Latin indices are Lorentzian and a to l run through $1, 2, 3$; indices m to z run through $0, 1, 2, 3$.

For construction of tensor method for the proof of the positive energy theorem in general relativity Nester [1] introduced the auxiliary conditions for the choice of special orthonormal frame on three-dimensional Riemannian manifold

$$d\widetilde{q}=0, \qquad d*q=0, \tag{5}$$

where

$$\widetilde{q}:=i_a d\theta^a, \qquad *q:=\theta_a\wedge d\theta^a.$$

For proof the statement that auxiliary conditions (3) are gauge it is necessary to prove the existence of the solution $||R^{a'}{}_b||\in SO(3)$ for the system of equations (5), where

$$q=i_a d\theta^a=q'+\theta_{m'}\wedge R^{m'}{}_b\, dR^b{}_{c'}\wedge\theta^{c'}, \tag{6}$$

$$q=\theta_a\wedge d\theta^a=\widetilde{q}\,'+R^{b'}{}_a R^a{}_{c',b}\theta^b. \tag{7}$$

The system of equations (6)–(7) is a nonlinear second-order elliptic system for the rotation $R^{a'}{}_b$. Nester proved the existence and uniqueness for the solution of the linearization of this system for geometries within a neighborhood of Euclidean space, and therefore the additional conditions (5) are gauge-fixing only asymptotically.

In paper [2] we have proved that conditions (5) are everywhere gauge on maximal hypersurface. Our purpose is to establish the existence of most wide as in [2] class of the conditions, under which auxiliary conditions (5) are gauge everywhere on Σ.

The method for the proof is based on the grounding for substitution of auxiliary conditions (5) by equivalent (up to sign) linear equations for $SU(2)$-spinor field, for which the theorems of existence are known. For these equations the new theorems about uniqueness and zeros of double-covariant system of equations in the bounded closed domain are also proved.

On the spaces, where the forms $\widetilde{q}$ and $*q$ are exact, the conditions (5) are replaced by their first integrals:

$$\widetilde{q}=-4d\ln\rho, \qquad *q=0.$$

Function ρ is arbitrary and everywhere on Σ_t is positive. Let us consider a case, when the form $\mathcal{K}\theta^3$ is exact, and introduce some function λ, which anywhere on Σ_t does not equal to zero and is defined by the relationship $d\ln\lambda:=4d\ln\rho=\mathcal{K}\theta^3$. Let us complement the triad θ^a to the tetrad defining θ^0 as following: $\theta^0\equiv n=Ndt$, here n is one form of the normal to Σ_t, and let us introduce the complex one-form $L=\frac{\lambda}{\sqrt{2}}(\theta^1+i\theta^2)$. Then one-form L satisfies the equation

$$\left\langle \widetilde{L},\, D\otimes L\right\rangle-\mathcal{K}L+3!\,i*(n\wedge D\wedge L)=0, \tag{8}$$

where $\widetilde{L}=|L|^{-1}*\left(L\wedge\overline{L}\right)$, and here it is taken into account that λ does not equal to zero. The form L is spatial and, therefore, it defines up to sign the $SU(2)$-spinor λ_A: $L=-\lambda_A\lambda_B$, which is a result of Sen [3] reduction of $SL(2,\mathbb{C})$-spinor on Σ_t according to the definition $\lambda^{A+}=\sqrt{2}n^{A\dot{A}}\lambda_{\dot{A}}$. Equation (8) is the "squared" Sen–Witten equation

$$\mathcal{D}^B{}_C\lambda^C=0, \tag{9}$$

where an action of the operator $\mathcal{D}_{AB}$ on $SU(2)$ spinor field is

$$\mathcal{D}_{AB}\lambda_C=D_{AB}\lambda_C+\frac{\sqrt{2}}{2}\mathcal{K}_{ABC}{}^D\lambda_D.$$

So, the question about existence of solution for nonlinear elliptic system (5) is reduced to the question about existence of solution for linear elliptic system (9) under the condition that this solution λ_A nowhere on Σ_t equals zero.

Further we will examine the question about conditions of existence and zeros absence for the solution of boundary value problem for general elliptic system of equations

$$\frac{1}{\sqrt{-h}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{-h}h^{\alpha\beta}\frac{\partial}{\partial x^\beta}u_A\right)+C_A{}^B u_B=0 \tag{10}$$

on bounded closed spherical-type domain Ω on Σ_t, where $h^{\alpha\beta}$ – metric tensor components, which are arbitrary real functions of independent variables x^α continuous in Ω; the quadratic form $h^{\alpha\beta}\xi_\alpha\xi_\beta$ is negatively defined. The unknown functions u_A are complex twice continuously differentiable functions of independent variables x^α. They are also the elements of vector space $\mathbb{C}^2$, in which the skew symmetric tensor ε^{AB} is defined, and the group $SU(2)$ acts. The matrix $C := ||C_A{}^B||$ is Hermitian, its elements are twice continuously differentiable, and $C_0{}^1 \neq 0$ in Ω. A system (10) is a generalization of a differential result of equation (9) and equations (1), (2) [2].

For strongly elliptic systems of second order equations the sufficient conditions for unique solvability of boundary value problem were obtained in [4]. Let us take into account that the system of equations (10) is covariant under arbitrary transformations of coordinates on Ω, and covariant under the local $SU(2)$ transformations. This allows to obtain the necessary and sufficient conditions for unique solvability of Dirichlet problem.

Let denote by

$$\Delta := C_1{}^1 - C_0{}^0 - \left[\left(C_1{}^1 - C_0{}^0\right)^2 + 4\left|C_0{}^1\right|^2\right]^{1/2}.$$

Theorem 1. *The boundary value problem for equation (10) is uniquely solvable in domain Ω if and only if in this domain there are exist functions of C^2 class which satisfy the inequalities*

$$\det\begin{pmatrix} (-h^{\alpha\beta}) & B^\beta_A \\ B^\alpha_A & \sum\limits_{\gamma=1}^{3}\frac{\partial B^\gamma_A}{\partial x^\gamma}+C_{A'} \end{pmatrix} > 0, \tag{11}$$

here

$$C_{0'} = \frac{4C_0{}^0\left|C_0{}^1\right|^4+\left(4\Delta\left|C_0{}^1\right|^2+C_1{}^1\Delta^2\right)\left(4\left|C_0{}^1\right|^2+\Delta^2\right)}{4\left|C_0{}^1\right|^2\left(4\left|C_0{}^1\right|^2+\Delta^2\right)}, \tag{12}$$

$$C_{1'} = \frac{\left(C_0{}^0\Delta^2-4\,\Delta\left|C_0{}^1\right|^2\right)\left(4\left|C_0{}^1\right|^2+\Delta^2\right)+4C_1{}^1\left|C_0{}^1\right|^4}{\left(4\left|C_0{}^1\right|^2+\Delta^2\right)}. \tag{13}$$

Proof. The system of equations (10) is covariant under the arbitrary transformations of coordinates and under the local transformations from spinor group $SU(2)$. This allows us to use them independently. Because the matrix C is Hermitian, there exists a matrix

$$R := ||R_A{}^B|| := \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \qquad \alpha\overline{\alpha}+\beta\overline{\beta}=1,$$

such that $C' = -\varepsilon R\varepsilon C R^{T+} = R^{T+}CR = \mathrm{diag}\,(C_{0'}, C_{1'})$. The matrix elements satisfy the conditions $\alpha\overline{\alpha}\left(1+\Delta^2/4\left|C_0{}^1\right|^2\right)=1$, $\beta=\alpha\Delta/2C_0{}^1$, and in the new spinor basis

$$u_{0'} = \overline{\alpha}\left(u_0+\frac{\Delta}{2\overline{C}_0{}^1}\,u_1\right), \qquad u_{1'} = \alpha\left(-\frac{\Delta}{2C_0{}^1}\,u_0+u_1\right). \tag{14}$$

The eigenvalues of matrix C are real, therefore, the system of equations (10) in the new spinor basis splits into the system of four independent equations. The coefficients before unknown functions are expressed as (12) and (13). Since $C_{0'}$ and $C_{1'}$ are scalars under the transformations of coordinates, we can apply the Skorobohat'ko theorem 1.16 [4] to each of equations. The conditions of the existence for each of equations are (11) with (12) or (13). This proves the Theorem. ■

Corollary 1. *The boundary value problem for equation (10) is uniquely solvable in domain Ω if in this domain the matrix C is positively defined.*

Theorem 2. *If the matrix C is positively defined in Ω, then a function $M = u_0\overline{u}_0 + u_1\overline{u}_1$ for any solution of class C^2 for equation (10) reaches the non-zero maximum only on the boundary of domain Ω.*

Proof. In arbitrary domain Ω in Riemannian space V^3 there exist solutions f_γ of class C^2 for a system of differential equations

$$h^{\alpha\beta}\frac{\partial f_\gamma}{\partial x^\alpha}\frac{\partial f_\delta}{\partial x^\beta}, \qquad \gamma \neq \delta.$$

Setting 3-orthogonal hypersurfaces $f_\alpha = \mathrm{const}$ as coordinate hypersurfaces $x^{\alpha'} = \mathrm{const}$ we will obtain the system of coordinates in which $h^{\alpha\beta} = 0$ at $\alpha \neq \delta$ [5].

Let us assume that the function M reaches the non-zero maximum in some intrinsic point of domain Ω. Then in this point $\frac{\partial M}{\partial x^\alpha} = 0$ and $h^{\alpha\alpha}\frac{\partial^2 M}{\partial x^{\alpha^2}} \geq 0$. But, from the other side, in the same point of maximum we have, taking into account that functions u_A and $C_A{}^B$ are scalars under arbitrary transformations of coordinate basis, the equation (10) is covariant with respect to them, and Hermitian matrix C is positively defined:

$$h^{\alpha\alpha}\frac{\partial^2 M}{\partial x^{\alpha^2}} = h^{\alpha\alpha}\frac{\partial u_A}{\partial x^\alpha}\frac{\partial \overline{u}_A}{\partial x^{\alpha^2}} - \sqrt{-h}C_A{}^B u_B \overline{u}^A - \sqrt{-h}\,\overline{C}_A{}^B \overline{u}_B u^A > 0.$$

The contradiction proves the statement of the theorem. ■

Theorem 2 generalizes the Bicadze extremum principle [6] onto the systems of elliptic equations with non-diagonal main parts of operators. It gives effective conditions of the knot points absence.

Corollary 2. *If the matrix C is positively defined in domain Ω, then in this domain the non-trivial solutions of class C^2 for equations (10) do not have the knot points.*

Reula has proved the existence of the C^∞ solution to Sen–Witten equation on Σ_t , if the initial data set $(\Sigma_t, h_{\mu\nu}, \mathcal{K}_{\pi\rho})$ is asymptotically flat [7].

The conditions, when initial data set is asymptotically flat and spinor u_A belongs to a certain Hilbert space, in this case substitute the conditions of Theorem 1. From Theorem 2 we obtain a condition for absence of knot points for Sen–Witten equation.

Theorem 3. *If matrix G with the elements*

$$G_A{}^B := \mathcal{D}_A{}^B K + \sqrt{2}\varepsilon_A^B\left(K^2 + \frac{1}{4}K_{\alpha\beta}K^{\alpha\beta} + \frac{1}{2}\mu\right)$$

is positively defined everywhere on Σ_t, then the solution of equation (10), which tends at infinity to a certain constant non-zero value, differs from zero everywhere on Σ_t.

From existence of the solution for Sen–Witten equation and from absence of knot points for this solution it follows that the Nester additional conditions are gauge on all hypersurfaces, on which the matrix G is positively defined. Therefore on such hypersurfaces there exists the Nester preferred orthonormal frame, and this is also the Sen–Witten frame. In particular, the Nester conditions are gauge, and the Nester frame is the Sen–Witten frame on maximal hypersurface.

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The Asymptotic Solutions of the Systems of Nonlinear Differential Equations

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The method of asymptotic integration of a singular perturbed nonlinear system of differential equations is offered.

In works [1, 2] the systems of singular perturbed linear differential equations were studied. The construction of asymptotic solution of nonlinear systems of differential equations were studied in works of W. Wasow, R. Langer, M. Iwano, A. Vasilieva, S. Lomov. In this work the method of asymptotic integration of the singular perturbed nonlinear system of differential equations is suggested.

Let us study the system of equations

$$\varepsilon\frac{dx}{dt} = A(t,\varepsilon)x + f(t,\varepsilon,x), \qquad x(0,\varepsilon) = x_0, \tag{1}$$

where ε $(0 < \varepsilon \le \varepsilon_0)$ is a small parameter, $f(t,\varepsilon,x)$, $x(t,\varepsilon)$, x_0 is n-dimensional vectors. We suppose to carry out such conditions:

1) vector $f(t,\varepsilon,x)$ has the decomposition into uniform convergent series

$$f(t,\varepsilon,x) = \sum_{|r|=2}^{\infty} a_r(t,\varepsilon)x^r, \tag{2}$$

where $a_r(t,\varepsilon)$ is n-dimensional vectors, $x^r = x_1^{r_1}x_2^{r_2}\cdots x_n^{r_n}$, $|r| = \sum\limits_{i=1}^{n} r_i$; x_i $(i = 1,\ldots,n)$, components of vector $x(t,\varepsilon)$;

2) the matrix $A(t,\varepsilon)$ and vectors $a_r(t,\varepsilon)$ have the decomposition using degrees of small parameters

$$A(t,\varepsilon) = \sum_{s=0}^{\infty}\varepsilon^s A_s(t), \qquad a_r(t) = \sum_{s=0}^{\infty}\varepsilon^s a_{rs}(t);$$

3) matrix $A_s(t)$ and vectors $a_{rs}(t)$ $(s = 0,1,\ldots)$ are infinite differentiable on the segment $[0;L]$;

4) solutions of characteristic equation

$$\det\|A_0(t) - \lambda(t)E\| = 0 \tag{3}$$

are simple on the segment $[0;L]$, where E is the identity matrix that has order n.

Let us use substitution into system (1)

$$x(t,\varepsilon) = U_m(t,\varepsilon)y(t,\varepsilon), \tag{4}$$

where $y(t,\varepsilon)$ is an n-dimensional vector, $U_m(t,\varepsilon)$ is an $n\times n$ matrix

$$U_m(t,\varepsilon) = \sum_{s=0}^{m}\varepsilon^s U_s(t)$$

the result

$$\varepsilon U_m(t,\varepsilon)y' = (A(t,\varepsilon)U_m(t,\varepsilon) - \varepsilon U_m'(t,\varepsilon))y(t,\varepsilon) + f(t,\varepsilon,U_m(t,\varepsilon)y), \tag{5}$$

here $()'$ means the derivative with respect to t.

We will construct matrix $U_s(t)$, $(s = 0,\dots,m)$ in the way that the matrix equation takes place:

$$A(t,\varepsilon)U_m(t,\varepsilon) - \varepsilon U_m'(t,\varepsilon) = U_m(t,\varepsilon)(\Lambda_m(t,\varepsilon) + \varepsilon_{m+1}C_m(t,\varepsilon)), \tag{6}$$

where $\Lambda_m(t,\varepsilon)$ is a diagonal matrix in form

$$\Lambda_m(t,\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s \Lambda_s(t),$$

$C_m(t,\varepsilon)$ is $n\times n$ matrix. Matrices $U_s(t)$, $\Lambda_s(t)$ $(s = 0,\dots,m)$ are obtained by using methods [1, 2] from equation (6). So, from system (1) we obtain system

$$\varepsilon y'(t,\varepsilon)y' = (\Lambda_m(t,\varepsilon)\varepsilon^{m+1}C_m(t,\varepsilon))y + U_m'(t,\varepsilon)f(t,\varepsilon,U_m(t,\varepsilon)y), \tag{7}$$

Let us substitute $y(t,\varepsilon) = z + q(t,\varepsilon,z)$ to (7), where $q(t,\varepsilon,z)$ has the development

$$q(t,\varepsilon,z) = \sum_{|r|=2}^{\infty} q_r(t,\varepsilon)z^r.$$

So system (7) has form

$$\begin{aligned}\varepsilon z'(t,\varepsilon) &= (E + Q_z(t,\varepsilon,z))^{-1}\Big(- \varepsilon q'(t,\varepsilon,z)\\ &\quad + (\Lambda_m(t,\varepsilon) + \varepsilon^{m+1}C_m(t,\varepsilon))z + (\Lambda_m(t,\varepsilon) + \varepsilon^{m+1}C_m(t,\varepsilon))q(t,\varepsilon,z)\Big)\\ &\quad + U_m^{-1}(t,\varepsilon)f(t,\varepsilon,U_m(t,\varepsilon)(z + q(t,\varepsilon,z)),\end{aligned} \tag{8}$$

where $Q_z(t,\varepsilon,z)$ is matrix that consist of partial derivative components of vector $q(t,\varepsilon,z)$.

Let us choose vector $q(t,\varepsilon,z)$ in a way that

$$\begin{aligned}&(E + Q_z(t,\varepsilon,z))^{-1}\Big(- \varepsilon q'(t,\varepsilon,z) + \Lambda_m(t,\varepsilon)(z + q(t,\varepsilon,z))\Big)\\ &\quad + U_m^{-1}(t,\varepsilon)f(t,\varepsilon,U_m(t,\varepsilon)(z + q(t,\varepsilon,z))) = (\Lambda_m(t,\varepsilon) + \varepsilon^{m+1}C_m(t,\varepsilon))z\end{aligned} \tag{9}$$

takes place. After multiplying (9) by matrix $E + Q_z(t,\varepsilon,z)$ and grouping similar terms with $\Lambda_m(t,\varepsilon)z$ we will obtain

$$\begin{aligned}\varepsilon q'(t,\varepsilon,z) &= (\Lambda_m(t,\varepsilon) + \varepsilon^{m+1}C_m(t,\varepsilon))q(t,\varepsilon,z)\\ &\quad + U_m^{-1}(t,\varepsilon)f(t,\varepsilon,U_m(t,\varepsilon)(z + q(t,\varepsilon,z)) - Q_z(t,\varepsilon,z)(\Lambda_m(t,\varepsilon) + \varepsilon^{m+1}C_m(t,\varepsilon))z.\end{aligned} \tag{10}$$

Let us present matrix $Q_z(t,\varepsilon,z)$ in the form

$$Q_z(t,\varepsilon,z) = \sum_{|r|=2}^{\infty} z^r q_r(t,\varepsilon)r_z,$$

where $r_z = \left(r_1 z_1^{-1}, \ldots, r_n z_n^{-1}\right)$. So, we have

$$Q_z(t,\varepsilon,z)\Lambda_m(t,\varepsilon)z = \sum_{|r|=2}^{\infty} \sum_{j=1}^{n} z^r q_r(t,\varepsilon) r_j \lambda_{mj}(t,\varepsilon),$$

$$Q_z(t,\varepsilon,z)C_m(t,\varepsilon)z = \sum_{|r|=2}^{\infty} \overline{g}_r(t,\varepsilon,z),$$

$$\overline{g}_r(t,\varepsilon,z) = z^r q_r(t,\varepsilon) r_z C_m(t,\varepsilon)z,$$

$$g(t,\varepsilon,z) = \left(U_m(t,\varepsilon)\left(z + \sum_{|s|=2}^{\infty} q_s(t,\varepsilon)z^s\right)\right)^r,$$

where $\lambda_{mj}(t,\varepsilon)$ are elements of the matrix $\lambda_m(t,\varepsilon)$. Decomposing functions $\overline{g}_r(t,\varepsilon,z)$, $g_r(t,\varepsilon,z)$ into power series, substituting to (10) and equating coefficient with similar degrees $z_1^{r_1}\cdots z_n^{r_n}$ we will obtain

$$\begin{aligned}\varepsilon q'(t,\varepsilon) &= \left(\Lambda_m(t,\varepsilon) - \sum_{j=1}^{n} r_j \lambda_{mj}(t,\varepsilon)\cdot E\right) q_r(t,\varepsilon)\\ &\quad + \varepsilon^{m+1} C_m(t,\varepsilon) q_r(t,\varepsilon) + V_r(t,\varepsilon,q_1,\ldots,q_{r-1}) + \varepsilon^{m+1}\overline{V}_r(t,\varepsilon,q_1,\ldots,q_{r-1}),\\ q_r(0,\varepsilon) &= 0, \end{aligned} \tag{11}$$

where $V_r(t,\varepsilon,q_1,\ldots,q_{r-1})$, $\overline{V}_r(t,\varepsilon,q_1,\ldots,q_{r-1})$ are expressed in terms of partial derivatives respectively to function $g_r(t,\varepsilon,z)$, $\overline{g}_r(t,\varepsilon,z)$.

System of equation has form

$$\begin{aligned}\varepsilon z'(t,\varepsilon) &= (\Lambda_m(t,\varepsilon) + \varepsilon^{m+1} C_m(t,\varepsilon))z,\\ z(0,\varepsilon) &= U_m^{-1}(0,\varepsilon)x_0\end{aligned} \tag{12}$$

approximate [1] m (11) we will write in the form

$$q_{rm}(t,\varepsilon) = -\exp\left(\frac{1}{\varepsilon}\int_0^t \Lambda_m(t,\varepsilon)dt\right)\overline{q}_{rm}(0,\varepsilon) + \overline{q}_{rm}(t,\varepsilon),$$

where

$$\overline{q}_{rm}(t,\varepsilon) = \sum_{s=\beta}^{m} \varepsilon^s q_{rs}(t)$$

the particular solution (11); $\beta = 0$ or $\beta = -1$.

Approximate m (12) we will write in the form

$$z_m(t,\varepsilon) = \exp\left(\frac{1}{\varepsilon}\int_0^t \Lambda_m(t,\varepsilon)dt\right) U_m^{-1}(0,\varepsilon)x_0.$$

So, approximate m (1) takes the form

$$\begin{aligned} x_m(t,\varepsilon) &= U_m(t,\varepsilon)\exp\left(\frac{1}{\varepsilon}\int_0^t \Lambda_m(t,\varepsilon)dt\right) U_m^{-1}(0,\varepsilon)x_0\\ &\quad + U_m(t,\varepsilon)\sum_{|r|=2}^{\infty}\left(\overline{q}_{rm}(t,\varepsilon) - \exp\left(\frac{1}{\varepsilon}\int_0^t \Lambda_m(t,\varepsilon)dt\right)\overline{q}_{rm}(0,\varepsilon)\right)\\ &\quad \times \left(\exp(\frac{1}{\varepsilon}\int_0^t \Lambda_m(t,\varepsilon)dt\right) U_m^{-1}(0,\varepsilon)x_0)^r. \end{aligned} \tag{13}$$

We proved that (13) consists of convergent series, and approximate m has an asymptotic property.

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Asynchronous Development of the Growing-and-Decaying Mode

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The solution to the Davey–Stewartson I equation is analyzed to show that the resonance between periodic soliton and growing-and-decaying mode exists. Under the quasi-resonant condition, the mode develops first in the one side region of the periodic soliton. The periodic soliton is accelerated as a result of the growth and decay of the mode existed in the region and the wave field shifts to the intermediate state, where only the periodic soliton exists. This intermediate state persists over a comparatively long time interval. After sufficiently long time, the mode starts to grow in the opposite side of the periodic soliton.

1 Introduction

A uniform train of weakly nonlinear deepwater waves is unstable to long wave modulational perturbations of the envelope, which is known as the Benjamin–Feir instability [1]. It is well known that the long time evolution of the unstable wave train is described by the nonlinear Schrödinger (NLS) equation [2, 3, 4]. The extension to the two-dimensional case was examined by Zakharov, Benney and Roskes and Davey and Stewartson [2, 5, 6]. The long time evolution of a two-dimensional wave-packet is described by the Davey–Stewartson (DS) equation [6]

$$\begin{aligned} & iu_t + pu_{xx} + u_{yy} + r|u|^2u - 2uv = 0, \\ & v_{xx} - pv_{yy} - r(|u|^2)_{xx} = 0, \end{aligned} \tag{1}$$

where $p = \pm 1$, r is constant. Equation (1) with $p = 1$ and $p = -1$ are called the DS I and DS II equations, respectively. The time evolution of the solution of the 1D-NLS equation with periodic boundary condition and with Benjamin–Feir unstable initial condition was studied numerically by Lake et al. [7]. They found that a modulated unstable wave train achieves a state of maximum modulation and returns to an unmodulated initial state. The nonlinear evolution of an unstable mode is described by the growing-and-decaying mode soluiton to the 1D-NLS equation [8].

The DS I equation has also the growing-and-decaying mode solution, which is given by [9]

$$u = u_0 e^{i\zeta} \frac{g}{f}, \qquad v = \, -2\,(\ln f)_{xx} \tag{2}$$

with

$$\begin{aligned} & f = 1 - e^{-\Omega t + \sigma} \cos\eta + \frac{M}{4} e^{-2\Omega t + 2\sigma}, \\ & g = 1 - e^{-\Omega t + \sigma + i\phi} \cos\eta + \frac{M}{4} e^{-2\Omega t + 2\sigma + 2i\phi}, \end{aligned}$$

where

$$\begin{aligned} & \zeta = kx + ly - \omega t, \qquad \omega = k^2 + l^2 - ru_0^2, \qquad \eta = \beta x + \delta y - \gamma t + \theta, \\ & \Omega = (\beta^2 + \delta^2) \cot\frac{\phi}{2}, \qquad \gamma = 2k\beta + 2l\delta, \qquad M = \frac{2}{1 + \cos\phi} > 1, \qquad \sin^2\frac{\phi}{2} = \frac{\delta^2 - \beta^2}{2ru_0^2}, \end{aligned}$$

σ and θ are arbitrary phase constants. The existence condition for the nonsingular solution is given by $M > 1$ for real ϕ, which is satisfied for

$$0 < \left(\delta^2 - \beta^2\right) < 2ru_0^2,$$

which is in agreement with the Benjamin–Feir unstable condition. This solution grows exponentially at initial stage, and reaches a state of maximum modulation and after reaching maximum modulation, demodulates and finally returns to an unmodulated initial state. Therefore, the solution (2) describes the nonlinear evolution of monochromatic perturbation with the Benjamin–Feir unstable condition in two-dimension.

The interactions between two-periodic solitons, between periodic soliton and line soliton and between periodic soliton and algebraic soliton to the DS equation have been investigated in detail [10, 11, 12]. It was shown that the periodic soliton resonances exist in each case. Pelinovsky pointed out the existence of the resonance between line soliton and growing-and-decaying mode [10]. The growing-and-decaying mode exists substantially only a finite period in time, but the resonance between line soliton and growing-and-decaying mode brings about the infinite phase shift to the line soliton. If the growing-and-decaying mode exists within only a finite time in reality, the mechanism bringing about the infinite phase shift to the line soliton is puzzle. Recently, we have investigated the time evolution of the quasi-resonant interaction between line soliton and growing-and-decaying mode and found the existence of an asynchronous development of the growing-and-decaying mode [13].

In this paper, it is shown that under the quasi-resonant condition for the interaction between periodic soliton and growing-and-decaying mode, the asynchronous development of the growing-and-decaying mode also exists.

2 Quasi-resonance between periodic soliton and growing-and-decaying mode

The interaction between periodic soliton and growing-and-decaying mode to the DS I equation is studied in this section. The solution describing the interaction can be obtained by the N-soliton solution of Satsuma and Ablowitz [14]. The solution consisting of a periodic soliton and growing-and-decaying mode is given by

$$u = u_0 e^{i\varsigma}\frac{g}{f}, \qquad v = -2\left(\ln f\right)_{xx} \tag{3}$$

with

$$\begin{aligned} f = {} & 1 - \frac{1}{L_1L_2}e^{\xi_1}\cos\eta_1 - e^{\xi_2}\cos\eta_2 + \frac{M_1}{4L_1^2L_2^2}e^{2\xi_1} + \frac{M_2}{4}e^{2\xi_2} \\ & - \frac{1}{4}e^{\xi_1+\xi_2}\left\{\frac{M_1}{L_1L_2}e^{\xi_1}\cos(\eta_2+\Psi_1-\Psi_2) + M_2e^{\xi_2}\cos(\eta_1+\Psi_1+\Psi_2)\right\} \\ & + \frac{1}{2L_1L_2}e^{\xi_1+\xi_2}\Big\{L_1\cos(\eta_1+\eta_2+\Psi_1) + L_2\cos(\eta_1-\eta_2+\Psi_2)\Big\} + \frac{M_1M_2}{16}e^{2(\xi_1+\xi_2)}, \end{aligned} \tag{4}$$

$$\begin{aligned} g = {} & 1 - \frac{1}{L_1L_2}e^{\xi_1+i\phi_{1r}}\cos(\eta_1+i\phi_{1i}) - e^{\xi_2+i\phi_2}\cos\eta_2 + \frac{M_1}{4L_1^2L_2^2}e^{2\xi_1+2i\phi_{1r}} + \frac{M_2}{4}e^{2\xi_2+2i\phi_2} \\ & - \frac{1}{4}e^{\xi_1+\xi_2+i(\phi_{1r}+\phi_2)}\left\{\frac{M_1}{L_1L_2}e^{\xi_1+i\phi_{1r}}\cos(\eta_2+\Psi_1-\Psi_2)\right. \\ & \left. + M_2e^{\xi_2+i\phi_2}\cos(\eta_1+i\phi_{1i}+\Psi_1+\Psi_2)\right\} \end{aligned}$$

$$+\frac{1}{2L_1L_2}e^{\xi_1+\xi_2+i(\phi_{1r}+\phi_2)}\Big\{L_1\cos(\eta_1+\eta_2+i\phi_{1i}+\Psi_1)$$
$$+L_2\cos(\eta_1-\eta_2+i\phi_{1i}+\Psi_2)\Big\}+\frac{M_1M_2}{16}e^{2(\xi_1+\xi_2)+2i(\phi_{1r}+\phi_2)}, \tag{5}$$

where

$$\xi_1=\alpha x+\kappa y-\Omega_1 t+\sigma_1, \qquad \xi_2=-\Omega_2 t+\sigma_2,$$
$$\eta_1=\beta_1 x+\delta_1 y-\gamma_1 t+\theta_1, \qquad \eta_2=\beta_2 x+\delta_2 y-\gamma_2 t+\theta_2,$$
$$\sin^2\frac{\phi_1}{2}=\frac{(\alpha+i\beta_1)^2-(\kappa+i\delta_1)^2}{2ru_0^2}, \qquad \sin^2\frac{\phi_2}{2}=\frac{\delta_2^2-\beta_2^2}{2ru_0^2},$$
$$\Omega_1=2k\alpha+2l\kappa-\Re\left\{\left\{(\alpha+i\beta_1)^2+(\kappa+i\delta_1)^2\right\}\cot\frac{\phi_1}{2}\right\},$$
$$\gamma_1=2k\beta_1+2l\delta_1-\Im\left\{\left\{(\alpha+i\beta_1)^2+(\kappa+i\delta_1)^2\right\}\cot\frac{\phi_1}{2}\right\},$$
$$\Omega_2=(\beta_2^2+\delta_2^2)\cot\frac{\phi_2}{2}, \qquad \gamma_2=2k\beta_2+2l\delta_2,$$
$$M_1=\frac{2ru_0^2|\sin\frac{\phi_1}{2}|^2\cosh\phi_{1i}-(\alpha^2+\beta_1^2)+(\kappa^2+\delta_1^2)}{2ru_0^2|\sin\frac{\phi_1}{2}|^2\cos\phi_{1r}-(\alpha^2+\beta_1^2)+(\kappa^2+\delta_1^2)}, \qquad M_2=\frac{2}{1+\cos\phi_1},$$
$$L_1e^{i\Psi_1}=\frac{2ru_0^2\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\cos\frac{\phi_1-\phi_2}{2}-i\{(\alpha+i\beta_1)\beta_2-(\kappa+i\delta_1)\delta_2\}}{2ru_0^2\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\cos\frac{\phi_1+\phi_2}{2}-i\{(\alpha+i\beta_1)\beta_2-(\kappa+i\delta_1)\delta_2\}},$$
$$L_2e^{i\Psi_2}=\frac{2ru_0^2\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\cos\frac{\phi_1-\phi_2}{2}+i\{(\alpha+i\beta_1)\beta_2-(\kappa+i\delta_1)\delta_2\}}{2ru_0^2\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\cos\frac{\phi_1+\phi_2}{2}+i\{(\alpha+i\beta_1)\beta_2-(\kappa+i\delta_1)\delta_2\}},$$

where we have assumed that ϕ_2 is real and θ_1, θ_2, σ_1 and σ_2 are arbitrary constants. When we consider the case: $0<\Omega_2$, $0<\alpha$, $0<\kappa$ and $0<\Omega_1$, the solutions long before and after the mode growing are given by

$$f=\frac{M_2}{4}e^{2\xi_2}\left\{1-e^{\xi_1}\cos(\eta_1+\Psi_1+\Psi_2)+\frac{M_1}{4}e^{2\xi_1}\right\}, \tag{6}$$
$$g=\frac{M_2}{4}e^{2(\xi_2+i\phi_2)}\left\{1-e^{\xi_1+i\phi_{1r}}\cos(\eta_1+i\phi_{1i}+\Psi_1+\Psi_2)+\frac{M_1}{4}e^{2(\xi_1+i\phi_{1r})}\right\}, \tag{7}$$

and

$$f=1-\frac{1}{L_1L_2}e^{\xi_1}\cos\eta_1+\frac{M_1}{4L_1^2L_2^2}e^{2\xi_1}, \tag{8}$$
$$g=1-\frac{1}{L_1L_2}e^{\xi_1+i\phi_{1r}}\cos(\eta_1+i\phi_{1i})+\frac{M_1}{4L_1^2L_2^2}e^{2(\xi_1+i\phi_{1r})}, \tag{9}$$

respectively, which are periodic soliton solutions. It is shown that the phase shift of the periodic soliton due to the growing-and-decaying mode is given by the amount $\ln(L_1L_2)$ (or $-\ln(L_1L_2)$). $(L_1L_2)=\infty$ and 0 may be thought of as resonance between periodic soliton and growing-and-decaying mode, the conditions of which are obtained by equating the denominator and numerator of L_1 or L_2 to zero, respectively: We now investigate the condition which L_1 becomes infinity, namely

$$D=2ru_0^2\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\cos\frac{\phi_1+\phi_2}{2}-i\left\{(\alpha+i\beta_1)\beta_2-(\kappa+i\delta_1)\delta_2\right\}=0. \tag{10}$$

When we express α, κ, β_1, δ_1, β_2 and δ_2 in term of ϕ_1, ϕ_2, θ_1 and θ_2 as follows,

$$\alpha + i\beta_1 = i\sqrt{2ru_0^2}\sin\frac{\phi_1}{2}\sinh\theta_1, \qquad \kappa + i\delta_1 = i\sqrt{2ru_0^2}\sin\frac{\phi_1}{2}\cosh\theta_1,$$
$$\beta_2 = \sqrt{2ru_0^2}\sin\frac{\phi_2}{2}\sinh\theta_2, \qquad \delta_2 = \sqrt{2ru_0^2}\sin\frac{\phi_2}{2}\cosh\theta_2.$$

Equation (10) is rewritten as

$$D = 2ru_0^2\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\left\{\cos\frac{\phi_1+\phi_2}{2} - \cosh(\theta_1-\theta_2)\right\}.$$

Therefore, the resonant condition is given by

$$\phi_2 = 2\theta_{1i} - \phi_{1r}, \qquad \theta_2 = \theta_{1r} + \frac{\phi_{1i}}{2}.$$

We study the time evolution of soliton in the following five periods in time. The solutions (4) and (5) are approximated in each period as follows:

(**p1**) $t \to -\infty$ (before the mode grows). The solution is given by equations (6) and (7), only the periodic soliton exists in the wave field.

(**p2**) $t \sim \dfrac{\sigma_2}{\Omega_2}$; ($e^{-\Omega_2 t+\sigma_2} \sim O(1)$). The solutions in the backward region and forward region of the periodic soliton are given by

$$f \simeq 1 - e^{\xi_2}\cos\eta_2 + \frac{M_2}{4}e^{2\xi_2}, \tag{11}$$
$$g \simeq 1 - e^{\xi_2+i\phi_2}\cos\eta_2 + \frac{M_2}{4}e^{2(\xi_2+i\phi_2)}, \tag{12}$$

and

$$f \simeq \frac{M_1M_2}{16}e^{2(\xi_1+\xi_2)}, \tag{13}$$
$$g \simeq \frac{M_1M_2}{16}e^{2(\xi_1+\xi_2)+2i\phi_{1r}+\phi_2)}, \tag{14}$$

respectively. The solutions corresponding to equations (11)–(12) and (13)–(14) denote the growing-and-decaying mode and uniform state, respectively. Therefore, in this period, the mode is growing only in the backward region of the periodic soliton, but the mode has not grown as yet in the forward region.

(**p3**) $t \sim \dfrac{\sigma_2 + \frac{1}{2}\ln L_1L_2}{\Omega_2}$; ($\sqrt{L_1L_2}e^{-\Omega_2 t+\sigma_2} \sim O(1)$)

$$f \simeq 1 + \frac{1}{2L_2}e^{\xi_1+\xi_2}\cos(\eta_1+\eta_2+\Psi_1) + \frac{M_1M_2}{16}e^{2(\xi_1+\xi_2)},$$
$$g \simeq 1 + \frac{1}{2L_2}e^{\xi_1+\xi_2+i(\phi_{1r}+\phi_2)}\cos(\eta_1+\eta_2+i\phi_{1i}+\Psi_1) + \frac{M_1M_2}{16}e^{2(\xi_1+\xi_2)+2i(\phi_{1r}+\phi_2)}.$$

Only the periodic soliton in the resonant state exists in the wave field.

(**p4**) $t \sim \dfrac{\sigma_2 + \ln L_1L_2}{\Omega_2}$; ($L_1L_2e^{\xi_2} \sim O(1)$). The solutions in the backward region and forward region of the periodic soliton are given by

$$f \simeq 1, \qquad g \simeq 1,$$

and

$$f \simeq \frac{M_1}{4L_1^2L_2^2}e^{2\xi_1}\left\{1 - L_1L_2e^{\xi_2}\cos(\eta_2 + \Psi_1 - \Psi_2) + \frac{M_2L_1^2L_2^2}{4}e^{2\xi_2}\right\},$$
$$g \simeq \frac{M_1}{4L_1^2L_2^2}e^{2(\xi_1 + i\phi_{1r})}\left\{1 - L_1L_2e^{\xi_2 + 2i\phi_2}\cos(\eta_2 + \Psi_1 - \Psi_2) + \frac{M_2L_1^2L_2^2}{4}e^{2(\xi_2 + i\phi_2)}\right\},$$

respectively. In this period, the mode is developed only in the forward region of the periodic soliton.

($\mathbf{p_5}$) $t \to +\infty$. The solution is given by equations (8) and (9) which is the periodic soliton after the grow and decay of the mode.

3 Conclusions

We have investigated the time evolution of the quasi-resonant interaction between periodic soliton and growing-and-decaying mode. Under the quasi-resonant condition, the mode develops first in the one side region of the periodic soliton. The periodic soliton is accelerated as a result of the grow and decay of the mode existed in the region and the wave field shifts to the intermediate state, where only the periodic soliton in the resonant state exists. This intermediate state persists over a comparatively long time interval. After sufficient long time, the mode starts to grow in the opposite side of the periodic soliton. The existence of soliton changes the evolution of the growing-and-decaying mode drastically as if the periodic soliton dominated the evolution of the instability in whole region of the wave field.

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Integrable Polynomial Potentials in N-Body Problems on the Line

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Integrable natural systems of n interacting particles on the line are investigated under assumption that the interacting potential is a polynomial. Restriction for degree of these potentials is obtained both for systems with pairwise interaction and for the case of lattices.

Dynamics of n equal pair-interactive particles on the line is described by the Hamiltonian system with the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i<j} V(x_i - x_j), \tag{1}$$

where the x_i and $p_i, i = 1, \ldots, n$, are the coordinates and momenta of the particles. We henceforth call the function V a potential. Complete integrability of this system was established in [1, 2] for the Weierstrass $\mathcal{P}$-function as the interaction potential. Moreover, this system possesses a complete collection of integrals which are polynomials in the momenta and are in involution. It is therefore natural to obtain a description of Hamiltonians (1) which admit integrals that are polynomials in the momenta. We are interested in considering the problem of integrability of such natural system of interacting particles in Liouville's sense for polynomial potential $V(z)$, such that $\deg V(z) = k > 2$.

Theorem 1. *Let the potential $V(z)$ admit an integral F, which is polynomial in the momenta. Then the potential z^k admits a nontrivial integral, which is also polynomial.*

Theorem 2. *The 3-body problem with the Hamiltonian (1) is integrable if and only if $k \leqslant 4$.*

Proof. The total momentum $P = \sum p_i$ is the first integral of the system under consideration. Therefore this system can be reduced to the system with two degrees of freedom and the Hamiltonian

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + V(x) + V\left(-\frac{x}{2} + \frac{y\sqrt{3}}{2}\right) + V\left(-\frac{x}{2} - \frac{y\sqrt{3}}{2}\right).$$

Now we shall use the Yoshida's theorem [3] on the nonintegrability of natural systems with homogeneous potential. According to his algorithm, we calculate the Kowalewski's indicators

$$\Delta\varrho_i = \left(1 + 8k\lambda_i/(k-2)^2\right)^{1/2},$$

where λ_i are the eigenvalues of the matrix $\Gamma = \frac{\partial^2 W}{\partial x^2}(c)$, $c \in \mathbb{C}^n$ is a nontrivial solution of the system of equations

$$\frac{\partial W}{\partial x_j}(c) = c_j, \qquad 1 \leqslant j \leqslant n. \tag{2}$$

In our case

$$W = x^k + \left(-\frac{x}{2} + \frac{y\sqrt{3}}{2}\right)^k + \left(-\frac{x}{2} - \frac{y\sqrt{3}}{2}\right)^k.$$

The solution of the system (2) is

$$c_1 = \left(2^{k-1}/k\left(1+2^{k-1}\right)\right)^{1/(k-2)}, \tag{3}$$

$$c_2 = 0. \tag{4}$$

The Kowalewski's indicators are

$$\Delta\varrho_1 = \frac{3k-2}{k-2} \in \mathbb{Q},$$

$$\Delta\varrho_2 = \left(1 + \frac{24k(k-1)}{(k-2)^2(1+2^{k-1})}\right)^{1/2}. \tag{5}$$

To show that $\Delta\varrho_2 \notin \mathbb{Q}$ consider the Diophantine equation

$$1 + \frac{24k(k-1)}{(k-2)^2(1+2^{k-1})} = \left(\frac{l}{(k-2)(1+2^{k-1})}\right)^{1/2}.$$

One can easily prove that for $k > 10$ $l \notin \mathbb{N}$, and it is easy to calculate l for $k \leqslant 10$ directly and check that l also is not natural. ■

The analogous result is established for the case of n pair-interactive particles on the line.

Theorem 3. *The n-body problem with the Hamiltonian (1) is nonintegrable for $k > 2$.*

Proof. First of all we reduce the system of n particles to the system with two degrees of freedom. Let the initial conditions of the dynamics are

$$x_1 = x_2 = \cdots = x_r = y, \tag{6}$$

$$x_{r+1} = \cdots = x_{2r} = -y, \tag{7}$$

$$x_{2r+1} = -x_{2r+2} = x, \tag{8}$$

$$\dot{x}_1 = \dot{x}_2 = \cdots = \dot{x}_r = p_y, \tag{9}$$

$$\dot{x}_{r+1} = \cdots = \dot{x}_{2r} = -p_y, \tag{10}$$

$$\dot{x}_{2r+1} = -\dot{x}_{2r+2} = p_x \tag{11}$$

for $n = 2(r+1)$, and additionally

$$x_{2r+3} = \dot{x}_{2r+3} = 0$$

for odd values of n ($n = 2(r+1)+1$). Then the reduced Hamiltonian can be written in the form

$$H = p_x^2 + rp_y^2 + W(x,y),$$

where

$$W(x,y) = 2r\left((x-y)^k + (x+y)^k\right) + (2x)^k + r^2(2y)^k$$

for $n = 2(r+1)$ and

$$W(x,y) = 2r\left((x-y)^k + (x+y)^k\right) + (2x)^k + r^2(2y)^k + 2x^k + 2ry^k$$

for $n = 2(r+1)+1$. The analogous Diophantine equations can be easily considered and one can prove that these equations do not have solutions for $k > 2$. ■

Consider now the problem of integrability of the system of $(n+1)$ interactive particles with the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{n+1} p_i^2 + \sum_{i=1}^{n} V(x_i - x_{i+1}) + \lambda V(x_{n+1} - x_1), \tag{12}$$

where λ can be equal to 0 or 1.

Theorem 4. *The system with the Hamiltonian (12) can be reduced to the system with two degrees of freedom and the Hamiltonian*

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + V(x) + \lambda V\left(\frac{x}{n} + \frac{y\sqrt{n^2-1}}{n}\right) + (n-1)V\left(\frac{x}{n} - \frac{y\sqrt{n^2-1}}{n(n-1)}\right). \tag{13}$$

Theorem 5. *The systems with the Hamiltonians (13) for* $\lambda \in \{0, 1\}$ *do not possess the additional first integral for* $k > 2$.

The proof of the Theorem 5 is based on considering the Kowalewski's indicators for the Hamiltonian (13). In this case also $\Delta\varrho_1 = \frac{3k-2}{k-2} \in \mathbb{Q}$ and it is proved that $\Delta\varrho_2 \notin \mathbb{Q}$ for values $k > 2$.

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Towards Classification of Separable Pauli Equations

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We extend our approach, used to classify separable Schrödinger equations [1], to the case of the (1+3)-dimensional Pauli equations for a spin-$\frac{1}{2}$ particle interacting with the electromagnetic field. As a result, we get eleven classes of the vector-potentials of the electromagnetic field providing separability of the corresponding Pauli equations. It is shown, in particular, that the necessary condition for the Pauli equation to be separable is that it must be equivalent to the system of two Schrödinger equations and, furthermore, the magnetic field must be independent of the spatial variables.

1 Introduction

The quantum mechanical system consisting of a spin-$\frac{1}{2}$ charged particle interacting with the electro-magnetic field is described in a non-relativistic approximation by the Pauli equation (see, e.g., [2])

$$(p_0 - p_a p_a + e\vec{\sigma}\vec{H})\psi(t,\vec{x}) = 0, \qquad (1)$$

where $\psi(t,\vec{x})$ is the two-component complex-valued function, e stands for the electric charge of particle. Here we use the notations

$$p_0 = i\frac{\partial}{\partial t} - eA_0(t,\vec{x}), \qquad p_a = -i\frac{\partial}{\partial x_a} - eA_a(t,\vec{x}), \qquad a = 1,2,3,$$

where $A = (A_0, A_1, A_2, A_3)$ is the vector-potential of the electro-magnetic field, $\vec{H} = \mathrm{rot}\,\vec{A}$ is the magnetic field, $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hereafter the summation over the repeated Latin indices from 1 to 3 is understood.

Clearly, system (1) with arbitrary functions A_0, A_a, $(a = 1,2,3)$ is not separable. On the other hand, there do exist configurations of the electro-magnetic field providing separability of the Pauli equation. So a natural question arises whether it is possible to get a systematic description of all the possible curvilinear coordinate systems and vector-potentials A such that equation (1) is integrable by the variable separation. One of the principal objectives of the present paper is to provide an efficient way for answering these kinds of questions for systems of partial differential equations (PDEs). The approach used is the further extension of the method developed in our paper [1], where the problem of separation of variables in the Schrödinger equation has been solved.

For a solution to be found we adopt the following separation Ansatz:

$$\psi(t,\vec{x}) = Q(t,\vec{x})\varphi_0(t)\prod_{a=1}^{3}\varphi_a\left(\omega_a(t,\vec{x}),\vec{\lambda}\right)\chi, \qquad (2)$$

where Q, φ_μ, $(\mu = 0,1,2,3)$ are non-singular 2×2 matrix functions of the given variables and χ is an arbitrary two-component constant column. What is more, the usual condition of commutativity of the matrices φ_μ is imposed, i.e.

$$[\varphi_\mu,\ \varphi_\nu] = \varphi_\mu\varphi_\nu - \varphi_\nu\varphi_\mu = 0, \qquad \mu,\nu = 0,1,2,3. \tag{3}$$

We say that the Pauli equation (1) is separable in a coordinate system t, $\omega_a = \omega_a(t,\vec{x})$, $(a = 1,2,3)$, if the separation Ansatz (2) reduces PDE (1) to four matrix ordinary differential equations (ODEs) for the functions φ_μ, $(\mu = 0,1,2,3)$

$$\begin{aligned} &i\varphi_0' = -\left(P_{00}(t) + P_{0b}(t)\lambda_b\right)\varphi_0,\\ &\varphi_a'' = \left(P_{a0}(\omega_a) + P_{ab}(\omega_a)\lambda_b\right)\varphi_a, \quad a = 1,2,3, \end{aligned} \tag{4}$$

where $P_{\mu\nu}$ $(\mu,\nu = 0,1,2,3)$ are some smooth 2×2 matrix functions of the given variables, λ_1, λ_2, λ_3 are separation constants and, what is more,

$$\operatorname{rank} \|P_{\mu a}\|^{3}_{\mu=0}{}^{3}_{a=1} = 6. \tag{5}$$

The condition (5) secures essential dependence of a solution with separated variables on the separation constants λ_1, λ_2, λ_3.

Next, we introduce the equivalence relation on the set of all coordinate systems providing separability of Pauli equation. We say that two coordinate systems t, ω_1, ω_2, ω_3 and $\tilde{t}$, $\tilde{\omega}_1$, $\tilde{\omega}_2$, $\tilde{\omega}_3$ are equivalent if the corresponding Ansätze (2) are transformed one into another by

- the continuous transformations from the Lie transformation group, admitted by the Pauli equation (1),
- the reversible transformations of the form

$$\begin{aligned} &t \to \tilde{t} = f_0(t), \qquad \omega_i \to \tilde{\omega}_a = f_a(\omega_a), \qquad a = 1,2,3,\\ &Q \to \tilde{Q} = Ql_0(t)l_1(\omega_1)l_2(\omega_2)l_3(\omega_3), \end{aligned} \tag{6}$$

  where $f_0,\ldots,f_3$ are some smooth functions and $l_0,\ldots,l_3$ are some smooth 2×2 matrix functions of the given variables.

This equivalence relation splits the set of all possible coordinate systems into equivalence classes. In a sequel, when presenting the lists of coordinate systems enabling us to separate variables in Pauli equation we will give only one representative for each equivalence class.

The principal steps of the procedure of variable separation in Pauli equation (1) are as follows:

1. We insert the Ansatz (2) into the Pauli equation and express the derivatives φ_0', φ_a'' in terms of the functions φ_0, φ_a, φ_a' $(a = 1,2,3)$ using equations (4).
2. We split the expression obtained by φ_0, φ_a, φ_a', λ_a $(a = 1,2,3)$ using the commutativity condition (3) and get an over-determined system of nonlinear PDEs for unknown functions A_0, A_a, Q, ω_a.
3. Integrating the obtained system yields all the possible configurations of the vector-potentials of the electro-magnetic field providing separability of the Pauli equation and the corresponding coordinate systems.

Having performed the first two steps of the above algorithm we obtain the system of nonlinear matrix PDEs

$$(i)\ \ \frac{\partial\omega_b}{\partial x_a}\frac{\partial\omega_c}{\partial x_a} = 0, \qquad b \neq c, \quad b,c = 1,2,3;$$

$$(ii)\quad \sum_{a=1}^{3} P_{ab}(\omega_a)\frac{\partial\omega_a}{\partial x_c}\frac{\partial\omega_a}{\partial x_c} = P_{0b}(t), \qquad b = 1,2,3;$$

$$(iii)\quad 2\left(\frac{\partial Q}{\partial x_b} - ieQA_b\right)\frac{\partial\omega_a}{\partial x_b} + Q\left(i\frac{\partial\omega_a}{\partial t} + \Delta\omega_a\right) = 0, \qquad a = 1,2,3;$$

$$(iv)\quad Q\sum_{a=1}^{3} P_{a0}(\omega_a)\frac{\partial\omega_a}{\partial x_b}\frac{\partial\omega_a}{\partial x_b} + i\frac{\partial Q}{\partial t} + \Delta Q - 2ieA_b\frac{\partial Q}{\partial x_b}$$
$$+\left(-P_{00}(t) - ie\frac{\partial A_b}{\partial x_b} - eA_0 - e^2A_bA_b + e\vec{\sigma}\vec{H}\right)Q = 0.$$

So the problem of variable separation in the Pauli equation reduces to integrating system of nonlinear PDEs for eight unknown functions A_0, A_1, A_2, A_3, Q, ω_1, ω_2, ω_3 of four variables t, $\vec{x}$. What is more, some coefficients are arbitrary matrix functions, which are to be determined while integrating system of PDEs (i)–(iv). We have succeeded in constructing the general solution of the latter, which yields, in particular, all the possible vector-potentials $A(t,\vec{x}) = (A_0(t,\vec{x}),\ldots,A_3(t,\vec{x}))$ such that Pauli equation (1) is solvable by the method of separation of variables. Due to the space limitations we are unable to present the full integration details, since the computations are very involved. The integration procedure is basically very much similar to that for classifying separable Schrödinger equations [1] (though the matrix case is considerably more difficult to handle). So that, we restrict ourselves to giving the list of the final results.

2 Principal results

Integration of the system PDEs (i)–(ii) yields the most general forms of coordinate systems t, $\vec{\omega}$ that provide separability of the Pauli equation. The general solution $\vec{\omega} = \vec{\omega}(t,\vec{x})$ of system of equations (i)–(ii) has been constructed in [1]. It is given implicitly within the equivalence relation (6) by the following formulae:

$$\vec{x} = \mathcal{T}(t)\mathcal{L}(t)\left(\vec{z}(\vec{\omega}) + \vec{v}(t)\right). \tag{7}$$

Here $\mathcal{T}(t)$ is the time-dependent 3×3 orthogonal matrix with the Euler angles $\alpha(t)$, $\beta(t)$, $\gamma(t)$:

$$\mathcal{T}(t) = \left(\begin{array}{lr} \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta\cos\gamma & \\ \sin\alpha\,\cos\beta + \cos\alpha\,\sin\beta\cos\gamma & \rightarrow \\ \sin\beta\,\sin\gamma & \end{array}\right.$$
$$\left.\begin{array}{lcc} & -\cos\alpha\,\sin\beta - \sin\alpha\,\cos\beta\cos\gamma & \sin\alpha\,\sin\gamma \\ \rightarrow & -\sin\alpha\,\sin\beta + \cos\alpha\,\cos\beta\cos\gamma & -\cos\alpha\,\sin\gamma \\ & \cos\beta\,\sin\gamma & \cos\gamma \end{array}\right); \tag{8}$$

$\vec{v}(t)$ stands for the vector-column whose entries $v_1(t)$, $v_2(t)$, $v_3(t)$ are arbitrary smooth functions of t; $\vec{z} = \vec{z}(\vec{\omega})$ is given by one of the eleven formulas

1. Cartesian coordinate system,
   $z_1 = \omega_1, \quad z_2 = \omega_2, \quad z_3 = \omega_3, \quad \omega_1,\omega_2,\omega_3 \in \mathbb{R}.$
2. Cylindrical coordinate system,
   $z_1 = e^{\omega_1}\cos\omega_2, \quad z_2 = e^{\omega_1}\sin\omega_2, \quad z_3 = \omega_3, \quad 0 \le \omega_2 < 2\pi, \quad \omega_1,\omega_3 \in \mathbb{R}.$
3. Parabolic cylindrical coordinate system,
   $z_1 = (\omega_1^2 - \omega_2^2)/2, \quad z_2 = \omega_1\omega_2, \quad z_3 = \omega_3, \quad \omega_1 > 0, \quad \omega_2,\omega_3 \in \mathbb{R}.$

4. Elliptic cylindrical coordinate system,
   $z_1 = a\cosh\omega_1\cos\omega_2,\quad z_2 = a\sinh\omega_1\sin\omega_2,\quad z_3 = \omega_3,$
   $\omega_1 > 0,\quad -\pi < \omega_2 \le \pi,\quad \omega_3 \in \mathbb{R},\quad a > 0.$
5. Spherical coordinate system,
   $z_1 = \omega_1^{-1}\operatorname{sech}\omega_2\cos\omega_3,\quad z_2 = \omega_1^{-1}\operatorname{sech}\omega_2\sin\omega_3,\quad z_3 = \omega_1^{-1}\tanh\omega_2,$
   $\omega_1 > 0,\quad \omega_2 \in \mathbb{R},\quad 0 \le \omega_3 < 2\pi.$
6. Prolate spheroidal coordinate system,
   $z_1 = a\operatorname{csch}\omega_1\operatorname{sech}\omega_2\cos\omega_3,\quad z_2 = a\operatorname{csch}\omega_1\operatorname{sech}\omega_2\sin\omega_3,$
   $z_3 = a\coth\omega_1\tanh\omega_2,\quad \omega_1 > 0,\quad \omega_2 \in \mathbb{R},\quad 0 \le \omega_3 < 2\pi,\quad a > 0.$ (9)
7. Oblate spheroidal coordinate system,
   $z_1 = a\csc\omega_1\operatorname{sech}\omega_2\cos\omega_3,\quad z_2 = a\csc\omega_1\operatorname{sech}\omega_2\sin\omega_3,$
   $z_3 = a\cot\omega_1\tanh\omega_2,\quad 0 < \omega_1 < \pi/2,\quad \omega_2 \in \mathbb{R},\quad 0 \le \omega_3 < 2\pi,\quad a > 0.$
8. Parabolic coordinate system,
   $z_1 = e^{\omega_1+\omega_2}\cos\omega_3,\quad z_2 = e^{\omega_1+\omega_2}\sin\omega_3,\quad z_3 = \left(e^{2\omega_1} - e^{2\omega_2}\right)/2,$
   $\omega_1, \omega_2 \in \mathbb{R},\quad 0 \le \omega_3 \le 2\pi.$
9. Paraboloidal coordinate system,
   $z_1 = 2a\cosh\omega_1\cos\omega_2\sinh\omega_3,\quad z_2 = 2a\sinh\omega_1\sin\omega_2\cosh\omega_3,$
   $z_3 = a(\cosh 2\omega_1 + \cos 2\omega_2 - \cosh 2\omega_3)/2,\quad \omega_1, \omega_3 \in \mathbb{R},\quad 0 \le \omega_2 < \pi,\quad a > 0.$
10. Ellipsoidal coordinate system,
   $$z_1 = a\,\frac{1}{\operatorname{sn}(\omega_1,\, k)}\operatorname{dn}(\omega_2,\, k')\operatorname{sn}(\omega_3,\, k),\quad z_2 = a\,\frac{\operatorname{dn}(\omega_1,\, k)}{\operatorname{sn}(\omega_1,\, k)}\operatorname{cn}(\omega_2,\, k')\operatorname{cn}(\omega_3,\, k),$$
   $$z_3 = a\,\frac{\operatorname{cn}(\omega_1,\, k)}{\operatorname{sn}(\omega_1,\, k)}\operatorname{sn}(\omega_2,\, k')\operatorname{dn}(\omega_3,\, k),$$
   $0 < \omega_1 < K,\quad -K' \le \omega_2 \le K',\quad 0 \le \omega_3 \le 4K,\quad a > 0.$
11. Conical coordinate system,
   $z_1 = \omega_1^{-1}\operatorname{dn}(\omega_2,\, k')\operatorname{sn}(\omega_3,\, k),\quad z_2 = \omega_1^{-1}\operatorname{cn}(\omega_2,\, k')\operatorname{cn}(\omega_3,\, k),$
   $z_3 = \omega_1^{-1}\operatorname{sn}(\omega_2,\, k')\operatorname{dn}(\omega_3,\, k),\quad \omega_1 > 0,\quad -K' \le \omega_2 \le K',\quad 0 \le \omega_3 \le 4K;$

and $\mathcal{L}(t)$ is the 3×3 diagonal matrix

$$\mathcal{L}(t) = \begin{pmatrix} l_1(t) & 0 & 0 \\ 0 & l_2(t) & 0 \\ 0 & 0 & l_3(t) \end{pmatrix},$$

where $l_1(t), l_2(t), l_3(t)$ are arbitrary non-zero smooth functions, that satisfy the following condition:

- $l_1(t) = l_2(t)$ for the partially split coordinate systems (cases 2–4 from (9)),
- $l_1(t) = l_2(t) = l_3(t)$ for non-split coordinate systems (cases 5–11 from (9)).

Here we use the standard notations for the trigonometric, hyperbolic and Jacobi elliptic functions, k $(0 < k < 1)$ being the module of the latter and $k' = \left(1 - k^2\right)^{1/2}$.

With this result in hand it is not difficult to integrate the remaining equations (*iii*) and (*iv*) from the system under study, since they can be regarded as the algebraic equations for the functions $A_a(t, \vec{x})$, $(a = 1, 2, 3)$ and $A_0(t, \vec{x})$, correspondingly. The principal results can be summarized as follows. The necessary condition for the Pauli equation (1) to be separable is that it is gauge equivalent to the Pauli equation with following space-like components of the vector-potential $A(t,\vec{x})$ of the electro-magnetic field

$$\vec{A}(t,\vec{x}) = \frac{1}{2}\begin{pmatrix} 0 & -H_3(t) & H_2(t) \\ H_3(t) & 0 & -H_1(t) \\ -H_2(t) & H_1(t) & 0 \end{pmatrix}\vec{x} = \frac{1}{2}\vec{H}(t)\times\vec{x}. \tag{10}$$

So that the magnetic field $\vec{H}$ is independent of the spatial variables and related to the Euler angles $\alpha(t)$, $\beta(t)$, $\gamma(t)$ of the matrix $\mathcal{T}(t)$ (8) through the following formulae:

$$\begin{aligned} eH_1 &= -\dot{\gamma}(t)\cos\alpha(t) - \dot{\beta}(t)\sin\alpha(t)\sin\gamma(t), \\ eH_2 &= -\dot{\gamma}(t)\sin\alpha(t) + \dot{\beta}(t)\cos\alpha(t)\sin\gamma(t), \\ eH_3 &= -\dot{\alpha}(t) - \dot{\beta}(t)\cos\gamma(t). \end{aligned}$$

Next, keeping in mind that $\vec{H}$ depends on t only, we make in (1) the change of variables

$$\psi = U(t)\tilde{\psi},$$

where $U(t)$ is a unitary 2×2 matrix function satisfying the matrix ODE

$$iU_t = (-e\vec{\sigma}\vec{H})U$$

with the initial condition $U(0) = I$. This transformation splits the separable Pauli equation into two Schrödinger equations, i.e. the term $e\vec{\sigma}\vec{H}$ is cancelled.

Summing up we conclude that the Pauli equation (1) admits separation of variables if and only if it is equivalent to the system of two Schrödinger equations. Moreover the space-like components A_1, A_2, A_3 of the vector-potential of the electro-magnetic field are linear in the spatial variables and given by (10).

The structure of the time-like component of the vector-potential $A(t,\vec{x})$ providing separability of Pauli equation is determined by the form of the corresponding coordinate systems $\omega_a(t,\vec{x})$, $a = 1,2,3$:

$$\begin{aligned} eA_0(t,\vec{x}) = {} & \sum_{a=1}^{3} F_{a0}(\omega_a)\frac{\partial\omega_a}{\partial x_b}\frac{\partial\omega_a}{\partial x_b} + T_0(t) - e^2 A_b A_b \\ & - \frac{1}{4}\sum_{a=1}^{3}\left(\ddot{l}_a l_a (z_a + v_a)^2 + 2l_a(l_a\ddot{v}_a + 2\dot{l}_a\dot{v}_a)(z_a+v_a) + l_a^2\dot{v}_a^2\right), \end{aligned} \tag{11}$$

where $F_{10}(\omega_1)$, $F_{20}(\omega_2)$, $F_{30}(\omega_3)$, $T_0(t)$ are arbitrary smooth functions defining the explicit form of the reduced equations (4); A_1, A_2, A_3 are given by (10); z_1, z_2, z_3 are the functions given in the list (9); and $l_a(t)$, $v_a(t)$, $\alpha(t)$, $\beta(t)$, $\gamma(t)$ are arbitrary smooth functions defining the form of the new coordinate system (7).

Thus there are eleven classes of separable Pauli equations corresponding to eleven classes of coordinate system (7). For instance, the general form of the time-like component of the vector-potential $A(t,\vec{x})$ providing separability of Pauli equation in the spherical coordinate system reads as

$$\begin{aligned} eA_0 = {} & T_0(t) + l_1^{-2}\omega_1^4 F_{10}(\omega_1) + l_1^{-2}\omega_1^2\cosh^2\omega_2\left(F_{20}(\omega_2) + F_{30}(\omega_3)\right) \\ & - e^2 A_b A_b - \frac{1}{4}\sum_{a=1}^{3}\left(\ddot{l}_1 l_1 (z_a+v_a)^2 + 2l_1(l_1\ddot{v}_a + 2\dot{l}_1\dot{v}_a)(z_a+v_a) + l_1^2\dot{v}_a^2\right), \end{aligned}$$

z_1, z_2, z_3 being given by the formulae 5 from (9).

The Pauli equation (1) for the class of functions $A_0(t,\vec{x})$, $\vec{A}(t,\vec{x})$ defined by (10), (11) under arbitrary $T_0(t)$, $F_{a0}(\omega_a)$ and arbitrarily fixed functions $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $v_a(t)$, $l_a(t)$, $a = 1,2,3$ separates in exactly one coordinate system. Properly specifying the functions $F_{\mu 0}(\omega_a)$, $\mu = 0,1,2,3$ may yield additional possibilities for variable separation in the corresponding Pauli equation. What we mean is that for some particular forms of the vector-potential $A(t,\vec{x})$ (10), (11) there might exist several coordinate systems (7) enabling to separate the corresponding Pauli equation. Note that the quantum mechanical models possessing this property are called super-integrable (see, e.g., [3]).

As an illustration, we consider the problem of separation of variables in the Pauli equation (1) for a particle moving in the constant magnetic field. Namely, we fix the following form of the vector-potential:

$$2e\vec{A} = \begin{pmatrix} 0 & -c & 0 \\ c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{x}, \qquad eA_0 = \frac{q}{|\vec{x}|} - \frac{c^2}{12}\left(x_1^2 + x_2^2 - 2x_3^2\right), \tag{12}$$

where $q = \text{const}$. As a direct check shows, this vector-potential satisfies the vacuum Maxwell equations without currents

$$\Box A_0 - \frac{\partial}{\partial t}\left(\frac{\partial A_0}{\partial t} + \operatorname{div}\vec{A}\right) = 0,$$
$$\Box \vec{A} + \operatorname{grad}\left(\frac{\partial A_0}{\partial t} + \operatorname{div}\vec{A}\right) = \vec{0},$$

where $\Box = \partial^2/\partial t^2 - \Delta$ is the d'Alembert operator. Therefore, vector-potential (12) is the natural generalization of the standard Coulomb potential, that is obtained from (12) under $c \to 0$.

The Pauli equation (1) with potential (12) separates in three coordinate systems

$$\vec{x} = \mathcal{T}(t)\vec{z},$$

where $\mathcal{T}$ is the time-dependent 3×3 orthogonal matrix (8), with the Euler angles

$$\alpha(t) = -ct, \qquad \beta = \text{const}, \qquad \gamma = \text{const}$$

and $\vec{z}$ is one of the following coordinate systems:

1) spherical (formula 5 from the list (9));
2) prolate spheroidal (formula 6 from the list (9));
3) conical (formula 11 from the list (9)).

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“Leonard Pairs” in Classical Mechanics

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We propose a concept of Leonard duality in classical mechanics. It is shown that Leonard duality leads to non-linear relations of the AW-type with respect to Poisson brackets.

1 Introduction

Let F, G, $\ldots$ be classical dynamical variables (DV) that can be represented as differentiable functions of the canonical finite-dimensional variables q_i, p_i, $i = 1, 2, \ldots, N$.

The Poisson brackets (PB) $\{F, G\}$ are defined as [1]

$$\{F, G\} = \sum_{i=1}^{N} \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}.$$

The PB satisfies fundamental properties [1]

(i) PB is a linear function in both F and G;

(ii) PB is anti-symmetric $\{F, G\} = -\{G, F\}$;

(iii) PB satisfies the Leibnitz rule $\{F_1F_2, G\} = F_1\{F_2, G\} + F_2\{F_1, G\}$;

(iv) for any dynamical variables F, G, H PB satisfies the Jacobi identity

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0.$$

PB are important in classical mechanics because they determine time dynamics: if the DV H is a Hamiltonian of the system, then for any DV G one has Poisson equation

$$\dot{G} = \{G, H\}.$$

In particular, the DV F is called an *integral* if it has zero PB with the Hamiltonian $\{F, H\} = 0$. In this case F does not depend on t.

In many problems of the classical mechanics DV form elegant algebraic structures which are closed with respect to PB.

The Poisson structures with *non-linear* PB were discussed in [9] and [6]. Sklyanin introduced [9] the so-called quadratic Poisson algebra consisting of 4 DV S_0, S_1, S_2, S_3 such that PB $\{S_i, S_k\}$ is expressed as a quadratic function of the generators S_i. The Sklyanin algebra appears quite naturally from theory of algebraic structures related to the Yang–Baxter equation in mathematical physics. Sklyanin also proposed to study general non-linear Poisson structures. Assume that there exists N dynamical variables F_i, $i = 1, 2, \ldots, N$ such that PB of these variables are closed in frames of the non-linear relations

$$\{F_i, F_k\} = \Phi_{ik}(F_1, \ldots, F_N), \qquad i, k = 1, 2, \ldots, N,$$

where $\Phi_{ik}(F_1, \ldots, F_N)$ are (nonlinear, in general) functions of N variables.

Several interesting examples of such non-linear Poisson structures are described in [6].

In [4] another example of such non-linear Poisson algebra was proposed. This example is connected with the property of "mutual integrability" and leads to the so-called classical AW-relations, where abbreviation AW means "Askey–Wilson algebra". Indeed, as was shown in [11] the operator (i.e. non-commutative) version of AW-relations has a natural representation in terms of generic Askey–Wilson polynomials, introduced in [2] (see also [7]).

In the present work we show that the property of "Leonard pairs" proposed in [5, 10] for matrices can be naturally generalized to the case of classical mechanics. For details and proofs of corresponding statements see [12].

Recall that two $N \times N$ matrices X, Y form the Leonard pair if there exists invertible matrices S and T such that the matrix $S^{-1}XS$ is diagonal whereas the matrix $S^{-1}YS$ is irreducible tri-diagonal and similarly, the matrix $T^{-1}YT$ is diagonal whereas the matrix $T^{-1}XT$ is irreducible tri-diagonal. We will call such the property "mutual tri-diagonality". Leonard showed [8] that the eigenvalue problem for a Leonard pair X, Y leads to the q-Racah polynomials (for definition see, e.g. [7]).

Terwilliger showed [5, 10] that a Leonard pair X, Y satisfies a certain algebraic relations with respect to commutators. In turn, the Terwilliger relations follow from to the so-called relations of the AW-algebra studied in [11] and [4].

We say that X and Y form a classical Leonard pair (CLP) whenever X, Y are of the form

$$X = \phi(x), \qquad Y = A_1(x)\exp(p) + A_2(x)\exp(-p) + A_3(x)$$

and

$$Y = \psi(Q), \qquad X = B_1(Q)\exp(P) + B_2(Q)\exp(-P) + B_3(Q),$$

where $\phi(x)$, $A_i(x)$, $\psi(x)$, $B_i(x)$ are some functions such that at least A_1 or A_2 are non-zero, and x, p, Q, P are some canonical variables such that

$$\{x, p\} = 1, \qquad \{Q, P\} = 1.$$

Note that the concept of the Leonard pair is closely related with the so-called "bispectrality problem" [3]. We thus arrive also at the classical analogue of the bispectral problem.

Assuming that X, Y are algebraically independent we can show that the following algebraic relations hold

$$\{Z, X\} = -1/2\, F_Y(X, Y) = -\left(Y\left(\alpha_1 X^2 + \alpha_3 X + \alpha_5\right) + \left(\alpha_2 X^2 + \alpha_6 X + \alpha_8\right)/2\right)$$

and

$$\{Y, Z\} = -1/2\, F_X(X, Y) = -\left(X\left(\alpha_1 Y^2 + \alpha_2 Y + \alpha_4\right) + \left(\alpha_3 Y^2 + \alpha_6 Y + \alpha_7\right)/2\right).$$

This is classical version of the AW-algebra introduced in [4] (i.e. Askey–Wilson algebra).

We mention also a remarkable property of the classical AW-algebra [4]. Assume that X is chosen as Hamiltonian: $H = X$. Then we have $\dot{Y} = \{Y, H\} = -Z$. Hence $\dot{Y}^2 = F(H, Y) + \alpha_9$ quadratic in Y. Hence $Y(t)$ is *elementary function* in the time t. This means that

$$Y(t) = G_1(H)\exp(\omega(H)t) + G_2(H)\exp(-\omega(H)t) + G_3(H)$$

or

$$Y(t) = G_1(H)t^2 + G_2(H)t + G_3(H),$$

where $G_i(H)$, $\omega(H)$ are some functions in the Hamiltonian H. Due to obvious symmetry between X, Y, the same property holds if one chooses Y as Hamiltonian: $H = Y$. In this case $X(t)$

behaves as elementary function in the time t. This property was called "mutual integrability" in [4]. It can be considered classical analogues of the property of "mutual tri-diagonality" [10, 5] in the "quantum" case.

It is interesting to note that when dynamics of the system is described by elliptic functions it is possible to generalize AW-algebra obtaining algebra with cubic non-linearity. We announce here the following result: Euler and Lagrange tops (which are known to be integrated in terms of Jacobi elliptic funcitons) have the same symmetry Poisson algebra with cubic non-linearity [13].

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Generalized Rayleigh–Schrödinger Perturbation Theory as a Method of Linearization of the so Called Quasi-Exactly Solvable Models

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Sextic oscillator in D dimensions is considered as a typical quasi-exactly solvable (QES) model. Usually, the QES N–plets of energies have to be computed using the nonlinear and coupled Magyari's algebraic equations. We propose and describe an alternative linear method which works with power series (in $1/\sqrt{D}$) in integer arithmetics.

1 Introduction

Sextic Hamiltonian in D dimensions

$$H = -\triangle + a\,|\vec{r}\,|^2 + b\,|\vec{r}\,|^4 + c\,|\vec{r}\,|^6, \qquad a = a(N)$$

enters many phenomenological and methodical considerations as a "next-to-solvable" model [1]. In fact, among all the real polynomial interactions, only the *harmonic and sextic* models can generate an arbitrary N–plet of bound state wavefunctions in an elementary form. All the similar models are often called quasi-exactly solvable (QES, cf. [2]).

Unfortunately, the close parallel between the sextic and harmonic oscillator is not too robust and breaks down in practical applications [3]. For example, the Rayleigh–Schrödinger unperturbed propagator ceases to be diagonal in the sextic case [4]. Moreover, the key weakness of *any* QES model lies in the nonlinearity of its secular equation which has the polynomial form of degree N [5]. Non-numerical determination of the sextic energies is only feasible at $N \le 4$. Otherwise, in a sharp contrast to harmonic case, the values of energies E_n are only available up to some rounding errors.

In order to refresh the parallels we shall describe a new approach to the sextic QES bound state problem. It is based on some surprising results of the symbolic manipulation experiments. They were performed in MAPLE using the technique of Groebner bases. We revealed that the QES energies become equidistant and proportional to integers in the limit of the large spatial dimensions $D \to \infty$. This feature is presented in Sections 2 and 3.

In the second step of our analysis one discovers that the systematic evaluation of the Rayleigh–Schrödinger corrections proves feasible in closed form. In spite of the non-diagonality of propagators, a merely slightly modified form of construction can be used. It gives the energy formula

$$E(\lambda) = E^{(0)} + \lambda\, E^{(1)} + \lambda^2\, E^{(2)} + \cdots + \lambda^K\, E^{(K)} + \mathcal{O}\left(\lambda^{K+1}\right), \qquad \lambda = 1/\sqrt{D}.$$

Its coefficients $E^{(k)}$ are obtainable *without any rounding errors* (cf. Sections 4 and 5 below).

2 An unusual solvable limit: Large dimensions D

All the sextic oscillator states are determined by the radial Schrödinger equation

$$\left[-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} + a\,r^2 + b\,r^4 + c\,r^6\right]\,\psi(r) = E\,\psi(r). \tag{1}$$

It contains the dimension D and the angular momenta $k = 0, 1, \dots$ in $\ell = k + (D-3)/2$. The elementary ansatz

$$\psi(r) = \sum_{n=0}^{\infty} h_n\, r^{2n+\ell+1} \exp\left(-\frac{1}{2}\beta\, r^2 - \frac{1}{4}\gamma\, r^4\right), \qquad c = \gamma^2 > 0, \quad b = 2\beta\gamma > 0 \tag{2}$$

converts this ordinary differential equation into the linear algebraic system characterized by the tridiagonal Hamiltonian matrix,

$$Q^{[N]}\,\vec{h} = E\,\vec{h}, \qquad Q^{[N]} = \left(\begin{array}{cccccc} B_0 & C_0 & & & & \\ A_1 & B_1 & C_1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & A_{N-2} & B_{N-2} & C_{N-2} \\ & & & A_{N-1} & B_{N-1} \end{array}\right), \tag{3}$$

where the dimension is to be infinite, $N \to \infty$, and the matrix elements are elementary,

$$\begin{array}{ll} A_n = \gamma\,(4n + 2\ell + 1) + a - \beta^2, & B_n = B_n(E) = \beta\,(4n + 2\ell + 3), \\ C_n = -2(n+1)\,(2n + 2\ell + 3), & n = 0, 1, \dots\ . \end{array} \tag{4}$$

The (quasi-)variational limit $N \to \infty$ gives the numerically correct spectrum [6]. For the sake of simplicity, let us now constrain our attention to the simplified model of Singh et al [5] characterized by the QES condition imposed upon the quadratic coupling $a = a(N)$,

$$a(N) = \frac{1}{4\gamma^2}\, b^2 - \gamma\,(4N + 2\ell + 1).$$

In this way one achieves the *rigorous* termination of the wavefunctions,

$$h_N = h_{N+1} = h_{N+2} = \cdots = 0. \tag{5}$$

The latter assumption merely changes the lower diagonal in equations (3) and (4) to the shorter formula $A_n = 4\gamma\,(n - N)$. Exact energies become available only at the first few integers $N \le 4$. Beyond $N = 4$, QES solutions remain numerical. Moreover, the intrinsic asymmetry of our Hamiltonian (3) causes a loss of precision which grows quickly with the degree N [6].

In such a setting we have noticed, purely empirically, that the solutions are getting simpler when the spatial dimensions grow, $D \gg 1$. In the leading-order approximation, the corresponding matrix Schrödinger equation becomes diagonally dominated,

$$\left(\begin{array}{ccccc} E - \beta D & 2D & & & \\ 4(N-1)\gamma & E - \beta D & 4D & & \\ & \ddots & \ddots & \ddots & \\ & & 6\gamma & E - \beta D & 2(N-1)D \\ & & & 4\gamma & E - \beta D \end{array}\right) \left(\begin{array}{c} h_0 \\ h_1 \\ \vdots \\ h_{N-2} \\ h_{N-1} \end{array}\right) = 0. \tag{6}$$

This enables us to evaluate the fully degenerate dominant eigenvalue,

$$E = \beta D - 2\sqrt{2\gamma D}\, z \tag{7}$$

where z is a constant.

3 The removal of degeneracy in sub-dominant approximation

Once we switch to the new energy variable z, we may pre-multiply equation (6) by a diagonal and regular matrix with elements ρ^j, where $\rho = \sqrt{D/(2\gamma)}$. This leads to the new, non-diagonal matrix Schrödinger equation. It determines the leading-order components of the renormalized Taylor coefficients $p_j = [D/(2\gamma)]^{j/2} h_j$ and has the following transparent form,

$$\left(\begin{array}{cccccc} 0 & 1 & & & & \\ (N-1) & 0 & 2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & 2 & 0 & (N-1) \\ & & & 1 & 0 \end{array} \right) \left(\begin{array}{c} p_0 \\ p_1 \\ \vdots \\ p_{N-2} \\ p_{N-1} \end{array} \right) = z \cdot \left(\begin{array}{c} p_0 \\ p_1 \\ \vdots \\ p_{N-2} \\ p_{N-1} \end{array} \right) . \tag{8}$$

In spite of the manifest asymmetry of this equation, all its eigenvalues remain strictly real. We computed these eigenvalues by symbolic manipulations in integer arithmetics and discovered that the underlying nonlinear secular equation is solvable exactly and completely. The N–plets of its energy roots proved nondegenerate, equidistant and extremely elementary,

$$(z_1, z_2, z_3, \dots, z_{N-1}, z_N) = (-N+1, -N+3, -N+5, \dots, N-3, N-1) . \tag{9}$$

This result is valid *at an arbitrary finite matrix size* N.

It is quite elementary to verify that also the respective left and right eigenvectors remain real. Up to their norm, all of them can be represented in terms of integers. Their components may be arranged in the rows and columns of certain square matrices,

$$P(0) = 1, \qquad P(1) = \frac{1}{\sqrt{2}} \left(\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right),$$

$$P(2) = \frac{1}{\sqrt{4}} \left(\begin{array}{ccc} 1 & 1 & 1 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{array} \right), \qquad P(3) = \frac{1}{\sqrt{8}} \left(\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 3 & 1 & -1 & -3 \\ 3 & -1 & -1 & 3 \\ 1 & -1 & 1 & -1 \end{array} \right),$$

$$P(4) = \frac{1}{\sqrt{16}} \left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 0 & -2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & -2 & 0 & 2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{array} \right)$$

etc. These matrices $P = P(N-1)$ are all asymmetric but idempotent, $P^2 = I$.

We may summarize that in the limit $D \to \infty$, the QES sextic model may be factorized easily. After a suitable normalization, all the components of the eigenvectors are integers.

4 An adapted Rayleigh–Schrödinger perturbation recipe

At the finite values of D and *starting directly from the second-order precision of the preceding section*, the routine perturbation theory becomes applicable since the unperturbed Hamiltonian remains diagonal and all its spectrum is safely non-degenerate.

At any $D \gg 0$ the Schrödinger equation (3) is an eigenvalue problem with the perturbed Hamiltonian of the two-term form,

$$H(\lambda) = H^{(0)} + \lambda H^{(1)} + \lambda^2 H^{(2)}, \qquad \lambda = 1/\sqrt{D}.$$

Both the perturbations are one-diagonal matrices which depend on the value of the angular momentum k,

$$\left(H^{(1)}\right)_{nn} = \frac{\beta}{\sqrt{2\gamma}}(2n+k), \qquad n = 0,1,\ldots,N-1,$$
$$\left(H^{(2)}\right)_{nn+1} = -(n+1)(2n+2k), \qquad n = 0,1,\ldots,N-2.$$

We may re-write our Schrödinger equation (3) in the textbook perturbation-series representation at any N,

$$\left(H^{(0)} + \lambda\, H^{(1)} + \lambda^2 H^{(2)}\right) \cdot \left(\psi^{(0)} + \lambda\,\psi^{(1)} + \cdots + \lambda^K \psi^{(K)} + \mathcal{O}\left(\lambda^{K+1}\right)\right)$$
$$= \left(\psi^{(0)} + \cdots + \lambda^K \psi^{(K)} + \mathcal{O}\left(\lambda^{K+1}\right)\right) \cdot \left(\varepsilon^{(0)} + \cdots + \lambda^K \varepsilon^{(K)} + \mathcal{O}\left(\lambda^{K+1}\right)\right). \tag{10}$$

Let us again concatenate the (lower-case) zero-order vectors $\vec{p} = \vec{p}^{(0)} \equiv \psi^{(0)}$ into an N by N matrix $P = P^{(0)}$, with all the eigenvalues arranged also in a diagonal matrix $\varepsilon^{(0)}$. In this way the zero-order equation $H^{(0)}\psi^{(0)} = \psi^{(0)}\varepsilon^{(0)}$ is satisfied identically. Indeed, in our compactified notation, it reads $P\varepsilon^{(0)}PP = P\varepsilon^{(0)}$ and we know that $P^2 = I$.

With the factorized $H^{(0)} = P\varepsilon^{(0)}P$, we shall use the same convention in all orders and concatenate the vectors $\vec{\psi}_j^{(k)}$, $j = 1,2,\ldots,N$ in the square matrix $\Psi^{(k)}$. In the first order of perturbation analysis this replaces the $\mathcal{O}(\lambda)$ part of equation (10) by the matrix relation

$$\varepsilon^{(1)} + P\,\Psi^{(1)}\varepsilon^{(0)} - \varepsilon^{(0)}P\,\Psi^{(1)} = P\,H^{(1)}\,P. \tag{11}$$

In the second order we get

$$\varepsilon^{(2)} + P\,\Psi^{(2)}\varepsilon^{(0)} - \varepsilon^{(0)}P\,\Psi^{(2)} = P\,H^{(2)}\,P + P\,H^{(1)}\,\Psi^{(1)} - P\,\Psi^{(1)}\varepsilon^{(1)} \tag{12}$$

etc. The available expressions occur on the right-hand side of these equations while the unknown quantities stand to the left. All the higher-order formulae have the same structure.

We may summarize that the diagonal part of equations (11) or (12) determines the energy corrections $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$, respectively. Non-diagonal components of these matrix relations are to be understood as a definition of the eigenvectors.

5 Merits of the method: an $N = 2$ illustration

One has to move up to the higher-order level for the elimination of the normalization ambiguities. This has been multiply clarified in the literature on perturbation theory [7]. Still, we should emphasize a user-friendliness of this normalization freedom within the framework of the present formalism. For illustration, let us consider just the s–wave problem in the $N = 2$ case. Immediately, our first-order formulae give the two energy corrections which are both equal to each other,

$$\varepsilon_{11}^{(1)} = \varepsilon_{22}^{(1)} = \beta/\sqrt{2\gamma}. \tag{13}$$

One discovers that the $\mathcal{O}(\lambda)$ level of precision provides just an incomplete information about the norms of the first-order wave functions. This is the well known normalization freedom manifesting itself in the present setting. On the $\mathcal{O}(\lambda)$ level of precision only two constraints $\Psi_{11}^{(1)} - \Psi_{21}^{(1)} = -\beta/\sqrt{2\gamma}$ and $\Psi_{12}^{(1)} + \Psi_{22}^{(1)} = \beta/\sqrt{2\gamma}$ are imposed upon the wavefunctions. Their definition must be completed in the subsequent order.

In any higher order computation, the use of the computerized symbolic manipulations is strongly recommended. Their implementation is trivial. The algorithm can be written in integer mathematics and generates, therefore, the perturbation series without any errors. This is our most important conclusion. One generalizes immediately the above leading-order results (7), (9) and (13) to the compact energy series for our particular sextic $k = N - 2 = 0$ illustration,

$$E_{1,2} = \frac{\beta}{\lambda^2} \pm \frac{2\sqrt{2\gamma}}{\lambda} + 2\beta \pm \frac{\beta^2}{\sqrt{2\gamma}}\,\lambda + 0\cdot\lambda^2 \mp \frac{\beta^4}{8\gamma\sqrt{2\gamma}}\,\lambda^3 + 0\cdot\lambda^4 + \mathcal{O}\left(\lambda^5\right). \tag{14}$$

One can observe the (complete) leading-order degeneracy of Section 2 as well as its immediate next-order removal (9) as discussed in Section 3. It is also amusing to notice the above, hand-evaluated and quite unexpected, degeneracy of the subsequent $\mathcal{O}(1)$ correction.

One can notice the existence of certain identically vanishing corrections here. In fact, their rigorous evaluation would not be possible within the standard framework of perturbation theory where the summations over the intermediate states must be computed in finite precision. Only within the present formalism which is able to work in integer arithmetics, the unusual feasibility of proving the *precise cancellation* of the series of corrections can be achieved. This is one of the less expected though most important merits of our present methodical proposal and construction.

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Conclusion

Any conference is definitively over only when editing the last volume of Proceedings is complete. And only now we can say that the Conference has finished. We hope that we did everything possible to make the Conference happen.

We would like to thank the Director of the Institute of Mathematics Academician A.M. Samoilenko and the staff of the Institute, without help of whom the conference would be impossible. We are indebted to the members of the Organizing Committee and especially to the members of the Local Organizing Committee for their enormous and devoted work that made the Conference successful.

Finally, we would like to express our sincere gratitude to all participants of the Fourth Conference "Symmetry in Nonlinear Mathematical Physics". We were happy to welcome all of them in Kyiv, especially those who participated in our previous meetings as well. We have awarded all our colleagues who participated in three of our conferences (first of all, Prof. B.K. Harrison who opened all four of our meetings) by memorable medals in order to distinguish those knights of Symmetry. These are Profs. M.B. Abd-el-Malek, P. Basarab-Horwath, J. Beckers, L. Berkovich, G. Goldin, B.K. Harrison, W. Klink, G. Svetlichny, M. Tajiri.

Unfortunately, not all participants were able to present their papers for publication in the Proceedings. Here are the titles of the talks given at the Conference that were not submitted to the Proceedings

O. Batsula, "Symmetry and dimensionality of space-time"

P. Casati, "A Lie-algebraic reduction scheme for the BKP, CKP, DKP hierarchies"

E. Corrigan, "Boundaries and boundary bound states in integrable quantum field theory"

E. Djeldubaev, "About identities in algebras generated by idempotents"

R. Golovnya, "On the problem of interaction of the particle of arbitrary spin"

J. Kubarski, "Representations of Lie algebroids and characteristic classes"

V. Lehenkyi, "Symmetries of terminal control problems and recursive principle of control"

S.-B. Liao, "Optimizing the renormalization group flow"

A. Lopatin, "Invariant criteria for existence of periodical solutions in linear systems"

N. Nekhoroshev, "Generalizations of Gordon's theorem"

A. Okninski, "Exactly linearizable maps and $SU(4)$ coherent states"

V. Ostrovskyi, "On centered one-parameter semigroups and representations of double commutator"

B. Palamarchuk, "Hidden symmetries of strong blast"

M. Pavlov, "Multi-Lagrangian representations for integrable systems, local and nonlocal Hamiltonian structures"

V. Repeta, "Construction of exact solutions of some system of differential equations"

M. Rosso, "Quantum groups and combinatorics on words"

S. Samokhvalov, "The infinite deformed groups of symmetries of the gauge theories"

V. Stogniy, "Symmetry properties and exact solutions of two-dimensional Fokker–Plank equations with homogeneous coefficients of drift and diffusion"

A. Stolin, "q-deformed power function over q-commuting variables and Drinfeld's problem of quantization of Lie bialgebras"

V. Tretyak, "Motion of interacting charged particles with accounting radiation reaction in weakly relativistic approximation"

M. Vybornov, "Affine Lie algebras, quiver varieties, and algebras of BPS states"

I. Yuryk, "Classification of maximal subalgebras of rank $n-1$ of the conformal algebra $AC(1,n)$"

O. Zaslavskii, "Quasi-exactly solvable Bose Hamiltonians"

R. Zhdanov, "Dunkl operator formalism and new solvable spin Calogero–Sutherland models"

K. Zheltukhin, "Construction of recursion operator for some equations of hydrodynamic type"

To conclude, we would like to reiterate our deep gratitude to all participants of the Conference "Symmetry in Nonlinear Mathematical Physics". And we invite everybody to participate in the next conference in these series planned for July 2003.

Anatoly NIKITIN,
March, 2002.

Наукове видання

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