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“Leonard Pairs” in Classical Mechanics

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We propose a concept of Leonard duality in classical mechanics. It is shown that Leonard duality leads to non-linear relations of the AW-type with respect to Poisson brackets.

1 Introduction

Let F, G, $\ldots$ be classical dynamical variables (DV) that can be represented as differentiable functions of the canonical finite-dimensional variables q_i, p_i, $i = 1, 2, \ldots, N$.

The Poisson brackets (PB) $\{F, G\}$ are defined as [1]

$$\{F,G\} = \sum_{i=1}^{N} \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}.$$

The PB satisfies fundamental properties [1]

(i) PB is a linear function in both F and G;

(ii) PB is anti-symmetric $\{F, G\} = -\{G, F\}$;

(iii) PB satisfies the Leibnitz rule $\{F_1F_2, G\} = F_1\{F_2, G\} + F_2\{F_1, G\}$;

(iv) for any dynamical variables F, G, H PB satisfies the Jacobi identity

$$\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0.$$

PB are important in classical mechanics because they determine time dynamics: if the DV H is a Hamiltonian of the system, then for any DV G one has Poisson equation

$$\dot{G} = \{G, H\}.$$

In particular, the DV F is called an *integral* if it has zero PB with the Hamiltonian $\{F, H\} = 0$. In this case F does not depend on t.

In many problems of the classical mechanics DV form elegant algebraic structures which are closed with respect to PB.

The Poisson structures with *non-linear* PB were discussed in [9] and [6]. Sklyanin introduced [9] the so-called quadratic Poisson algebra consisting of 4 DV S_0, S_1, S_2, S_3 such that PB $\{S_i, S_k\}$ is expressed as a quadratic function of the generators S_i. The Sklyanin algebra appears quite naturally from theory of algebraic structures related to the Yang–Baxter equation in mathematical physics. Sklyanin also proposed to study general non-linear Poisson structures. Assume that there exists N dynamical variables F_i, $i = 1, 2, \ldots, N$ such that PB of these variables are closed in frames of the non-linear relations

$$\{F_i, F_k\} = \Phi_{ik}(F_1, \ldots, F_N), \qquad i, k = 1, 2, \ldots, N,$$

where $\Phi_{ik}(F_1, \ldots, F_N)$ are (nonlinear, in general) functions of N variables.

Several interesting examples of such non-linear Poisson structures are described in [6].

In [4] another example of such non-linear Poisson algebra was proposed. This example is connected with the property of "mutual integrability" and leads to the so-called classical AW-relations, where abbreviation AW means "Askey–Wilson algebra". Indeed, as was shown in [11] the operator (i.e. non-commutative) version of AW-relations has a natural representation in terms of generic Askey–Wilson polynomials, introduced in [2] (see also [7]).

In the present work we show that the property of "Leonard pairs" proposed in [5, 10] for matrices can be naturally generalized to the case of classical mechanics. For details and proofs of corresponding statements see [12].

Recall that two $N \times N$ matrices X, Y form the Leonard pair if there exists invertible matrices S and T such that the matrix $S^{-1}XS$ is diagonal whereas the matrix $S^{-1}YS$ is irreducible tri-diagonal and similarly, the matrix $T^{-1}YT$ is diagonal whereas the matrix $T^{-1}XT$ is irreducible tri-diagonal. We will call such the property "mutual tri-diagonality". Leonard showed [8] that the eigenvalue problem for a Leonard pair X, Y leads to the q-Racah polynomials (for definition see, e.g. [7]).

Terwilliger showed [5, 10] that a Leonard pair X, Y satisfies a certain algebraic relations with respect to commutators. In turn, the Terwilliger relations follow from to the so-called relations of the AW-algebra studied in [11] and [4].

We say that X and Y form a classical Leonard pair (CLP) whenever X, Y are of the form

$$X = \phi(x), \qquad Y = A_1(x)\exp(p) + A_2(x)\exp(-p) + A_3(x)$$

and

$$Y = \psi(Q), \qquad X = B_1(Q)\exp(P) + B_2(Q)\exp(-P) + B_3(Q),$$

where $\phi(x)$, $A_i(x)$, $\psi(x)$, $B_i(x)$ are some functions such that at least A_1 or A_2 are non-zero, and x, p, Q, P are some canonical variables such that

$$\{x,p\} = 1, \qquad \{Q,P\} = 1.$$

Note that the concept of the Leonard pair is closely related with the so-called "bispectrality problem" [3]. We thus arrive also at the classical analogue of the bispectral problem.

Assuming that X, Y are algebraically independent we can show that the following algebraic relations hold

$$\{Z,X\} = -1/2\, F_Y(X,Y) = -\left(Y\left(\alpha_1 X^2 + \alpha_3 X + \alpha_5\right) + \left(\alpha_2 X^2 + \alpha_6 X + \alpha_8\right)/2\right)$$

and

$$\{Y,Z\} = -1/2\, F_X(X,Y) = -\left(X\left(\alpha_1 Y^2 + \alpha_2 Y + \alpha_4\right) + \left(\alpha_3 Y^2 + \alpha_6 Y + \alpha_7\right)/2\right).$$

This is classical version of the AW-algebra introduced in [4] (i.e. Askey–Wilson algebra).

We mention also a remarkable property of the classical AW-algebra [4]. Assume that X is chosen as Hamiltonian: $H = X$. Then we have $\dot{Y} = \{Y,H\} = -Z$. Hence $\dot{Y}^2 = F(H,Y) + \alpha_9$ quadratic in Y. Hence $Y(t)$ is *elementary function* in the time t. This means that

$$Y(t) = G_1(H)\exp(\omega(H)t) + G_2(H)\exp(-\omega(H)t) + G_3(H)$$

or

$$Y(t) = G_1(H)t^2 + G_2(H)t + G_3(H),$$

where $G_i(H)$, $\omega(H)$ are some functions in the Hamiltonian H. Due to obvious symmetry between X, Y, the same property holds if one chooses Y as Hamiltonian: $H = Y$. In this case $X(t)$

behaves as elementary function in the time t. This property was called "mutual integrability" in [4]. It can be considered classical analogues of the property of "mutual tri-diagonality" [10, 5] in the "quantum" case.

It is interesting to note that when dynamics of the system is described by elliptic functions it is possible to generalize AW-algebra obtaining algebra with cubic non-linearity. We announce here the following result: Euler and Lagrange tops (which are known to be integrated in terms of Jacobi elliptic funcitons) have the same symmetry Poisson algebra with cubic non-linearity [13].

[1] Arnold V.I., Mathematical methods of classical mechanics, Moscow, Nauka, 1989 (in Russian).

[2] Askey R. and Wilson J., Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials, *Mem. Amer. Math. Soc.*, 1985, V.54, N 319, 1–55.

[3] Duistermaat J.J. and Grünbaum F.A., Differential equations in the spectral parameters, *Commun. Math. Phys.*, 1986, V.103, 177–240.

[4] Granovskii Ya.I., Lutzenko I.M. and Zhedanov A.S., Mutual integrability, quadratic algebras, and dynamical symmetry, *Ann. Physics*, 1992, V.217, 1–20.

[5] Ito T., Tanabe K. and Terwilliger P., Some algebra related to P- and Q-polynomial association schemes, Preprint, 1999.

[6] Karasev M.V. and Maslov V.P., Nonlinear Poisson brackets. Geometry and quantization, Moscow, Nauka, 1991 (in Russian).

[7] Koekoek R. and Swarttouw R.F., The Askey scheme of hypergeometric orthogonal polynomials and its q-analogue, Report 94-05, Faculty of Technical Mathematics and Informatics, Delft University of Technology, 1994.

[8] Leonard D.A., Orthogonal polynomials, duality and association schemes, *SIAM J. Math. Anal.*, 1982, V.13, 656–663.

[9] Sklyanin E.K., On some algebraic structures connected with the Yang-Baxter equation, *Funkc. Anal. i ego Prilozh.*, 1982, V.16, N 4, 27–34 (in Russian).

[10] Terwilliger P., Two linear transformations each tridiagonal with respect to an eigenbasis of the other, Preprint, 1999.

[11] Zhedanov A.S., "Hidden symmetry" of the Askey–Wilson polynomials, *Teoret. Mat. Fiz.*, 1991, V.89, 190–204 (in Russian).

[12] Korovnichenko A.E. and Zhedanov A.S., Dual algebras with non-linear Poisson brackets, in Integrable Structures of Exactly Solvable Two-Dimensional Models of Quantum Field Theory, Editors S. Pakuliak and G. von Gehlen, NATO Sci. Series, Kluwer, 2001, 265–272.

[13] Zhedanov A.S. and Korovnichenko A.E., "Leonard pairs" in classical mechanics, to appear.